Nonlinear Dynamics and Models A course in Xi'an Jiaotong University

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by

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Nonlinear dynamics is developed in three different directions. Geometric qualitative theory of phase portraits, analysis (normal form, averaging, perturbative approach, asymptotics) and numerical simulations. This course is of basic level for undergraduate students who followed fundamental courses of linear algebra, differential calculus, topology and provides them an introduction to nonlinear dynamics. Nonlinear dynamics applications are ubiquitous in many sciences including physics, mechanics, biology and economics. In return, this field has received important feedback from modeling. We also include in this course presentations of such models.

1. Geometric qualitative theory of phase portraits

1.1. Continuous dynamics on a metric space.

1.1.1. *Definition of a continuous dynamics, invariant sets.*

Definition 1. *Let E be a metric space and $f : \mathbb{R} \times E \to E$, $f : (t,p) \mapsto f(t,p)$ a continuous map. The map f defines a topological continuous dynamical system if:*

(i) $f(0,p) = p$

(ii) $f(t_1, f(t_2,p)) = f(t_1 + t_2, p)$.

The variable t is called the time. The set of points

$$\{f(t,p), t \in \mathbb{R}\} = f(\mathbb{R}, p)$$

is the orbit or the trajectory of the point p. The set

$$\{f(t,p), t \in \mathbb{R}_+\},$$

respectively

$$\{f(t,p), t \in \mathbb{R}_-\},$$

is the positive (resp. negative) orbit of p. The set of points

$$\{f(t,p), T_1 < t < T_2, \} = f(]T_1, T_2[, p)$$

is an arc of the orbit of p

Definition 2. *A subset $S \subset X$ is said to be invariant by the dynamical system if for all $p \in S$, $f(t,p) \in S$, for all $t \in \mathbb{R}$.*

Definition 3. *A subset Σ is said to be a minimal invariant set if it is non-empty, closed and invariant and if it does not strictly contain any subset with the same properties.*

1.1.2. *Lagrange stability, Poisson stability, Lyapunov stability, non-wandering points.*

Definition 4. *A solution $f(t, p)$ is positively (resp. negatively) stable in Lagrange sense if the closure of $f([0, +\infty[, p)$ (resp. of $f(] - \infty, 0], p)$) is compact. It is said to be stable in Lagrange sense if the closure of the orbit $f(R, p)$ is compact.*

Definition 5. *A point p is positively stable in Poisson sense if for all neighborhood U of p, and for all $T > 0$, there exists a $t \geq T$ so that $f(t, p) \in U$.*

Definition 6. *A point p is stable (positively stable, negatively stable) in Lyapunov's sense relatively to B if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $q \in B$ so that $d(p, q) < \delta$: $d(f(t, p), f(t, q)) < \epsilon$, for all t, resp. $t \geq 0$, $t < 0$.*

Definition 7. *A point p is said to be wandering if there exists a neighborhood U of p and a positif number T so that for all $t \geq T$, $f(t, U) \cap U = \emptyset$.*

The set of wandering points is invariant and open. Its complement called the set of non-wandering points is a closed invariant set denoted M_1.

A non-wandering point is thus characterized by the fact that for any neighborhood U of the point, there exists values of t arbitrarily large so that $U \cap f(t, U) \neq \emptyset$. For instance, a stable point in Poisson sense is non-wandering. Reciprocally, a non-wandering point is not necessarily stable.

Consider a compact metric space E on which a continuous dynamics is defined. Let M_1 be the set of non-wandering points. Consider the restriction of the dynamics to M_1. This restricted dynamics displays a set of non-wandering points M_2. In general, we can associate to the dynamical system an infinite sequence of sets:

$$M_1 \subset M_2 \subset ... \subset M_n \subset ...$$

1.1.3. *Limit points.* Let $f(t,p)$ be a continuous dynamical system defined on a metric space E. Let $f(R+,p)$ (resp. $f(R-,p)$ be the positive (resp negative) orbit of the point p that we assume to be relatively compact.

Definition 8. *Let* $0 \leq t_1 < t_2 < ... < t_n < ...,$ *be an increasing non-bounded sequence of values of* t, $\lim_{n\to+\infty} t_n = +\infty$. *The* ω*-limit set of a point* p *is the set of points* q *so that there exists an increasing non-bounded sequence* $0 \leq t_1 < t_2 < ... < t_n < ..., \lim_{n\to+\infty} t_n = +\infty$ *so that* $q = \lim f(t_n,p), \lim_{n\to+\infty}$. *The* α*-limit set of* p *is defined similarly by changing* t *into* $-t$.

Lemma 1. *If* p' *belongs to the trajectory of* p *then the* ω *and* α*-limit sets of* p' *are the same as those of* p.

This lemma whose easy proof is left to the reader justifies the notation $\omega(\gamma)$ and $\alpha(\gamma)$ for the ω or α-set of any point of a trajectory γ.

Proposition 2. *The sets* $\omega(\gamma)$ *et* $\alpha(\gamma)$ *are invariant closed sets contained in the closure of* γ.

Proof. Let q_n be a sequence of points in $\omega(\gamma)$ which converges to q. there is a sequence t^n_m so that : $f(t^n_m,p) \to q_n$. Choose $m(n)$ so that $t_n = t^n_{m(n)} > n$ et $d(f(t_n,p),q_n) < \frac{1}{n}$. This implies that $d(f(t_n,p),q) \to 0$ and $q \in \omega(\gamma)$. The invariance by f is obvious. $\square$

Theorem 3. *Let* $f(t,p)$ *be a continuous dynamical system defined on a metric space* E. *Assume that the orbit* γ *is relatively compact. Then* $\omega(\gamma)$ *(resp.* $\alpha(\gamma)$ *is non-empty, compact and connected.*

Proof. Let t_n be a real sequence which tends to infinity and $p \in \gamma$. As the sequence $f(t_n, p)$ is contained in a compact, there exists a convergent subsequence. Let q be the limit of this subsequence, as $q \in \omega(p)$ it follows that $\omega(\gamma)$ is not empty. The set $\omega(\gamma)$ is closed and contained in a compact, hence it is compact. Assume it would not be connex and that there would exist two disjoint closed sets A and B so that $\omega(\gamma) = A \cup B$. Let $d = d(A, B) > 0$. There exists a sequence $t'_n \to +\infty$ so that $f(t'_n, p) \to a \in A$ and another sequence $t"_n \to +\infty$ so that $\phi(t"_n, p) \to b \in B$. Let us consider a new sequence t_n whose terms of even order satisfy $d(f(t_n, p), A) < d/2$ and terms of odd order $d(f(t_n, p), A) > d/2$. The function $\phi(t) = d(f(t, p), A)$ is continuous on the interval (t_n, t_{n+1}) and on this interval it takes values above and below $d/2$. Bu the intermediate value theorem, there is τ_n so that $d(f(\tau_n, p), A) = d/2$. Extraction of a subsequence $f(\tau_n, p)$ yields a convergent sequence of limit $q*$. THis limit point is such that $q* \in \omega(p)$, $d(q*, A) = d/2$ and $d(q*, B) \geq d(A, B) - d(q*, A) = d/2$. It follows that Il s'ensuit que $q*$ does not belong neither to A nor to B, which is a contradiction. □

1.1.4. *Recurrent solutions in Birkhoff's sense.*

Definition 9. *A trajectory $u(t) = f(t, p)$ is said periodic if it is not constant and there is a $T > 0$ such that $u(t + T) = u(t)$.*

Proposition 4. *A relatively compact trajectory is periodic if and only if $u(R+)$ (resp. $u(R-)$) is closed.*

Proof. Obviously if u is periodic, $u(R+) = u([0, T])$ is closed. Conversely if $u(R+)$ is closed, then $\omega(p) \subset u(R+)$. Since $\omega(p) \neq \emptyset$, $u(R+) \cap \omega(p) \neq \emptyset$, hence $u(R) \subset$

$\omega(p)$. Thus $u(-1) \in \omega(p) \subset u(R+)$ and there is a $\tau \in R+$ so that $u(-1) = u(\tau)$, then $u(t+1+\tau) = u(t)$. $\square$

Definition 10. *A point p is said to be recurrent if $p \in \omega(p)$. The set of recurrent point is noted* R.

Lemma 5. *If $p \in$ R, then $f(t,p) \in$ R.*

Definition 11. *Solution $u(t) = f(t,p)$ is said to be recurrent if $p \in \omega(p)$.*

Observe that a periodic orbit is recurrent.

Lemma 6. *If $u(t)$ is recurrent then there is a sequence $t_n \to +\infty$ such that $u(t+t_n) \to u(t)$.*

Proposition 7. *For all $p \in E$ whose orbit is relatively compact, $\omega(p) \cap \mathbf{R}$ contains a non-empty compact minimal invariant set K.*

Proof. Consider $\omega(p)$, which is non-empty. By Zorn's lemma, it contains a minimal (for the inclusion) compact invariant set K. For every $y \in K$, $\omega(y) \subset K$. As K is minimal, $\omega(y) = K$, hence $y \in \omega(y)$ is recurrent. So $K \subset \omega(p) \cap \mathbf{R}$ is compact and invariant. $\square$

Define the set of translated of a solution $u(t) = f(t,p)$ as the subset $\Gamma = \{t \mapsto u(t+a), a \in \mathbb{R}\}$ of $C(\mathbb{R}, E)$.

Definition 12. *The solution $u(t)$ is almost-periodic if the set Γ is equicontinuous in $C(\mathbb{R}, E)$.*

Note that the family Γ is equicontinuous and bounded, hence the Ascoli's theorem implies that it is precompact.

Proposition 8. *A almost-periodic solution $u(t)$ is recurrent.*

Proof. Consider the sequence of translated $u(t+n)$, accordingly to Ascoli's theorem, there is a subsequence $u(t+n_k)$ which is convergent. Let $u(t+a)$ be its limit. Then the sequence $u(t+n_k-a)$ converges to $u(t)$. So that $u(t)$ is recurrent. □

1.1.5. *Recurrence theorem of Poincaré-Carathéodory.* In this part, we focus on $X \subset E = \mathbb{R}^n$ compact and assume defined a probability measure μ on X which is preserved by a continuous dynamical system $f(t,p)$ defined on X.

Theorem 9. *For almost all $p \in X$, the trajectory $f(t,p)$ is recurrent. More precisely, there is a set $A \subset X$ of full measure $\mu(A) = 1$ and a sequence $m_k \to +\infty$ so that for all $x \in A$, $u(t) = f(t,x)$ is such that* $\lim u(t+m_k) = u(t)$.

Proof. Take any $\epsilon > 0$ and set

$$A_\epsilon = \{x \in X, \quad \text{for} \quad \text{all} \quad \mathrm{k} \in \mathbb{N}, \mathrm{d(f(k,x),x)} \geq \epsilon\}. \tag{1}$$

Assume that $\mu(A_\epsilon) > 0$, then there exists $x_0 \in X$ such that $\mu(C) > 0$, $C = A_\epsilon \cap \overline{B}(x_0, \frac{\epsilon}{3})$. Write $C_n = f(n, C)$, for all $n \in \mathbb{N}$, $\mu(C_n) = \mu(C) > 0$. Assume $C_p \cap C_m \neq \emptyset, p < m$, then set $\xi \in C \cap C_{m-p}$. This yields $\xi = f(m-p, \eta), d(\eta, x) \leq \epsilon/3, d(\xi, x) \leq \epsilon/3$, $d(\xi, \eta) \geq \epsilon > 2\epsilon/3$. Contradiction shows that $C_p \cap C_m = \emptyset$ for all $p \neq m$. Thus this yields

$$\mu(X) \geq \Sigma_n \mu(C_n) = +\infty. \tag{2}$$

This last contradiction displays $\mu(A_\epsilon) = 0$ for all ϵ.

Finally, set $A = \cup A_{1/k}$. This displays $\mu(A) = 0$ and if $x \in X - A$, for every k, there exists m_k such that $d(f(m_k, x), x) \leq 1/k$. □

1.1.6. *Absorbing sets and attractors.*

Definition 13. *Let $f : E \times \mathbb{R} \to E$ be a continuous dynamical system defined on a metric space E. Let B be a bounded set of E and U an open neighborhood of B. The set B is called an absorbing bounded set of U if the orbit of any point of any bounded set of U enters B after some time (which depends on the bounded set).*

Definition 14. *A set A is an attractor of the dynamical system if A is the ω-limit set of one of its neighborhoods U. The basin of attraction of A is the reunion of all the open sets U so that $\omega(U) = A$.*

Theorem 10. *Let U be an open set and B a bounded subset. Assume there is t_0 so that $\cup_{t \geq t_0} f(t, B)$ is relatively compact in E, then $A = \omega(B)$ is a compact attractor in U. If E is a Banach space, U convex and connex, then A is connex.*

1.2. **Differentiable vector fields, stationary points and periodic orbits.**

1.2.1. *Fundamental theorems of differential equations.*

Theorem 11. *Let E be a Banach space, $\mathbb{R} \times D \subset E$ open, $(x_0, t_0) \in D$, $a, b \in \mathbb{R}$, $B =\mid t - t_0 \mid \leq a, \mid\mid x - x_0 \mid\mid \leq b$. Assume that f is defined, continuous on D and Lipschitz in x of constant k on B. Let:*

$$M = \mathrm{Max}_{(t,x) \in B} \mid\mid f(x, t) \mid\mid,$$

$$A = \mathrm{Min}(a, \frac{b}{M}).$$

The differential equation

$$x' = f(t, x),$$

displays a unique solution $x(t,t_0,x_0)$ *on* $[t_0-A,t_0+A]$ *so that* $x(t_0,t_0,x_0)=x_0$. *Furthermore, this solution satisfies*

$$\| x(t,t_0,x_0)-x_0 \| \leq MA$$

for all $t\in[t_0-A,t_0+A]$.

Proof. Note that the solution writes

$$x(t,t_0,x_0)=x_0+\int_{t_0}^{t} f(s,x(s,t_0,x_0)ds. \tag{3}$$

Consider the sequence of functions:

$$\begin{array}{l} x_0(t)=x_0, \\ x_1(t)=x_0+\int_{t_0}^{t} f(s,x_0(s))ds, \\ \ldots \\ x_{m+1}(t)=x_0+\int_{t_0}^{t} f(s,x_m(s))ds, \ldots \end{array} \tag{4}$$

We prove that this sequence converges uniformly on $[t_0-A,t_0+A]$. Indeed it is more convenient to proceed first on $[t_0,t_0+A]$ and extends to the whole interval. We check that $x_m(t)$ is defined for all $m>0$ and yields

$$\mid x_m(t)-x_0 \mid \leq M \mid t-t_0 \mid . \tag{5}$$

This is obvious for $m=0$. Assume this is true untill $m=q$, then

$$\mid x_q(t)-x_0 \mid \leq MA \leq b, \tag{6}$$

hence:

$$x_{q+1}(t)=x_0+\int_{t_0}^{t} f(s,x_q(s))ds \tag{7}$$

is defined, continuous and so that

$$\mid x_{q+1}(t)-x_0 \mid \leq M \mid t-t_0 \mid \leq MA. \tag{8}$$

Consider then

$$\begin{array}{l} d_m(t)=\mid x_{m+1}(t)-x_m(t) \mid \leq \int_{t_0}^{t} \mid f(s,x_m(s)-f(s,x_{m-1}(s) \mid ds \\ \leq k\int_{t_0}^{t} d_{m-1}(s)ds. \end{array} \tag{9}$$

Furthermore as

$$d_0(t) = \mid x_1(t) - x_0 \mid \leq M \mid t - t_0 \mid, \tag{10}$$

assume inductively that

$$d_m(t) \leq \frac{M}{k} k^{m+1} \frac{(t-t_0)^{m+1}}{(m+1)!}, \tag{11}$$

then it follows that

$$d_{m+1}(t) \leq \frac{M}{k} \frac{k^{m+2}}{(m+1)!} \int_{t_0}^{t} (s-t_0)^{m+1} ds = \frac{M}{k} k^{m+2} \frac{(t-t_0)^{m+2}}{(m+2)!}. \tag{12}$$

This yields the normal convergence of the series:

$$x_0(t) + \Sigma_{m=0}^{+\infty} [x_{m+1}(t) - x_m(t)], \tag{13}$$

Indeed, it displays

$$\Sigma_{m=0}^{+\infty} d_m(t) \leq \frac{M}{k} \Sigma_{m=0}^{+\infty} \frac{k^{m+1} A^{m+1}}{(m+1)!} \leq \frac{M}{k} [\exp(kA) - 1]. \tag{14}$$

At this point, this secure the existence of the solution. Check now the unicity. Assume *ad absurdum* that there exist two continuous distinct solutions $x(t)$ and $\overline{x}(t)$ defined on $[t_0 - r, t_0 + r] \subset [t_0 - A, t_0 + A]$ so that $x(t_0) = \overline{x}(t_0) = x_0$. As $x(t)$ and $\overline{x}(t)$ are continuous, for all δ arbitrarily small, there exists a B so that:

$$\mid x(t) - \overline{x}(t) \mid \leq B, \tag{15}$$

for all $t \in [t_0, t_0 + r - \delta]$. This displays

$$\mid x(t) - \overline{x}(t) \mid \leq \int_{t_0}^{t} \mid f(s, x(s)) - f(s, \overline{x}(s)) \mid \leq kB(t - t_{0)}. \tag{16}$$

We can then deduce inductively that for all m,

$$\mid x(t) - \overline{x}(t) \mid \leq [k^m/m!] B (t - t_0)^m, \tag{17}$$

tends to 0 as $m \to +\infty$. This yields that $\mid x(t) - \overline{x}(t) \mid$ vanishes identically on $[t_0, t_0 + r - \delta]$. As δ is arbitrarily small, unicity follows on $[t_0, t_0 + r]$ then the same can be shown on $[t_0 - r, t_0]$. $\square$

Corollary 12. *The solution $x(t, t_0, x_0)$ depends continuously on (t, t_0, x_0).*

Proof. Back to the proof, one can easily check by recurrence that $x_m(t, t_0, x_0)$ depends continuously of its arguments. The corollary follows as a uniform limit of continuous functions is continuous. $\square$

Corollary 13. *If $f(t, x, \lambda)$ is Lipschitz in λ, then the solution is continuous in λ.*

Proof. Suffices to consider

$$\begin{aligned} &x' = f(t, x, \lambda), \\ &\lambda' = 0. \end{aligned} \tag{18}$$

$\square$

Definition 15. *If $x(t)$ is a solution of the differential equation defined on (a, b) and $\overline{x}(t)$ is a solution of the differential equation defined on (α, β), $\overline{x}(t)$ is called an extension of $x(t)$ if $(a, b) \subset (\alpha, \beta)$ and if $\overline{x}(t) \mid_{(a,b)} = x(t)$.*

Definition 16. *If $x(t)$ is a solution of the differential equation defined on (a, b) so that for all extensions $\overline{x}(t)$ defined on (α, β), with $(a, b) \subset (\alpha, \beta)$, necessarily $\overline{x}(t) = x(t)$ et $(\alpha, \beta) = (a, b)$, then solution $x(t)$ is said maximal.*

The local existence theorem yields existence of solution $x(t)$ defined on an interval $[-A, +A]$ so that $x(0) = x_0$.

