Homogenization of a model for the propagation of sound in the lungs

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December 13, 2012

Abstract

In this paper, we are interested in the mathematical modeling of the propagation of sound waves in the lung parenchyma, which is a foam–like elastic material containing millions of air–filled alveoli. In this study, the parenchyma is governed by the linearized elasticity equations and the air by the acoustic wave equations. The geometric arrangement of the alveoli is assumed to be periodic with a small period $\varepsilon > 0$. We consider the time–harmonic regime forced by vibrations induced by volumic forces. We use the two–scale convergence theory to study the asymptotic behavior as ε goes to zero and prove the convergence of the solutions of the coupled fluid–structure problem to the solution of a linear–elasticity boundary value problem.

Keywords: acoustic–elastic interaction, Periodic homogenization, Two scale convergence method

1 Introduction and motivation

Lung sounds provide a cheap, non–invasive diagnostic technique which is often used for the detection of some pathologies of the respiratory system [28, 30]. Some diseases are associated with changes in the structure of the lung at various scales. Medical doctors have developed a good empirical understanding of the relation between the characteristics of the lung sounds they can hear, for example thanks to the stethoscope, and the underlying pathologies. But one lacks a precise physical understanding of the generation and propagation of sound waves through the respiratory system and the lung tissue, as well as of the changes in acoustic properties associated with underlying lung diseases. Another factor of interest is the need for understanding the propagation of pressure waves due to high–velocity impacts on the chest, thought to be responsible for lung contusions [23].

The lung tissue (called the parenchyma) is a very complex structure similar to a foam. Indeed, the lungs contain up to 300 million air pockets called the alveoli, connected by a bifurcating network of airways and embedded in an elastic matrix of connective tissue. The acoustic

*This work was partially funded by the ANR-08-JCJC-013-01 (M3RS) project headed by C. Grandmont.

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properties of this media are the consequence of this very complex, porous microstructure. Nevertheless, it is hard to describe accurately the properties of such porous media and, in practice, macroscopic models of reduced complexity are used. Our goal is to obtain macroscopic models based on more detailed tissue mechanics and geometry that are expected to further improve the understanding of experimental studies [28].

Current models for the acoustic properties of the lung parenchyma are usually based on the work by Rice [29], modeling the parenchyma as a homogeneous mixture of tissue and non–communicating air bubbles. When the sound wavelength is greatly superior to the size of the air bubbles, averaging the properties of the medium over volume leads to consider the porous medium as an elastic one. In this case, the speed of sound is independent of the frequency and given by the Wood formula $c = (K/\rho)^{1/2}$, where K is the effective volumetric stiffness of the medium and ρ its average density. When the volumetric proportion of the tissue phase is h, and under adiabatic conditions, the effective stiffness can be found by the following averaging process:

$$\frac{1}{K} = \frac{1-h}{\gamma P} + \frac{h}{K_s},$$

where γ is the ratio of specific heats of the air, P is the gas pressure and K_t is the stiffness of the tissue structure. The average density is given by

$$\rho = (1-h)\rho_g + h\rho_s,$$

where ρ_g is the density of the air phase and ρ_s the density of the tissue phase. Experimental measurements of the speed of sound in the low–frequency range (100 Hz to 1000 Hz) presented in [25, 29] show a good agreement with Woods' formula. However this homogeneous elastic representation is not valid as the frequency increases and the wavelength approaches the size of the alveoli as studied in [23] on a one dimensional model.

Other acoustic models of the lung's parenchyma have been proposed, mainly to study the effects of air communication between alveoli, which may be an important phenomemum at very low frequencies [13]. The main wave propagation models for such porous media go back to the work of Biot [8, 9]. Biot's equations were first introduced to characterize the flow of a viscous fluid through a porous elastic frame as well as the associated acoustic phenomena [10]. This model has been then derived using general homogenization theory [5, 31, 12]. More recently, assuming periodicity, the model as been obtained in [1, 22] by an asymptotic process using two–scale homogenization theory [26, 2]. Moreover, concerning the lung tissue modeling, the homogenization approach has been used by Owen and Lewis [27] to study high–frequency ventilation, and Siklosi et al. [32] to study the lungs of fetal sheep.

Here, we propose to derive rigorously, thanks to the homogenization theory, the non–dissipative model developped by Rice [29] for the propagation of low–frequency sound in a domain Ω modeling the parenchyma. We assume that this domain is occupied by an elastic deformable structure (the lung tissue [33]) and closed pockets filled with a compressible inviscid fluid (the air). Moreover, we assume that the size of the alveoli is small compared to the wavelength, i.e. that the macroscale and microscale are well separated, and we use the two–scale homogenization technique in order to investigate the asymptotic behavior of this medium as the size of the alveoli tends to zero. Consequently we have to find the homogenized limit of a fluid–structure interaction problem where the structure is elastic and the enclosed air is compressible and inviscid. Note that when the model includes a viscous fluid, the effective material obtained by homogenization usually depends strongly upon the contrast of property between the viscosity of the fluid and the elasticity of the structure, ranging from a viscoelastic material when this contrast is small to material with a diphasic macroscopic behavior when the contrast is strong [22]. The case of an inviscid but incompressible fluid can be found in [20]. In this work, since there is no viscosity, the main difficulty to deal with the absence of space derivatives of the fluid velocity in the linearized compressible Euler equations. As a result, the result depends strongly upon the geometry of the micro–structure and specifically the connectedness of the fluid part. Here, in this paper, we assume that the alveoli are disconnected. This is based on the common assumption [29, 23] that air does not communicate freely between neighbouring alveoli at frequencies above a few hundred hertz under normal circumstances. This hypothesis has been validated by a number of experimental studies, see e.g. [25, 13]. Moreover, the space repetition of the alveoli suggests us to consider an idealized medium containing a periodic arrangement of disconnected pores with a small period $\varepsilon > 0$.

The material we study behaves like a closed foam. We consider time–harmonic solutions to understand the behavior of the material in response to a harmonic forcing. Such a material was studied in the static case in [7], and we will see that we recover the same model in the vanishing frequency limit. To obtain a homogenized system, we pass to the limit as ε goes to zero and we use the two–scale convergence theory. In the case of a vanishing viscosity of order ε^2 and an connected incompressible fluid, the limit of the time–hamonic system was studied in [4]. The harmonic non–dissipative case brings some specific difficulties since the problem set in the frequency domain, of a Helmhotz nature, is not coercive. This means that the standard two–scale homogenization procedure cannot be applied directly and we have to use some nonstandard arguments to study the convergence.

In the limit, we obtain a homogeneous, non dispersive elastic medium, as expected [28]. We can recover the effective coefficients by computing the solutions of cell problems. Interestingly, on one hand, the averaging effects on the fluid pressure give rise to a nonlocal term in the formulation of the cell problems, and we obtain the same elastic tensor as in [7]. On the other hand, the macroscopic effect of the gaseous bubbles is mainly a modification of the bulk modulus (compressibility) of the limit material.

The paper is organized as follows. First, we detail the geometry and derive the equations of the model. Then, we study the well–posedness of the coupled elastic–acoustic problem for a fixed value of the micro–scale parameter ε and show that it verifies a Fredholm Alternative Principle (Proposition 5). In section 3.1, we analyze the asymptotic behavior of the displacement field, using homogenization techniques and an argument by contradiction. The main result of the paper is the convergence Theorem 2, which describes both the two–scale convergence of the displacement field and the homogenized problem (73).

2 Description of the coupling of the elastic and acoustic equations in a perforated domain

2.1 Geometric setting

We consider that the lung tissue occupies a smooth domain Ω of $\mathbb{R}^d$ with $d = 2$ or 3. This domain is filled homogeneously with a porous medium modeling the air–filled alveoli embedded in the elastic structural matrix. We assume that the alveoli are periodically distributed and of size $\varepsilon > 0$. More precisely, we define an open periodic unit cell $\mathcal{Y}$ representing the geometry of an alveolus. By rescaling, we normalize $\mathcal{Y}$ so that $|\mathcal{Y}| = 1$ and we define the associated periodic array $\mathbf{Z}$ of $\mathbb{R}^d$, which is the discrete set of translation vectors such that $\mathcal{Y} + \mathbf{Z}$ is a tiling of the whole space. The standard example is $\mathcal{Y} = (-1/2, 1/2)^d$ and $\mathbf{Z} = \mathbb{Z}^d$. We can also study for example a honeycomb as presented in Figure 1, where $\mathcal{Y}$ is an hexagon with side $a > 0$ such that

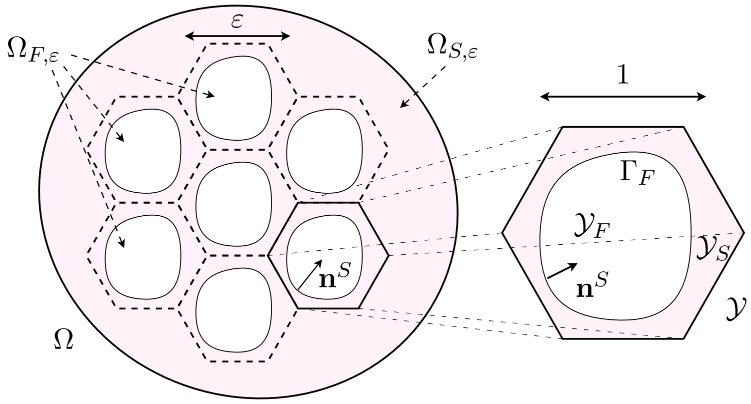

Figure 1: Domain Ω and reference cell $\mathcal{Y}$

its volume is 1 and $\mathbf{Z}$ the discrete lattice with basis $(0, a)$ and $\big(\sqrt{3}a/2, a/2\big)$ in $\mathbb{R}^2$, or a paving based on the truncated octahedron in $3D$ which is a standard representation of the alveoli [19]. The reference unit cell is supposed to be divided between an elastic and a fluid (acoustic) part $\mathcal{Y}_S$ and $\mathcal{Y}_F$, where $\overline{\mathcal{Y}_F} \subset \mathring{\mathcal{Y}}$ is smooth, simply connected, and locally lies on one side only of its boundary. The boundary $\Gamma_F = \partial\mathcal{Y}_F$ is the interface between the two components of $\mathcal{Y}$. For the sake of simplicity, we suppose that the barycenter of $\mathcal{Y}_F$ is at the origin of $\mathbb{R}^d$.

Next, for any given small parameter $\varepsilon > 0$, we introduce the following notations:

- For a given a multi–index $\mathbf{k} \in \mathbf{Z}$, let

$$\mathcal{Y}_\varepsilon^{\mathbf{k}} = \varepsilon(\mathcal{Y} + \mathbf{k}), \quad \mathcal{Y}_{F,\varepsilon}^{\mathbf{k}} = \varepsilon(\mathcal{Y}_F + \mathbf{k}), \quad \mathcal{Y}_{S,\varepsilon}^{\mathbf{k}} = \varepsilon(\mathcal{Y}_S + \mathbf{k}), \quad \Gamma_{F,\varepsilon}^{\mathbf{k}} = \varepsilon(\Gamma_F + \mathbf{k}), \tag{1}$$

  that are, a translation by $\mathbf{k}$ and a rescaling by ε of the unit cell $\mathcal{Y}$ and of the fluid and structure part as well as of the fluid–structure boundary.

- Introducing the multi–index set

$$\mathbf{Z}_\varepsilon^\Omega = \{\mathbf{k} \in \mathbf{Z} | \overline{\mathcal{Y}_\varepsilon^{\mathbf{k}}} \subset \Omega\},$$

  we define the periodically perforated structure domain, the fluid domain and the interior interface respectively as

$$\Omega_{S,\varepsilon} = \Omega \setminus \bigcup_{\mathbf{k}\in\mathbf{Z}_\varepsilon^\Omega} \overline{\mathcal{Y}_F{}^{\mathbf{k}}}, \qquad \Omega_{F,\varepsilon} = \bigcup_{\mathbf{k}\in\mathbf{Z}_\varepsilon^\Omega} \mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}, \qquad \Gamma_\varepsilon^I = \bigcup_{\mathbf{k}\in\mathbf{Z}_\varepsilon^\Omega} \Gamma_{F,\varepsilon}^{\mathbf{k}}. \tag{2}$$

- Let $\mathbf{n}^S$ and $\mathbf{n}_\varepsilon^S$ be unit normal vectors on the fluid–structure cell interface Γ_F and interior interface Γ_ε^I respectively, pointing in each case to the exterior of the structure represented respectively by $\mathcal{Y}_S$ and $\Omega_{S,\varepsilon}$.

- Let χ_F, χ_S be the characteristic functions of $\mathcal{Y}_F$ and $\mathcal{Y}_S$ respectively, and $\chi_{F,\varepsilon}$, $\chi_{S,\varepsilon}$, $\chi_{F,\varepsilon}^{\mathbf{k}}$, $\chi_{S,\varepsilon}^{\mathbf{k}}$ the characteristic functions of $\Omega_{F,\varepsilon}$, $\Omega_{S,\varepsilon}$, $\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}$ and $\mathcal{Y}_{S,\varepsilon}^{\mathbf{k}}$ respectively.

- The subscript $\#$ on the functional spaces' name denotes the property of periodicity with respect to $\mathbf{Z}$, in the sense that $C_\#^\infty(\mathcal{Y})$ is the space of $\mathbf{Z}$–periodic functions on $\mathbb{R}^d$ indefinitely differentiable on $\mathbb{R}^d$, and $H_\#^1(\mathcal{Y})$ and $L_\#^2(\mathcal{Y})$ are the closure of $C_\#^\infty(\mathcal{Y})$ respectively in the H^1– and the L^2–norm. Moreover, $H_\#^1(\mathcal{Y}_S)$ and $L_\#^2(\mathcal{Y}_S)$ are defined as the spaces of restrictions of functions in $H_\#^1(\mathcal{Y})$ and $L_\#^2(\mathcal{Y})$ to $\mathcal{Y}_S + \mathbf{Z}$.

Note that due to the choice of $\mathbf{Z}_\varepsilon^\Omega$, no hole intersects the exterior boundary of Ω. For this reason, $\partial\Omega_{S,\varepsilon} = \partial\Omega$ does not depend on ε. This will make the homogenization process, as ε goes to zero, more convenient but not fundamentally different from a case where the holes are allowed to sometimes intersect the exterior boundary.