Theorem 14. *There exists a unique maximal solution $x'(t) = f(t, x)$, so that $x(0) = x_0$ defined on an interval $]\alpha(x_0), \omega(x_0)[$.*

Proof. Let (α, β) be the reunion of all intervals I which contains 0 so that the equation displays a solution so that $x(0) = x_0$ defined on I. Define a function $x(t)$ on (α, β) as follows. Given $t \in (\alpha, \beta)$, there exists I so that $t \in I$ on which there is a solution $u(t)$ of the equation. Set $x(t) = u(t)$. Necessary to check that $x(t)$ is well defined. Assume there are two intervals I_1 and I_2 so that $t \in I_1 \cap I_2$ and two solutions u_1 and u_2. From the local uniqueness, there exists an interval $]-A, +A[$ on which $u_1(t) = u_2(t)$. Let $I*$ be the largest interval contained in $I_1 \cap I_2$ so that $u_1 = u_2$. If $I*$ is a proper sub-interval of $I_1 \cap I_2$, for one of its extremities τ, by continuity of u_1 and u_2,

$$\mathrm{Lim}_{t\to\tau} u_1(t) = \mathrm{Lim}_{t\to\tau} u_2(t) = u_0. \tag{19}$$

From the local existence and unicity $u_1(t) = u_2(t)$ on an interval $]\tau - \alpha, \tau + \alpha[$, contradicts maximality of $I*$. Hence $u_1(t) = u_2(t)$ on $I_1 \cap I_2$, and $x(t)$ is well defined. This function $x(t)$ is a solution of the differential equation because in all points it coincides with a solution $u(t)$ of the differential equation. This maximal interval of existence is open by the possibility of an extension at the extremity of the interval. □

Theorem 15. *Let $x(t)$ be the maximal solution defined on (α, β). Assume that $\beta < +\infty$, for a compact $K \subset \Omega$, then there exists a $t \in (\alpha, \beta)$ so that $x(t) \notin K$.*

Proof. Assume that $x(t) \in K$ for all $t \in (\alpha, \beta)$. If $\alpha < t_1 < t_2 < \beta$, then

$$\mid x(t_1) - x(t_2) \mid \leq \int_{t_1}^{t_2} \mid f(x(s) \mid \leq M(t_1 - t_2), \tag{20}$$

and this yields existence of $\mathrm{Lim}_{t\to\beta_-} x(t) = x_1$, $x_1 \in K$. Set then $u(t) = x(t)$ for $t \in (\alpha, \beta)$ and $u(\beta) = x_1$. This function displays

$$u(t) = x_0 + \int_0^t f(u(s))ds, \tag{21}$$

and is thus differentiable in β with $u'(\beta) = f(u(\beta)$. It is then possible to extend the solution and there is a contradiction. □

Corollary 16. *For any maximal solution contained in a compact,*

$(\alpha, \beta) =]-\infty, +\infty[$.

Consider more particularly the autonomous case, also named vector fields

$$x'(t) = f(x). \tag{22}$$

Corollary 17. *If the solutions of an autonomous differential equation remain in a compact set K, then they define an associated topological dynamical system on K.*

Proof. Consider an initial point x_0 of K, the solution passing by x_0 exists for all values of $t \in \mathbb{R}$. The unicity of this solution yields:

$$x(t + t') = x(t, x(t'), \tag{23}$$

and this defines a topological dynamical system:

$$f : \mathbb{R} \times K \to K, f : (t, p) \mapsto f(t, p) = x(t, p). \tag{24}$$

□

1.2.2. *Gronwall's Lemma.*

Theorem 18. *Let $\phi(t)$ be a continuous function, with positive values defined on an interval $t_0 \leq t \leq t_0 + T$ which displays the inequality:*

$$\phi(t) \leq \delta_1 \int_{t_0}^{t} \phi(s) ds + \delta_2 (t - t_0) + \delta_3, \tag{25}$$

then it satisfies the inequality:

$$\phi(t) \leq (\frac{\delta_2}{\delta_1} + \delta_3) \exp[\delta_1 (t - t_0)] - \frac{\delta_2}{\delta_1}, \tag{26}$$

for all $t \in [t_0, t_0 + T]$.

Proof. Write

$$\psi(t) = \phi(t) + \frac{\delta_2}{\delta_1}, \tag{27}$$

which displays

$$\psi(t) \leq \delta_1 \int_{t_0}^{t} \psi(s)ds + \frac{\delta_2}{\delta_1} + \delta_3. \tag{28}$$

This implies:

$$\frac{\delta_1 \psi(t)}{\delta_1 \int_{t_0}^{t} \psi(s)ds + \frac{\delta_2}{\delta_1} + \delta_3} \leq \delta_1. \tag{29}$$

Then, by integration,

$$\log[\delta_1 \int_{t_0}^{t} \psi(s)ds + \frac{\delta_2}{\delta_1} + \delta_3] - \log(\frac{\delta_2}{\delta_1} + \delta_3) \leq \delta_1(t - t_0), \tag{30}$$

this yields

$$\delta_1 \int_{t_0}^{t} \psi(s)ds + \frac{\delta_2}{\delta_1} + \delta_3 \leq (\frac{\delta_2}{\delta_1} + \delta_3)\mathrm{e}^{\delta_1(t-t_0)}. \tag{31}$$

Application of the same inequality again, displays

$$\psi(t) \leq (\frac{\delta_2}{\delta_1} + \delta_3)\mathrm{e}^{\delta_1(t-t_0)}, \tag{32}$$

which provides the expected inequality. □

Consider now a differential equation depending of a parameter λ:

$$\frac{dx}{dt} = f(x, t, \lambda), \tag{33}$$

so that f is Lipschitz of coefficient K relatively to x uniformly in λ and $t \in [-a, +a]$.

Let $\phi = \phi(t, t_0, x_0)$ be the unique maximal solution so that $\phi(t_0, t_0, x_0) = x_0$, defined on the interval $I(t_0, x_0) = (\omega_-(t_0, x_0, \lambda), \omega_+(t_0, x_0, \lambda))$.

Gronwall Lemma allows to show the

Theorem 19. *For all $t \in I(t_0, x_0) \cap I(t_0, y_0)$, the inequality:*

$$\mid \phi(t, t_0, x_0) - \phi(t, t_0, y_0) \mid \leq \exp(K \mid t - t_0 \mid) \mid x_0 - y_0 \mid, \tag{34}$$

stands.

This shows that the solution is a differentiable function of the initial data x_0 as well as of t_0 if the differential equation $f(x,t)$ is differentiable.

1.2.3. *Differentiable dynamical system associated to a vector field, transversal section and first return mapping.* We have thus obtain that the dynamical system associated to the solutions of a differentiable vector field $f(x)$ (on a compact set) is not only continuous but also differentiable.

Definition 17. *The mapping $\phi_t : x_0 \mapsto x(t)$ which associates to the initial value x_0 the value of the maximal solution $x(t)$ at time t, is called the flow at time t of the vector field.This flow is said complete if this mapping is defined for all $t \in [-\infty, +\infty]$.*

For instance the flow is complete if the solutions remain in a compact set K. It is possible to prove that if the vector field $f(x)$ is of class C^k, the flow is also a mapping of class C^k. Assume further that $f(x)$ is at least differentiable.

Definition 18. *The orbit (or integral curve) γ of the vector field $f(x)$ passing by the point x_0 is the differentiable curve $t \mapsto x(t)$. This curve is oriented by the sense of variation of t (time). The tangent to the orbit at the point $x(t)$ is given by the vector $f(x(t))$. The positive orbit is $\gamma_+ = \{x(t), t \geq 0\}$ and the negative orbit is $\gamma_- = \{x(t), t \leq 0\}$.*

Definition 19. *The phase portrait of a vector field $f(x)$ is the partition of the space into its orbits.*

Definition 20. *Given a vector field $f(x)$, a point x_0 is said to be stationary if $f(x_0) = 0$. A stationary point is a fixed point of the flow. A point which is not stationary is said to be regular.*

Definition 21. *Let f be a differentiable vector field defined on an open set U of $\mathbb{R}^n$. Let A be an open set of $\mathbb{R}^{n-1}$. A transverse section of the vector field is a differentiable mapping $\sigma : A \to U$ such that for all points $a \in A$:*

$$D\sigma(a)(R^{n-1})\bigoplus X(S(a)) = \mathbb{R}^n, \tag{35}$$

where D denotes the differential.

Definition 22. *Two vector fields f and g are said to be topologically (resp. C^k) conjugate if there exists an homeomorphism (resp. diffeomorphism of class C^k) which sends the orbits of f on those of g with preservation of the time t. If $\phi(t, x)$ (resp. $\psi(t, x)$) denotes the flow of f (resp. g). This writes*

$$h(\phi(t, x)) = \psi(t, h(x)). \tag{36}$$

Theorem 20. *Let f be a vector field of class C^k defined on an open set U of $\mathbb{R}^n$ and p a regular point. Let $S : A \to \Sigma$ be a transverse section to f so that $S(0) = p$. There exists a neighborhood V of p and a diffeomorphism h of class C^k, $h : V \to (-\epsilon, +\epsilon)\times B$ where B is an open ball of $\mathbb{R}^{n-1}$ centered at the origin so that:*

(i) $h(\Sigma \cap V) = 0\times B$

(ii) h is a C^k conjugacy between the vector field f restricted to V eand the constant vector field g

$$g : (-\epsilon, +\epsilon)\times B \to \mathbb{R}^n, \quad g = (1, 0, 0, ..., 0) \in \mathbb{R}^n. \tag{37}$$

Proof. Let $\phi(t,p)$ be the flow at time t applied to the point p. Define the mapping $F:(t,u)\mapsto\phi(t,S(u))$. The differential $DF(0)$ is an isomorphism and by the inverse local theorem, F is a local diffeomorphism. Choose a neighborhood $(-\epsilon,+\epsilon)\times B$ of the origin so that F defines a diffeomorphism onto its image $V=F((-\epsilon,+\epsilon)\times B)$. Set $h=F^{-1}\mid V$ which displays

$$\begin{aligned}&h(\Sigma\cap V)=0\times B,\\ &Dh^{-1}(t,u).Y(t,u)=\tfrac{dF(t,u)}{dt}=X(F(t,u))=X(h^{-1}(t,u)).\end{aligned}\tag{38}$$

□

Corollary 21. *Let Σ be a transverse section of f. Let p be a point of Σ. There exists $\epsilon(p)$ and a neighborhood V of p and a function $\tau:V\to\mathbb{R}$ of class C^k such that $\tau(V\cap\Sigma)=0$ and*

(i) For all point $q\in V$, the integral curve $\phi(t,q)$ of $f|V$ exists for $t\in(-\epsilon,+\epsilon)$.

(ii) The point q belongs to Σ if and only if $\tau(q)=0$.

(iii) $\xi(q)=\phi(\tau(q),q)$ is the only intersection of the orbit passing by q with Σ. The mapping $\xi:V\to\Sigma$ is of class C^k, $D\xi(q)$ is surjective and $q\in V$ and $D\xi(q).v=0$ are equivalent to $v=\alpha f(q)$ for some $\alpha\in\mathbb{R}$.

Proof. This result is obvious for the constant vector field g and it is preserved by conjugacy. □

Hereafter the theorem and its corollary will be referred as the Flow-Box theorem.

Definition 23. *A periodic orbit is an orbit of the vector field passing by a regular point x_0 so that the solution exists for all $t\in\mathbb{R}$ and there exists a number $T\neq 0$ so that*

$$x(t+T)=x(t).\tag{39}$$

Lemma 22. *The set* $P = \{c \in \mathbb{R}, x(t+c) = x(t), \quad \text{for} \quad \text{all} \quad t \in \mathbb{R}\}$ *is an additive subgroup of* $\mathbb{R}$ *which is closed in* $\mathbb{R}$*.*

Proof. If $c, d \in P$, then $c + d, -c \in P$, because

$$\begin{aligned} x(t+c+d) &= x(t+c) = x(t) \\ x(t-c+c) &= x(t-c) = x(t), \end{aligned} \tag{40}$$

and the continuity of $t \mapsto x(t)$ yields that if a convergence sequence $c_n \to c$ belongs to P, then $c \in P$. □

Lemma 23. *Any additive subgroup* $P \neq 0$ *of* $\mathbb{R}$ *is either of the form* $\tau\mathbb{Z}$*, with* $\tau > 0$ *or is dense in* $\mathbb{R}$*.*

Proof. Suppose $P \neq 0$, then $P \cap \mathbb{R}_+ \neq 0$ because there exists $c \in P, c \neq 0$ and either c or $-c$ belongs to $P \cap \mathbb{R}_+$. Consider $\tau = \inf(P \cap \mathbb{R}_+)$. If $\tau = 0$, given any $\epsilon > 0$, and $t \in \mathbb{R}$, there exists $c_0 \in P \cap \mathbb{R}_+$ so that $0 < c_0 < \epsilon_0$. The set $c_0\mathbb{Z}$ divides $\mathbb{R}$ into intervals of length less than ϵ. It follows that for any $t \in \mathbb{R}$ there exists an element $c \in P$ such that $\mid t - c \mid < \epsilon$. This means that P is dense in $\mathbb{R}$. Consider now the case $\tau > 0$. If there exists a $c \in P - \tau\mathbb{Z}$, there is a unique $K \in \mathbb{Z}$ so that $K\tau < c < (K+1)\tau$ and then $0 < c - K\tau < \tau$ and $c - K\tau \in P \cap \mathbb{R}_+$. This contradiction yields $P = \tau\mathbb{Z}$. □

Definition 24. *The minimal period of the periodic orbit is the smallest real positive number* T_0 *so that* $x(t + T_0) = x(t)$ *for all* t*.*

From the two previous lemmas, follows the fact that the set of real numbers T so that $x(t + T) = x(t)$ called the periods of the periodic orbit is the set of multiples of the minimal period $T_0\mathbb{Z}$.

Let γ be a periodic orbit of a vector field f of class C^k passing by point x_0. Let $\Sigma = \{x \mid (x - x_0).f(x_0) = 0\}$ be a germ of hyperplane transverse at γ at the point x_0. Denote $\phi_t(x)$ the flow of the vector field at time t applied at point p. The periodic orbit is identified with $\gamma = \{x \mid x = \phi_t(x_0), 0 \leq t \leq T\}$.

Theorem 24. *There exists $\delta > 0$ and a differentiable function $x \mapsto \tau(x)$ of class C^k defined on $x \in \Sigma, \quad \mid x - x_0 \mid < \delta$, so that $\phi_{\tau(x)}(x) \in \Sigma$. This function $x \mapsto \tau(x)$ is called the function of the time of first return.*

Proof. Given any point $q \in \gamma$, there exists a neighborhood of q defined by theorem 20. As the periodic orbit γ is compact, it is possible to cover γ by a finite number of intervals $]-\epsilon_i, +\epsilon_i[\times B(0, \rho_i)$. The minimum $\rho = \min_i \rho_i$, defines a neighborhood of γ on which the flow is defined for all $t \in [0, A], A > T$. We can then define a C^k funtion $F(t, x) = [\phi_t(x) - x_0].f(x_0)$. The periodicity displays $F(T, x_0) = 0$. Furthermore, as x_0 is necessarily a regular point,

$$\frac{\partial F(T, x_0)}{\partial t} = \frac{\partial \phi_t(x_0)}{\partial t} \mid_{t=T} .f(x_0) = f(x_0).f(x_0) = \mid f(x_0) \mid^2 \neq 0. \tag{41}$$

By the implicit function theorem, there exists a unique function $\tau(x)$ of class C^k so that $\tau(x_0) = T$ and $F(\tau(x), x) = 0$, hence $\phi_{\tau(x)}(x) \in \Sigma$. $\square$

Definition 25. *Define the first-return mapping $P : x \mapsto P(x) = \phi_{\tau(x)}(x)$. This mapping is also of class C^k.*

2. Stability and Hyperbolicity

2.1. **Lyapunov Stability.**

2.1.1. *Linear Systems.* Consider a linear differential equation:

$$\dot{x} = A.x, \tag{42}$$

where x is a vector in $\mathbb{R}^n$ and A a matrix $n \times n$.

Recall that the solution $x(t)$ with initial condition $x(0) = x_0$ is given in that case by

$$x(t) = \exp(tA).x_0. \tag{43}$$

Let $w_j = u_j + \mathrm{i}v_j$ an eigenvector associated to an eigenvalue $\lambda_j = a_j + \mathrm{i}b_j$ of A, $(j = 1, ..., n)$. Assume that the k first eigenvalues of A and that $2(m-k)$ eigenvalues are complex conjugates. Assume that the real vectors $u_1, ..., u_k; u_{k+1}, v_{k+1}; ...u_m, v_m$ display a basis of $\mathbb{R}^n$, $n = 2m - k$.

Definition 26. *The stable space E^s, unstable space E^u and neutral space E^n are defined respectively by*

E^s = subspace generated by u_j, v_j so that $a_j < 0$

E^u = subspace generated by u_j, v_j so that $a_j > 0$

E^n = subspace generated by u_j, v_j so that $a_j = 0$.

Theorem 25. *The following properties are equivalent*

(a) Eigenvalues of A have negative real parts.