As the material presents two characteristic length scales (macroscopic and microscopic), we introduce finally two sets of spatial variables: the ordinary position vector $\mathbf{x} \in \Omega$, and the position vector in a stretched coordinate system $\mathbf{y} = \varepsilon^{-1}\mathbf{x}$. The variable $\mathbf{x}$ is called the *slow variable* and the variable $\mathbf{y}$ the *fast variable*, and as ε goes to zero we expect the two sets of variables to become independent. To make a difference between differentiation with respect to either set of variables $\mathbf{x}$ or $\mathbf{y}$, we will use a subscript as in $\nabla_{\mathbf{x}}$ or $\mathrm{div}_{\mathbf{y}}$ when there is a doubt. When necessary, we will use the Einstein convention of repeated indexes to write summations.

2.2 Acoustic–Elastic interaction

Following [24], we write the model equations for the propagation of sound waves through our perforated material. As a first step, we describe the equations governing this propagation in the time domain for a given parameter ε. As we are studying sound waves, the perturbation or displacement from the rest configuration of the structure or air is the relevant unknown to consider. This perturbation is supposed to be small, so one can consider the linearized models to describe the behavior of both structure and air parts of the material to understand the wave propagation. As a second step, the signal will be represented by a harmonic superposition of monochromatic waves, for which every excitation source and every unknown obeys a harmonic dependence of frequency ω. Our goal is then to obtain a homogenized system in the asymptotic limit where ε goes to zero, describing the effective equation satisfied by the pressure wave for each value of ω.

Let us write the equations describing the mechanical behavior of the material. For simplicity, we adopt a Lagrangian point of view and denote $\mathbf{U}_\varepsilon$ the time–dependent displacement field throughout the structure and air parts of the domain Ω. We begin by describing the equations modeling the behavior of the structure part. Assuming that the wall material behaves like a linearized elastic medium, the stress tensor satisfies Hooke's law:

$$\sigma_\varepsilon(\mathbf{U}_\varepsilon) = \lambda\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)\mathrm{div}(\mathbf{U}_\varepsilon)\mathrm{Id} + \mu\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)e(\mathbf{U}_\varepsilon),$$

where $\lambda > 0$, $\mu > 0$ are the Lamé parameters, Id the identity matrix, and $e(\mathbf{U}_\varepsilon)$ is the linearized Cauchy strain tensor:

$$e(\mathbf{U}_\varepsilon) = \frac{1}{2}\left(\nabla\mathbf{U}_\varepsilon +^T \nabla\mathbf{U}_\varepsilon\right).$$

Note that we allow λ and μ to vary through the domain, for example to model a pathology where the parenchyma is locally rigidified. Moreover, to model variations both at the macroscopic level and at the alveolar, microscopic level, we allow a dependence on both the slow variable $\mathbf{x}$ and the fast variable $\mathbf{y} = \varepsilon^{-1}\mathbf{x}$. We assume that λ and μ are essentially bounded, continuous in the x variable on $\overline{\Omega}$ and periodic in the $\mathbf{y}$ variable (this is the right regularity for the two–scale convergence method, and continuity in at least one variable is necessary for $\mathbf{x} \mapsto \mu(\mathbf{x}, \varepsilon^{-1}\mathbf{x})$ to be measurable, see [2]). Moreover, μ is supposed to be uniformly bounded away from 0, consequently there exists a constant $\mu_0 > 0$ independent of $(\mathbf{x}, \mathbf{y})$ such that:

$$\forall \mathbf{x} \in \Omega,\ \forall \mathbf{y} \in \mathcal{Y},\ \mu(\mathbf{x}, \mathbf{y}) \geq \mu_0 > 0. \tag{3}$$

Suppose that the material reacts to a volumic force $\mathbf{F}_\varepsilon$. The Newton law then yields the equations for the linearized elastic material, with ρ_S denoting the density:

$$\rho_S \frac{\partial^2 \mathbf{U}_\varepsilon}{\partial t^2} - \mathrm{div}(\sigma_\varepsilon(\mathbf{U}_\varepsilon)) = \mathbf{F}_\varepsilon, \qquad \text{in } \Omega_{S,\varepsilon}. \tag{4}$$

Finally we also impose homogeneous Dirichlet boundary conditions on the outer boundary $\partial\Omega$:

$$\mathbf{U}_\varepsilon = \mathbf{0}, \qquad \text{on } \partial\Omega. \tag{5}$$

Let us now describe the fluid behavior. The fluid domain $\Omega_{F,\varepsilon}$ is filled with air which is considered as an inviscid, irrotational, compressible perfect gas. We consider only small perturbations with respect to a reference equilibrium state in each hole, with the reference pressure being the atmospheric pressure P_0 and a constant equilibrium density ρ_0, under a potential volumic excitation force ∇G_ε. Following [24], a complete description of the behavior of the gas is given by two conservation laws and an appropriate state law of the gas, using three unknowns: the displacement $\mathbf{U}_\varepsilon$, the absolute pressure P_ε and the gas density ρ_ε.
The momentum conservation law for an inviscid, irrotational gas writes:

$$\rho_\varepsilon \frac{\partial^2 \mathbf{U}_\varepsilon}{\partial t^2} + \nabla P_\varepsilon = \nabla G_\varepsilon, \qquad \text{in } \Omega_{F,\varepsilon}. \tag{6}$$

The continuity equation, or mass conservation law, writes:

$$\frac{\partial \rho_\varepsilon}{\partial t} + \mathrm{div}\left(\rho_\varepsilon \frac{\partial \mathbf{U}_\varepsilon}{\partial t}\right) = 0, \qquad \text{in } \Omega_{F,\varepsilon}. \tag{7}$$

To close the system, we make the assumption that the air compression associated with the propagation of sound waves is an adiabatic process. This is an usual assumption regarding sound propagation, and it is motivated by the difference in characteristic times between the heat dissipation process and the short timescale associated with the propagating waves. Pressure and density are then linked by the following relation:

$$P_\varepsilon = P_0 \left(\frac{\rho_\varepsilon}{\rho_0}\right)^\gamma, \qquad \text{in } \Omega_{F,\varepsilon}, \tag{8}$$

where γ is the adiabatic index of the air ($\gamma \approx 1.4$). Let us now linearize the equations (6), (7), (8) around the reference state following our assumption of small perturbation from rest:

$$\rho_0 \frac{\partial^2 \mathbf{U}_\varepsilon}{\partial t^2} + \nabla P_\varepsilon = \nabla G_\varepsilon \qquad \text{in } \Omega_{F,\varepsilon}, \tag{9a}$$

$$\frac{\partial \rho_\varepsilon}{\partial t} + \rho_0 \mathrm{div}\left(\frac{\partial \mathbf{U}_\varepsilon}{\partial t}\right) = 0 \qquad \text{in } \Omega_{F,\varepsilon}. \tag{9b}$$

$$P_\varepsilon - P_0 = c^2(\rho_\varepsilon - \rho_0) \qquad \text{in } \Omega_{F,\varepsilon}, \tag{9c}$$

where we have introduced $c = \sqrt{\gamma \frac{P_0}{\rho_0}}$, the sound speed in the air. We eliminate the density ρ_ε by combining (9b) and (9c), and we find that the displacement and pressure in the fluid are solution to the coupled system of equations:

$$\rho_0 \frac{\partial^2 \mathbf{U}_\varepsilon}{\partial t^2} + \nabla P_\varepsilon = \nabla G_\varepsilon, \qquad \text{in } \Omega_{F,\varepsilon}. \tag{10a}$$

$$\frac{1}{c^2}\frac{\partial P_\varepsilon}{\partial t} + \rho_0 \mathrm{div}\left(\frac{\partial \mathbf{U}_\varepsilon}{\partial t}\right) = 0, \qquad \text{in } \Omega_{F,\varepsilon}. \tag{10b}$$

Let us now describe the coupling conditions between the fluid and the structure. The first condition expresses the continuity of the normal component of the strain tensor at the interface:

$$-P_\varepsilon \mathbf{n}_\varepsilon^S = \sigma_\varepsilon(\mathbf{U}_\varepsilon|_{\Omega_{S,\varepsilon}})\mathbf{n}_\varepsilon^S \qquad \text{on } \Gamma_\varepsilon^I. \tag{11}$$

Moreover, because the air is inviscid, there is no constraint on the tangential component of the trace of the velocity at the interface. Rather, we have slip boundary conditions, meaning that the normal component of the displacement is continuous:

$$\mathbf{U}_\varepsilon|_{\Omega_{S,\varepsilon}} \cdot \mathbf{n}_\varepsilon^S = \mathbf{U}_\varepsilon|_{\Omega_{F,\varepsilon}} \cdot \mathbf{n}_\varepsilon^S \qquad \text{on } \Gamma_\varepsilon^I. \tag{12}$$

Finally our coupled fluid–structure interaction problem is described by equations (4), (10) and the boundary conditions (5), (11) and (12) complemented with initial conditions. By construction, this coupled system is now linear and its behavior can be understood by harmonic superposition technique. We thus assume that both G_ε, $\mathbf{F}_\varepsilon$ and the initial conditions are coherent with a time–harmonic forcing along the mode $e^{i\omega t}$. This leads to assume that the unknowns write:

$$\begin{aligned} \mathbf{U}_\varepsilon(\mathbf{x},t) &= \mathbf{u}_\varepsilon(\mathbf{x})e^{i\omega t} \quad \text{in } \Omega, & P_\varepsilon(\mathbf{x},t) &= p_\varepsilon(\mathbf{x})e^{i\omega t} \quad \text{in } \Omega_{F,\varepsilon},\\ \mathbf{F}_\varepsilon(\mathbf{x},t) &= \mathbf{f}_\varepsilon(\mathbf{x})e^{i\omega t} \quad \text{in } \Omega, & g_\varepsilon(\mathbf{x},t) &= G_\varepsilon(\mathbf{x})e^{i\omega t} \quad \text{in } \Omega. \end{aligned}$$

Note that the fields $\mathbf{u}_\varepsilon$, p_ε, $\mathbf{f}_\varepsilon$, g_ε will be complex–valued in what follows. In particular the Hilbert spaces we consider will be complex–valued spaces unless it is otherwise specified. We denote by $\mathrm{Re}\,(\cdot)$ and $\mathrm{Im}\,(\cdot)$ respectively the real and imaginary part of a complex argument.

2.3 Harmonic formulation

Taking into account this time dependency, the behavior of the coupled fluid and structure for some frequency ω is described by the complex displacement / pressure field $(\mathbf{u}_\varepsilon, p_\varepsilon)$ solving the following system:

$$-\rho_S\omega^2\mathbf{u}_\varepsilon - \mathrm{div}\sigma_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{f}_\varepsilon \qquad \text{in } \Omega_{S,\varepsilon}, \tag{13a}$$

$$-\rho_0\omega^2\mathbf{u}_\varepsilon + \nabla p_\varepsilon = \nabla g_\varepsilon \qquad \text{in } \Omega_{F,\varepsilon}, \tag{13b}$$

$$\frac{1}{c^2}p_\varepsilon + \rho_0\mathrm{div}(\mathbf{u}_\varepsilon) = 0 \qquad \text{in } \Omega_{F,\varepsilon}, \tag{13c}$$

$$-p_\varepsilon\mathbf{n}_\varepsilon^S = \sigma_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n}_\varepsilon^S \qquad \text{on } \Gamma_\varepsilon^I, \tag{13d}$$

$$\mathbf{u}_\varepsilon|_{\Omega_{S,\varepsilon}} \cdot \mathbf{n}_\varepsilon^S = \mathbf{u}_\varepsilon|_{\Omega_{F,\varepsilon}} \cdot \mathbf{n}_\varepsilon^S \qquad \text{on } \Gamma_\varepsilon^I, \tag{13e}$$

$$\mathbf{u}_\varepsilon = \mathbf{0} \qquad \text{on } \partial\Omega. \tag{13f}$$

Remember that we have assumed that $\mathbf{u}_\varepsilon$ is irrotational in $\Omega_{F,\varepsilon}$, this has lead to (13b). To write this system in a more suitable form for further analysis, let us introduce a velocity potential ϕ_ε defined up to a constant in each hole, such that

$$\nabla\phi_\varepsilon = i\omega\mathbf{u}_\varepsilon. \tag{14}$$

We choose to work with the potential that has zero mean in each hole to fix the constant. By combining the three relations (13b), (13c) and (14), we see that:

$$\nabla\left(-\omega^2\phi_\varepsilon - c^2\Delta\phi_\varepsilon - i\omega g_\varepsilon/\rho_0\right) = 0.$$

To get rid of the gradient in this equation we need to introduce a constant $C_\varepsilon^{\mathbf{k}}$ on each connected component of $\Omega_{F,\varepsilon}$, depending only on the hole index $\mathbf{k}$. This leads to the following Hehlmoltz equation set on each hole $\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}$:

$$-\omega^2\phi_\varepsilon - c^2\Delta\phi_\varepsilon = i\omega\frac{g_\varepsilon + C_\varepsilon^{\mathbf{k}}}{\rho_0}. \tag{15}$$

Moreover, the boundary condition (13e) together with (14) imply that the following compatibility condition is satisfied:

$$\int_{\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}} \left(i\omega\frac{g_\varepsilon + C_\varepsilon^{\mathbf{k}}}{\rho_0}\right) = c^2\int_{\Gamma_{F,\varepsilon}^{\mathbf{k}}}\frac{\partial\phi_\varepsilon}{\partial\mathbf{n}_\varepsilon^S} = i\omega c^2\int_{\Gamma_{F,\varepsilon}^{\mathbf{k}}}\mathbf{u}_\varepsilon\cdot\mathbf{n}_\varepsilon^S. \tag{16}$$

From (16) the constant $C_\varepsilon^{\mathbf{k}}$ appearing in equation (15) can be determined and satisfies:

$$C_\varepsilon^{\mathbf{k}} = \frac{1}{|\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}|}\left(\rho_0 c^2\int_{\Gamma_{F,\varepsilon}^{\mathbf{k}}}\mathbf{u}_\varepsilon\cdot\mathbf{n}_\varepsilon^S - \int_{\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}} g_\varepsilon\right). \tag{17}$$

We next define a function C_ε of $L^2(\Omega)$, constant in each cell $\mathcal{Y}_\varepsilon^{\mathbf{k}}$ by

$$C_\varepsilon(\mathbf{x}) = \begin{cases} C_\varepsilon^{\mathbf{k}} & \text{if } \mathbf{x}\in\mathcal{Y}_\varepsilon^{\mathbf{k}} \text{ for some } \mathbf{k}\in\mathbf{Z}_\varepsilon^\Omega,\\ 0 & \text{else.}\end{cases} \tag{18}$$

We now eliminate the fluid pressure from the equations. From (13c) and (14) we derive

$$i\omega p_\varepsilon = -\rho_0 c^2\Delta\phi_\varepsilon,$$

which combines with (15) yields

$$p_\varepsilon = -i\omega\rho_0\phi_\varepsilon + g_\varepsilon + C_\varepsilon. \tag{19}$$

Bringing together (15), (17) and (19), we write a new, equivalent system of equations describing the behavior of our coupled fluid–structure material. The new unknowns are the structure displacement and the fluid velocity potential $(\mathbf{u}_\varepsilon, \phi_\varepsilon)$. Note that the displacement field $\mathbf{u}_\varepsilon$ is defined only on $\Omega_{S,\varepsilon}$ from now on.

$$-\rho_S\omega^2\mathbf{u}_\varepsilon - \mathrm{div}\sigma_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{f}_\varepsilon \quad \text{in } \Omega_{S,\varepsilon}, \tag{20a}$$

$$-\omega^2\phi_\varepsilon - c^2\Delta\phi_\varepsilon = i\omega\frac{g_\varepsilon + C_\varepsilon}{\rho_0} \quad \text{in } \Omega_{F,\varepsilon}, \tag{20b}$$

$$\sigma_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n}_\varepsilon^S = -\left(-i\omega\rho_0\phi_\varepsilon + g_\varepsilon + C_\varepsilon\right)\mathbf{n}_\varepsilon^S \quad \text{on } \Gamma_\varepsilon^I, \tag{20c}$$

$$i\omega\mathbf{u}_\varepsilon\cdot\mathbf{n}_\varepsilon^S = \frac{\partial\phi_\varepsilon}{\partial\mathbf{n}_\varepsilon^S} \quad \text{on } \Gamma_\varepsilon^I, \tag{20d}$$

$$\mathbf{u}_\varepsilon = \mathbf{0} \quad \text{on } \partial\Omega, \tag{20e}$$

with C_ε defined by (18).