(b) There exists positive constants M, c positive such that for all $x_0 \in \mathbb{R}^n$ and for all $t \in \mathbb{R}$

$$\mid \exp(tA).x_0 \mid \leq M \mid x_0 \mid \exp(-ct). \tag{44}$$

(c) For all $x_0 \in \mathbb{R}^n$*,* $\lim_{t\to+\infty} \exp tA.x_0 = 0$*.*

Proof. It is obvious that (a) implies (b) which implies (c). Assume that (c) does not imply (a) and let $\lambda = a + ib$ be an eigenvalue with $a \geq 0$. Then there exists a vector x_0 so that $\lim_{t\to+\infty} \exp(tA).x_0 \neq 0$. In fact any non-zero real vector x_0 of the subspace generated by the real parts and imaginary parts of the eigenvector associated to λ displays this property. □

With this theorem, it is possible to deduce easily the

Corollary 26. *The vector space* $\mathbb{R}^n$ *is decomposed as a direct sum*

$$\mathbb{R}^n = E^s \bigoplus E^u \bigoplus E^n, \tag{45}$$

of subspaces invariant by the flow $t \mapsto \exp(tA)$

2.1.2. *Stable and asymptotically stable solutions.*

Definition 27. *A solution* $x(t)$ *of a differential equation* $\frac{dx}{dt} = f(x,t)$ *is stable if for all* $\epsilon > 0$*, there is a* $\delta > 0$ *so that if another solution* $y(t)$ *so that for* $t = t_0$ $\| (x-y)(t) \| \leq \delta$*, then* $\| (x-y)(t) \| \leq \epsilon$ *for all* $t \geq t_0$*. If furthermore* $\| y - x \| \to 0$ *when* $t \to +\infty$ *the solution* $x(t)$ *is said asymptotically stable.*

In the autonomous case (vector field), if $x(t) = x_0$ is a stationary point, then x_0 is a stable (resp. asymptotically stable) stationary point.

2.1.3. *Lyapunov-Poincaré theorem.*

Theorem 27. *Consider a differential equation*

$$\frac{dx}{dt} = A.x + f(x,t), \tag{46}$$

A is a $n\times n$ matrix, $f(x,t)$ is continuous on $\| x \|\leq \rho$, $\quad t\geq 0$ so that $\frac{\|f(x,t)\|}{\|x\|}\to 0$ when $x\to 0$, uniformly relatively to $t\geq 0$. If the real parts of the eigenvalues of A are strictly negative, the solution $x=0$ is asymptotically stable.

Proof. Let

$$m=\mathrm{Sup}_{\|x\|\leq\rho,t\geq 0}\| f(x,t)\|, \tag{47}$$

c $\| c \|<\rho$ and d a positive number so that $\| c \| +d\leq\rho$, this displays

$$\mathrm{Sup}_{\|x\|\leq\rho,t\geq 0}\| A.x+f(x,t)\|\leq\| A\|\rho+m=m'. \tag{48}$$

By the fundamental theorem of existence of solutions of differential equations, if $0<t_0<\frac{d}{m'}$, there exists a unique solution $x(t)$ on $t\in[0,t_0]$ so that $x(0)=c$, $x(t)\in\| x \|\leq\rho$.

The matrix solution $Y(t)$of the linear equation

$$\frac{dY}{dt}=A.Y,\qquad Y(0)=I, \tag{49}$$

tends to 0 as $t\to+\infty$ and

$$\int_0^{+\infty}\| Y(t)\| dt<+\infty. \tag{50}$$

The solution $y(t)$ of $\frac{dy}{dt}=A.y$ so that $y(0)=c$ is given by $y(t)=Y(t).c$. This yields:

$$\| y\|\leq\| Y\| .\| c\|\leq a\| c\|, \tag{51}$$

for some a independent of A that we can assume larger than 1.

The solution $x(t)$ displays:

$$x(t)=y(t)+\int_0^t Y(t-\tau)f(x(\tau),\tau)d\tau. \tag{52}$$

For c small enough, for all, $t\in[0,t_0]$ this yields

$$\| x(t)\|<2a\| c\| . \tag{53}$$

Indeed, let

$$\epsilon < \frac{1}{2}(\int_0^{+\infty} \parallel Y(\tau) \parallel)^{-1}, \tag{54}$$

and let η so that $\parallel x \parallel\leq \eta$ implies $\parallel f(x,\tau) \parallel\leq \epsilon \parallel x \parallel$, this displays

$$\begin{array}{l} \parallel x(t) \parallel\leq\parallel y(t) \parallel + \int_0^\tau \parallel Y(t-\tau) \parallel \epsilon \parallel x(\tau) \parallel\leq \\ a \parallel c \parallel + \frac{1}{2}\mathrm{Max}_{t\in[0,t_0]} \parallel x(t) \parallel, \end{array} \tag{55}$$

hence

$$\frac{1}{2} \parallel x(t) \parallel\leq a \parallel c \parallel . \tag{56}$$

If the vector c is choosen so that

$$\parallel c \parallel +d < \rho, \qquad \parallel c \parallel< \frac{\eta}{2a}, \qquad 2a \parallel c \parallel +d < \rho, \tag{57}$$

, it follows that i$\parallel x(t_0) \parallel +d < \rho$. The solution $x(t)$ extends to $t \in [t_0, 2t_0]$ and satifies the same estimates. By successive extensions, we check that $x(t)$ exists of all $t > 0$ and $\parallel x(t) \parallel +d < \rho$. This shows the stability. For the asymptotic stability, change x into u so that $x = u\mathrm{e}^{\lambda t}$ where λ is a negative real number larger to the maximum of the real parts of the eigenvalues of A. The function u solves the equation:

$$\frac{du}{dt} = (A - \lambda I)u + \mathrm{e}^{-\lambda t} f(u\mathrm{e}^{\lambda t}, t). \tag{58}$$

As $\parallel u \parallel\leq \eta$ then $\parallel \mathrm{e}^{\lambda t} u \parallel\leq \eta$ and so $\parallel \mathrm{e}^{-\lambda t} f(u\mathrm{e}^{\lambda t}, t) \parallel\leq \mathrm{e}^{-\lambda t}\epsilon \parallel u\mathrm{e}^{\lambda t} \parallel= \epsilon \parallel u \parallel$. It is then possible to apply to the equation in u what was proved for the equation in x because all eigenvalues of $A - \lambda$ have negative real parts. If $u(0) = x(0)$ is small, $u(t)$ is bounded and $x(t) \to 0$ as $t \to +\infty$. □

Definition 28. *A linear time-dependent differential equation*

$$\frac{dy}{dt} = A(t)y, \tag{59}$$

is said to be reducible it there exists a (time-dependent) linear change of variables $y = Q(t)x$, with $Q(t)$ differentiable and invertible with

$$\begin{aligned} &\sup_t \| Q(t) \| < +\infty, \\ &\sup_t \| Q^{-1}(t) \| < +\infty, \end{aligned} \tag{60}$$

so that

$$\frac{dx}{dt} = Bx, \tag{61}$$

where B is a constant matrix.

The Poincaré-Lyapunov theorem extends immediately to differential equations

$$\frac{dx}{dt} = A(t).x + f(x,t), \tag{62}$$

with $A(t)$ a reducible matrix.

2.1.4. *Lyapunov theorem and Lasalle invariance principle.* Denote here more precisely a vector field defined on an open set U of $\mathbb{R}^n$ by its components:

$$\frac{dx_i}{dt} = f_i(x_1, ..., x_n), \qquad i = 1, ..., n, \tag{63}$$

and let $x_0 \in U, f(x_0) = 0$ be stationary point.

Theorem 28. *Assume there exists a function V of class C^1 defined on a neighborhood of x_0, so $V(x_0) = 0$, $V(x) > 0$ if $x \neq x_0$ and*

$$\frac{dV}{dt} = \Sigma_i f_i(x) \frac{\partial V}{\partial x_i} \leq 0, \tag{64}$$

the stationary point x_0 is stable. If furthermore $\frac{dV}{dt}(x) < 0$ for all $x \neq x_0$, the stationary point x_0 is asymptotically stable. Such a function V is called a Lyapunov function for the vector field.

Proof. Let $\epsilon > 0$ and $\overline{B(x_0, \epsilon)}$ the closed ball centered at x_0 of radius ϵ. Set

$$\begin{aligned} &S_\epsilon = (x \in R^n / \mid x - x_0 \mid = \epsilon), \\ &m_\epsilon = \mathrm{Min}(V(x), x \in S_\epsilon). \end{aligned} \tag{65}$$

As S_ϵ is compact, V which is continuous displays a minima on S_ϵ and $m_\epsilon > 0$. As V is continuous and $V(x_0) = 0$, there exists δ so that: $\mid x - x_0 \mid < \delta$ implies $V(x) < m_\epsilon$. Let x be a point of the ball $\overline{B(x_0, \delta)}$. Assume there may exist a value $t_1 > 0$ so that $\mid \phi_{t_1}(x) - x_0 \mid = \epsilon$. This would yield $V(\phi_{t_1}(x)) \geq m_\epsilon$ in contradiction with

$$V(\phi_{t_1}(x)) \leq V(x) < m_\epsilon. \tag{66}$$

Now let $t > 0$ a real positive number and t_k a sequence which increases to $+\infty$. There is a rank k_0 so that $t < t_{k_0}$ and for all $k > k_0$, then

$$V(\phi_t(x)) > V(\phi_{k_0}(x)) > V(\phi_k(x). \tag{67}$$

Taking the limit $k \to +\infty$, displays

$$V(\phi_t(x)) > V(\phi_{k_0}(x)) \geq V(y_0), \tag{68}$$

which yields $V(\phi_t(x) > V(y_0)$ for all $t > 0$.

Now, as $y_0 \neq x_0$, function V is strictly decreasing along the orbit of y_0. For all $s > 0$, this displays

$$V(\phi_s(y_0) < V(y_0). \tag{69}$$

Then by continuity, if t_k is large enough, $V(\phi_{s+t_k}(x)) < V(y_0)$. Contradiction shows the asymptotic stability. $\square$

The following is called the "Lasalle invariance principle":

Theorem 29. *Let V be decreasing along the orbits of a dynamical system. Let p be a point so that the positive orbit $\gamma_+(p)$ through p is relatively compact. Then* $\mathrm{Lim}_{t\to+\infty} V(\phi_t(p) = c$ *exists and for all $q \in \omega$-limit de p, $V(q) = c$. In other words, V is constant on the ω-limit of p.*

Proof. The function $t \mapsto V(\phi(t,p))$ is decreasing and bouded so that $\mathrm{Lim}_{t\to+\infty} V(\phi_t(p)) = c$ exists. If $y \in \omega(p)$, there exists a sequence $t_n \to +\infty$ so that $y = \mathrm{Lim}\phi_{t_n}(p)$, and thus $V(y) = \mathrm{Lim} V(\phi_{t_n}(p)) = c$. $\square$

Theorem 30. *Consider the differential equation*

$$\frac{dx}{dt} = A.x + f(x), \tag{70}$$

so that A is a $n \times n$-matrix, f is of class C^2 so that $f(x) = O(\mid x \mid^2)$. Assume that the eigenvalues of A have negative real parts. Then the vector field displays a Lyapunov function on a neighborhood of the origin.

Proof. Consider the linear differential equation

$$\frac{dx}{dt} = A.x. \tag{71}$$

For this equation, there exists $\alpha > 0$ so that for all solutions $x(t, x_0)$ for initial data $x(0) = x_0$:

$$\| x(t) \| \leq C \| x(0) \| \mathrm{e}^{-\alpha t}. \tag{72}$$

Introduce the quadratic form

$$W(\xi) = \int_0^{+\infty} \| x(t,\xi) \|^2 \, dt. \tag{73}$$

This quadratic form is positive definite and there exist constants (μ, ν) so that:

$$\mu \| \xi \|^2 \leq W(\xi) \leq \nu \| \xi \|^2, \tag{74}$$

for all ξ. This yields:

$$\begin{aligned} &W(x(\tau,\xi)) = \textstyle\int_0^{+\infty} \| x(t+\tau,\xi) \|^2 = \int_\tau^{+\infty} \| x(t,\xi) \|^2 \, dt, \\ &\dot{W} = \tfrac{d}{dt} W(x(t,\xi)) \mid_{t=0} = - \| x(t,\xi) \|^2 |_{t=0} = \\ &- \| \xi \|^2 \leq -\tfrac{1}{\mu} W. \end{aligned} \tag{75}$$

Consider now the full nonlinear system. The derivative of the function W relatively to this vector field displays:

$$\dot{W} \leq -\frac{1}{\mu} W + \sum_i f_i(x) \frac{\partial W}{\partial x_i}. \tag{76}$$

Each components f_i of the vector field verifies:

$$\mid f_i(x) \mid \leq k \mid\mid x \mid\mid^2 \leq \frac{k}{\mu} W(x). \tag{77}$$

As W is a quadratic form,

$$\mid \frac{\partial W}{\partial x_i} \mid \leq l \sqrt{W(x)}, \tag{78}$$

and there is a constant q so that

$$\sum_i f_i(x) \frac{\partial W}{\partial x_i} \leq q W(x)^{3/2}. \tag{79}$$

Consider a neighborhood of the origin so that $W(x) < b$ and choose c so that

$$c \leq b, \quad q\sqrt{c} \leq \frac{1}{2\mu}. \tag{80}$$

This displays

$$\dot{W} \leq -\frac{1}{2\mu} W, \tag{81}$$

and this shows that the quadratic form W is a Lyapunov function. □

2.2. **Hyperbolic stationary points and hyperbolic periodic orbits.**

2.2.1. *Invariant manifolds of hyperbolic stationary points.*

Definition 29. *A stationary point x_0 of a vector field $\frac{dx}{dt} = f(x)$ is said hyperbolic if all eigenvalues λ_j of the Jacobian matrix $A = (\frac{\partial f_i}{\partial x_j})(x_0)$ have non-zero real parts.*

Theorem 31. *Let $\frac{dx}{dt} = f(x)$ be a vector field defined on a neighborhood U of $\mathbb{R}^n$ of the origin. Assume that 0 is a hyperbolic stationary point of the vector field. Assume that the Jacobian matrix of f at the origin displays k eigenvalues of strictly* *negative real parts and $n-k$ eigenvalues of strictly positive parts. Then there exists an invariant manifold S of dimension k tangent to the stable space of the linear part of f at the origin so that: for all $t \geq 0$, $\phi_t(S) \in S$ and $x_0 \in S$, $\lim_{t\to\infty}\phi_t(x_0) = 0$. There exists an invariant manifold U of dimension $n-k$ tangent to the unstable space of the linear part of f at the origin so that: for all $t \leq 0$, $\phi_t(S) \in S$ and for $x_0 \in U$, $\lim_{t\mapsto -\infty}\phi_t(x_0) = 0$.*

Proof. After a linear change of coordinates, assume:

$$\dot{x} = Ax + F(x), \quad f(x) = 0(\mid x \mid^2, \tag{82}$$

$A = (P, Q)$ with P a matrix $k \times k$ with eigenvalues $\lambda_1, ..., \lambda_k$ of negative real parts and Q a matrix $(n-k)\times(n-k)$ with eigenvalues $\lambda_{k+1}, ..., \lambda_n$ of positive real parts. Set $U(t) = (\exp(tP), 0)$ and $V(t) = (0, \exp(tQ))$. Let α so that

$$\mathrm{Re}(\lambda_j) < -\alpha, \qquad j = 1, ..., k. \tag{83}$$

There are constants K and σ so that

$$\begin{aligned} &\| U(t) \| < K\exp(-(\alpha+\sigma)t), t \geq 0,\\ &\| V(t) \| < K\exp(\sigma t), t \leq 0. \end{aligned} \tag{84}$$

Introduce the Duhamel equation depending of a parameter $a \in \mathbb{R}^n$:

$$u(t,a) = U(t)a + \int_0^t U(t-s)F(u(s,a))ds - \int_t^\infty V(t-s)F(u(s,a))ds. \tag{85}$$

As F is of class C^1 and $DF(0) = 0$, for all ϵ there is δ so that if

$$\mid x \mid \leq \delta, \mid y \mid \leq \delta, \tag{86}$$

then

$$\mid F(x) - F(y) \mid \leq \epsilon \mid x - y \mid . \tag{87}$$

Consider the sequence of functions $t \mapsto u_j(t,a)$ defined as

(88)

$$\begin{array}{l} u_0(t,a) = 0, \\ u_{j+1}(t,a) = U(t)a + \int_0^t U(t-s)F(u_j(s,a))ds - \int_t^\infty V(t-s)F(u_j(s,a))ds. \end{array}$$

Assume by induction that if $\frac{\epsilon K}{\sigma} < \frac{1}{4}$, and $2K \mid\mid a \mid\mid < \delta$, then

$$| u_j(t,a) - u_{j+1}(t,a) | \leq \frac{K \mid a \mid \exp(-\alpha t)}{2^{j-1}}. \tag{89}$$

is true up to order m. Then, this yields:

$$\begin{array}{l} | u_{m+1}(t,a) - u_m(t,a) | \leq \int_0^t \mid\mid U(t-s) \mid\mid \epsilon \mid u_m(s,a) - u_{m-1}(s,a) \mid ds \\ + \int_t^\infty \mid\mid V(t-s)\epsilon \mid u_m(s,a) - u_{m-1}(s,a) \mid ds \\ \leq \epsilon \int_0^t K\exp(-(\alpha+\sigma)(t-s)) \frac{K|a|\exp(-\alpha s)}{2^{m-1}} ds \\ + \epsilon \int_0^\infty K\exp(\sigma(t-s)) \frac{K|a|\exp(-\alpha s)}{2^{m-1}} ds \\ \leq \frac{\epsilon K^2 |a| \exp(-\alpha t)}{\sigma 2^{m-1}} + \frac{\epsilon K^2 |a| \exp(-\alpha t)}{\sigma 2^{m-1}} \\ leq(\frac{1}{4} + \frac{1}{4}) \frac{K|a|\exp(-\alpha t)}{2^{m-1}} = \frac{K|a|\exp(-\alpha t)}{2^m}. \end{array} \tag{90}$$

These estimates show that the sequence of functions $t \mapsto u_j(t,a)$ converges uniformly and that its limit displays

$$| u(t,a) | \leq 2K \mid a \mid \exp(-\alpha t). \tag{91}$$

Note that the $n-k$ last components of a do not intervene. It is possible to assume they are nul.This yields

$$\begin{array}{l} u_j(0,a) = a_j, \quad j = 1, ..., k, \\ u_j(0,a) = -(\int_0^\infty V(-s)F(u(s,a_1,...,a_k,0))ds)_j, \quad j = k+1,...,n. \end{array} \tag{92}$$

Define the functions

$$\psi_j(a_1,...,a_k) = u_j(0,a_1,...,a_k), \quad j = k+1,...,n. \tag{93}$$

The stable manifold is defined as a graph:

$$y_j = \psi_j(y_1,...,y_k), \quad j = k+1,...,n. \tag{94}$$

Indeed, if $y \in S$, set $y = u(0,a)$ then $y(t) = \phi_t(y) = u(t,a)$ and $\lim_{t \mapsto \infty}(y(t)) = 0$. Existence of the unstable manifold can be shown analogously after changing t into $-t$ and permuting U and V in the Duhamel integral equation. $\square$

Definition 30. *The global stable (resp. unstable) manifold of an hyperbolic stationary point is the set of point $\phi(t,m), t \in \mathbb{R}, m \in S$ resp. set of points $\phi(t,m), t \in \mathbb{R}, m \in U$. In other words, the maximal positively invariant set which contains S (resp negatively invariant set which contains U).*

Definition 31. *A point p is said to belong to an heteroclinic connexion if there are two stationary points (x_0, x_1) so that $\lim_{t\to-\infty}\phi(t,p) = x_0$ and $\lim_{t\to+\infty}\phi(t,p) = x_1$. It is said to belong to an homoclinic connexion if there is a stationary point x_0 so that $\lim_{t\to-\infty}\phi(t,p) = x_0$ and $\lim_{t\to+\infty}\phi(t,p) = x_0$.*

2.2.2. *Orbital stability.*

Definition 32. *Let γ be a periodic orbit with initial data x_0 of a vector field. It is said to be (orbitally) stable if for all ϵ, there exists a δ so that the solution $y(t)$ with initial data y_0), such that $\mid x_0 - y_0 \mid < \epsilon$, exists for all values of t and*

$$d(y(t), \gamma) < \epsilon. \tag{95}$$

The periodic orbit is said asymptotically (orbitally) stable if it is (orbitally) stable and

$$\lim_{t\to+\infty} d(x'(t), C) = 0. \tag{96}$$

In the following, it will be noted “stable” or “unstable” in short for meaning orbitally stable (resp. unstable).

2.2.3. *Floquet theory of periodic orbits.* Let γ be a periodic orbit of a vector field $\dot{x} = f(x)$. Linearisation of the vector field along γ displays $\dot{x} = A(t)x$, where $A(t) = Df(\gamma(t))$ is a $n\times n$-matrix, T-periodic. Let $\Phi(t)$ be the fundamental solution of this linear system

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t), \qquad \Phi(0) = I. \tag{97}$$

The solution $x(t)$ so that $x(0) = x_0$ is

$$x(t) = \Phi(t).x_0. \tag{98}$$

Note that $t \mapsto \Phi(t)$ is no longer associated to a one-parameter group but to a process (in Daffermos' sense). More precisely let $U_{t_0}(t)$ be the solution of the matrix differential equation

$$\begin{array}{l} \frac{dU}{dt} = A(t)U(t), \\ U_{t_0}(t_0) = I. \end{array} \tag{99}$$

Unicity of the solution displays

$$U_0(t) = U_{t_0}(t)U_0(t_0), \tag{100}$$

and yields in particular the invertibility of $U_0(t_0)$. It follows that the fundamental matrix $\Phi(t)$ is invertible for all t.

Lemma 32. *Any invertible matrix Φ is the exponential of a matrix B (not unique).*

Proof. □

Theorem 33. *The fundamental solution $\Phi(t)$ is the product*

$$\Phi(t) = Q(t)\exp(tB), \tag{101}$$

of a differentiable T-periodic matrix $Q(t)$ by the exponential of tB, B, constant matrix.

Proof. Consider the fundamental matrix $\Phi(t)$ at $t = T$. As it is invertible, it writes:

$$\Phi(T) = \exp(TB). \tag{102}$$

Matrix $\Phi(t+T)$ solves:

$$\Phi'(t+T) = A(t+T)\Phi(t+T) = A(t)\Phi(t+T), \tag{103}$$

and displays

$$\Phi(t+T)_{|t=0} = \exp(TB). \tag{104}$$

Matrix $\Phi(t).\exp(TB)$, is also solution of the differential equation with the same initial data. This yields:

$$\Phi(t+T) = \Phi(t).\exp(TB). \tag{105}$$

Matrix $Q(t) = \Phi(t)\exp(-tB)$ is differentiable and displays:

$$Q(t+T) = \Phi(t+T).\exp[-(t+T)B] = \Phi(t).\exp(-tB) = Q(t), \tag{106}$$

and thus, it is T-periodic. □

Proposition 34. *A differentiable T-periodic matrix $P(t)$ is reducible in Lyapunov's sense*

Proof. The equation

$$\frac{dy}{dt} = P(t)y, \tag{107}$$

turns into

$$\frac{dx}{dt} = Bx, \tag{108}$$

by change of variables $y = Q(t).x$ □

Definition 33. *The characteristic exponents of the periodic orbit γ are the eigenvalues λ_j of the matrix B. The Floquet multipliers of the periodic orbit γ are* $\exp(\lambda_j)$.

Let x_0 be a point of a periodic orbit γ and Σ a transverse section to γ at x_0. Assume origin of coordinates at the point x_0 and Σ is an hyperplane orthogonal to γ en x_0. Let $\phi_t(x)$ be the flow near 0 and $D\phi_t(x)$ the differential of this mapping which displays

$$\begin{aligned} &\tfrac{\partial D\phi_t(x)}{\partial t} = Df(\phi_t(x))D\phi_t(x), \\ &D\phi_0(0) = I. \end{aligned} \tag{109}$$

This identifies $D\phi_t(0)$ with the fundamental solution $\Phi(t)$. This yields:

$$D\phi_t(0) = Q(t).\exp(tB), \tag{110}$$

and thus, in particular $D\phi_T(0) = \exp(TB)$.

Theorem 35. *One of the characteristic exponents λ_j is always 0. It is possible to choose the coordinates so that the differential of the first return map $DP(0)$ is identified with the $(n-1)\times(n-1)$-matrix extracted of $D\phi_T(0)$ by erasing the last line and the last column.*

Proof. The flow restricted to the periodic orbit $\gamma(t) = \phi_t(0)$ displays

$$\gamma'(t) = f(\gamma(t)), \tag{111}$$

and thus

$$\gamma''(t) = Df(\gamma(t))\gamma'(t). \tag{112}$$

This yields:

$$\gamma'(t) = \Phi(t)f(0), \tag{113}$$

and as $\gamma'(T) = f(0)$, $D\phi_T(0)f(0) = f(0)$. Thus $f(0)$ is an eigenvector of $\exp(TB)$ of eigenvalue 1. So, one of the eigenvalues of B is 0. Choose the ordering of the eigenvalues so that $\lambda_n = 0$. Choose coordinates so that $f(0) = (0, ..., 1)^t$, and thus the last column of the matrix $\exp(TB)$ becomes $(0, ..., 1)^t$. Denote

$$h(x) = \phi_{\tau(x)}(x). \tag{114}$$

The first-return map identifies with h restricted to Σ: $P = h(x) \mid \Sigma$. Differential of h displays:

$$Dh(x) = \frac{\partial \phi_{\tau(x)}(x)}{\partial t} D\tau(x) + D\phi_{\tau(x)}(x). \tag{115}$$

Setting $x = 0$, yields

$$D(h(0)) = f(0)D\tau(0) + D\phi_T(0). \tag{116}$$

Matrix $f(0)D\tau(0)$ displays non-zero terms only on the last line. Restriction of $D(h(0))$ to Σ corresponds to keep only the upper-left block $(n-1) \times (n-1)$ which coincides with $D\phi_T(0) = \exp(TB)$.

□

Definition 34. *Periodic orbit γ is said hyperbolic if all the eigenvalues $\lambda_j, \quad j = 1, ..., n-1$ of B displays*

$$\mathrm{Re}(\lambda_j) \neq 0, j = 1, ..., n-1. \tag{117}$$

Definition 35. *Periodic orbit γ is said stable (resp. unstable) if all eigenvalues $\lambda_j, j = 1, ..., n-1$ displays*

$$\mathrm{Re}(\lambda_j)) < 0 (\text{resp.} > 0), \quad 0, j = 1, ..., n-1. \tag{118}$$

Consider again the fundamental solution $\Phi(t)$ of the linear equation $\frac{d\Phi(t)}{dt} = A(t)\Phi(t)$. The following proposition is due to Liouville:

Proposition 36. *The determinant $\Delta(t) = \det\Phi(t)$ is given by the formula:*

$$\Delta(t) = \exp(\int_0^t \mathrm{Tr} A(s) ds). \tag{119}$$

Proof. Denote $e_i, i = 1, ..., n$ the vectors of a basis of $\mathbb{R}^n$. As $\Phi(t)$ is always invertible, the vectors $\Phi(t).e_i$ define also a basis of $\mathbb{R}^n$. In this basis, the matrix $A(t)$ writes:

$$A(t)\Phi.e_i = \Sigma_{j=1}^n A_{ij}\Phi.e_j. \tag{120}$$

Derivative relatively to t of $\Delta(t)$ writes now:

$$\frac{\Delta(t)}{dt} = \Sigma_i \det(\Phi.e_1, ..., \frac{d\Phi.e_i}{dt}, ...\Phi.e_n) = \Sigma_i A_{ii}(t)\Delta(t). \tag{121}$$

This displays $\Delta(t)$ as the solution of the differential equation:

$$\frac{\Delta(t)}{dt} = \mathrm{Tr}A(t)\Delta(t), \tag{122}$$

so that $\Delta(0) = 1$. This yields

$$\Delta(t) = \exp(\int_0^t \mathrm{Tr}A(s)ds). \tag{123}$$

$\square$

In particular, this displays that if a periodic orbit is stable, then

$$\int_0^T \mathrm{Div}X(x(s))ds \leq 0. \tag{124}$$

2.2.4. *Invariant manifolds of hyperbolic periodic orbits.*

Theorem 37. *Let γ be an hyperbolic periodic orbit of a vector field. Let λ_j be the eigenvalues of the matrix B defined by Floquet theory ($\lambda_n = 1$). Assume that $k, 0 \leq k \leq n-1$ characteristic exponants λ_j display negative real parts and $n-k$ positive real parts. Then there exists $\delta > 0$ so that:*

$$(\gamma) = \{x, \mid x - x_0 \mid < \delta, d(\phi_t(x), \gamma) \to 0, x \to \infty\}, \tag{125}$$

is a manifold of dimension $k+1$, left invariant by positive flow and called the stable manifold of the periodic orbit. Similarly,

$$U(\gamma) = \{x, \mid x - x_0 \mid < \delta, d(\phi_t(x), \gamma) \to 0, x \to -\infty\}, \tag{126}$$

is a manifold of dimension $n-k+1$ invariant by the negative flow and called the unstable manifold of the periodic orbit. The stable and the unstable manifolds intersects transversally along the periodic orbit.

Proof. Proceed near $\gamma(t)$ and write $x=\gamma(t)+v$ in the equation $\dot{x}=f(x)$. This displays

$$\dot{v}=Df(\gamma(t)).v+G(t,v), \tag{127}$$

with $G(t,v)$ T-periodic in t and $G(t,v)=O(\|\ v\ \|^2)$, O uniform in t. Write $A(t)=Df(\gamma(t))$, and let $U(t)$ be the fundamental solution of the linear differential equation $\frac{dU}{dt}=A(t)U$. Floquet theory displays $U(t)=Q(t)\mathrm{exp}(tB)$, with $Q(t)$ T-periodic and B constant. Writing $v=Q(t).y$ displays

$$\dot{y}=B.y+F(t,y), \tag{128}$$

with $F(t,y)$ T-periodic in t and $F(t,v)=O(\|\ v\ \|^2)$, O uniform in t. Introduce the Duhamel equation where the matrix B separates in blocks:

$$B=\begin{pmatrix}0 & 0 & 0\\ 0 & B_+ & 0\\ 0 & 0 & B_-\end{pmatrix}, \tag{129}$$

as follows:

(130)

$$w(t,a)=\mathrm{exp}[t.\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & B_-\end{pmatrix}]a+\int_0^t \mathrm{exp}[(t-s)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & B_--\end{pmatrix}]F(s,w(s,a))ds$$
$$-\int_t^{+\infty}\mathrm{exp}[(t-s)\begin{pmatrix}0 & 0 & 0\\ 0 & B_+ & 0\\ 0 & 0 & 0\end{pmatrix}]F(s,w(s,a))ds.$$

The proof goes on quite similarly to the case of the stationary points. $\square$

As in the case of hyperbolic stationary points, the existence of these manifolds is proved only locally. It is convenient to define the global stable and unstable manifolds as the maximal possible extension pushed by the flow of the vector field.

2.2.5. *Asymptotic phase.*

Theorem 38. *Under the same asumptions than in the previous theorem, there exists α so that* $\mathrm{Re}(\lambda_j) < -\alpha, j = 1, ..., k$ *and* $\mathrm{Re}(\lambda_j) > \alpha$. *There exists a K, such that for all $x \in S(\gamma)$, there exists an asymptotic phase t_0 so that for all $t \geq 0$,*

$$| \phi_t(x) - \gamma(t - t_0) | < K \mathrm{e}^{-\alpha \frac{t}{T}}. \tag{131}$$

Analogously, for all $x \in U(\gamma)$,there exists an asymptotic phase t_0 such that for all $t \leq 0$,

$$| \phi_t(x) - \gamma(t - t_0) | < K \mathrm{e}^{\alpha \frac{t}{T}}. \tag{132}$$

Proof. Assume to simplify that the orbit is attractive. Let $\gamma : t \mapsto \gamma(t)$ the periodic orbit. Assume that $\gamma(0) = 0$, and consider Σ a transversal section at $\gamma(0)$. The first return map writes

$$P : x_0 \mapsto x_1 = A.x_0 + D(x_0), \quad x_0 \in \Sigma, \tag{133}$$

where $a = || A || < e^{\alpha}$, D with its differential at 0. For all ϵ small enough, $| x_0 || < \epsilon$, $|| x_1 || < e^{\alpha} || x_0 ||$. More generally $x_n = P^n(x_0)$, displays $|| x_n || < e^{\alpha n} || x_0 ||$. Given ϵ there exists $\delta = \delta(\epsilon)$ so that if $d(\gamma, x_0) < \delta$, there exists a value $\tau_0 = \tau_0(x_0)$ so that the flow $\phi_t(x_0)$ exists for all $0 \leq t \leq \tau_0$, $\phi_{\tau_0}(x_0) \in \Sigma$, and $|| \phi_{\tau_0}(x_0 || < \epsilon$.Consider again the notation $x \mapsto \tau(x)$ for the function first return time. Introduce $\tau_1 = \tau(x_0)$, and $\tau_n = \tau_{n-1} + \tau(x_n)$. As the function τ is of class C^1 and that $x_n \to 0$, there exits L_0 so that

$$| \tau(x_{n-1}) - T | < L_0 || x_{n-1} || < L_0 e^{\alpha(n-1)} || x_0 || . \tag{134}$$

The series $(\tau_n - nT) - (\tau_{n-1} - (n-1)T)$ is thus normally convergent. This yields convergence of the sequence $\tau_n - nT$. Let t_0 be its limit. Considering a partial sum of the series displays:

$$|\tau_n - (nT + t_0)| \leq L_1 \mathrm{e}^{\alpha n} \| x_0 \|, \tag{135}$$

with$L_1 = L_0/(1 - e^{\alpha})$. This yields

$$\| \phi_{t+\tau(n)}(x_0) - \phi_{t+nT+t_0}(x_0) \| \leq L_3 \mathrm{e}^{\alpha n} \| x_0 \| . \tag{136}$$

Furthermore, it yields

$$\| \phi_{t+\tau_n}(x_0) - \gamma(t) \| = \| \phi_t(x_n) - \phi_t(0) \| \leq L_2 \| x_0 \| \leq L_2 \mathrm{e}^{\alpha n} \| x_0 \| . \tag{137}$$

It then follows:

$$\| \phi_{t+nT+t_0}(x_0) - \gamma(t) \| \leq (L_2 + L_3) \mathrm{e}^{\alpha n} \| x_0 \| . \tag{138}$$

Changing $t + nT$ into t, finally yields:

$$\| \phi_{t+t_0}(x_0) - \gamma(t) \| \leq (L_2 + L_3) \mathrm{e}^{\alpha \frac{t}{T}} \| x_0 \| . \tag{139}$$

□

2.2.6. *Persistence of hyperbolic stationary points and hyperbolic orbits.* Consider a family of differentiable vector fields which depends differentiably of a parameter λ:

$$\frac{dx}{dt} = f(x, \lambda). \tag{140}$$

Assume that if $\lambda = \lambda_0$, the vector field displays a stationary point x_0:

$$f(x_0, \lambda_0) = 0. \tag{141}$$

By the implicit function theorem, if

$$\mathrm{DetJac}_x f(x_0, \lambda_0) \neq 0, \tag{142}$$

There exists a solution $x(\lambda)$ which depends C^1 in λ to the equations:

$$f(x(\lambda), \lambda) = 0. \tag{143}$$

In this case, it is said that the stationary point is persitent to a small deformation of the parameter. If furthermore, the stationary point x_0 is hyperbolic, then as the eigenvalues $\mu(\lambda)$ of the matrix

$$\mathrm{Jac}_x f(x(\lambda), \lambda), \tag{144}$$

depends continuously of λ, there exists a neighborhood of λ_0 in the parameter space so that for all λ in this neighborhood,

$$\mathrm{Re}(\mu(\lambda)) \neq 0. \tag{145}$$

There is thus persistence of hyperbolic stationary points.

Proposition 39. *Assume that the vector field displays for $\lambda = \lambda_0$ an hyperbolic periodic orbit γ_{λ_0} Then there exists a neighborhood of λ_0 so that for all λ in this neighborhood, the vector fielf displays an hyperbolic periodic orbit γ_λ which tends to γ_{λ_0} as $\lambda \to \lambda_0$.*

Proof. Let Σ a transverse section to the flow of the vector field $\frac{dx}{dt} = f(x, \lambda_0)$, near γ_{λ_0} and let

$$P : \Sigma \to \Sigma, \quad P : u \mapsto P(u, \lambda_0), \tag{146}$$

be the first return mapping. The regularity theorems of the solutions of differential equations depending of a parameter show that the vector field $\frac{dx}{dt} = f(x, \lambda)$, displays a return mapping

$$u \mapsto P(u, \lambda), \tag{147}$$

whicg depends differentiably of (u, λ). Existence of the periodic orbit γ_0 displays the existence of a solution u_0 to the equation

$$P(u, \lambda_0) - u = 0. \tag{148}$$

The fact that the orbit is assumed to be hyperbolic yields

(149) $$\mathrm{DetJac}_u[P(u,\lambda)-u]\mid_{u=u_0,\lambda=\lambda_0}\neq 0.$$

The implicit function theorem implies existence of a solution $u(\lambda)$ to:

(150) $$P(u(\lambda),\lambda)-u(\lambda)=0.$$

Continuity of the eigenvalues of $\mathrm{Jac}P(u(\lambda),\lambda)$ in function of λ implies that the orbit remains hyperbolic. □

3. Planar vector fields

In this paragraph, notation is slightly changed as follows: a vector field of the plane is given by:

$$\begin{aligned} \tfrac{dx}{dt} &= f(x,y), \\ \tfrac{dy}{dt} &= g(x,y), \end{aligned} \tag{151}$$

where the two function $(x, y) \mapsto f(x, y)$, $(x, y) \mapsto g(x, y)$ assumed to be at least of class C^1 are called the components of the vector field. The local solution $(x(t), y(t))$ with initial point $(x(0), y(0))$ determines a local flow $(x_0, y_0) \mapsto (x(t), y(t)$ at time t. Another possible notation for a planar vector field is

$$X = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}, \tag{152}$$

which enables a coordinate-free approach. Note that given a differentiable function $(x, y) \mapsto F(x, y)$ the time-derivative of F computed along the flow $\frac{d}{dt}F(x(t), y(t))$ is

$$f(x(t),y(t))\frac{\partial F}{\partial x}(x(t),y(t)) + g(x(t),y(t))\frac{\partial F}{\partial y}(x(t),y(t)). \tag{153}$$

This justifies the notation used for X which is seen as a derivation of the algebra of differential functions in the sense that:

$$X : F \mapsto X.F = f(x,y)\frac{\partial F}{\partial x} + g(x,y)\frac{\partial F}{\partial y}, \tag{154}$$

displays

$$\begin{aligned} &X.(\lambda F + \mu G) = \lambda X.F + \mu X.G, \lambda, \mu \in \mathbb{R}, \\ &X.(FG) = (X.F)G + F(X.G). \end{aligned} \tag{155}$$

3.1. **Stationary points, first integral.** A stationary point is thus, in this new notation a point (x_0, y_0) such that $f(x_0, y_0) = g(x_0, y_0) = 0$.