Now we are going to write the variational formulation of this problem. Let us define the complex Hilbert spaces (bold face letters indicate spaces of vector–valued functions):

$$\mathbf{H}_0^1(\Omega_{S,\varepsilon}) = \left\{\mathbf{v}_\varepsilon\in\mathbf{H}^1(\Omega_{S,\varepsilon}), \mathbf{v}_\varepsilon|_{\partial\Omega} = \mathbf{0}\right\},$$

$$H^1_{mean}(\Omega_{F,\varepsilon}) = \left\{ \psi \in H^1(\Omega_{F,\varepsilon}), \forall \mathbf{k} \in \mathbf{Z}^\Omega_\varepsilon, \int_{\mathcal{Y}^{\mathbf{k}}_{F,\varepsilon}} \psi = 0 \right\}.$$

The norms associated to functional spaces on $\Omega_{F,\varepsilon}$ are to be understood as broken norms. For instance, for $\psi \in L^2(\Omega_{F,\varepsilon})$,$\|\psi\|_{L^2(\Omega_{F,\varepsilon})} = \left(\sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} \|\psi\|^2_{L^2(\mathcal{Y}^{\mathbf{k}}_{F,\varepsilon})} \right)^{1/2}$.

Let us also define the L^2–projector Π_ε onto the space of functions that are constant on each cell $\mathcal{Y}^{\mathbf{k}}_\varepsilon$, by

$$\Pi_\varepsilon(\phi) = \sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} \frac{1}{|\mathcal{Y}^{\mathbf{k}}_\varepsilon|} \left(\int_{\mathcal{Y}^{\mathbf{k}}_\varepsilon} \phi \right) \chi_{\mathcal{Y}^{\mathbf{k}}_\varepsilon} = \sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} \frac{1}{\varepsilon^d} \left(\int_{\mathcal{Y}^{\mathbf{k}}_\varepsilon} \phi \right) \chi_{\mathcal{Y}^{\mathbf{k}}_\varepsilon}. \tag{21}$$

Using the operator Π_ε and (17), we can rewrite the L^2 function C_ε introduced in (18) as:

$$C_\varepsilon = \rho_0 c^2 \sum_{k\in\mathbf{Z}^\Omega_\varepsilon} \frac{1}{\varepsilon^d |\mathcal{Y}_F|} \left(\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} \mathbf{u}_\varepsilon \cdot \mathbf{n}^S_\varepsilon \right) \chi_{\mathcal{Y}^{\mathbf{k}}_\varepsilon} - \frac{1}{|\mathcal{Y}_F|} \Pi_\varepsilon \left(\chi_{F,\varepsilon} g_\varepsilon \right).$$

By taking a couple of test functions $(\mathbf{v}, \psi)$ in $\mathbf{H}^1_0(\Omega_{S,\varepsilon}) \times H^1_{mean}(\Omega_{F,\varepsilon})$ and using $\mathbf{v}$ as a test function in equation (20a) and ψ in equation (20b), the weak formulation of (20) reads as follows: for $\mathbf{f}_\varepsilon \in \mathbf{L}^2(\Omega)$ and $g_\varepsilon \in H^1(\Omega)$, find $(\mathbf{u}_\varepsilon, \phi_\varepsilon) \in \mathbf{H}^1_0(\Omega_{S,\varepsilon}) \times H^1_{mean}(\Omega_{F,\varepsilon})$ such that for any $(\mathbf{v}, \psi) \in H^1_0(\Omega)^d \times H^1_{mean}(\Omega_{F,\varepsilon})$,

$$\left\{ \begin{aligned} &\int_{\Omega_{S,\varepsilon}} -\rho_S \omega^2 \mathbf{u}_\varepsilon \cdot \overline{\mathbf{v}} + \sigma_\varepsilon(\mathbf{u}_\varepsilon) : \overline{e(\mathbf{v})} + \rho_0 \int_{\Gamma^I_\varepsilon} i\omega \left(\overline{\psi} \mathbf{u}_\varepsilon \cdot \mathbf{n}^S_\varepsilon - \phi_\varepsilon \overline{\mathbf{v} \cdot \mathbf{n}^S_\varepsilon} \right) \\ &\qquad + \rho_0 c^2 \sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} \frac{1}{\varepsilon^d |\mathcal{Y}_F|} \left(\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} \mathbf{u}_\varepsilon \cdot \mathbf{n}^S_\varepsilon \right) \overline{\left(\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} \mathbf{v} \cdot \mathbf{n}^S_\varepsilon \right)} \\ &\qquad + \rho_0 \int_{\Omega_{F,\varepsilon}} -\frac{\omega^2}{c^2} \phi_\varepsilon \overline{\psi} + \nabla \phi_\varepsilon \cdot \overline{\nabla \psi} \\ &\quad = \int_{\Omega_{S,\varepsilon}} \mathbf{f}_\varepsilon \cdot \overline{\mathbf{v}} - \int_{\Gamma^I_\varepsilon} \left(g_\varepsilon - \frac{1}{|\mathcal{Y}_F|} \Pi_\varepsilon(\chi_{F,\varepsilon} g_\varepsilon) \right) \overline{\mathbf{v} \cdot \mathbf{n}^S_\varepsilon} \\ &\qquad + \int_{\Omega_{F,\varepsilon}} \frac{i\omega}{c^2} \left(g_\varepsilon - \frac{1}{|\mathcal{Y}_F|} \Pi_\varepsilon(\chi_{F,\varepsilon} g_\varepsilon) \right) \overline{\psi}. \end{aligned} \right. \tag{22}$$

Note that the trace of $\Pi_\varepsilon(\chi_{F,\varepsilon} g_\varepsilon)$ over $\Gamma^I_\varepsilon = \cup_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} \Gamma^{\mathbf{k}}_{F,\varepsilon}$ makes sense since its restriction to each $\mathcal{Y}^{\mathbf{k}}_\varepsilon$ is constant and thus belongs to $H^1(\mathcal{Y}^{\mathbf{k}}_\varepsilon)$.

Remark 1. *If we take the frequency ω to be zero, we recover precisely the static model studied in detail in [7].*

Remark 2. *Note, the presence of the unusual term*

$$\sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} \frac{1}{\varepsilon^d |\mathcal{Y}_F|} \left(\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} \mathbf{u}_\varepsilon \cdot \mathbf{n}^S_\varepsilon \right) \overline{\left(\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} \mathbf{v} \cdot \mathbf{n}^S_\varepsilon \right)},$$

which for instance appeared in the model studied in [7]. It is a local term at the macroscopic scale and nonlocal at the microscopic (alveolar) scale, and it comes from an average pressure in each hole that mathematically was expressed by the compatibility condition (16).

We are going to study this system and its limit as ε goes to zero. But first let us introduce some useful notations and tools.

2.4 A few useful definitions and results

Let us describe here a few definitions and results we will frequently use in what follows and, in particular, the framework of two–scale homogenization laid out by G. Nguetseng [26] and G. Allaire [2]. Since we want to pass to the limit as ε goes to zero, we have to pay special attention to the dependency of the various constants with respect to ε: it is indeed crucial to get uniform bounds in order to obtain the compactness properties of the weak or two–scale topologies. Consequently we will first define extension operators for functions defined on the domains $\Omega_{S,\varepsilon}$ or $\Omega_{F,\varepsilon}$ to functions defined on the whole domain Ω, whose norms are independent of ε. Next, we will derive Poincaré and Korn inequalities on $\Omega_{S,\varepsilon}$. Finally, after recalling the two–scale convergence properties, we will study the well-posedness of (22) for a given $\varepsilon > 0$ and derive some uniform energy bounds.

2.4.1 Extension operators

As is standard when dealing with porous multiscale domains, we need extension operators from $\Omega_{S,\varepsilon}$ and $\Omega_{F,\varepsilon}$ onto Ω since convergence cannot be described in parameter dependent domains. We define two extension operators:

- An extension operator in $\mathcal{L}\left(\mathbf{H}^k(\Omega_{S,\varepsilon}), \mathbf{H}^k(\Omega)\right)$ for $k = 0, 1$, denoted by $\widehat{\cdot}$, such that for some $C > 0$ independent of ε and depending only on Ω and $\mathcal{Y}$, for all $\mathbf{u}_\varepsilon \in \mathbf{H}^1(\Omega_{S,\varepsilon})$,

$$\widehat{\mathbf{u}_\varepsilon} = \mathbf{u}_\varepsilon \text{ in } \Omega_{S,\varepsilon},$$
$$\|\widehat{\mathbf{u}_\varepsilon}\|_{\mathbf{L}^2(\Omega)} \leq C\|\mathbf{u}_\varepsilon\|_{\mathbf{L}^2(\Omega_{S,\varepsilon})}, \quad \|\nabla\widehat{\mathbf{u}_\varepsilon}\|_{\mathbf{L}^2(\Omega)} \leq C\|\nabla\mathbf{u}_\varepsilon\|_{\mathbf{L}^2(\Omega_{S,\varepsilon})}. \tag{23}$$

  The construction of such an operator can be found e.g. in [17].

- An extension still denoted by $\widehat{\cdot} : H^1_{mean}(\Omega_{F,\varepsilon}) \to H^1_0(\Omega)$ that we are going to construct in the following Lemma.

Lemma 1. *There exists an extension operator* $\widehat{\cdot}:\ H^1_{mean}(\Omega_{F,\varepsilon}) \mapsto H^1_0(\Omega)$ *for every* $\varepsilon > 0$*, such that* $\forall\phi_\varepsilon \in H^1_{mean}(\Omega_{F,\varepsilon})$ *we have the property*

$$|\widehat{\phi_\varepsilon}|_{H^1(\Omega)} \leq C|\phi_\varepsilon|_{H^1(\Omega_{F,\varepsilon})},$$

where the constant C depends only on $\mathcal{Y}$ and not on ε.

Proof. First of all, let us consider a linear continuous extension operator from $H^1_{mean}(\mathcal{Y}_F)$ (defined as the set of functions in $H^1(\mathcal{Y}_F)$ with zero average) to the space $H^1_0(\mathcal{Y})$. As an example, we define for any $\phi \in H^1_{mean}(\mathcal{Y}_F)$ its harmonic extension $E(\phi) \in H^1_0(\mathcal{Y})$ by solving the Poisson problem

$$\begin{cases} -\Delta\psi = 0 & \text{in } \mathcal{Y}_S, \\ \psi = \phi|_{\Gamma_F} & \text{on } \Gamma_F, \\ \psi = 0 & \text{on } \partial Y. \end{cases}$$

It is well–known that for some constant C depending only on $\mathcal{Y}_S$,

$$\|\psi\|_{H^1(\mathcal{Y}_S)} \leq C\|\phi\|_{H^{1/2}(\Gamma_F)}.$$

Thanks to both the trace inequality and the Poincaré–Wirtinger inequality in $H^1_{mean}(\mathcal{Y}_F)$, we have

$$\|\psi\|_{H^1(\mathcal{Y}_S)} \leq C\|\phi\|_{H^1(\mathcal{Y}_F)} \leq C|\phi|_{H^1(\mathcal{Y}_F)},$$

where C depends only on $\mathcal{Y}$, $\mathcal{Y}_F$, $\mathcal{Y}_S$. The function $E(\phi)$ on $\mathcal{Y}$ defined as

$$E(\phi)(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}) & \text{if } \mathbf{x} \in \mathcal{Y}_F, \\ \psi(\mathbf{x}) & \text{if } \mathbf{x} \in \mathcal{Y}_S, \end{cases}$$

belongs to $H_0^1(\mathcal{Y})$ and the following estimate holds for some constant C, depending only on $\mathcal{Y}$, $\mathcal{Y}_S$ and $\mathcal{Y}_F$:

$$|E(\phi)|_{H^1(\mathcal{Y})} \leq C|\phi|_{H^1(\mathcal{Y}_F)}. \tag{24}$$

Now let $\phi_\varepsilon \in H^1_{mean}(\Omega_{F,\varepsilon})$. For each $\mathbf{k} \in \mathbf{Z}_\varepsilon^\Omega$, we have $\phi_\varepsilon|_{\mathcal{Y}^{\mathbf{k}}_{F,\varepsilon}}(\varepsilon \cdot + \mathbf{k}) \in H^1_{mean}(\mathcal{Y}_F)$. Let us define:

$$\widehat{\phi_\varepsilon}(\mathbf{x}) = \begin{cases} E\left(\phi_\varepsilon|_{\mathcal{Y}^{\mathbf{k}}_{F,\varepsilon}}(\varepsilon \cdot + \mathbf{k})\right)(\varepsilon^{-1}(\mathbf{x}-\mathbf{k})) & \text{if } \mathbf{x} \in \mathcal{Y}^{\mathbf{k}}_\varepsilon,\ \mathbf{k} \in \mathbf{Z}^\Omega_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Because the traces of $\widehat{\phi}$ coincide on each side of $\partial\mathcal{Y}^{\mathbf{k}}_\varepsilon$ with 0, $\widehat{\phi_\varepsilon}$ belongs globally to $H^1_0(\Omega)$. We have the estimate:

$$\begin{aligned} |\widehat{\phi_\varepsilon}|^2_{H^1(\Omega)} &= \sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} \int_{\mathcal{Y}^{\mathbf{k}}_\varepsilon} |\nabla\widehat{\phi_\varepsilon}|^2 \\ &= \sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} \varepsilon^d \int_{\mathcal{Y}} |\varepsilon^{-1}\nabla\left(E\left\{\phi_\varepsilon(\varepsilon\cdot+\mathbf{k})\right\}(\mathbf{y})\right)|^2 \\ &= \varepsilon^{d-2} \sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} |E\left\{\phi_\varepsilon(\varepsilon\cdot+\mathbf{k})\right\}|^2_{H^1_0(\mathcal{Y})} \\ &\leq C^2\varepsilon^{d-2} \sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} |\phi_\varepsilon(\varepsilon\cdot+\mathbf{k})|^2_{H^1(\mathcal{Y}_F)} \\ &\leq C^2|\phi_\varepsilon|^2_{H^1(\Omega_{F,\varepsilon})}, \end{aligned} \tag{25}$$

where C is the same constant as in (24) and thus is independent of ε. This concludes the proof of the Lemma. □