Definition 36. *A stationary point is said to be elementary (or simple) if the Jacobian matrix:*

$$\mathrm{Jac}(f,g)(x_0),y_0) = (\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x})(x_0,y_0), \tag{156}$$

is not 0.

Definition 37. *Let λ and μ be the two eigenvalues of the matrix* $\mathrm{Jac}(f,g)(x_0,y_0)$*, assume that $\lambda.\mu \neq 0$. It the two eigenvalues are real and of opposite signs, the stationary point is called a saddle. If the two eigenvalues are real of same sign, the stationary point is called an attractive (or stable) node if $\lambda < \mu < 0$ and is called a repulsive (or unstable) node if $\lambda > \mu > 0$. It the two eigenvalues are complex conjugate and if the real part if negative (resp. positive), the stationary point is called an attractive or stable (resp. repulsive or unstable) focus.*

In these previous cases, the stationary point is hyperbolic. The previous theorem of the existence of stable and unstable manifolds applies. In particular, in the saddle case, there are two invariant manifolds (curves) one stable and one unstable which are defined locally near the stationary point.

In the case, where the two eigenvalues are purely imaginary the nature of the stationary point, which is not hyperbolic, cannot be decided only on the linear approximation of the vector field near the stationary point. A special study has to be done and it is called the center-focus problem.

Definition 38. *A differentiable function $(x,y) \mapsto H(x,y)$ is a first integral of the vector field if*

$$f(x,y)\frac{\partial H}{\partial x} + g(x,y)\frac{\partial H}{\partial y} = 0. \tag{157}$$

The function H is then constant along the orbits of the vector field.

Definition 39. *The hamitonian vector field X_H associated to the function H is the vector field defined as:*

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}. \tag{158}$$

It is easily verified that the Hamiltonian vector field displays the function H as first integral. The stationary points of X_H are exactly the critical points of the Hamiltonian H.

3.2. **Periodic orbits.** Periodic orbits of planar vector fields are traditionally called cycles.

Definition 40. *A limit cycle is a periodic orbit which is isolated among the set of periodic orbits.*

Recall to study the solutions near a periodic orbit $\gamma(t)$, it is useful to introduce the matrix $A(t) = Df(\gamma(t))$ and Liouville formula yields:

$$\det\Phi(T) = \det\exp(TB) = \exp\int_0^T \mathrm{Tr}A(s)ds. \tag{159}$$

In dimension 2, the differential of the first-return map $DP(0)$ is the matrix extracted from $\Phi(t)$ by erasing the last line and the last column, it yields the derivative of P at 0:

$$P'(0) = \exp\int_0^T \mathrm{Div}X(x(s), y(s))ds. \tag{160}$$

Thus, an hyperbolic limit cycle is such that $P'(0) = \exp\int_0^T \mathrm{Div}X(x(s), y(s))ds \neq 0$. It is necessarily attractive (or repulsive) both on each side of the limit cycle. There are although limit cycles which are not hyperbolic.

A planar vector field which displays a C^1 first integral may have periodic orbits (cycles) but there are no isolated periodic orbits (limit cycles).

3.3. The Poincaré-Bendixson theorem.

Definition 41. *A singular cycle* Γ *of the vector field is an invariant compact set which is a finite union of stationary points* $\{p_1, ..., p_k\}$ *and of orbits so that the* ω*-limit and* α*-limit of the orbits are among the stationary points* $p_k, k = 1, ..., s$.

Theorem 40. *Consider a planar vector field*

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y), \end{aligned} \tag{161}$$

with components f, g *of class* C^1. *Let* $\gamma_m = \{\phi(t, m), t \in \mathbb{R}\}$ *be a solution defined for all values of* $t > 0$, *so that the orbit of* m : $\gamma_m^+ = (\phi(t, m), \quad t \geq 0)$ *is precompact. Assume that the vector field displays a finite number of stationary points in* $\omega(m)$. *Then:*

a) If there are no stationary points in $\omega(m)$, $\omega(m)$ *is a cycle.*

c) If there are no regular point in $\omega(m)$, *then* $\omega(m)$ *is a stationary point.*

b) It there are both stationary points and regular points in $\omega(m)$, *then* $\omega(m)$ *is a singular cycle.*

Proof is consequence of several lemmas of independent interest.

Lemma 41. *Let* Σ *be a transversal section to the flow near an orbit* $\gamma = \{\phi(t, q), t \in \mathbb{R}\}$ *and let* $p \in \Sigma \cap \omega(\gamma)$. *There exists a sequence* $\phi(\tau_n)(q)$, $n \to \infty$ *of points in* Σ, *so that*

$$p = \lim_{n \to \infty} \phi(\tau_n, q). \tag{162}$$

Proof. As p is a regular point, there exists a neighborhood V of p and a function $\tau : V \to \mathbb{R}$ defined by the Flow-Box theorem. As $p \in \omega(\gamma)$, there exists a sequence t_n such that $t_n \to \infty$ and $\phi(t_n, q) \to p$ as $n \to \infty$. Assume the points of the sequence $\phi(t_n, q)$ are inside V (up to some rank). Set:

$$\tau_n = t_n + \tau(\phi(t_n, q)), \tag{163}$$

and so

$$\phi(\tau_n, q) = \phi(\tau(\phi(t_n, q)), \phi(t_n, q)) \in \Sigma. \tag{164}$$

Furthermore, continuity of τ yields:

$$\lim_{n\to\infty} \phi(\tau_n, q) = \phi(\tau(\phi(t_n, q)), \phi(t_n, q)) = \phi(0, p) = p. \tag{165}$$

□

Section Σ is an interval, hence there is a natural ordering on Σ associated to its orientation.

Lemma 42. *Let Σ be a transversal section. A positive orbit $\gamma^+(p) = \{\phi(t, p), t \geq 0\}$, intersects Σ in a monotonous sequence of points $p_1, p_2, ..., p_n, ...$.*

Proof. With the natural ordering, the set

$$D = \{t \geq 0, \phi(t, p) \in \Sigma\} = \{0 < t_1 < t_2 < ... < t_n < ...\}. \tag{166}$$

Set $p_1 = p$, $p_2 = \phi(t_1, p)$ and inductively $p_n = \phi(t_{n-1}, p)$. If $p_1 = p_2$, the orbit is periodic and $p_n = p$, for all n. If $p_1 \neq p_2$, say for instance that $p_1 < p_2$. If there exists $p_3 \neq p_2$, then necessarily $p_2 < p_3$.

As Σ is connected, the orbits cut the section in "the same direction" for instance from left to right. Consider the Jordan curve obtained by the segment p_1p_2 on Σ

followed by the arc of solution $\{\phi(t,p), 0 \leq t \leq t_1\}$ between p_1 and p_2. By Jordan's theorem, this closed curve displays an interior and an exterior. The orbit γ enters the interior in p_1 and cannot get out. It then follows that $p_1 < p_2 < p_3$. Inductively this yields the result. □

Lemma 43. *For all point $p \in V$, the set $\omega(p)$ contains at most one point in Σ.*

Proof. By the first lemma, a point of $\Sigma \cap \omega(p)$ is necessarily the limit of points of the orbit on Σ. By the second lemma, this sequence is monotonous. It is then necessarily convergent and any subsequence would converge to the same and unique point. □

Lemma 44. *Let p be a point of V so that $\gamma^+(p)$ is precompact. Let γ be an orbit contained in $\omega(p)$. If $\omega(\gamma)$ contains a regular point, then γ is a closed orbit and $\omega(p) = \gamma$.*

Proof. Let $q \in \omega(\gamma)$ be a regular point. Consider the neighborhood U of q with an associated transversal section σ. From first lemma there is a sequence $t_n \to \infty$ so that $\gamma(t_n) \in \sigma$. As $\gamma(t_n) \in \omega(p)$, by the second lemma, this sequence is reduced to a single point and the orbit γ is periodic.

Let $\overline{p}$ be an arbitrary point of γ. Let $V_{\overline{p}}$ the neighborhood and $\sigma_{\overline{p}}$ the transverse section provided by the Flow-Box theorem. There is the obvious inclusion

$$V_{\overline{p}} \cap \gamma \subset V_{\overline{p}} \cap \omega(p). \tag{167}$$

Assume there is a point $q' \in V_{\overline{p}} \cap \omega(p)$ which does not belong to γ. Then, there exists a $t \in \mathbb{R}$ so that $\phi(t,q') \in \omega(p) \cap \Sigma_{\overline{p}}$ and$\phi(t,q') \neq \overline{p}$. This is a contraction because these two points belong both to $\omega(p)$ and $\sigma_{\overline{p}}$, d'après le lemme 3. This displays the equality

$$V_{\overline{p}} \cap \gamma = V_{\overline{p}} \cap \omega(p). \tag{168}$$

Define then the open set $U = \cup_{\overline{p}\in\gamma} V_{\overline{p}}$ so that $U \cap \omega(p) = U \cap \gamma = \gamma$ which yields that γ is open $\omega(p)$. As γ is closed and $\omega(p)$ is connected, this displays the equality $\gamma = \omega(p)$. □

Finally, the proof of the Poincaré-Bendixson theorem can be achieved as follows:

Proof. Assume first that all points of $\omega(p)$ are regular. Consider $q \in \omega(p)$ a regular point. The orbit γ_q is contained in $\omega(p)$. As $\omega(p)$ is compact, $\omega(\gamma_q)$ is not empty. As $\omega(\gamma_q)$ is contained in $\omega(p)$ it contains only regular points. From previous lemma, follows that $\omega(p) = \gamma_q$ is a periodic orbit.

If $\omega(p)$ contains only stationary points, as it is connected then $\omega(p)$ is a stationary point.

Assume now that $\omega(p)$ contains both regular and stationary points. Let γ be an orbit contained in $\omega(p)$ which is not a stationary point. As both $\omega(\gamma)$ and $\alpha(\gamma)$ cannot contain regular points and are connected, they are stationary points and $\omega(p)$ is a singular cycle.

□

Proposition 45. *Assume that $\Gamma = \omega(\gamma)$ contains a regular point p_0, then there exists a transverse section σ to the vector field $p_0 \in \Gamma$ and a first-return mapping P differentiable on σ with $P(p_0) = p_0$.*

Proof. Let U a neighborhood of p_0 and a transversal section Σ given by the Flow-Box theorem. From previous lemma, it is known that p_0 is a limit of points in $\gamma \cap \Sigma$. Let $q_0 \in \gamma \cap \Sigma$ be one of them and choose an ordering on Σ between q_0 and p_0. By Jordan's theorem, for all points q between q_0 and p_0, there exists a return $P(q)$ on Σ. The differentiability of this mapping P on $]p_0, q_0[$ is secured by the regularity theorems of the flow. $\square$

In the following we denote by X a planar vector field defined on a relatively compact open set. The vector space $E(U)$ of C^1 vector fields defined on U is equiped with the C^1-norm.

Definition 42. *Let $X \in E(U)$ be a vector field defined on U. A compact invariant set Γ of X is a limi-periodic set if there exists a sequence of vector fields X_n defined on U, which converges to X, $X_n \to X$ and a sequence of periodic orbits γ_n of X_n which tends to Γ in the sense of the Hausdorff topology on compact sets.*

Proposition 46. *Let Σ be a transverse section to the vector field X and Γ be a limit-periodic set X then Γ intersects Σ in at most a point.*

Proof. If Γ would intersect Σ in two distinct points, as $X_n \to X$, for n large enough, section Σ would also be transverse to X_n and the periodic orbit γ_n would also cut the section in more than a point. $\square$

Proposition 47. *If all points of a limit-periodic set Γ are regular, then Γ is a periodic orbit.*

Proof. Let $q \in \Gamma$ a regular point, $\gamma(q) \subset \Gamma$ and $\omega(\gamma(q)) \subset \Gamma$ is non-empty and union of regular points. Let $p \in \omega(\gamma(q))$ and Σ a transverse section by p. There is a sequence of points of $\Gamma(q)\cap\Sigma$ which converges to p. As these points are in Γ,$\gamma(q)$ is a periodic orbit. Let $\overline{p}$ be a point of $\gamma=\gamma(q)$ anf $V_{\overline{p}}$ a neighborhood of $\overline{p}$ given by the Flow-Box theorem. Assume there may exist a point $q\in V_{\overline{p}}\cap\Gamma$ which does not belong to $V_{\overline{p}}\cap\gamma$. In that case, there would exist two distinct points in $\Sigma\cap\Gamma$. Connexity of Γ implies that $\Gamma=\gamma$ is a periodic orbit. □

Proposition 48. *A limit-periodic set of a planar vector field can be a stationary point, a periodic orbit or an union of orbits whose α-limit and ω-limit sets are stationary points.*

Proof. This follows the lines of the previous proof of the Poincaré-Bendixson theorem. □

3.4. **Perturbative theory of periodic orbits.** In this section, it is convenient to associate to a planar vector field $X=f(x,y)\frac{\partial}{\partial x}+g(x,y)\frac{\partial}{\partial y}$ a 1-form

$$\omega=f(x,y)dy-g(x,y)dx. \tag{169}$$

This 1-form vanishes identically along the solutions of the vector field X. For instance, the 1-form corresponding to the Hamiltonian vector field associated with H is $\omega=dH$.

Assume that the connected components of the level sets $H=c, c_0<c<c_1$ are compact and non critical.

Lemma 49. *The connected components of the level sets $H=c, c_0<c<c_1$ are periodic orbits of the Hamiltonian vector field X_H.*

Proof. Consider p a point on the curve $H=c$. Let γ be the orbit of p for the vector field X_H. Consider $\omega(\gamma)\subset H^{-1}(c)$. By the Poincaré-Bendixson theorem, $\omega(\gamma)=\gamma$

is a periodic orbit. It is closed in $H = c$ and obviously open, by the fundamental theorem of existence of local solutions. As $H = c$ is connected, γ is equal to the connected component of $H = c$ $\square$

Consider now a perturbation of the Hamiltonian vector field:

$$X_\epsilon = X_H + \epsilon X, \tag{170}$$

so that the associated 1-form is

$$\omega_\epsilon = dH + \epsilon\omega. \tag{171}$$

Let Σ be a transversal section to the flow of X_H to a periodic orbit $H = c$. For ϵ small enough, Σ remains transversal to the perturbed flow and there is a return map $L_\epsilon : \Sigma \to \Sigma$. To simplify the notations, we take as coordinate on the transversal section, the value of the Hamiltonian. Let γ_ϵ be the arc of solution of $\omega_\epsilon = 0$ between the point $c \in \Sigma$ and the point of first return of the flow $F(c, \epsilon)$. This displays:

$$\int_{\gamma_\epsilon} \omega_\epsilon = 0. \tag{172}$$

Assume that both H and X are C^1. Then the first return map is also C^1 $L_\epsilon : c \mapsto c + L_1(c)\epsilon + O(\epsilon)$ and at first order in ϵ this yields

$$L_1(c) = \int_{H=c} \omega. \tag{173}$$

Assume now (in view for instance to application to polynomial vector fields) that both H and X are analytic. Then the return map is itself analytic. The periodic orbits displaid by the perturbed vector field in the neighborhood of $H = c$ are in one-to-one correspondence with the solutions of the analytic equations:

$$L(c, \epsilon) - c = 0. \tag{174}$$

Two situations can occur. Assume first that the solutions of $L_1(c) = 0$ are isolated. Then by the Weierstrass preparation theorem, the isolated zeros of the abelian integral gives birth to periodic orbits of the perturbed vector field for ϵ small. No other periodic orbits may occur in that case. In contrast, the function $c \mapsto L_1(c)$ may be identically zero. In that case, it is useful to introduce the following condition (*-property):

Definition 43. *The function H displays the *-property near $H = c_0$ if for all polynomial form ω such that $\int_{H=c} \omega = 0$ for all c close to c_0, there exists two polynomials g and R so that*

$$\omega = gdH + dR. \tag{175}$$

If this condition is satisfied, then considering

$$(1 - \epsilon g)\omega_\epsilon = dH + \epsilon(gdH + dR) - \epsilon gdH + o(\epsilon^2) = d(H + \epsilon R), \tag{176}$$

with:

$$\int_{\gamma_\epsilon} (1 - \epsilon g)\omega_\epsilon = 0, \tag{177}$$

yields

$$\begin{array}{l} L(c,\epsilon) = c + \epsilon^2 L_2(c) + o(\epsilon^3), \\ L_2(c) = -\int_{H=c} g\omega. \end{array} \tag{178}$$

Again, two possibilites occur. Either the abelian integral $\int_{H=c} g\omega$ displays isolated zeros and then the possible isolated periodic orbits of the perturbed vector field can be born near the zeros of this integral (by Weierstrass preparation theorem) or this abelian integral vanishes identically. Again, in that last case, the $*$-property yields existence of two polynomials g_2 and R_2 so that with $g_1 = g$,

$$g_1\omega = g_2 dH + dR_2, \tag{179}$$

then considering the 1-form $[1-\epsilon g_1+\epsilon^2 g_2](dH+\epsilon\omega)$ yields the expression of the third derivative

$$L_3(c)=\int_{H=c} g_2\omega. \tag{180}$$

In general, this process allows to compute the first derivative of the return map which is not identically zero. In case it never stops, it yields both a first integral

$$F=H+\epsilon R_1+\epsilon^2 R_2+... \tag{181}$$

and an integrating factor

$$1-\epsilon g_1+\epsilon^2 g_2+... \tag{182}$$

In this case, of course no limit cycle can appear.

3.5. **The center-focus problem.** Consider the set of polynomial planar vector fields X_n of type:

$$\begin{array}{l} \dot{x}=-y+\sum_{i,j/i+j=d} a_{i,j}x^iy^j=-y+P(x,y), \\ \dot{y}=x+\sum_{i,j/i+j=d} b_{i,j}x^iy^j=-y+Q(x,y). \end{array} \tag{183}$$

The coefficients of the vector fields (a,b) can take any real values and thus (a,b) should be considered as a point in the vector space $\mathbb{R}^{2(d+1)}$. Write the vector field in polar coordinates (r,θ):

$$x=r\text{cos}(\theta), y=r\text{sin}(\theta). \tag{184}$$

This displays:

$$\begin{array}{l} 2r\dot{r}=2(x\dot{x}+y\dot{y}), \\ r\dot{r}=xP+yQ=r^{d+1}A(\theta), \\ \dot{\theta}=(x\dot{y}-y\dot{x})/(x^2+y^2)=1+r^{d-1}B(\theta), \end{array} \tag{185}$$

where $A(\theta)$ and $B(\theta)$ are two trigonometric polynomials (in $\cos(\theta), \sin(\theta)$) linear in the parameters (a, b).

This yields:

$$dr/d\theta = r^d A(\theta)/[1 + r^{d-1} B(\theta)], \tag{186}$$

and thus

$$dr/d\theta = \sum_{k=0}^{\infty} (-1)^k r^{k(d-1)+d} A(\theta) B(\theta)^k. \tag{187}$$

This equation may be rewritten as:

$$dr/d\theta = \sum_{k \geq d} r^k R_k(\theta), \tag{188}$$

where the coefficients $R_k(\theta)$ are trigonometric polynomials in $(\cos(\theta), \sin(\theta))$ and polynomials in the parameters (a, b). To simplify the notations, the dependence on the parameters (a, b) is not made explicit. Look for a solution $r = r(\theta)$ so that $r(0) = r_0$, given as an expansion:

$$r = r_0 + v_2(\theta) r_0^2 + ... + v_k(\theta) r_0^k + ... \tag{189}$$

The identification of the unknown coefficients v_k can be done:

$$\begin{array}{l} v_2(\theta) = ... = v_{d-1}(\theta) = 0, \\ v'_d(\theta) = R_d(\theta), \\ v'_k(\theta) = \sum_{i=2}^{k} B_{ik}[v_d(\theta), ..., v_{k-1}(\theta)] R_i(\theta), k \geq d+1. \end{array} \tag{190}$$

The polynomial $B_{ik}[a_d, ..., a_{k-1}]$ is the coefficient of X^{k-i} in $(X + a_d X^d + ... + a_p X^p + ...)^i$ and thus it displays integer coefficients.

The equation determines inductively the functions $v_k(\theta)$:

$$v_k(\theta) = \int_0^\theta [\sum_{i=2}^{k} B_{ik}[v_d(\phi), ..., v_{k-1}(\phi)]R_i(\phi)]d\phi. \tag{191}$$

Two facts can be easily derived from this construction:

i) $v_k(\theta)$ is polynomial in θ (of degree less than k) and in $(sin(\theta), cos(\theta))$.

ii) $v_k(\theta)$ is polynomial in the parameters (a, b) of the vector field. Thus in particular the coefficients $v_k(2\pi)$ of the return mapping are polynomials in (a, b).

Definition 44. *A planar vector field displays a center at the origin*$(0, 0)$ *if all orbits of a neighborhood of the origin are periodic.*

What precedes show in particular the following

Theorem 50. *The set of polynomial vector field of X_n which displays a center at the origin is an algebraic manifold.*

Proof. This set is defined by the simultaneous vanishing of the coefficients $v_k(2\pi)$ of the return mapping which are polynomials in (a, b). □

Let $f_\lambda(x) = \sum a_k(\lambda)x^k$ be an analytic series in x with polynomial coefficients in the parameters $\lambda = (\lambda_0, ..., \lambda_D)$. Denote $\mid a_k \mid$ (norm of the polynomial $a_k(\lambda)$) as the sum of the absolute value of the coefficients.

Definition 45. *The series f_λ is called an A_0-series if the following two conditions are satisfied:*

There are positive constants K_1, K_2, K_3, K_4 such that:

1- $deg(a_k) \leq K_1 k + K_2$,

2- $\mid a_k \mid \leq K_3 K_4^k$.

A_0-series form a subring of the ring of formal power series in x with polynomial coefficients in λ. All the usual analytic operations, like substitution to a given analytic function, composition, inversion, etc... transform A_0-series into themselves.

Lemma 51. *An A_0-series $f_\lambda(x)$ converges in the disc $D(0,R)$ of radius* $R = [1/(K_4(1+ \mid \lambda \mid))^{K_1}]$.

Proof. This is an easy fact proved by majorizing series. $\square$

In the following, we also denote by f_λ the complex analytic function defined for all $\lambda \in \mathbb{C}^D$ on the disc $D(0,R)$ by the A_0-series.

Proposition 52. *For all θ, the series $\sum_{k\geq d} x^k R_k(\theta)$, is an A_0-series with $K_1 = 1/(d-1)$, $K_2 = -1/(d-1)$, $K_3 = 1/[2(d+1)]^{1/d-1}$, $K_4 = [2(d+1)]^{1/d-1}$.*

Proof. This is indeed a simple consequence of the previous computations. For all θ, the norms of the polynomials $A(\theta)$ and $B(\theta)$ (seen as polynomials in (a,b)),

$$\begin{aligned} A(\theta) &= (xP + yQ)[cos(\theta), sin(\theta)], \\ B(\theta) &= (xQ - yP)[cos(\theta), sin(\theta)], \end{aligned} \tag{192}$$

are bounded by:

$$\mid A(\theta) \mid \leq 2(d+1), \mid B(\theta) \mid \leq 2(d+1). \tag{193}$$

Write:

$$dr/d\theta = \sum_{k\geq d} r^k R_k(\theta) = \sum_{j\geq 0} (-1)^j r^{d+j(d-1)} A(\theta) B(\theta)^j. \tag{194}$$

Denote $k = d + j(d-1)$, then this yields:

$$deg[R_k(\theta)] \leq 1 + j \leq 1 + [(k-d)/(d-1)] \leq [(k-1)/(d-1)]. \tag{195}$$

Furthermore, the norm of $R_k(\theta)$ as polynomial in the parameters (a, b) is estimated by:

$$| R_k(\theta) |=| A(\theta)B(\theta)^j |\leq| A(\theta) || B(\theta) |^j\leq [2(d+1)]^{[(k-1)/(d-1)]}. \tag{196}$$

Theorem 53. *For all θ, the series $\sum_{k\geq d} x^k v_k(\theta)$, is an A_0-series with $K_1' = 1/(d-1)$, $K_2' = 0$, $K_3' = [2\pi K_3/4K_4^2][K_4 - 2C + ((K_4 - 2C)^2 - K_4^2]^2$, $1/K_4' = |K_4 - 2C + ((K_4 - 2C)^2 - K_4^2|/[2K_4^2]$, $C = K_4 + 2\pi K_3 K_4^2$.*

Proof. First observe that $deg(v_d(\theta)) = deg(R_d(\theta)) = 1$ (The degree as polynomial in the parameters). Thus we have $deg(v_d(\theta)) \leq d/(d-1)$. Assume inductively that:

$$deg(v_j(\theta)) \leq j/(d-1) for j = d, ..., k-1. \tag{197}$$

The recurrency relation displays:

$$\deg[v_k(\theta)] \leq max[\deg(B_{ik}[v_d(\phi)), ..., \deg(v_{k-1}(\phi)]R_i(\phi))], i = 2, ..., k. \tag{198}$$

This shows that:

$$deg[B_{ik}[v_d(\phi), ..., v_{k-1}(\phi)]R_i(\phi)] \leq K_1(k-i) + K_1 i + K_2 \leq K_1 k. \tag{199}$$

This shows the first part of the theorem on the bound of the degrees of the coefficients $v_k(\theta)$. For the second part of the proof related to the bound on the norms of the coefficients, we use standard methods of majorant series. □

Definition 46. *The formal series $\Psi(x) = \sum_{k\geq 1} \Psi_k x^k$ with positive coefficients dominates the formal series $\Phi(x) = \sum_{k\geq 1} \Phi_k x^k$ with positive coefficients if and only if for all coefficients $\Phi_k \leq \Psi_k, k \geq 1$.*

The series $x+\sum_{k\geq 2}\mid v_k(\theta)\mid x^k$ is dominated by the series $x+\sum_{k\geq 2}\Psi_k x^k$ so that:

$$\psi_k = 2\pi \sum_{i=d}^{k} B_{ik}(\psi_d, ..., \psi_{k-1}) \mid R_i \mid . \tag{200}$$

Denote

$$\bar{R}(x) = \sum_{i\geq 2} K_3 K_4^i x^i = K_3 (K_4 x)^2/[1-K_4 x], \tag{201}$$

This yields:

$$\sum_{i\geq d} \mid R_i \mid x^i \quad \text{is} \quad \text{dominated} \quad \text{by} \quad \bar{R}(x). \tag{202}$$

The series $\Psi(x) = x+\sum_{k\geq 2}\Psi_k x^k$ is then dominated by the solution $\bar{\Psi}(x)$ solution of the equation:

$$\bar{\Psi}(x) - x = 2\pi \bar{R}[\bar{\Psi}(x)] = 2\pi K_3 [K_4 \bar{\Psi}(x)]^2/[1-K_4 \bar{\Psi}(x)]. \tag{203}$$

At this point we have obtained that $\bar{\Psi}(x)$ is a solution of an algebraic equation of degree two. Estimates of the constant K_4' is then obtained by the distance to the first zero of the discriminant and constant K_3' is then adjusted from the first term of the development. $\square$

Definition 47. *The center set C is the set of values of parameters (a, b) so that the corresponding vector field X has a center at the origin.*

From now on, it is appropriated to change of notations and denote $L_k(a, b) = v_k(2\pi)$ the coefficients of the return mapping to emphasize their dependence in terms of the parameters (a, b). We denote:

$$r \mapsto L(r) = r + L_d(a, b) r^d + ... + L_k(a, b) r^k + ... \tag{204}$$

the return mapping defined for $\theta = 2\pi$.

Definition 48. *The Bautin ideal is the ideal generated in the ring* $\mathbb{R}[a, b]$ *by the coefficients* $L_k(a, b)$.

Definition 49. *The Bautin index is the first integer* k_0 *so that the polynomials* $L_d(a, b), ..., L_{k_0}(a, b)$ *generate the Bautin ideal.*

Note that the existence of the Bautin index just follows from the fact that the ring $\mathbb{R}[a, b]$ is Noetherian.

The local Hilbert's 16th problem is finding a bound depending only on d to the number of limit cycles of X in a neighborhood of the origin. Isolated periodic orbits of X defined in the domain of definition of the return mapping correspond exactly to the isolated solutions of the equation:

$$L(r) - r = 0. \tag{205}$$

We gradually change into notations more pertinent to the general algebraic geometry setting. Let $\Phi(x, \lambda)$ be an analytic series in x with polynomial coefficients in the parameter $\lambda = (\lambda_1, ..., \lambda_D)$.

Definition 50. *The Bautin ideal of the series* $\Phi(x, \lambda)$ *is the ideal generated in the ring* $\mathbb{R}[\lambda]$ *by the coefficients* $\Phi_k(x, \lambda)$. *The center set of the series* $\Phi(x, \lambda)$ *is the zero set of its Bautin ideal. The Bautin index d of the series* $\Phi(x, \lambda)$ *is the minimal integer d such that the coefficients* $\Phi_1(x, \lambda), ...\Phi_d(x, \lambda)$ *generate the Bautin ideal of the series* $\Phi(x, \lambda)$.

Definition 51. *Two series* $\Phi(x, \lambda)$ *and* $\Psi(x, \lambda)$ *with polynomial coefficients in the parameters* λ *are said to be* ϕ*-equivalent if for all integers* $k \geq 1$, *the polynomial* $\Phi_k(\lambda) - \Psi_k(\lambda)$ *belongs to the ideal generated by* $(\Phi_1(\lambda), ..., \Phi_{k-1}(\lambda))$.

Let $\Phi(x, \lambda)$ be an analytic series in x with polynomial coefficients in the parameters λ and with Bautin index d. Assume that $\Phi(0, \lambda) = 0$ and $\Phi(x, 0) = 0$.

Theorem 54. *There is a ball $B \in \mathbb{R}^D$, centered at 0, in the space $\mathbb{R}^D$ and an interval I containing 0 such that for all $\lambda \in B$, the number of zeros of $\Phi(x, \lambda)$ contained in I is less than or equal to d.*

Proof. Using the definition of the Bautin index d, we write:

$$\Phi(x, \lambda) = \sum_{i=1}^{d} \Phi_i(\lambda)[1 + \Psi_i(x, \lambda)]x^i, \tag{206}$$

with $\Psi_i(x, \lambda)$ analytic in x, with polynomial coefficients in λ such that $\Psi_i(0, \lambda) = \Psi_i(x, 0) = 0$. Assume that B and I are small enough so that (for instance) $\mid \Psi_i(x, \lambda) \mid \leq 1/2$ on $I \times B$. Then divide $\Phi(x, \lambda)$ by $[1 + \Psi_d(x, \lambda)]$ and write:

$$[\Phi(x, \lambda)]/[1 + \Psi_d(x, \lambda)] = \Phi_1(\lambda) + \Phi_2(\lambda)[1 + \Psi_i'(x, \lambda)]x^2 + ... \tag{207}$$

Then from Rolle's lemma, the number of zeros of $\Phi(x, \lambda)$ is less than 1+ number of zeros of the derivative $[\Phi(x, \lambda)]/[1 + \Psi_d(x, \lambda)]'$. Write then this derivative as

$$\Phi_2(\lambda)[1 + \Psi_i^{(2)}(x, \lambda)]x + ...\Phi_{d-1}(\lambda)[1 + \Psi_i^{(2)}(x, \lambda)]x^{d-1}. \tag{208}$$

Then repeat the process (sometimes referred to as the division-derivation algorithm). We obtain the result by an easy induction. □

It is posible to formulate this last result in terms of projection of analytic sets. Let $\Phi : \mathbb{R} \times \mathbb{R}^D \mapsto \mathbb{R}$ be an analytic series with polynomial coefficients:

$$\Phi(x, \lambda) = x + \Phi_2(\lambda)x^2 + ... + \Phi_k(\lambda)x^k + ... \tag{209}$$

We consider the subset $\Sigma \subset \mathbb{R} \times \mathbb{R}^D$ defined as the zero-set of $\Phi(x, \lambda) - x$:

$$\Sigma = [(x, \lambda)/\Phi(x, \lambda) - x = 0]. \tag{210}$$

Let $\pi : \Sigma \mapsto \mathbb{R}^D$ be the restriction to Σ of the natural projection $\pi : \mathbb{R} \times \mathbb{R}^D \mapsto \mathbb{R}^D$. The center set associated to the analytic series $\Phi(x, \lambda) - x$ is the set $C \subset \mathbb{R}^D$ of parameters λ such that the fiber of the projection $\pi^{-1}(\lambda)$ is contained in the set Σ. The last theorem can be reformulated as follows

Theorem 55. *There is a neighborhood $I \times B$ of $(0, 0)$ in $\mathbb{R} \times \mathbb{R}^D$ such that for all points λ of B the number of isolated points of the fibers $\pi^{-1}(\lambda)$ restricted to Σ is less than the Bautin index d of the analytic series $\Phi(x, \lambda) - x$.*

3.6. **The Poincaré compactification of polynomial planar vector fields.** Consider a polynomial vector field of the plane:

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y), \end{aligned} \tag{211}$$

where f et g are two polynomials so that $n = \max(degf, degg)$. The Poincaré transformation maps the plane $\mathbb{R}^2$ on the half-sphere of $\mathbb{R}^3$:

$$\{(u, v, w); u^2 + v^2 + w^2 = 1, \quad w > 0, \tag{212}$$

by:

$$\begin{aligned} x &= \tfrac{u}{w}, \\ u &= \tfrac{x}{\sqrt{1+x^2+y^2}}, \\ v &= \tfrac{y}{\sqrt{1+x^2+y^2}}, \\ w &= u = \tfrac{1}{\sqrt{1+x^2+y^2}}. \end{aligned} \tag{213}$$

This yields:

$$\begin{aligned} \dot{u}w - u\dot{w} &= w^2 f(\tfrac{u}{w}, \tfrac{v}{w}), \\ \dot{v}w - v\dot{w} &= w^2 g(\tfrac{u}{w}, \tfrac{v}{w}). \end{aligned} \tag{214}$$

This displays:

$$-\dot{w}w^{n-2} = w^n u f(\frac{u}{w}, \frac{v}{w}) + w^n v g(\frac{u}{w}, \frac{v}{w}) = u f_1(u, v, w) + v g_1(u, v, w), \tag{215}$$

where f_1 et g_1 are polynomials. Function $t \mapsto w^{n-1}$ is strictly positive so that it is possible to divide the time t by w^{n-1} and this yields:

$$\begin{aligned} \dot{u} &= (v^2 + w^2) f_1(u, v, w) - uv g_1(u, v, w), \\ \dot{v} &= -uv f_1(u, v, w) + (u^2 + w^2) g_1(u, v, w), \\ \dot{w} &= -uw f_1(u, v, w) - vw g_1(u, v, w). \end{aligned} \tag{216}$$

This vector field displays the first integral $u^2 + v^2 + w^2$. Compactification of the plane produces a circle ta infinity which is globally invariant and identified to the equator ($w = 0$) of the sphere. Setting $u = \cos\theta$ and $v = \sin\theta$, the system displays at infinity $w = 0$,:

$$\dot{\theta} = -\sin\theta f_1(\cos\theta, \sin\theta, 0) + \cos\theta g_1(\cos\theta, \sin\theta, 0). \tag{217}$$

The zeroes of the second member of the equation are called stationary points at infinity of the Poincaré compactification. These stationary points can be studied with the same methods as the usual ones. Note that they appear by symmetric pairs. Their study is usefull to understand the asymptotics of unbounded trajectories. For instance,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x, \end{aligned} \tag{218}$$

displays

$$\begin{aligned} \dot{u} &= (1 - 2u^2) v \\ \dot{v} &= (1 - 2v^2) u \\ \dot{w} &= -2uvw, \end{aligned} \tag{219}$$

It yields four stationary points at infinity which are symmetric pairs $\pm\frac{1}{\sqrt{2}}(1, 1, 0)$ and $\pm\frac{1}{\sqrt{2}}(1, -1, 0)$. One is an attractive node and the other is a repulsive node.