2.4.2 Korn and Poincaré inequalities

The L^2–norm of the Cauchy stress tensor $e(\mathbf{u})$ will appear naturally when we compute energy bounds for our solutions. To deduce H^1–bounds, we need the Korn inequality. This result is well–known in the case of a bounded open set Ω with Dirichlet boundary conditions. Once again, here we pay special attention to the dependency of the constants on ε. It is well–known that the Korn inequality holds on Ω [15]: there exists $K > 0$ depending only on Ω, such that

$$\|e(\mathbf{u})\|_{\mathbf{L}^2(\Omega)} \geq K_1|\mathbf{u}|_{\mathbf{H}^1(\Omega)}, \qquad \forall\mathbf{u} \in \mathbf{H}^1_0(\Omega). \tag{26}$$

The Poincaré inequality also holds on Ω: there exists $K_2 > 0$ depending only on Ω, such that

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq K_2|\mathbf{u}|_{\mathbf{H}^1(\Omega)}, \qquad \forall\mathbf{u} \in \mathbf{H}^1_0(\Omega). \tag{27}$$

Using the extension operator $\mathbf{u} \mapsto \widehat{\mathbf{u}}$ we can easily extend, uniformly with respect to ε, the Korn and the Poincaré inequality to $\Omega_{S,\varepsilon}$ using the property (23):

Lemma 2. *(Korn inequality on $\Omega_{S,\varepsilon}$) There exists a constant α, depending only on Ω and $\mathcal{Y}$, such that:*

$$\forall \varepsilon > 0, \quad \forall \mathbf{u}_\varepsilon \in \mathbf{H}^1_0(\Omega_{S,\varepsilon}) \quad \|e(\mathbf{u}_\varepsilon)\|_{\mathbf{L}^2(\Omega_{S,\varepsilon})} \geq \alpha |\mathbf{u}_\varepsilon|_{\mathbf{H}^1(\Omega_{S,\varepsilon})}. \tag{28}$$

Lemma 3. *(Poincaré inequality on $\Omega_{S,\varepsilon}$) There exists a constant β, depending only on Ω and $\mathcal{Y}$, such that:*

$$\forall \varepsilon > 0, \quad \forall \mathbf{u}_\varepsilon \in \mathbf{H}^1_0(\Omega_{S,\varepsilon}) \quad \|\mathbf{u}_\varepsilon\|_{\mathbf{H}^1(\Omega_{S,\varepsilon})} \leq \beta |\mathbf{u}_\varepsilon|_{\mathbf{H}^1(\Omega_{S,\varepsilon})}. \tag{29}$$

Remark 3. *To sum things up, $|\cdot|_{\mathbf{H}^1_0(\Omega_{S,\varepsilon})}$, $\|\cdot\|_{\mathbf{H}^1_0(\Omega_{S,\varepsilon})}$, $\|e(\ \cdot\ )\|_{\mathbf{L}^2(\Omega_{S,\varepsilon})}$, $|\widehat{\cdot}|_{\mathbf{H}^1_0(\Omega)}$, $\|\widehat{\cdot}\|_{\mathbf{H}^1_0(\Omega)}$, $\|e(\widehat{\cdot})\|_{\mathbf{L}^2(\Omega)}$ are all equivalent norms on $\mathbf{H}^1_0(\Omega_{S,\varepsilon})$, uniformly with respect to ε.*

On $H^1_{mean}(\Omega_{F,\varepsilon})$, we also have a Poincaré inequality. Let ϕ_ε belongs to $H^1_{mean}(\Omega_{F,\varepsilon})$. By rescaling each $\mathcal{Y}^{\mathbf{k}}_\varepsilon$ to $\mathcal{Y}$ and applying the Poincaré inequality for $E\left(\phi_\varepsilon(\varepsilon \cdot + \mathbf{k})\right) \in H^1_0(\mathcal{Y})$, using (25), we have

$$\begin{aligned}
\|\phi_\varepsilon\|^2_{L^2(\mathcal{Y}^{\mathbf{k}}_{F,\varepsilon})} \leq \|\widehat{\phi_\varepsilon}\|^2_{L^2(\mathcal{Y}^{\mathbf{k}}_\varepsilon)} &= \varepsilon^d \|E\left(\phi_\varepsilon(\varepsilon \cdot + \mathbf{k})\right)\|^2_{L^2(\mathcal{Y})} \\
&\leq C\varepsilon^d \left\|\nabla\left(E(\phi_\varepsilon(\varepsilon \cdot + \mathbf{k}))\right)\right\|^2_{L^2(\mathcal{Y})} \\
&\leq C\varepsilon^d \left\|\nabla\left(\phi_\varepsilon(\varepsilon \cdot + \mathbf{k})\right)\right\|^2_{L^2(\mathcal{Y}_F)} \\
&\leq C\varepsilon^{d+2} \left\|(\nabla\phi_\varepsilon)(\varepsilon \cdot + \mathbf{k})\right\|^2_{L^2(\mathcal{Y}_F)} \\
&\leq C\varepsilon^2 |\phi_\varepsilon|^2_{H^1(\mathcal{Y}^{\mathbf{k}}_{F,\varepsilon})}, \qquad \forall \phi_\varepsilon \in H^1_{mean}(\Omega_{F,\varepsilon}),
\end{aligned}$$

where the constant C depends only on $\mathcal{Y}$, $\mathcal{Y}_F$, $\mathcal{Y}_S$. Summing these inequalities over $\mathbf{k}$ we get

Lemma 4. *(Poincaré inequality on $\Omega_{F,\varepsilon}$) There exists a constant γ depending only on $\mathcal{Y}$ such that:*

$$\forall \varepsilon > 0, \quad \forall \phi_\varepsilon \in H^1_{mean}(\Omega_{F,\varepsilon}), \quad \|\phi_\varepsilon\|_{L^2(\Omega_{F,\varepsilon})} \leq \|\widehat{\phi_\varepsilon}\|_{L^2(\Omega)} \leq \gamma\varepsilon |\phi_\varepsilon|_{H^1(\Omega_{F,\varepsilon})}. \tag{30}$$

2.4.3 Two–scale convergence

Our objective in this paper is the study of the behavior of the solutions $\mathbf{u}_\varepsilon$ and ϕ_ε of the problem (22) as the parameter ε tends to zero. To achieve this, we will use the two–scale homogenization and for the sake of completeness, we recall here the definition of two–scale convergence, see [26, 2]. Note that we could also use the closely related periodic unfolding method, see [16].

Definition 1. *We say that a sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega)$ two–scale converges to some function $u \in L^2(\Omega; L^2_\#(\mathcal{Y}))$, and we note $u_\varepsilon \twoheadrightarrow u$, if for all admissible test functions $v \in L^2(\Omega, C_\#(\mathcal{Y}))$,*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(\mathbf{x}) v\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) d\mathbf{x} = \int_\Omega \int_\mathcal{Y} u(\mathbf{x}, \mathbf{y}) v(\mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x}. \tag{31}$$

This definition can be extended in an obvious way to complex, vector– and tensor–valued functions in $L^2(\Omega)$, $\mathbf{L}^2(\Omega)$ or $\mathbf{L}^2(\Omega)^d$ by changing the product to the sesquilinear scalar product in $\mathbb{C}$ or the scalar product for vectors, the tensorial product for matrices respectively.

Remark 4. *The question of determining which test functions are admissible is a delicate one and has been adressed on [2]. In particular, some amount of continuity in one variable or the other is necessary to ensure the measurability of $\mathbf{x} \mapsto v\left(\mathbf{x}, \mathbf{x}/\varepsilon\right)$. For example, any $v \in L^2_\#(\mathcal{Y}, C(\overline{\Omega}))$, such as $1_{\overline{\Omega}}(\mathbf{x})\chi_F(\mathbf{y})$, is an admissible test function for the two–scale convergence.*

The two–scale homogenization method relies on the following Proposition, see [2] for proofs:

Proposition 1. *1. Let u_ε be a bounded sequence in $L^2(\Omega)$, there exists $u(\mathbf{x},\mathbf{y}) \in L^2(\Omega\times\mathcal{Y})$ such that up to a subsequence still denoted by u_ε, $u_\varepsilon \twoheadrightarrow u$.*

2. *Let u_ε be a bounded sequence in $H^1(\Omega)$ that converges weakly to a limit u in $H^1(\Omega)$. Then, u_ε two–scale converges to $u(\mathbf{x})$ and there exists a function $u^1(\mathbf{x},\mathbf{y})$ in $L^2(\Omega; H^1_\#(\mathcal{Y})/\mathbb{R})$ such that up to a subsequence, ∇u_ε two–scale converges to $\nabla_x u(\mathbf{x}) + \nabla_y u^1(\mathbf{x},\mathbf{y})$.*

3. *Let u_ε and $\varepsilon\nabla u_\varepsilon$ be two bounded sequences in $L^2(\Omega)$. Then, there exists a function $u(\mathbf{x},\mathbf{y})$ in $L^2(\Omega; H^1_\#(\mathcal{Y}))$ such that up to a subsequence, $u_\varepsilon \twoheadrightarrow u(\mathbf{x},\mathbf{y})$ and $\varepsilon\nabla u_\varepsilon \twoheadrightarrow \nabla_y u(\mathbf{x},\mathbf{y})$.*

2.5 Gårding's inequality and well–posedness

Let us now study the variational problem (22) for any given $\varepsilon > 0$. In a standard way, using the fact that $\lambda \geq 0$, property (3) on μ and Korn's inequality derived at Lemma 2, we obtain for all functions $\mathbf{v}_\varepsilon$ in $\mathbf{H}^1_0(\Omega_{S,\varepsilon})$ the inequality:

$$\int_{\Omega_{S,\varepsilon}} \sigma_\varepsilon(\mathbf{v}_\varepsilon) : e(\mathbf{v}_\varepsilon) \geq \mu_0 \|e(\mathbf{v}_\varepsilon)\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} \geq \mu_0\alpha^2 |\mathbf{v}_\varepsilon|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})}. \tag{32}$$

We define the sesquilinear form on $\mathbf{H}^1_0(\Omega_{S,\varepsilon}) \times H^1_{mean}(\Omega_{F,\varepsilon})$ appearing in the left–hand side of (22) by:

$$\begin{aligned} a^\omega_\varepsilon\left((\mathbf{u}_\varepsilon,\phi_\varepsilon);(\mathbf{v},\psi)\right) = &\int_{\Omega_{S,\varepsilon}} -\rho_S\omega^2 \mathbf{u}_\varepsilon\cdot\overline{\mathbf{v}} + \sigma_\varepsilon(\mathbf{u}_\varepsilon) : \overline{e(\mathbf{v})} \\ &+ \rho_0 \int_{\Gamma^I_\varepsilon} i\omega\left(\overline{\psi}\mathbf{u}_\varepsilon\cdot\mathbf{n}^S_\varepsilon - \phi_\varepsilon\overline{\mathbf{v}\cdot\mathbf{n}^S_\varepsilon}\right) \\ &+ \rho_0 c^2 \sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} \frac{1}{\varepsilon^d|\mathcal{Y}_F|}\left(\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} \mathbf{u}_\varepsilon\cdot\mathbf{n}^S_\varepsilon\right)\overline{\left(\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} \mathbf{v}\cdot\mathbf{n}^S_\varepsilon\right)} \\ &+ \rho_0\int_{\Omega_{F,\varepsilon}} -\frac{\omega^2}{c^2}\phi_\varepsilon\overline{\psi} + \nabla\phi_\varepsilon\cdot\overline{\nabla\psi}. \end{aligned} \tag{33}$$

The analysis proceeds by the use of the Fredholm alternative to obtain a criterium for the well–posedness of the variational problem (22). We show that the alternative holds by proving in the following Lemma that the sesquilinear form a^ω_ε defined by (33) satisfies Gårding's inequality, which is known to be a sufficient condition for the alternative to hold (see [21]).