With the Airy system:

$$\dot{x} = x^2 - y, \dot{y} = \epsilon, \tag{220}$$

the Poincaré transformation yields:

$$\begin{aligned}
\dot{u} &= u\mu + u^2 - vw \\
\dot{v} &= v\mu + \epsilon w^2 \\
\dot{w} &= w\mu \\
\mu &= -(u^3 - uvw + \epsilon vw^2.
\end{aligned} \tag{221}$$

This displays as stationary points at infinity $(1, 0, 0)$ which is an attractive node, its symmetric $(-1, 0, 0)$ which is a repulsive node, $(0, 1, 0)$ and $(0, -1, 0)$ which are repulsive but not elementary.

3.7. **Structural stability and dangerous boundaries of stability domains.** Consider the set $C^1(K)$ of planar vector fields of class C^1 defined on a compact $K \subset \mathbb{R}^2$. Assume that the boundary ∂K is a smooth curve not tangent to the vector field. This set, endowed with the norm

$$|| X ||= Sup_{x\in K}(|| X || + || \frac{\partial X}{\partial x} ||, \tag{222}$$

becomes a Banach space.

Definition 52. *A vector field X is said to be rough (structurally stable) if for all ϵ there is a δ so that:*

1) All Y in a δ-neighborhood of X is topologically equivalent to X and moreover

2) The homeomorphism, which establishes the topological equivalence, is ϵ-close to the identity.

Lemma 56. *A structurally stable vector field X displays hyperbolic stationary points.*

Proof. Assume that the vector field X displays a non elementary stationary point which, after translation is the origin. The Taylor developement of X near the origin writes:

$$\begin{aligned}
f(x, y) &= f_2(x, y) + f_3(x, y) + ... \\
g(x, y) &= g_2(x, y) + g_3(x, y) + ...
\end{aligned} \tag{223}$$

Then consider a pertubation by an arbitrarily small linear vector field:

$$\begin{aligned} \xi &= \epsilon\lambda x \\ \eta &= \epsilon\mu y, \end{aligned} \tag{224}$$

With ϵ arbitrarily small. The Poincaré-Lyapunov theorem tells that the perturbed vector field displays, depending of sign of λ and μ either a saddle or a node and there is no structural stability. Assume that the vector field would display a linear center at the origin, the same pertubation would yield a stable or unstable focus. This shows that stationary points of a structurally stable vector field are necessarily hyperbolic. □

Consider a family of vector fields depending of p parameters $\epsilon = (\epsilon_1, ..., \epsilon_p)$:

$$\begin{aligned} \tfrac{dx}{dt} &= f(x, y, \epsilon), \\ \tfrac{dy}{dt} &= g(x, y, \epsilon), \end{aligned} \tag{225}$$

The notion of safe and dangerous boundaries of stability was proposed by Bautin who studied boundaries of stability domains for stationary points. It was latter extended to stability domains of periodic orbits.

Definition 53. *A point ϵ_0 on the boundary of stability of the stationary point O_ϵ(resp. periodic orbit) is said to be safe if O_{ϵ_0} is asymptotically stable (resp. orbitally asymptotically stable). It is said to be dangerous if $O(\epsilon_0)$ is Lyapunov unstable.*

Definition 54. *A stability boundary is dynamically definite if upon crossing the boundary the behaviour of the representative solution is uniquely defined (The unstable set of the stationary point contains at most an attractor). In contrast, if the new regime is not well defined, such a boundary is dynamically indefinite.*

For instance,

$$\dot{x} = \epsilon x - x^3, \tag{226}$$

displays a safe boundary which is dynamically indefinite at $\epsilon = 0$.

Take a family which displays for $\epsilon > 0$ a repulsive limit cycle around an attractive focus. As $\epsilon \to 0$, this limit cycle shrinks to 0 and this boundary is dangerous as the only point of that boundary is an unstable focus. This dangerous boundary may be dynamically undefined if, for instance, the system displays a singular cycle with two attractive nodes and two saddle nodes.

The classification of the "principal" cases of stability boundaries displays 11 + 2 cases). We list the cases which can occur in planar systems.

(1) Let O_ϵ be a stationary point and assume that O_0 displays a couple of purely imaginary complex eigenvalues. In this case, the system X_ϵ writes (generically):

$$\begin{array}{l} \dot{x} = \rho(\epsilon)x - \omega(\epsilon)y + [L_1(\epsilon)x - \Omega_1(\epsilon)y](x^2+y^2) + ... \\ \dot{y} = \omega(\epsilon)x + \rho(\epsilon)y + [\Omega_1(\epsilon)x + L_1(\epsilon)y](x^2+y^2) + ... \\ \omega(0) \neq 0, \quad \rho(0) = 0, \quad \rho(\epsilon)\epsilon > 0, \quad \text{if} \epsilon \neq 0. \end{array} \tag{227}$$

If $L_1(0) < 0$ then this boundary is safe. This is the so-called Andronov-Poincaré-Hopf bifurcation: when ϵ increases from 0, a unique stable cycle emerges from the weak attractive focus O_0.

(2) A stable stationary point can loose its stability in a saddle-node or saddle-focus bifurcation. This means that a stable node (or focus) approaches another stationary point (saddle) and at the bifurcation they merge into a single stationary point. At the bifurcation the stationary point O_0 is no longer stable and this is a dangerous boundary.

(3) The limit of a periodic orbit is a (homoclinic) singular cycle Γ composed of a saddle-node stationary point and its separatrix. The stability boundary in that

case is safe since the curve Γ is stable. Beyond the bifurcation, a stable stationary point appears.

4. Fast-slow systems

In this part, we use Landau notations for asymptotics.

Definition 55. *If ϵ is a small parameter, f et g are functions of ϵ, denote*

$$f(\epsilon) = O(g(\epsilon)), \tag{228}$$

if there exists K so that for ϵ small enough,

$$\mid f \mid / \mid g \mid < K. \tag{229}$$

Denote

$$f(\epsilon) = o(g(\epsilon)), \tag{230}$$

if furthermore

$$\lim_{\epsilon \to 0} \frac{f(\epsilon)}{g(\epsilon)} = 0. \tag{231}$$

Definition 56. *A function $F(\epsilon)$ displays an asymptotic development*

$$\sum_{n=0}^{\infty} a_n f_n(\epsilon), \tag{232}$$

if for all pour tout N,

$$F(\epsilon) = \sum_{0}^{N} a_n f_n(\epsilon) + o(f_N(\epsilon)). \tag{233}$$

4.1. Basic theorems of asymptotics.

Definition 57. *A slow-fast system of type $m + k$, defined on an open set U de $\mathbb{R}^n = \mathbb{R}^m \bigoplus \mathbb{R}^k$ is given by*

$$\begin{aligned} &\epsilon \dot{x} = f(x, y) \\ &\dot{y} = g(x, y), (x, y) \in \mathbb{R}^m \bigoplus \mathbb{R}^k. \end{aligned} \tag{234}$$

The variables $x \in \mathbb{R}^m$ are called fast variables, variables y are slow. Parameter ϵ, which measures the ratio of time scales between x and y is assumed to be small. Change of time t into ϵt yields a new system (defined on another time sale):

$$\begin{aligned} &\dot{x} = f(x, y) \\ &\dot{y} = \epsilon g(x, y), \\ &(x, y) \in \mathbb{R}^m \bigoplus \mathbb{R}^k. \end{aligned} \tag{235}$$

Take the singular limit $\epsilon \to 0$. The first system tends to a differential system with constraints:

$$\dot{y} = g(x, y), f(x, y) = 0. \tag{236}$$

The second system displays the so-called fast system:

$$\dot{x} = f(x, y), \dot{y} = 0. \tag{237}$$

Definition 58. *The set defined by the equations $f(x, y) = 0$ is called the critical set.*

Definition 59. *Points of the critical set where $D_x f(x, y)$ is singular are called fold points.*

To keep the course self-contained, we stick to a basic level which suffices to analyse the relaxation oscillations.

Consider differential equations

$$\frac{dx}{d\tau} = F(x, \tau, \epsilon), \quad x(0, \epsilon) = x_0, \tag{238}$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

(i) F is C^0 and uniformly bounded in G,

$$G = \{x \in \overline{D}\} \times \{0 \leq \tau < A\} \times \{0 \leq \epsilon \leq \epsilon_0\}, \tag{239}$$

with D open and bounded in $\mathbb{R}^n$.

(ii) F is Lipschitz-continuous in $x \in G$.

(iii) $D_0 \subset D$ compact so that the distance $\delta(D_0, \partial D)$ to the boundary of D is bounded independently of ϵ.

Theorem 57. *Consider two function F_1 and F_2 satisfying the conditions above. Let assume that the two solutions*

$$\begin{aligned} x^1(\tau,\epsilon) &= x_0^1 + \int_0^\tau F^1(x^1(u,\epsilon),u,\epsilon)du \\ x^2(\tau,\epsilon) &= x_0^2 + \int_0^\tau F^2(x^2(u,\epsilon),u,\epsilon)du, \end{aligned} \tag{240}$$

display:

(i) $x_0^1, x_0^2 \in D_0$, $\mid x_0^1 - x_0^2 \mid \leq \delta_0(\epsilon) = o(1)$.

(ii) For all $x \in \overline{D}$, $0 \leq \tau \leq A$,

$$\mid F_1 - F_2 \mid \leq \delta_f(\epsilon) = o(1). \tag{241}$$

(iii)Solution $x^2(\tau,\epsilon)$ exists for $0 \leq \tau \leq T$, $T < A$ and $x^2 \in D_0$.

Then solution $x^1(\tau,\epsilon)$ exists for $0 \leq \tau \leq T$ and in this interval:

$$\mid x^1(\tau,\epsilon) - x^2(\tau,\epsilon) \mid = o(\delta_0(\epsilon)) + o(\delta_f(\epsilon)). \tag{242}$$

Proof. The complete proof is not included here but it follows the lines of the regularity theorems of solutions of differential equations proved in the first chapter. □

Theorem 58. *Suppose that x^2 is an approximation of x^1: $\mid x^1(\tau,\epsilon) - x^2(\tau,\epsilon) \mid = o(1)$ for all $0 \leq \tau \leq T$ where T can be an arbitrarily large positive number, then there exists an order function $\delta(\epsilon) = o(1)$ so that $\mid x^1(\tau,\epsilon) - x^2(\tau,\epsilon) \mid = o(1)$ for all $0 \leq t \leq \frac{1}{\delta(\epsilon)}$. Similarly if $\mid x^1(\tau,\epsilon) - x^2(\tau,\epsilon) \mid = o(1)$ for $0 < d \leq \tau \leq T$ where d can be an arbitrarily small number, then there exists an order function $\eta(\epsilon) = o(\epsilon)$ so that the approximation holds for $\eta(\epsilon) \leq \tau \leq T$.*

4.2. **Relaxation oscillations.** In 1920, Balthazar van der Pol introduced the famous differential equation now bearing his name:

$$\frac{d^2x}{d\tau^2} - \mu(1-x^2)\frac{dx}{d\tau} + x = 0, \tag{243}$$

where μ is a positive parameter. Relaxation oscillations arise when μ is large. Change time $\tau = \mu t$, $\mu = \epsilon^{-1/2}$, displays:

$$\epsilon\frac{d^2x}{dt^2} - (1-x^2)\frac{dx}{dt} + x = 0. \tag{244}$$

Introducing $f(x) = -\int_0^x (1-u^2)du = -x + \frac{x^3}{3}$ yields the planar vector field:

$$\begin{array}{l} \epsilon\frac{dx}{dt} = y - f(x) \\ \frac{dy}{dt} = -x. \end{array} \tag{245}$$

In this part, we explain how asymptotic developments and the two technical theorems of the first paragraph can be used to analyse this system.

4.2.1. *The fast part of the orbit.* Consider an initial data $(x_0, f(x_0)$ so that $y_0 - f(x_0) \neq o(1)$. Transform t into $\tau = \frac{t}{\epsilon}$. Equation becomes:

$$\begin{array}{l} \frac{dx}{d\tau} = y - f(x) \\ \frac{dy}{d\tau} = -\epsilon x. \end{array} \tag{246}$$

Use previous regularity theorem and get

$$\begin{array}{l} y(\tau) - y_0 = o(\epsilon) \\ x(\tau) - \overline{x}(\tau) = o(\epsilon), \end{array} \tag{247}$$

where $\overline{x}(\tau)$ solves:

$$\frac{d\overline{x}}{d\tau} = y_0 - f(\overline{x}), \quad \overline{x}(0) = x_0. \tag{248}$$

Introduce $\overline{x}_0$ such that $y_0 = f(\overline{x}_0)$. For large values of τ, independently of ϵ, one has:

$$\overline{x}(\tau) = \overline{x}_0 + O(\exp(-f'(\overline{x}_0)\tau). \tag{249}$$

Trajectories, thus enter a $o(1)$ neighborhood of the critical set $y = f(x)$.

4.2.2. *The stable slow curves.* To study behaviour of trajectories in a $o(1)$ neighborhood of $y = f(x)$, introduce

$$y = f(x) + \sigma(\epsilon)\Phi. \tag{250}$$

This displays:

$$\begin{aligned} \tfrac{d\Phi}{d\tau} &= -f'(x)\Phi - \tfrac{\epsilon}{\sigma(\epsilon)}x \\ \tfrac{dx}{d\tau} &= \sigma(\epsilon)\Phi. \end{aligned} \tag{251}$$

Take for $\sigma(\epsilon)$ any order so that: $\sigma(\epsilon) = o(1)$ and $\frac{\epsilon}{\sigma(\epsilon)} = o(1)$. The approximation theorem shows easily that the solution enters a smaller order neighborhood. This allows to reduce to order $\sigma(\epsilon) = \epsilon$. The approximation theorem then yields that all trajectories starting in D_0 are attracted within time $t - t_0 = o(1)$ to the curve

$$\begin{aligned} &y = f(x) + \epsilon[\Phi_0(x) + o(1)], \\ &\Phi_0(x) = -\tfrac{x}{f'(x)}, \end{aligned} \tag{252}$$

which is called a stable slow curve. Time evolution along this curve follows from

$$\frac{dx}{d\tau} = \Phi_0(x) + o(1). \tag{253}$$

4.2.3. *Leaving the stable slow curve at the fold.* In this part, we consider the equation

$$\begin{aligned} &\epsilon\tfrac{dx}{dt} = y - f(x) \\ &\tfrac{dx}{dt} = -(x + \alpha), \\ &0 < \alpha < 1, \alpha \neq o(1), \alpha + 1 \neq o(1), \\ &f(x) = \tfrac{1}{2}x^2 + x^2 O(x). \end{aligned} \tag{254}$$

Introduce scalings:

$$\xi = x\epsilon^{-1/3}, \eta = y\epsilon^{-2/3}, \tag{255}$$

which yields:

$$\begin{aligned} \epsilon^{2/3}\tfrac{d\eta}{dt} &= -(\alpha + \epsilon^{2/3}\xi \\ \epsilon^{2/3}\tfrac{d\xi}{dt} &= \eta - \tfrac{1}{2}\xi^2 - \xi^2 O(\epsilon^{1/3}\xi). \end{aligned} \tag{256}$$

Trajectories are given by:

$$\frac{d\xi}{d\eta} = -\frac{\eta - \frac{1}{2}\xi^2 - \xi^2 O(\epsilon^{1/3}\xi)}{-(\alpha + \epsilon^{2/3}\xi}, \tag{257}$$

whose solutions are approximated by those of a Ricatti equation:

$$\alpha \frac{d\xi}{d\eta} = -\eta + \frac{1}{2}\xi^2. \tag{258}$$

Asymptotics of the Airy function shows that after time interval $t_1 - t_0 = O(\epsilon^{1/3})$ the flow escapes toward the second attractive branch of the critical curve.

4.3. **Existence of solutions asymptotic to the unstable part of the critical set.** Consider now

$$\begin{aligned} \epsilon \tfrac{dx}{dt} &= y - f(x) \\ \tfrac{dy}{dt} &= -(x+\alpha), \end{aligned} \tag{259}$$

with $\alpha = o(1)$, where $f(x)$ is a polynomial so that its derivative writes $f'(x) = xg(x)$, $g(0) = 1$. Note that $f(x) = \frac{1}{2}x^2 + o(x^3)$.

First assume that $g(x) = 1$. Observe that for $\alpha = 0$, the curve $y = f(x) - \epsilon$ is a solution. Try to find a curve $y - f(x) = -\epsilon + \epsilon^2\Phi_1(x,\epsilon)$ for $\alpha \neq 0$. The equations yield:

$$\begin{aligned} \epsilon \tfrac{d\Phi_1}{dt} &= -x\Phi_1 - \tfrac{\alpha}{\epsilon} \\ \tfrac{dx}{dt} &= -1 + \epsilon\Phi_1. \end{aligned} \tag{260}$$

Such a function Φ_1 should solve:

$$\epsilon[-1 + \epsilon\Phi_1]\frac{d\Phi_1}{dx} = -x\Phi_1 - \frac{\alpha}{\epsilon}. \tag{261}$$

The objective is to study perturbations of the phase portrait for $\alpha \neq 0$ and find out under which conditions on α, there exists solutions $\Phi_1(x,\epsilon)$ that remain bounded for $\epsilon \to 0$ and are defined for $x \in [x_0, x_1]$ with $x_0 > 0$ and $x_1 < 0$. The solution Φ_1 should solve the following integral equation:

$$\begin{aligned} &\Phi_1(x) = \mathrm{e}^{\frac{Q(x)}{\epsilon}}[\Phi_1(x_0)\mathrm{e}^{-Q(x_0)} + \tfrac{\alpha}{\epsilon^2}\textstyle\int_{x_0}^{x} \mathrm{e}^{-\frac{Q(u)}{\epsilon}}du] \\ &Q(x) = \tfrac{1}{2}x^2 + \epsilon^2\Phi_1(x). \end{aligned} \tag{262}$$

Introduce the new function:

$$\psi_1(x) = \Phi_1(x_0)\mathrm{e}^{\frac{1}{\epsilon}[\frac{x^2-x_0^2}{2}]} + \frac{\alpha^2}{\epsilon^2}\mathrm{e}^{\frac{1}{2\epsilon^2}x^2}\int_{x_0}^{x}\mathrm{e}^{\frac{1}{2\epsilon^2}u^2}du. \tag{263}$$

Let V be the vector space of continuous functions $v(x, \epsilon)$ defined on $[x_0, x_1]$ bounded for $\epsilon \to 0$. Consider the operator L on V defined by

$$\begin{aligned} L.v &= \mathrm{e}^{\frac{Q(x)}{\epsilon}}[\Phi_1(x_0)\mathrm{e}^{-Q(x_0)} + \tfrac{\alpha}{\epsilon^2}\int_{x_0}^{x} \mathrm{e}^{-\frac{Q(u)}{\epsilon}} du] \\ Q(x) &= \tfrac{1}{2}x^2 + \epsilon^2 v(x). \end{aligned} \tag{264}$$

By easy estimates, one gets

$$L.v = \Psi_1(x)[1 + O(\epsilon)]. \tag{265}$$

If ψ_1 is bounded ($\epsilon \to 0$), then $L.v$ is bounded and L is strictly contracting. Hence there exists a unique solution Φ_1 to $L.v = v$ and furthermore $\Phi_1 = \psi_1[1 + O(\epsilon)]$. The converse is also true.