Lemma 5. *The sesquilinear form $a^\omega_\varepsilon(\cdot,\cdot)$ verifies Gårding's inequality on the space $\mathbf{H}^1_0(\Omega_{S,\varepsilon}) \times H^1_{mean}(\Omega_{F,\varepsilon})$: for all $\omega \geq 0$, there exists constants C, $\gamma > 0$, both independent on ε but dependent on ω, such that for any $\varepsilon > 0$ and $(\mathbf{v}_\varepsilon,\psi_\varepsilon) \in \mathbf{H}^1_0(\Omega_{S,\varepsilon}) \times H^1_{mean}(\Omega_{F,\varepsilon})$,*

$$\begin{aligned} \mathrm{Re}\left(a^\omega_\varepsilon\left((\mathbf{v}_\varepsilon,\psi_\varepsilon);(\mathbf{v}_\varepsilon,\psi_\varepsilon)\right)\right) + C\left(\|\mathbf{v}_\varepsilon\|^2_{\mathbf{L}^2(\Omega_{S,\varepsilon})} + \|\psi_\varepsilon\|^2_{L^2(\Omega_{F,\varepsilon})}\right) \\ \geq \gamma\left(\|\mathbf{v}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \|\psi_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})}\right). \end{aligned} \tag{34}$$

Proof. We follow the same steps as in [18], pp 63–64 for the proof of this inequality. Let $\varepsilon > 0$,

$\omega \in \mathbb{R}$, $\mathbf{v}_\varepsilon \in \mathbf{H}^1_0(\Omega_{S,\varepsilon})$ and $\psi_\varepsilon \in H^1_{mean}(\Omega_{F,\varepsilon})$. We compute from (33):

$$\begin{aligned}a_\varepsilon^\omega\left((\mathbf{v}_\varepsilon,\psi_\varepsilon);(\mathbf{v}_\varepsilon,\psi_\varepsilon)\right) = &\int_{\Omega_{S,\varepsilon}} -\rho_S\omega^2|\mathbf{v}_\varepsilon|^2 + \sigma_\varepsilon(\mathbf{v}_\varepsilon) : \overline{e(\mathbf{v}_\varepsilon)} \\ &+\rho_0 c^2 \sum_{\mathbf{k}\in\mathbf{Z}_\varepsilon^\Omega} \frac{1}{\varepsilon^d|\mathcal{Y}_F|}\left|\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} \mathbf{v}_\varepsilon\cdot\mathbf{n}_\varepsilon^S\right|^2 + \rho_0\int_{\Omega_{F,\varepsilon}} -\frac{\omega^2}{c^2}|\psi_\varepsilon|^2 + |\nabla\psi_\varepsilon|^2 \\ &+\rho_0\int_{\Gamma_\varepsilon^I} i\omega\left(\overline{\psi_\varepsilon}\mathbf{v}_\varepsilon\cdot\mathbf{n}_\varepsilon^S - \psi_\varepsilon\overline{\mathbf{v}_\varepsilon\cdot\mathbf{n}_\varepsilon^S}\right).\end{aligned}$$

Taking the real part of the previous equality and using the coercivity of the stress tensor operator (32), it follows that:

$$\begin{aligned}&\operatorname{Re}\left(a_\varepsilon^\omega\left((\mathbf{v}_\varepsilon,\psi_\varepsilon);(\mathbf{v}_\varepsilon,\psi_\varepsilon)\right)\right) \\ &\qquad + \left(\mu_0\alpha^2 + \rho_S\omega^2\right)\|\mathbf{v}_\varepsilon\|^2_{\mathbf{L}^2(\Omega_{S,\varepsilon})} + \rho_0\left(\frac{\omega^2}{c^2}+1\right)\|\psi_\varepsilon\|^2_{L^2(\Omega_{F,\varepsilon})} \\ &\quad \geq \mu_0\alpha^2\|\mathbf{v}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \rho_0\|\psi_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})} - 2\rho_0\omega\left|\int_{\Gamma_\varepsilon^I}\overline{\psi_\varepsilon}\mathbf{v}_\varepsilon\cdot\mathbf{n}_\varepsilon^S\right|.\end{aligned}$$

If ω is equal to zero, we have proved the Gårding inequality (34). Else, we bound the last term as follows. Using the divergence theorem, the Cauchy-Schwartz inequality and the extension operator properties, see (23), we have

$$\begin{aligned}\left|\int_{\Gamma_\varepsilon^I}\overline{\psi_\varepsilon}\ \widehat{\mathbf{v}_\varepsilon}\cdot\mathbf{n}_\varepsilon^S\right| &= \left|\int_{\Omega_{F,\varepsilon}}\operatorname{div}(\widehat{\mathbf{v}_\varepsilon})\ \overline{\psi_\varepsilon} + \int_{\Omega_{F,\varepsilon}}\nabla\overline{\psi_\varepsilon}\cdot\widehat{\mathbf{v}_\varepsilon}\right| \\ &\leq C\left(|\mathbf{v}_\varepsilon|_{\mathbf{H}^1(\Omega_{S,\varepsilon})}\|\psi_\varepsilon\|_{L^2(\Omega_{F,\varepsilon})} + |\psi_\varepsilon|_{H^1(\Omega_{F,\varepsilon})}\|\mathbf{v}_\varepsilon\|_{\mathbf{L}^2(\Omega_{S,\varepsilon})}\right),\end{aligned}$$

where C is a constant independent of ε and ω. Hence, by Young's inequality, we get for any constants $\delta_1,\delta_2>0$:

$$\begin{aligned}2\rho_0\omega\left|\int_{\Gamma_\varepsilon^I}\overline{\psi_\varepsilon}\ \widehat{\mathbf{v}_\varepsilon}\cdot\mathbf{n}_\varepsilon^S\right| \leq \rho_0 C\omega\Big(&\delta_1|\mathbf{v}_\varepsilon|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \delta_1^{-1}\|\psi_\varepsilon\|^2_{L^2(\Omega_{F,\varepsilon})} \\ &+\delta_2|\psi_\varepsilon|^2_{H^1(\Omega_{F,\varepsilon})} + \delta_2^{-1}\|\mathbf{v}_\varepsilon\|^2_{\mathbf{L}^2(\Omega_{S,\varepsilon})}\Big),\end{aligned}$$

and choosing respectively $\delta_1 = \frac{\mu_0\alpha^2}{2\rho_0 C\omega}$ and $\delta_2 = \frac{1}{2C\omega}$ we obtain

$$\begin{aligned}2\rho_0\omega\left|\int_{\Gamma_\varepsilon^I}\overline{\psi_\varepsilon}\mathbf{v}_\varepsilon\cdot\mathbf{n}_\varepsilon^S\right| \leq &\frac{\mu_0\alpha^2}{2}|\mathbf{v}_\varepsilon|^2_{\mathbf{H}^1(\Omega_{F,\varepsilon})} + \frac{2\rho_0^2C^2\omega^2}{\mu_0\alpha^2}\|\psi_\varepsilon\|^2_{L^2(\Omega_{F,\varepsilon})} \\ &+\frac{\rho_0}{2}|\psi_\varepsilon|^2_{H^1(\Omega_{F,\varepsilon})} + 2\rho_0C^2\omega^2\|\mathbf{v}_\varepsilon\|^2_{\mathbf{L}^2(\Omega_{S,\varepsilon})}.\end{aligned}$$

Finally we have the estimate:

$$\begin{aligned}&\operatorname{Re}\left(a_\varepsilon^\omega\left((\mathbf{v}_\varepsilon,\psi_\varepsilon);(\mathbf{v}_\varepsilon,\psi_\varepsilon)\right)\right) + \left(\mu_0\alpha^2+\rho_S\omega^2\right)\|\mathbf{v}_\varepsilon\|^2_{\mathbf{L}^2(\Omega_{S,\varepsilon})} \\ &\quad +\rho_0\left(\frac{\omega^2}{c^2}+1\right)\|\psi_\varepsilon\|^2_{\mathbf{L}^2(\Omega_{F,\varepsilon})} + 2\rho_0C^2\omega^2\left(\|\mathbf{v}_\varepsilon\|^2_{\mathbf{L}^2(\Omega_{S,\varepsilon})} + \frac{\rho_0}{\mu_0\alpha^2}\|\psi_\varepsilon\|^2_{L^2(\Omega_{F,\varepsilon})}\right) \\ &\qquad \geq \frac{\mu_0\alpha^2}{2}\|\mathbf{v}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \frac{\rho_0}{2}\|\psi_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})}.\end{aligned}$$

Consequently, we have proved that a_ε^ω satisfies (34) for all $\omega \geq 0$. □

Gårding's inequality (34) is then a sufficient condition for the Fredholm Alternative Principle to hold for the problem (22) (see [21]). Thus the following proposition hold true:

Proposition 2. *Either the problem* (22) *is well–posed, or there exists a nonzero solution* $(\mathbf{u}_\varepsilon, \phi_\varepsilon)$ *to the homogeneous adjoint problem:*

$$\overline{a_\varepsilon^\omega}\left((\mathbf{v}_\varepsilon, \psi_\varepsilon); (\mathbf{u}_\varepsilon, \phi_\varepsilon)\right) = 0 \quad \forall (\mathbf{v}_\varepsilon, \psi_\varepsilon) \in \mathbf{H}_0^1(\Omega_{S,\varepsilon}) \times H^1_{mean}(\Omega_{F,\varepsilon}).$$

Remark 5. *Note that the existence of nonzero solutions* $(\mathbf{u}_\varepsilon, \phi_\varepsilon)$ *is effective since this is the case when, e.g.,* ω *is an eigenvalue for the elasticity problem in* $\Omega_{S,\varepsilon}$ *with the boundary conditions* $\sigma_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{n}_\varepsilon^S = 0$ *such that the associated eigenmode* $\mathbf{u}_\varepsilon$ *satisfies at the same time the additional condition* $\mathbf{u}_\varepsilon \cdot \mathbf{n}_\varepsilon^S = \mathbf{0}$ *on* Γ_ε^I*, see [24, 18]. The associated* ϕ_ε *is then equal to zero. Since we cannot control the apparition of these eigenmodes as* ε *varies, we have to be careful about the well–posedness of* (22)*.*

2.6 Energy estimates

Let us assume that (22) has a solution $(\mathbf{u}_\varepsilon, \phi_\varepsilon)$ in $\mathbf{H}_0^1(\Omega_{S,\varepsilon}) \times H^1_{mean}(\Omega_{F,\varepsilon})$. In this section we are looking for *a priori* bounds for $(\mathbf{u}_\varepsilon, \phi_\varepsilon)$ independent on ε. Nevertheless, it is not possible to obtain directly *a priori* estimates uniform in ε because ω could be an eigenvalue for the harmonic problem (22), see e.g. Remark 5. Yet we can prove the following "energy" estimate:

Lemma 6. *Let* $(\mathbf{u}_\varepsilon, \phi_\varepsilon) \in \mathbf{H}_0^1(\Omega_{S,\varepsilon}) \times H^1_{mean}(\Omega_{F,\varepsilon})$ *be a solution of problem* (22)*. There exists a constant* $C(\omega) > 0$*, independent of* ε *(but depending on* ω*) such that:*

$$\begin{aligned} \|\mathbf{u}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \|\phi_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})} \le C(\omega) \Big(& \|\mathbf{u}_\varepsilon\|^2_{\mathbf{L}^2(\Omega_{S,\varepsilon})} + \|\phi_\varepsilon\|^2_{L^2(\Omega_{F,\varepsilon})} \\ & + \|\mathbf{f}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} + \|g_\varepsilon\|^2_{H^1(\Omega)} \Big). \end{aligned} \tag{35}$$

Proof. These estimates are obtained, thanks to standard arguments, by choosing $\mathbf{u}_\varepsilon$ and ϕ_ε as test functions in the variational formulation (22). This leads to:

$$\begin{aligned} a_\varepsilon^\omega\left((\mathbf{u}_\varepsilon, \phi_\varepsilon); (\mathbf{u}_\varepsilon, \phi_\varepsilon)\right) = \int_{\Omega_{S,\varepsilon}} \mathbf{f}_\varepsilon \cdot \overline{\mathbf{u}_\varepsilon} - \int_{\Gamma_\varepsilon^I} \left(g_\varepsilon - \frac{1}{|\mathcal{Y}_F|} \Pi_\varepsilon(\chi_{F,\varepsilon} g_\varepsilon) \right) \overline{\mathbf{u}_\varepsilon} \cdot \mathbf{n}_\varepsilon^S \\ + \frac{i\omega}{c^2} \int_{\Omega_{F,\varepsilon}} \left(g_\varepsilon - \frac{1}{|\mathcal{Y}_F|} \Pi_\varepsilon(\chi_{F,\varepsilon} g_\varepsilon) \right) \overline{\phi_\varepsilon}. \end{aligned}$$

We denote by g_ε^0 the L^2–function defined by:

$$g_\varepsilon^0 = g_\varepsilon - \frac{1}{|\mathcal{Y}_F|} \Pi_\varepsilon(\chi_{F,\varepsilon} g_\varepsilon).$$

Note that $\nabla \left(g_\varepsilon^0|_{\mathcal{Y}_\varepsilon^{\mathbf{k}}} \right) = \nabla \left(g_\varepsilon|_{\mathcal{Y}_\varepsilon^{\mathbf{k}}} \right)$, and

$$\|g_\varepsilon^0\|_{L^2(\Omega)} \le \left(1 + \frac{1}{|\mathcal{Y}_F|} \right) \|g_\varepsilon\|_{L^2(\Omega)}. \tag{36}$$

Thanks to Proposition 5, there exists two constants $C, \gamma > 0$, independent of ε such that

$$\begin{aligned} \operatorname{Re} \left(\int_{\Omega_{S,\varepsilon}} \mathbf{f}_\varepsilon \cdot \overline{\mathbf{u}_\varepsilon} - \sum_{\mathbf{k} \in \mathbf{Z}_\varepsilon^\Omega} \int_{\Gamma_{F,\varepsilon}^{\mathbf{k}}} g_\varepsilon^0 \overline{\mathbf{u}_\varepsilon} \cdot \mathbf{n}_\varepsilon^S + \frac{i\omega}{c^2} \int_{\Omega_{F,\varepsilon}} g_\varepsilon^0 \overline{\phi_\varepsilon} \right) \\ + C \left(\|\mathbf{u}_\varepsilon\|^2_{\mathbf{L}^2(\Omega_{S,\varepsilon})} + \|\phi_\varepsilon\|^2_{L^2(\Omega_{F,\varepsilon})} \right) \ge \gamma \left(\|\mathbf{u}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \|\phi_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})} \right). \end{aligned} \tag{37}$$

To control the first term of estimate (37), we use the Cauchy–Schwartz and Young inequalities. We obtain

$$\left|\int_{\Omega_{S,\varepsilon}} \mathbf{f}_\varepsilon \cdot \overline{\mathbf{u}_\varepsilon}\right| \leq \frac{1}{2}\left(\|\mathbf{u}_\varepsilon\|^2_{\mathbf{L}^2(\Omega_{S,\varepsilon})} + \|\mathbf{f}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)}\right), \tag{38}$$

$$\left|\frac{i\omega}{c^2}\int_{\Omega_{F,\varepsilon}} g^0_\varepsilon \cdot \overline{\phi_\varepsilon}\right| \leq \frac{\omega}{2c^2}\left(\|\phi_\varepsilon\|^2_{L^2(\Omega_{F,\varepsilon})} + \|g^0_\varepsilon\|^2_{L^2(\Omega)}\right), \tag{39}$$

and, for all $\delta > 0$:

$$\begin{aligned}\left|\sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon}\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} g^0_\varepsilon \overline{\mathbf{u}_\varepsilon}\cdot\mathbf{n}^S_\varepsilon\right| &= \left|\sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon}\int_{\mathcal{Y}^{\mathbf{k}}_{F,\varepsilon}} g^0_\varepsilon\ \overline{\operatorname{div}\widehat{\mathbf{u}_\varepsilon}} + \nabla g^0_\varepsilon\cdot\overline{\widehat{\mathbf{u}_\varepsilon}}\right| \\ &\leq \frac{\delta}{2}\|\widehat{\mathbf{u}_\varepsilon}\|^2_{\mathbf{H}^1(\Omega_{F,\varepsilon})} + \frac{1}{2\delta}\left(\|g^0_\varepsilon\|^2_{L^2(\Omega_{F,\varepsilon})} + \sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon}\|\nabla g^0_\varepsilon\|^2_{L^2(\mathcal{Y}^{\mathbf{k}}_{F,\varepsilon})}\right).\end{aligned}$$

Thanks to the properties of the extension operator (see Lemma 1) and to (36), there exists a constant C_1 independent of ε and ω such that

$$\left|\sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon}\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} g^0_\varepsilon \overline{\mathbf{u}_\varepsilon}\cdot\mathbf{n}^S_\varepsilon\right| \leq \frac{C_1\delta}{2}\delta\|\mathbf{u}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \frac{C_1}{2\delta}\sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon}\|g_\varepsilon\|^2_{H^1(\mathcal{Y}^{\mathbf{k}}_{F,\varepsilon})}.$$

Hence choosing $\delta = \gamma/C_1$ we get the estimate

$$\left|\int_{\Gamma^I_\varepsilon} g^0_\varepsilon \overline{\mathbf{u}_\varepsilon}\cdot\mathbf{n}^S_\varepsilon\right| = \left|\sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon}\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} g^0_\varepsilon \overline{\mathbf{u}_\varepsilon}\cdot\mathbf{n}^S_\varepsilon\right| \leq \frac{\gamma}{2}\|\mathbf{u}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \frac{C_1^2}{2\gamma}\|g_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})}. \tag{40}$$

Combining finally (37), (38), (39) and (40) we can conclude that for some constant $C > 0$ independent of ε (but depending on ω), the following estimate holds true:

$$\begin{aligned}\frac{\gamma}{2}\left(\|\mathbf{u}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \|\phi_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})}\right) \leq C\Big(&\|\mathbf{u}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} + \|\phi_\varepsilon\|^2_{L^2(\Omega_{F,\varepsilon})} \\ &+\|\mathbf{f}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} + \|g_\varepsilon\|^2_{H^1(\Omega)}\Big).\end{aligned}$$

This ends the proof of estimate (35). □

3 Two–scale homogenization of the coupled model

In this section we study the asymptotic as ε goes to zero. Nevertheless the standard scheme to obtain the homogenized limit does not apply here. Indeed, usually we follow the steps:

- existence of a solution for a given ε,
- *a priori* bounds, independent of ε,
- two–scale convergence up to a subsequence by the use of Proposition 1,
- identification of the two–scale homogenized problem.

However the problem presented here satisfies neither the first point, because of the two valid statements in the Fredholm alternative, nor the second point since we only have a Gårding inequality and not a coercivity property. In fact, it so happens that for some values of the frequency ω, depending on ε, our problem is not well–posed due to the occurence of so–called traction–free oscillations as explained in Remark 5.

A way to deal with this difficulty is to make the hypothesis that the required well–posedness and boundedness results are true for ε small enough, and perform the homogenization process according to the usual theory. Then, by studying the resulting homogenized problem, it is possible to get a better understanding of the Fredholm alternative for the coupled problem (22) as ε goes to zero. This kind of arguments was already used in [11, 6, 4]. In fact, we show that away from the discrete set of eigenvalues of the homogenized problem, the coupled problem (22) is well–posed for ε small enough. Moreover, when the homogenized problem has a unique solution and due to the linear character of the system, the solutions of the problem (22) also satisfy *a priori* bounds uniform in ε. This allows us to prove that the initial assumption (well-posedness and *a proiri* estimates for ε small enough) holds true for all values of ω distinct from the spectrum of the homogenized problem.

Let us now present the main result of this section, which will allow us to pass to the limit and obtain, as the main conclusion of the paper, the homogenized behavior of the material.

Theorem 1. *There is a discrete set Λ, such that for any $\omega \in \mathbb{R} \setminus \Lambda$, there exists $\varepsilon_0(\omega)$ and $C(\omega)$ in $\mathbb{R}_+^*$ such that for any $0 < \varepsilon < \varepsilon_0$, the problem (22) is well–posed for any data $(\mathbf{f}_\varepsilon, g_\varepsilon) \in \mathbf{L}^2(\Omega) \times H^1(\Omega)$, and its solution $(\mathbf{u}_\varepsilon, \phi_\varepsilon)$ satisfies the a priori bounds:*

$$\|\mathbf{u}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \|\phi_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})} \leq C(\omega) \left(\|\mathbf{f}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} + \|g_\varepsilon\|^2_{H^1(\Omega)} \right). \tag{41}$$

The proof of this result is detailed in Section 3.2, but we need to identify and study the homogenized problem first.

3.1 Two–scale problem identification

In this whole section, we fix $\omega \in \mathbb{R}$ and a sequence of data $(\mathbf{f}_\varepsilon, g_\varepsilon)_{\varepsilon>0} \subset \mathbf{L}^2(\Omega) \times H^1(\Omega)$, that converges strongly to $(\mathbf{f}, g) \in \mathbf{L}^2(\Omega) \times H^1(\Omega)$. We assume that there exists $C > 0$ such that, for ε small enough, the variational problem (22) with data $(\mathbf{f}_\varepsilon, g_\varepsilon)$ has at least one solution $(\mathbf{u}_\varepsilon, \phi_\varepsilon)$, such that the following bound holds uniformly in ε:

$$\|\mathbf{u}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \|\phi_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})} \leq C \left(\|\mathbf{f}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} + \|g_\varepsilon\|^2_{H^1(\Omega)} \right). \tag{42}$$

Remark 6. *Note that these assumptions reflect the conclusions of Theorem 1, which we prove later on in Section 3.2.*

Using the two–scale convergence framework, we are going to investigate the asymptotics of problem (22) and identify the homogenized two–scale problem. Since the sequence $(\mathbf{f}_\varepsilon, g_\varepsilon)_{\varepsilon>0}$ converges strongly in $\mathbf{L}^2(\Omega) \times H^1(\Omega)$ it is bounded uniformly in ε in $\mathbf{L}^2(\Omega) \times H^1(\Omega)$. So from (42), there exists $C > 0$ independent of ε such that

$$\|\mathbf{u}_\varepsilon\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon})} + \|\phi_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})} \leq C.$$

Thanks to the properties of the extension operators introduced in Section 2.4 and to (30), we have then for some constant $C > 0$:

$$\|\widehat{\mathbf{u}_\varepsilon}\|^2_{\mathbf{H}^1(\Omega)} + \frac{1}{\varepsilon^2}\|\widehat{\phi_\varepsilon}\|^2_{L^2(\Omega)} + |\widehat{\phi_\varepsilon}|^2_{H^1(\Omega)} \leq C. \tag{43}$$

Thanks to Proposition 1, we know that there exists a subsequence, still indexed by ε for simplicity, and three functions: $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $\mathbf{u}^1 \in \mathbf{L}^2(\Omega; H^1_\#(\mathcal{Y})/\mathbb{C})$ and $\phi \in L^2(\Omega; H^1_\#(\mathcal{Y}))$, such that $\widehat{\mathbf{u}_\varepsilon}$, $\widehat{\phi_\varepsilon}$ and their gradients two–scale converge:

$$
\begin{aligned}
&\widehat{\mathbf{u}_\varepsilon} \twoheadrightarrow \mathbf{u} \text{ in } \mathbf{L}^2(\Omega\times\mathcal{Y}), \qquad && \nabla\widehat{\mathbf{u}_\varepsilon} \twoheadrightarrow \nabla_x \mathbf{u} + \nabla_{\mathbf{y}}\mathbf{u}^1 \text{ in } \mathbf{L}^2(\Omega\times\mathcal{Y}),\\
&\widehat{\phi_\varepsilon}/\varepsilon \twoheadrightarrow \phi \text{ in } L^2(\Omega\times\mathcal{Y}), && \nabla\widehat{\phi_\varepsilon} \twoheadrightarrow \nabla_{\mathbf{y}}\phi \text{ in } \mathbf{L}^2(\Omega\times\mathcal{Y}).
\end{aligned}
\tag{44}
$$

We are now going to identify the homogenized problem, satisfied by $\mathbf{u}$, $\chi_S \mathbf{u}^1$ and $\chi_F \phi$.

3.1.1 Identification of the homogenized problem

To pass to the limit in the variational formulation we shall use well chosen test functions:

- $\mathbf{v}_\varepsilon(\mathbf{x}, \mathbf{x}/\varepsilon) = \mathbf{v}(\mathbf{x}) + \varepsilon \mathbf{v}^1(\mathbf{x}, \mathbf{x}/\varepsilon)$ with $\mathbf{v} \in \boldsymbol{\mathcal{D}}(\Omega)$ and $\mathbf{v}^1 \in \boldsymbol{\mathcal{D}}(\Omega, C^\infty_\#(\mathcal{Y}))$, and
- $\psi_\varepsilon(\mathbf{x}, \mathbf{x}/\varepsilon) = \varepsilon\psi(\mathbf{x}, \mathbf{x}/\varepsilon)$ with $\psi \in \mathcal{D}\left(\Omega, C^\infty(\mathcal{Y}_F) \cap H^1_{mean}(\mathcal{Y}_F)\right)$.

We can then pass to the limit as ε goes to zero in the weak formulation (22), which writes:

$$
\begin{aligned}
\int_{\Omega_{S,\varepsilon}} &-\rho_S \omega^2 \mathbf{u}_\varepsilon \cdot \overline{\mathbf{v}_\varepsilon} + \sigma_\varepsilon(\mathbf{u}_\varepsilon) : \overline{e(\mathbf{v}_\varepsilon)} + \rho_0 \int_{\Gamma^I_\varepsilon} i\omega \left(\overline{\psi_\varepsilon}\mathbf{u}_\varepsilon \cdot \mathbf{n}^S_\varepsilon - \phi_\varepsilon \overline{\mathbf{v}_\varepsilon}\cdot \mathbf{n}^S_\varepsilon\right)\\
&+ \rho_0 c^2 \sum_{\mathbf{k}\in\mathbf{Z}^\Omega_\varepsilon} \frac{1}{\varepsilon^d |\mathcal{Y}_F|} \left(\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} \mathbf{u}_\varepsilon \cdot \mathbf{n}^S_\varepsilon\right) \overline{\left(\int_{\Gamma^{\mathbf{k}}_{F,\varepsilon}} \mathbf{v}_\varepsilon \cdot \mathbf{n}^S_\varepsilon\right)}\\
&+ \rho_0 \int_{\Omega_{F,\varepsilon}} -\frac{\omega^2}{c^2}\phi_\varepsilon \overline{\psi_\varepsilon} + \nabla\phi_\varepsilon \cdot \overline{\nabla\psi_\varepsilon}\\
= \int_{\Omega_{S,\varepsilon}} &\mathbf{f}_\varepsilon \cdot \overline{\mathbf{v}_\varepsilon} + \int_{\Omega_{F,\varepsilon}} \nabla g_\varepsilon \cdot \overline{\mathbf{v}_\varepsilon}\\
&+ \int_{\Omega_{F,\varepsilon}} \left(g_\varepsilon - \frac{1}{|\mathcal{Y}_F|}\Pi_\varepsilon(\chi_{F,\varepsilon} g_\varepsilon)\right) \overline{\left(\operatorname{div}(\mathbf{v}_\varepsilon) - \frac{i\omega}{c^2}\psi_\varepsilon\right)}.
\end{aligned}
\tag{45}
$$

It is straightforward to pass to the limit in most terms of the identity. For instance, we write

$$
\int_{\Omega_{S,\varepsilon}} \sigma_\varepsilon(\mathbf{u}_\varepsilon) : e(\mathbf{v}_\varepsilon) = \int_\Omega \chi_{S,\varepsilon}\, \lambda\, \partial_i \mathbf{u}_{\varepsilon,i}\, \partial_i \mathbf{v}_{\varepsilon,i} + \chi_{S,\varepsilon}\, \mu\, \partial_i \mathbf{u}_{\varepsilon,i}\, \partial_j \mathbf{v}_{\varepsilon,j},
$$

where $\mathbf{v}_{\varepsilon,j}$ denotes the jth component of the vector $\mathbf{v}_\varepsilon$. As the functions $\chi_{S,\varepsilon}\, \lambda\, \partial_i \mathbf{v}_{\varepsilon,j}$ are admissible test functions in the sense of two–scale convergence, we can pass to the limit as ε goes to zero, and we obtain:

$$
\int_{\Omega_{S,\varepsilon}} \sigma_\varepsilon(\mathbf{u}_\varepsilon) : e(\mathbf{v}_\varepsilon) \to \int_\Omega \int_{\mathcal{Y}_S} \left(\sigma_{\mathbf{x}}(\mathbf{u}) + \sigma_{\mathbf{y}}(\mathbf{u}^1)\right) : \left(e_{\mathbf{x}}(\mathbf{v}) + e_{\mathbf{y}}(\mathbf{v}^1)\right),
$$

where we denote by $\sigma_{\mathbf{x}}(\cdot)$ and $\sigma_{\mathbf{y}}(\cdot)$ respectively the following tensor–valued operators:

$$
\sigma_{\mathbf{x}}(\mathbf{v})(\mathbf{x},\ \mathbf{y}) = \lambda(\mathbf{x},\mathbf{y})\operatorname{div}_{\mathbf{x}}(\mathbf{v})(\mathbf{x},\mathbf{y})\mathrm{Id} + \mu(\mathbf{x},\mathbf{y}) e_{\mathbf{x}}(\mathbf{v})(\mathbf{x},\mathbf{y}) \text{ for } \mathbf{v} \in \mathbf{H}^1(\Omega, L^2_\#(\mathcal{Y})),
$$

$$
\sigma_{\mathbf{y}}(\mathbf{v})(\mathbf{x},\ \mathbf{y}) = \lambda(\mathbf{x},\mathbf{y})\operatorname{div}_{\mathbf{y}}(\mathbf{v})(\mathbf{x},\mathbf{y})\mathrm{Id} + \mu(\mathbf{x},\mathbf{y}) e_{\mathbf{y}}(\mathbf{v})(\mathbf{x},\mathbf{y}) \text{ for } \mathbf{v} \in \mathbf{H}^1_\#(\mathcal{Y}, L^2(\Omega)).
$$