What are now the conditions on α so that the function $\psi_1(x)$ remains bounded as $\epsilon \to 0$? Consider $x_0 > 0$ and $x < 0$. Elementary asymptotics yield

$$\psi_1(x) = \Phi_1(x_0)\mathrm{e}^{\frac{1}{\epsilon}[\frac{x^2 - x_0^2}{2}]} + \frac{\alpha}{\epsilon^{3/2}}\mathrm{e}^{\frac{1}{2\epsilon}x^2}[-\sqrt{2\pi} + o(\mathrm{e}^{-\frac{1}{2\epsilon}x^2}) + o(\mathrm{e}^{-\frac{1}{2\epsilon}x_0^2})]. \tag{266}$$

In order to keep the second term bounded, one must take:

$$\alpha = \sigma\epsilon^{3/2}\mathrm{e}^{-\frac{k}{2\epsilon}}, \tag{267}$$

where k and σ are constants. If σ is positive, the stationary point is an unstable focus. The first term is dominant when $x_0^2 < k^2$ whereas the second term is dominant if $x_0^2 > k^2$.

4.4. **Enhanced delay.** The classical transcritical bifurcation occurs when the parameter λ in the equation:

$$\dot{x} = -\lambda x + x^2 \tag{268}$$

crosses $\lambda = 0$. Equation displays two equilibria, $x = 0$ and $x = \lambda$. For $\lambda > 0$, $x = 0$ is stable and $x = \lambda$ is unstable. After the bifurcation, $\lambda < 0$, $x = 0$ is stable and $x = \lambda$ is unstable. The two axis have "exchanged" their stability.

The terminology "Dynamical Bifurcation" refers to the situation where the bifurcation parameter is replaced by a slowly varying variable. In the case of the transcritical bifurcation, this yields:

$$\begin{array}{rcl} \dot{x} & = & -yx + x^2 \\ \dot{y} & = & -\varepsilon \end{array} \tag{269}$$

where ε is assumed to be small. This yields:

$$\dot{x} = -(-\varepsilon t + y_0)x + x^2, \qquad (y_0 = y(0)) \tag{270}$$

which is an integrable equation of Bernoulli type. Its solution is:

$$x = \frac{x_0 \exp[-Y(t)]}{1 - x_0 \int_0^t \exp[-Y(u)]du}, \qquad (x_0 = x(0)) \tag{271}$$

where:

$$Y(t) = \int_0^t y(s)ds = \int_0^t (-\varepsilon s + y_0)ds = -\varepsilon \frac{t^2}{2} + y_0 t \tag{272}$$

If we fix an initial data (x_0, y_0), $y_0 > 0$, $0 < x_0 < y_0/2$, and we consider the solution with this initial data we find easily that it takes time $t = y_0/\varepsilon$ to reach the axis $y = 0$. If x_0 is quite small, that means the orbit stays closer and closer of the attractive part of the critical manifold until it reaches the axis $x = y$ and then coordinate x starts increasing. But now consider time cy_0/ε, $1 \geq c \geq 2$. Then a straightforward computation shows that:

$$Y(t) = c(1 - \frac{c}{2})\frac{y_0^2}{\varepsilon} = \frac{k}{\varepsilon} \tag{273}$$

and:

$$x(t) = O\left(\frac{x_0 \mathrm{e}^{-\frac{k}{\varepsilon}}}{1 - 2\frac{x_0}{y_0}}\right) < x_0 \tag{274}$$

This shows that, despite the repulsiveness of the axis $x = 0, y < 0$, the orbit remains for a very long time close to $x = 0$, indeed $x(t) < x_0$. Note that after a larger time, the orbit blows away from this repulsive axis. This phenomenon, although quite simply explained, is of the same nature as the existence of trajectories lingering along the unstable critical set that was presented previously.

Consider next the equation:

$$\begin{array}{rcl} \varepsilon\dot{x} & = & (1-x^2)(x-y) \\ \dot{y} & = & x \end{array} \tag{275}$$

The fast dynamics displays the invariant lines $x=-1$, $x=1$, and $y=x$. A quick analysis shows that, as the slow variable y varies, the fast system undergoes two transcritical bifurcations near the points $(-1,-1)$ and $(1,1)$. As we recalled in the first paragraph, a typical orbit near $(-1,-1)$ first displays a "delay" along the repulsive part $(x=-1, y<-1)$ of the slow manifold. Then, by hysteresis, it jumps to the attractive part $(x=1, y<1)$ till it reaches the other transcritical bifurcation where it again displays another delay along the repulsive part $(x=1, y>1)$. Then it jumps again to $(x=1, y>-1)$ and starts again. There is such a mechanism of successive enhancements of the delay after several turns generated by the hysteresis. After this intuitive explanation, we give now a formal proof of the:

Theorem 59. *For all initial data inside the strip $-1<x<1$, for all δ and for all T, the corresponding orbit spends a time larger than T within a distance less than δ to the repulsive part of the critical set.*

Proof. Inside the strip $|x|<1$, it is convenient to use the variable u: $x=\tanh u$. The system yields the equations:

$$\begin{array}{rcl} \dot{u} & = & \tanh u - y \\ \dot{y} & = & \varepsilon \tanh u \end{array} \tag{276}$$

Consider the function:

$$\Phi(u,y) = \frac{1}{2}(\tanh u - y)^2 + \varepsilon \ln(\cosh u) \tag{277}$$

and its time derivation along the flow. This displays:

$$\frac{d}{dt}\Phi(u,y) = (\frac{\dot{u}}{\cosh u})^2 \tag{278}$$

Hence the function Φ is strictly increasing along the flow – Lyapunov function for the flow. Note now that if $(u(t), y(t))$ is a solution then $(-u(t), -y(t))$ is also a solution. To study the orbits of the system, we can restrict to initial data $u = u_0 \geq 0$ and $y = y_0$. The first step of the proof is to show that all orbits intersect both axes $u = 0$ and $y = u$ in infinitely many points.

Assume first $y_0 \geq \tanh u_0$. Then $\dot{u}(0) \leq 0$. But:

$$\frac{d}{dt}(y - \tanh u) = \varepsilon \tanh u + \frac{y - \tanh u}{\cosh^2 u} \tag{279}$$

shows that $y - \tanh u$ grows hence remains positive. Assume that u would remain always positive. Then, as $y - \tanh u > 0$, u is monotone decreasing. Hence there exists l such that $u \to l$ as $t \to +\infty$.

If $l > 0$, $\dot{y} = \varepsilon \tanh u$ implies (via the mean value theorem) $y \to +\infty$ but then $\dot{u} \to -\infty$ and (mean value theorem) contradiction with $u > 0$.

If $l = 0$, $\dot{u} + y \to 0$. But y is monotone increasing. If $y \to +\infty$, then $\dot{u} \to -\infty$ and again contradiction. If y tends to a finite limit m, then $\dot{u} \to -m$ and again contradiction. Hence all orbits with initial data (u_0, y_0) with $y_0 \geq \tanh u_0 \geq 0$ intersect the axe $u = 0$.

Consider now the case $y_0 < \tanh u_0$. The variable u is first strictly increasing (as soon as $y < \tanh u$), hence positive and so y is increasing. Assume that $\tanh u - y$ would remain positive along the orbit. Then as $t \to +\infty$, u would tend to a limit m (eventually $m = +\infty$).As $\dot{y} \to m$, mean value theorem would imply $y \to +\infty$ and again a contradiction with $\dot{u} \to -\infty$. So the orbit necessarily intersects the axe $y = \tanh u$ and ultimately the axe $u = 0$ by the preceding argument. By symmetry, we can also show the existence of two sequences of times (t_n) and (θ_n) such that:

$$\begin{array}{rcl} t_n & < & \theta_n < t_{n+1} \\ x(t_n) & = & x(t_{n+1}) = 0 \\ x(\theta_n) & = & y(\theta_n) \\ y(t_n) & = & (-1)^n a_n, \quad a_n > 0 \end{array} \tag{280}$$

In the second part of the proof we show that the sequence a_n is unbounded.

Consider now the function:

$$2\Phi(u, y) = w(x, y) = (x - y)^2 - \varepsilon \ln \left|1 - x^2\right| \tag{281}$$

which satisfies:

$$\dot{w} = 2(1 - x^2)(x - y)^2 \tag{282}$$

As w is strictly increasing, this yields:

$$w(t_n) = a_n^2 < w(\theta_n) = -\varepsilon \ln(1 - \xi_n^2) < w(t_{n+1}) = a_{n+1}^2 \tag{283}$$

with $\xi_n = x(\theta_n)$. Integration along the flow of $\dot{(w)} = 2\dot{x}(x - y)$ yields:

$$a_{n+1}^2 - a_n^2 = \int_{t_n}^{t_{n+1}} 2\dot{x}(x - y)dt = 2\int_{t_n}^{t_{n+1}} x\dot{y}dt \tag{284}$$

$$= 2\varepsilon \int_{t_n}^{t_{n+1}} x^2 dt \leq 2\varepsilon(t_n - t_{n+1}) \tag{285}$$

This shows that if (t_n) converges to a finite limit then so does (a_n) and (ξ_n). Now integration along the flow of $\dot{y} = \varepsilon u$ yields:

$$a_n + a_{n+1} \leq \varepsilon \int_{t_n}^{t_{n+1}} |\tanh u|\, dt \leq \varepsilon(t_{n+1} - t_n) \tag{286}$$

and this shows that the sequence of times (t_n) is necessarily unbounded.

Now assume that the sequence (a_n) would be bounded. Then, as the sequence (t_n) tends to $+\infty$, the function w would be bounded on the orbit. But then so would be both the two functions $(x - y)^2$ and $-\ln(1 - x^2)$. But then there would exist a constant α such that $(1 - x^2) \geq \alpha$ along the orbit and

$$\dot{w} \geq 2\alpha(x - y)^2 \geq 2\alpha w \tag{287}$$

hence $\mathrm{e}^{-2\alpha t}w(t)$ increasing and contradiction with the fact that w would be bounded. Last step is classical in slow-fast dynamics, there is for all orbits inside the strip, all δ and all T a part of the orbit which remains at a distance less than δ of the repulsive parts of the boundary of the strip for a time larger than T. □

Theorem 60. *Given any initial point (x_0, y_0) outside the strip $|x| \leq 1$, the corresponding orbit is asymptotic to $y = x$.*

Proof. We can always assume that $x_0 > 1$ because the system is symmetric relatively to the origin. The equations yield:

$$\varepsilon \frac{dx}{dy} = \left(x - \frac{1}{x}\right)(y - x) \tag{288}$$

So if $y \geq x$ and (x_0, y_0), (x_1, y_1) are two points on the same orbit with $x_0 < x_1$, we get:

$$\left(x_0 - \frac{1}{x_0}\right)(y - x) \leq \varepsilon \frac{dx}{dy} \leq \left(x_1 - \frac{1}{x_1}\right)(y - x) \tag{289}$$

Set:

$$\alpha_i = \frac{1}{\varepsilon}(x_i - \frac{1}{x_i}), \quad i = 0, 1 \tag{290}$$

this yields:

$$\frac{dx}{dy} \quad - \quad \alpha_0(y - x) \geq 0 \tag{291}$$

$$\frac{dx}{dy} \quad - \quad \alpha_1(y - x) \leq 0 \tag{292}$$

hence:

$$\frac{d}{dy}(\mathrm{e}^{\alpha_0 y}x) \quad \geq \quad \alpha_0 y \mathrm{e}^{\alpha_0 y} \tag{293}$$

$$\frac{d}{dy}(\mathrm{e}^{\alpha_1 y}x) \quad \leq \quad \alpha_1 y \mathrm{e}^{\alpha_1 y} \tag{294}$$

Integration between y_0 and y_1 yields:

$$e^{\alpha_0 y_0}\left(y_0 - x_0 - \frac{1}{\alpha_0}\right) \geq e^{\alpha_0 y_1}\left(y_1 - x_1 - \frac{1}{\alpha_0}\right) \tag{295}$$

and

$$e^{\alpha_1 y_0}(y_0 - x_0 - \frac{1}{\alpha_1}) \leq e^{\alpha_1 y_1}(y_1 - x_1 - \frac{1}{\alpha_1}) \tag{296}$$

The second inequality shows that if $y_0 = x_0$, then:

$$e^{\alpha_1 y_0} \geq e^{\alpha_1 y_1}(1 - \alpha_1(y_1 - x_1)) \tag{297}$$

and thus that the orbit stays above the line $y = x$. If we now choose x_0 (and α_0) large enough, the first inequality displays:

$$(y_1 - x_1) \leq \frac{1}{2\alpha_0} \tag{298}$$

This proves that the orbit is asymptotic to $y = x$. $\square$

5. Models in Biology

Two example of models are considered. The first relates to modeling neurosciences and the second relates to population dynamics.

5.1. **Excitable Dynamics and Physiology.**

5.1.1. *The FitzHugh-Nagumo equations.*

The following equation (or planar vector field) was introduced independently by FitzHugh and Nagumo in relation with an approximation of the Hodgkin-Huxley equations for electrophysiology. They can be written as

$$\begin{aligned} \epsilon \tfrac{dx}{dt} &= y - f(x) + I \\ \tfrac{dy}{dt} &= -x + c, \end{aligned} \tag{299}$$

where $f(x)$ is a cubic polynomial, I, c are parameters. Polynomial f can be conveniently scaled to $f(x) = -x + \frac{x^3}{3}$ so that the case $(I, c) = (0, 0)$ corresponds to van der Pol equation. The parameter ϵ is small so that the FitzHugh-Nagumo equation is a fast-slow system. The critical set is given by the cubic curve $y = f(x) - I$ which is translated up or down parallel to the y-axis as the parameter I varies. The critical set displays two folds in $x = -1$ and $x = +1$. First assume that $I = 0$.

Proposition 61. *The vector field displays a single stationary point $(c, f(c))$ which is a focus if $-2\sqrt{\epsilon} < f'(c) < 2\sqrt{\epsilon}$ and a node if $f'(c) < -2\sqrt{\epsilon}$ or $f'(c) > 2\sqrt{\epsilon}$, stable if $f'(c) > 0$, unstable if $f'(c) < 0$.*

Proof. The eigenvalues of the jacobian matrix at the stationary point solve:

$$\lambda^2 + \frac{f'(c)}{\epsilon}\lambda + \frac{1}{\epsilon} = 0. \tag{300}$$

□

5.1.2. *Excitability.* Excitability is a very important notion introduced in electrophysiology in relation with theso-called action potential. This notion, which is deeply non linear, finds a simple interpretation with the FitzHugh-Nagumo equation using the fast-slow asymptotics. We assume here that $f'(c) > 2\sqrt{\epsilon}$, hence the stationary point is a stable node. We also assume that $c < -1, f(c) > f(1)$ so that the stationary point belongs to the stable left branch of the cubic. In relation with electrophysiology, the variable x is the membrane potential of the axon of a neuron (like in Hodgkin-Huxley case) or of a cardiac cell. The variable y is so-called a gate variable of a channel. Consider an initial data with the gate variable fixed $y = y_0, f(c) < y_0 < f(-1)$. The unstable branch of the cubic is a threshold. If the initial potential x_0 is below the threshold $(x_0 < \alpha, f(\alpha) = y_0)$ then the solution jumps left to a neighborhood of the left attracting branch, then moves slowly down the stationary point. If the initial potential is above the threshold, the solution jumps right to a neighborhood of the right attracting branch then slowly down the branch till it reaches a neighborhood of the fold. The local analysis near that point showed that the orbit follows by a fast part moving to the left branch of the cubic. It arrives close to a point below the value c and hence it follows a slow part close to the left attracting branch untill it reaches the stationary point. To summarize, if the initial data is above the threshold, the solutions undergo a large excursion in the phase portrait before coming to the equilibrium. This is called excitability.

5.2. **Ecology and Evolutionary Dynamics.**

5.2.1. *Replicator dynamics.* Consider a population of different phenotypes $x_i, i = 1, ..., n$. Each phenotype comes with a function $w_i(x)$ called fitness of the i^{th} phenotype. Assume the state of the population evolves according to:

$$\dot{x}_i = [w_i(x) - \overline{w}(x)]x_i, \quad i = 1, ..., n \tag{301}$$

where $\overline{w}(x) = \Sigma_{j=1}^n x_j W_j(x)$ is the mean fitness of the population. Note that the simplex $S_{n-1} = \{x \in \mathbb{R}_+^n, \Sigma_{j=1}^n x_j = 1\}$ is left invariant by the dynamics.

5.2.2. *Model of cyclic competition.* The model of cyclic competition is given by the equations:

$$\begin{array}{l} \frac{dx_1}{dt} = x_1(1 - x_1 - \alpha x_2 - \beta x_3) \\ \frac{dx_2}{dt} = x_2(1 - x_2 - \alpha x_3 - \beta x_1) \\ \frac{dx_3}{dt} = x_3(1 - x_3 - \alpha x_1 - \beta x_2), \\ 0 < \beta < 1 < \alpha, \quad \alpha + \beta \geq 2. \end{array} \tag{302}$$

The phase portrait is limited to $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$. Introduce $S = x_1 + x_2 + x_3$ whose derivative along the flow is

$$\dot{S} = S - (x_1^2 + x_2^2 + x_3^2) - (\alpha + \beta)(x_1 x_2 + x_2 x_3 + x_1 x_3). \tag{303}$$

In the case $\alpha + \beta = 2$, this yields $\dot{S} = S - S^2$. In that case, both $S = 0$ (the origin) and $S = 1$ (the simplex S_2) are invariant. In that case, the function P/S^3, with $P = x_1 x_2 x_3$, is another first integral. The phase portrait in restriction to the triangle S_2 is formed of periodic orbits surrounding a center. In general, the system displays a repulsive focus inside an attractive singular cycle contained in the planes $P = 0$.