The main difficulty consists in dealing with the nonstandard terms supported by the interior boundary Γ_ε^I, which are:

$$\rho_0 c^2 \sum_{\mathbf{k}\in\mathbf{Z}_\varepsilon^\Omega} \frac{1}{\varepsilon^d |\mathcal{Y}_F|} \left(\int_{\Gamma_{F,\varepsilon}^{\mathbf{k}}} \mathbf{u}_\varepsilon \cdot \mathbf{n}_\varepsilon^S \right) \overline{\left(\int_{\Gamma_{F,\varepsilon}^{\mathbf{k}}} \mathbf{v}_\varepsilon \cdot \mathbf{n}_\varepsilon^S \right)} + \rho_0 \int_{\Gamma_\varepsilon^I} i\omega \left(\overline{\psi_\varepsilon} \mathbf{u}_\varepsilon - \phi_\varepsilon \overline{\mathbf{v}_\varepsilon} \right) \cdot \mathbf{n}_\varepsilon^S,$$

and also with the term

$$\int_{\Omega_{F,\varepsilon}} \left(g_\varepsilon - \frac{1}{|\mathcal{Y}_F|} \Pi_\varepsilon(\chi_{F,\varepsilon} g_\varepsilon) \right) \overline{\left(\operatorname{div}(\mathbf{v}_\varepsilon) - \frac{i\omega}{c^2} \psi_\varepsilon \right)}.$$

First, we consider the product of integrals on the boundary of the holes as in [3, 7]. We write:

$$\begin{aligned} &\sum_{\mathbf{k}\in\mathbf{Z}_\varepsilon^\Omega} \varepsilon^{-d} \left(\int_{\Gamma_{F,\varepsilon}^{\mathbf{k}}} \mathbf{u}_\varepsilon \cdot \mathbf{n}_\varepsilon^S \right) \overline{\left(\int_{\Gamma_{F,\varepsilon}^{\mathbf{k}}} \mathbf{v}_\varepsilon \cdot \mathbf{n}_\varepsilon^S \right)} \\ &\qquad = \sum_{\mathbf{k}\in\mathbf{Z}_\varepsilon^\Omega} \int_{\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}} \operatorname{div}\, \widehat{\mathbf{u}_\varepsilon}(\mathbf{x}) \overline{\left(\varepsilon^{-d} \int_{\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}} \operatorname{div}\, \mathbf{v}_\varepsilon(\mathbf{x}') \mathrm{d}\mathbf{x}' \right)} \mathrm{d}\mathbf{x} \\ &\qquad = \int_\Omega \chi_{F,\varepsilon} \operatorname{div}\, \widehat{\mathbf{u}_\varepsilon}\ \overline{\Pi_\varepsilon(\chi_{F,\varepsilon} \operatorname{div}\, \mathbf{v}_\varepsilon)}, \end{aligned}$$

where Π_ε is defined by (21). To study the convergence of this product involving the projector Π_ε, we need the following strong convergence result, which is proved in [3, 7]:

Lemma 7. *Let $\psi \in C^\infty(\overline{\Omega}; C_\#^\infty(\mathcal{Y}))$, then*

$$\Pi_\varepsilon \left(\psi \left(\cdot, \frac{\cdot}{\varepsilon} \right) \chi_{F,\varepsilon} \right) \to \int_{\mathcal{Y}_F} \psi(\cdot, \mathbf{y}) \mathrm{d}\mathbf{y} \qquad \textit{strongly in } L^2(\Omega).$$

Consequenlty, since $\operatorname{div}\, \mathbf{v}_\varepsilon = \operatorname{div}_\mathbf{x} \mathbf{v} + \varepsilon \operatorname{div}_\mathbf{y} \mathbf{v}^1 + \operatorname{div}_\mathbf{x} \mathbf{v}^1$, we obtain immediately:

$$\Pi_\varepsilon(\chi_{F,\varepsilon} \operatorname{div}\, \mathbf{v}_\varepsilon) \to \int_{\mathcal{Y}_F} \operatorname{div}_\mathbf{x} \mathbf{v} + \operatorname{div}_\mathbf{y} \mathbf{v}^1 \qquad \text{strongly in } L^2(\Omega).$$

Moreover, for any $w \in \mathcal{D}(\Omega)$, the function

$$\chi_F(\mathbf{y}) w(\mathbf{x}) \in C(\overline{\Omega}, L_\#^2(\mathcal{Y}))$$

is an admissible test function for the two–scale convergence, see Remark 4. Hence, by definition of the two–scale convergence we obtain

$$\chi_{F,\varepsilon} \operatorname{div}\, \widehat{\mathbf{u}_\varepsilon} \rightharpoonup \int_{\mathcal{Y}_F} \operatorname{div}_\mathbf{x} \mathbf{u} + \operatorname{div}_\mathbf{y} \mathbf{u}^1 \qquad \text{weakly in } L^2(\Omega).$$

Combining these two results, we see that $\int_\Omega \left(\chi_{F,\varepsilon} \operatorname{div}\, \widehat{\mathbf{u}_\varepsilon}\ \overline{\Pi_\varepsilon(\chi_{F,\varepsilon} \operatorname{div}\, \mathbf{v}_\varepsilon)} \right)$ converges to

$$\frac{1}{|\mathcal{Y}_F|} \int_\Omega \left(\int_{\mathcal{Y}_F} (\operatorname{div}_\mathbf{x} \mathbf{u} + \operatorname{div}_\mathbf{y} \mathbf{u}^1) \mathrm{d}\mathbf{y} \right) \overline{\left(\int_{\mathcal{Y}_F} (\operatorname{div}_\mathbf{x} \mathbf{v} + \operatorname{div}_\mathbf{y} \mathbf{v}^1) \mathrm{d}\mathbf{y}' \right)}.$$

Another nonstandard term corresponds to the integral over the interior boundary. This one is easier to deal with since it can be rewritten as a standard bilinear form using the Stokes formula. Indeed, we obtain:

$$\int_{\Gamma_\varepsilon^I} \overline{\psi}\mathbf{u}_\varepsilon \cdot \mathbf{n}_\varepsilon^S - \phi_\varepsilon \overline{\mathbf{v}_\varepsilon} \cdot \mathbf{n}_\varepsilon^S = \int_{\Omega_{F,\varepsilon}} \left(\nabla\phi_\varepsilon \cdot \overline{\mathbf{v}_\varepsilon} + \phi_\varepsilon \mathrm{div}\ \mathbf{v}_\varepsilon - \overline{\nabla\psi_\varepsilon} \cdot \mathbf{u}_\varepsilon - \overline{\psi_\varepsilon}\mathrm{div}\ \mathbf{v}_\varepsilon\right).$$

Because $\widehat{\phi_\varepsilon}$ and ψ_ε converge strongly to 0 in $L^2(\Omega)$, see (44), passing to the two–scale limit yields:

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon^I} \overline{\psi}\mathbf{u}_\varepsilon \cdot \mathbf{n}_\varepsilon^S - \phi_\varepsilon \overline{\mathbf{v}} \cdot \mathbf{n}_\varepsilon^S = \int_\Omega \int_{\mathcal{Y}_F} \nabla_{\mathbf{y}}\phi \cdot \overline{\mathbf{v}} - \overline{\nabla_{\mathbf{y}}\psi} \cdot \mathbf{u}.$$

Finally, let us compute the limit of the term:

$$\int_{\Omega_{F,\varepsilon}} \left(g_\varepsilon - \frac{1}{|\mathcal{Y}_F|}\Pi_\varepsilon(\chi_{F,\varepsilon} g_\varepsilon)\right) \overline{\left(\mathrm{div}(\mathbf{v}_\varepsilon) - \frac{i\omega}{c^2}\psi_\varepsilon\right)}.$$

Let $g_\varepsilon^0 = \left(g_\varepsilon - \frac{1}{|\mathcal{Y}_F|}\Pi_\varepsilon(g_\varepsilon \chi_{F,\varepsilon})\right) \in L^2(\Omega_{F,\varepsilon})$. This function has zero mean over each pore $\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}$, and its restriction to each fluid cell $\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}$ belongs to H^1. Thus g_ε^0 belongs to $H^1_{mean}(\Omega_{F,\varepsilon})$. Moreover $\nabla(g_\varepsilon^0|_{\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}}) = \nabla(g_\varepsilon|_{\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}})$. Consequently from Lemma 4, we deduce that for some $C > 0$ independent of ε,

$$\|\widehat{g_\varepsilon^0}\|_{L^2(\Omega)} \leq C\varepsilon \|\nabla g_\varepsilon^0\|_{\mathbf{L}^2(\Omega_{F,\varepsilon})} \leq C\varepsilon\|\nabla g_\varepsilon\|_{\mathbf{L}^2(\Omega)}.$$

Since the sequence $(g_\varepsilon)_{\varepsilon>0}$ is strongly convergent in $H^1(\Omega)$, $\|\nabla g_\varepsilon\|_{\mathbf{L}^2(\Omega)}$ is bounded independently of ε. Thus $\widehat{g_\varepsilon^0}$ converges strongly to 0 in $L^2(\Omega)$. Hence

$$\lim_{\varepsilon\to 0} \int_{\Omega_{F,\varepsilon}} g_\varepsilon^0 \overline{\left(\mathrm{div}(\mathbf{v}_\varepsilon) - \frac{i\omega}{c^2}\psi_\varepsilon\right)} = \lim_{\varepsilon\to 0} \int_\Omega \chi_{F,\varepsilon}\widehat{g_\varepsilon^0} \overline{\left(\mathrm{div}(\mathbf{v}_\varepsilon) - \frac{i\omega}{c^2}\psi_\varepsilon\right)} = 0.$$

We can now pass to the two–scale limit in every term of identity (45). We deduce that $\mathbf{u}$, $\mathbf{u}^1$ and ϕ are solutions of the following two–scale variational formulation: for all $\mathbf{v} \in \boldsymbol{\mathcal{D}}(\Omega)$, $\mathbf{v}^1(\mathbf{x},\mathbf{y}) \in \boldsymbol{\mathcal{D}}(\Omega, C^\infty_\#(\mathcal{Y}))$ and $\psi \in \mathcal{D}\left(\Omega, C^\infty(\mathcal{Y}_F) \cap H^1_{mean}(\mathcal{Y}_F)\right)$,

$$\begin{aligned}
&\int_\Omega \int_{\mathcal{Y}_S} -\rho_S\omega^2 \mathbf{u}\cdot\overline{\mathbf{v}} + \left(\sigma_{\mathbf{x}}(\mathbf{u}) + \sigma_{\mathbf{y}}(\mathbf{u}^1)\right) : \overline{(e_{\mathbf{x}}(\mathbf{v}) + e_{\mathbf{y}}(\mathbf{v}^1))} \\
&\quad + \frac{\rho_0 c^2}{|\mathcal{Y}_F|}\int_\Omega \left(\int_{\mathcal{Y}_F} (\mathrm{div}_{\mathbf{x}}\mathbf{u} + \mathrm{div}_{\mathbf{y}}\mathbf{u}^1)\mathrm{d}\mathbf{y}\right) \overline{\left(\int_{\mathcal{Y}_F} (\mathrm{div}_{\mathbf{x}}\mathbf{v} + \mathrm{div}_{\mathbf{y}}\mathbf{v}^1)\mathrm{d}\mathbf{y}'\right)}\mathrm{d}\mathbf{x} \\
&\quad + \rho_0 \int_\Omega\int_{\mathcal{Y}_F} \nabla_{\mathbf{y}}\phi \cdot \overline{\nabla_{\mathbf{y}}\psi} + \rho_0 i\omega \int_\Omega\int_{\mathcal{Y}_F} \left(\nabla_{\mathbf{y}}\phi\cdot\overline{\mathbf{v}} - \overline{\nabla_{\mathbf{y}}\psi}\cdot\mathbf{u}\right) \\
&\quad = \int_\Omega\int_{\mathcal{Y}} (\mathbf{f}\chi_S + \nabla g\chi_F)\cdot\overline{\mathbf{v}}.
\end{aligned} \tag{47}$$

Remark 7. *As a consequence of Lemma 7, we can show that*

$$\int_{\mathcal{Y}_F} \phi = 0 \ a.e. \ in \ \Omega. \tag{48}$$

To prove this, we note that $\chi_{F,\varepsilon}\phi_\varepsilon/\varepsilon$ *converges weakly to* $\int_{\mathcal{Y}_F} \phi$ *in* $L^2(\Omega)$ *by definition of the two–scale convergence. Moreover, since* $\phi_\varepsilon \in H^1_{mean}(\Omega_{F,\varepsilon})$,

$$\Pi_\varepsilon(\chi_{F,\varepsilon}\phi_\varepsilon/\varepsilon) = \sum_{\mathbf{k}\in\mathbf{Z}_\varepsilon^\Omega} \frac{1}{|\mathcal{Y}_\varepsilon^{\mathbf{k}}|}\left(\int_{\mathcal{Y}_{F,\varepsilon}^{\mathbf{k}}} \phi_\varepsilon/\varepsilon\right)\chi_{\mathcal{Y}_\varepsilon^{\mathbf{k}}} = 0. \tag{49}$$

But for any test function $\psi \in C^\infty(\overline{\Omega})$ we have

$$\int_{\Omega_{F,\varepsilon}} \Pi_\varepsilon(\chi_{F,\varepsilon}(\phi_\varepsilon/\varepsilon))\psi = \int_\Omega \chi_{F,\varepsilon}(\phi_\varepsilon/\varepsilon)\Pi_\varepsilon(\chi_{F,\varepsilon}\psi).$$

Applying Lemma 7 yields

$$\int_\Omega \chi_{F,\varepsilon}(\phi_\varepsilon/\varepsilon)\Pi_\varepsilon(\chi_{F,\varepsilon}\psi) \to \int_\Omega \left(\int_{\mathcal{Y}_F} \phi\right) |\mathcal{Y}_F|\psi, \text{ as } \varepsilon \to 0.$$

Consequently, thanks to this convergence and (49), we obtain that for any test function $\psi \in C^\infty(\overline{\Omega})$, $\int_\Omega \left(\int_{\mathcal{Y}_F} \phi\right)\psi = 0$ and (48) follows.

Remark 8. *Let us make a few comments on the homogenized model described by the system (47). At first glance, the only remaining inertia term seems to be $\rho_S\omega^2\mathbf{u}$, so it seems that there is no added mass effect from the fluid on the structure. However, we will see that we have the relationship*

$$\nabla_{\mathbf{y}}\phi = i\omega\mathbf{u},$$

so the effective density is equal to the average density of the mixture.

On the other hand, there is no impact from the micro–structure geometry on the effective density of the homogenized material because $\mathbf{u}^1$ does not appear in the inertia terms. This means, for example, that there is no possibility of a band gap effect as in [6] as the mass does not depend on the frequency ω.

Remark 9. *When ω is zero, the fluid and the structure decouple. We then have*

$$\rho_0 \int_\Omega \int_{\mathcal{Y}_F} \nabla_{\mathbf{y}}\phi \cdot \overline{\nabla_{\mathbf{y}}\psi} = 0 \quad \forall \psi \in \mathcal{D}\left(\Omega, C^\infty(\mathcal{Y}_F) \cap H^1_{mean}(\mathcal{Y}_F)\right).$$

Since $\mathcal{D}\left(\Omega, C^\infty(\mathcal{Y}_F) \cap H^1_{mean}(\mathcal{Y}_F)\right)$ is dense in $L^2(\Omega; H^1_{mean}(\mathcal{Y}_F))$, one can take $\psi = \phi$ as a test function to obtain $\nabla_y\phi = 0$ in $\Omega \times \mathcal{Y}_F$. Moreover $\int_{\mathcal{Y}_F} \phi = 0$ a.e. in Ω (see Remark 7), so we find that $\phi|_{\mathcal{Y}_F} = 0$ a.e. in Ω. Our homogenized model then reduces to the same homogenized two–scale system found in the static case in [7].

The next step is to decompose this two–scale problem on $\Omega \times \mathcal{Y}$ into cell problems for ϕ and $\mathbf{u}^1$ where we use the macroscopic displacement $\mathbf{u}$ as a slow–varying parameter, and an effective homogenized problem on $\mathbf{u}$. Solving the cell problems yields explicit corrector functions, which can be reinjected in (47) to write the homogenized coefficients for the macroscopic problem.

3.1.2 Fluid cell problem

Choosing $\mathbf{v} = \mathbf{0}$ and $\mathbf{v}^1 = \mathbf{0}$, we recover the following variational problem for the homogenized fluid velocity potential ϕ. The restriction $\phi|_{\mathcal{Y}_F} \in L^2(\Omega, H^1_{mean}(\mathcal{Y}_F))$ verifies:

$$\rho_0 \int_\Omega \int_{\mathcal{Y}_F} \nabla_{\mathbf{y}}\phi \cdot \overline{\nabla_{\mathbf{y}}\psi} = \rho_0 i\omega \int_\Omega \int_{\mathcal{Y}_F} \overline{\nabla_{\mathbf{y}}\psi} \cdot \mathbf{u} \quad \forall \psi \in \mathcal{D}\left(\Omega, C^\infty(\mathcal{Y}_F) \cap H^1_{mean}(\mathcal{Y}_F)\right).$$

Since $\mathbf{u}$ does not depend on the $\mathbf{y}$ variable and $\mathcal{Y}_F$ is strictly included in $\mathcal{Y}$, it implies that $\nabla_{\mathbf{y}}\phi = i\omega\mathbf{u}$ a.e. in $\Omega \times \mathcal{Y}_F$. This determines uniquely $\phi|_{\mathcal{Y}_F}$ as a function of $\mathbf{u}$. Remember that we have chosen originally the origin as the barycenter of $\mathcal{Y}_F$, hence this yields

$$\phi = i\omega\mathbf{y} \cdot \mathbf{u} \quad \text{and} \quad \nabla_{\mathbf{y}}\phi = i\omega\mathbf{u}, \qquad \text{on } \Omega \times \mathcal{Y}_F. \tag{50}$$

Remark 10. *We see that the limit velocity of the fluid coincides locally with the limit velocity of the structure. This result is mainly a consequence of the completely disconnected geometry of the fluid domain: since the pores are closed, there is no independent motion of the gas with respect to the structure.*

3.1.3 Elastic cell problem

From (47), by taking $\mathbf{v} = \mathbf{0}$ and $\psi = 0$ we obtain that for a.e. $\mathbf{x} \in \Omega$ and for all $\mathbf{v}^1 \in \boldsymbol{C}^\infty_\#(\mathcal{Y})$,

$$\int_{\mathcal{Y}_S} \left(\sigma_{\mathbf{x}}(\mathbf{u}) + \sigma_{\mathbf{y}}(\mathbf{u}^1)\right) : \overline{e_{\mathbf{y}}(\mathbf{v}^1)} = \frac{\rho_0 c^2}{|\mathcal{Y}_F|} \left(\int_{\mathcal{Y}_F} \mathrm{div}_{\mathbf{x}}\mathbf{u} + \mathrm{div}_{\mathbf{y}}\mathbf{u}^1 \mathrm{d}\mathbf{y}\right) \overline{\left(\int_{\Gamma_F} \mathbf{v}^1 \cdot \mathbf{n}_S\right)}. \tag{51}$$

The strong formulation associated with (51) is

$$\begin{cases} -\mathbf{div}_{\mathbf{y}}\left(\sigma_{\mathbf{y}}(\mathbf{u}^1)\right) = \mathbf{div}_{\mathbf{y}}\left(\sigma_{\mathbf{x}}(\mathbf{u})\right), & \text{in } \mathcal{Y}_S, \\ \sigma_{\mathbf{y}}(\mathbf{u}^1)\mathbf{n}_S - \dfrac{\rho_0 c^2}{|\mathcal{Y}_F|}\left(\displaystyle\int_{\Gamma_F} \mathbf{u}^1 \cdot \mathbf{n}_S\right)\mathbf{n}_S = \rho_0 c^2 \mathrm{div}_{\mathbf{x}}(\mathbf{u})\mathbf{n}_S - \sigma_{\mathbf{x}}(\mathbf{u})\mathbf{n}_S, & \text{on } \Gamma_F, \\ \mathbf{u}^1 \text{ is } \mathcal{Y}\text{–periodic.} \end{cases}$$

Remark 11. *Note that there is no dependence on ω in the structure cell problem, so the homogenized material's elastic behavior is independent of frequency.*

Remark 12. *The cell problem is nonstandard as there is a nonlocal term in the boundary conditions, as in the static case [7] which corresponds to the case $\omega = 0$.*

Since this problem is linear, we are going to take advantage of the superposition principle to express $\mathbf{u}^1$ in terms of $\mathbf{u}$. We define the classical auxiliary functions $\mathbf{p}^{kl} \in \mathbf{H}^1(\mathcal{Y}_S)$ by:

$$\mathbf{p}^{kl}(\mathbf{y}) = \frac{1}{2}\left(y_k \mathbf{e}^l + y_l \mathbf{e}^k\right) \qquad \text{for } 1 \le k, l \le d, \tag{52}$$

where the vectors $\mathbf{e}^k$ for $1 \le k \le d$ are the unit vectors of $\mathbb{R}^d$ whose components are $\mathbf{e}^k_l = \delta_{kl}$ for $1 \le k, l \le d$. Using the superposition principle in the local problem (51), we decompose $\mathbf{u}^1|_{\Omega\times\mathcal{Y}_S}$ as follows:

$$\mathbf{u}^1(\mathbf{x}, \mathbf{y}) = e_{\mathbf{x}}(\mathbf{u})_{kl}(\mathbf{x})\boldsymbol{\chi}^{kl}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \Omega,\ \mathbf{y} \in \mathcal{Y}_S, \tag{53}$$

where the functions $\boldsymbol{\chi}^{kl} \in \mathbf{L}^\infty(\Omega, H^1_\#(\mathcal{Y}_S, \mathbb{R})/\mathbb{R})$, $1 \le k, l \le d$ are solutions of the cell problems

$$\begin{cases} -\mathbf{div}_{\mathbf{y}}\left(\sigma_{\mathbf{y}}(\mathbf{p}^{kl} + \boldsymbol{\chi}^{kl})\right) = \mathbf{0}, & \text{in } \mathcal{Y}_S, \\ \sigma_{\mathbf{y}}(\mathbf{p}^{kl} + \boldsymbol{\chi}^{kl})\mathbf{n}_S - \dfrac{\rho_0 c^2}{|\mathcal{Y}_F|}\left(\displaystyle\int_{\Gamma_F} (\mathbf{p}^{kl} + \boldsymbol{\chi}^{kl}) \cdot \mathbf{n}_S\right)\mathbf{n}_S = \mathbf{0}, & \text{on } \Gamma_F, \\ \boldsymbol{\chi}^{kl} \text{ is } \mathcal{Y}\text{–periodic.} \end{cases} \tag{54}$$

Remark 13. *The functions $\boldsymbol{\chi}^{kl}$ are called the correctors for the homogenized problem (47). The cell problems (54) have only real coefficients and data; therefore, the family of correctors $(\boldsymbol{\chi}^{kl})_{kl}$ are in fact $\mathbb{R}^d$–valued functions by opposition to the complex–valued displacement. This will be important when computing the homogenized coefficients, see Proposition 3.*

The necessary compatibility conditions for existence of solutions of (54), or more generally for any problem of the form

$$\begin{aligned}
-\mathbf{div}_{\mathbf{y}}\left(\sigma_{\mathbf{y}}(\mathbf{u})\right) &= \mathbf{F}, && \text{in } \mathcal{Y}_S \\
\sigma_{\mathbf{y}}(\mathbf{u})\mathbf{n}_S - \frac{\rho_0 c^2}{|\mathcal{Y}_F|}\left(\int_{\Gamma_F} \mathbf{u}\cdot\mathbf{n}_S\right)\mathbf{n}_S &= \mathbf{G}, && \text{on } \Gamma_F \\
\mathbf{u} \text{ is } \mathcal{Y}\text{–periodic},
\end{aligned} \tag{55}$$

reads, since $\int_{\Gamma_F} \mathbf{n}_S = \mathbf{0}$:

$$\int_{\mathcal{Y}_S} \mathbf{F} + \int_{\Gamma_F} \mathbf{G} = \mathbf{0}. \tag{56}$$

In our case, it writes:

$$\begin{aligned}
\int_{\mathcal{Y}_S} \mathbf{div}_{\mathbf{y}}\left(\sigma_{\mathbf{y}}(\mathbf{p}^{kl})\right) + \int_{\Gamma_F}\left(\rho_0 c^2 \mathrm{div}_{\mathbf{y}}(\mathbf{p}^{kl})\mathbf{n}_S - \sigma_{\mathbf{y}}(\mathbf{p}^{kl})\mathbf{n}_S\right) \\
= \int_{\Gamma_F} \sigma_{\mathbf{y}}(\mathbf{p}^{kl})\mathbf{n}_S - \int_{\Gamma_F} \sigma_{\mathbf{y}}(\mathbf{p}^{kl})\mathbf{n}_S = \mathbf{0}.
\end{aligned}$$

Thus, the compatibility conditions are satisfied, and the local problems (54) as well as (51) are well posed. Notice that the function $\mathbf{p}^{kl} + \boldsymbol{\chi}^{kl}$, which appears in the cell problem (54), describes the microstructure's response to a spatially slowly varying strain. We will need the following technical result for such functions, a special version of Korn's inequality, which is proved in the annex:

Lemma 8. *Consider the space of real–valued functions on $\mathcal{Y}_S$ defined as follows:*

$$\mathbf{V} = Span\left\{(\mathbf{p}^{kl})_{1\leq k,l\leq d}\right\} + \mathbf{H}^1_{\#}(\mathcal{Y}_S,\mathbb{R})/\mathbb{R} \subset \mathbf{H}^1(\mathcal{Y}_S,\mathbb{R})/\mathbb{R}, \tag{57}$$

where the family $(\mathbf{p}_{kl})_{1\leq k,l\leq d}$ is defined by (52). Then the following Korn's inequality holds in $\mathbf{V}$: there exists $C > 0$ depending only on $\mathcal{Y}_S$ such that

$$\|\boldsymbol{\phi}\|_{\mathbf{H}^1(\mathcal{Y}_S)} \leq C\|e(\boldsymbol{\phi})\|_{L^2(\mathcal{Y}_S)} \qquad \forall \boldsymbol{\phi} \in \mathbf{V}. \tag{58}$$

3.1.4 Homogenized problem

Thanks to the expressions of ϕ given by (50) and of $\mathbf{u}^1$ parameterized by $\mathbf{u}$ given by (53), we can eliminate $\mathbf{u}^1$ and ϕ from the two–scale system (47) to obtain the homogenized variational formulation satisfied by the displacement $\mathbf{u}$. We obtain, for any $\mathbf{v} \in \boldsymbol{\mathcal{D}}(\Omega)$,

$$\begin{aligned}
\int_\Omega -(|\mathcal{Y}_S|\rho_S + |\mathcal{Y}_F|\rho_0)\omega^2 \mathbf{u}\cdot\overline{\mathbf{v}} + \left(\int_{\mathcal{Y}_S} \sigma_{\mathbf{x}}(\mathbf{u}) + e_{\mathbf{x}}(\mathbf{u})_{kl}\sigma_{\mathbf{y}}(\boldsymbol{\chi}^{kl})\right) : \overline{e_{\mathbf{x}}(\mathbf{v})} \\
+ \rho_0 c^2 \int_\Omega \left(|\mathcal{Y}_F|\mathrm{div}_{\mathbf{x}}\mathbf{u} - e_{\mathbf{x}}(\mathbf{u})_{kl}\int_{\Gamma_F} \boldsymbol{\chi}^{kl}\cdot\mathbf{n}_S\right)\overline{\mathrm{div}_{\mathbf{x}}\mathbf{v}}\mathrm{d}\mathbf{x} \\
= \int_\Omega (|\mathcal{Y}_S|\mathbf{f} + |\mathcal{Y}_F|\nabla g)\cdot\overline{\mathbf{v}}.
\end{aligned} \tag{59}$$

Now, this formulation motivates the introduction of the *homogenized coefficients*, respectively the homogenized density, elastic tensor and stress of the effective material:

$$\rho^* = |\mathcal{Y}_S|\rho_S + |\mathcal{Y}_F|\rho_0, \tag{60}$$

$$\mathcal{A}^*_{ijkl} = \int_{\mathcal{Y}_S} \left(\sigma_{\mathbf{y}}(\mathbf{p}^{kl} + \boldsymbol{\chi}^{kl})_{ij} - \rho_0 c^2 \delta_{ij} \mathrm{div}_{\mathbf{y}} \boldsymbol{\chi}^{kl} \right) + \rho_0 c^2 |\mathcal{Y}_F| \delta_{ij} \delta_{kl}, \tag{61}$$

$$\sigma^*(\mathbf{u}) = \left(\mathcal{A}^*_{ijkl} e(\mathbf{u})_{kl}\right)_{1\le i,j\le d} = \mathcal{A}^* e(\mathbf{u}). \tag{62}$$

Finally, by density of test functions $\mathbf{v} \in \boldsymbol{\mathcal{D}}(\Omega)$ in $\mathbf{H}^1_0(\Omega)$, $\mathbf{u}$ is a solution of the following variational problem on $\mathbf{H}^1_0(\Omega)$: find $\mathbf{u} \in \mathbf{H}^1_0(\Omega)$ such that for any $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$,

$$\int_\Omega -\rho^* \omega^2 \mathbf{u} \cdot \overline{\mathbf{v}} + \mathcal{A}^* e(\mathbf{u}) : \overline{e(\mathbf{v})} = \int_\Omega (|\mathcal{Y}_S|\mathbf{f} + |\mathcal{Y}_F| \nabla g) \cdot \overline{\mathbf{v}}. \tag{63}$$

Remark 14. *Let us make some comments on the properties of the homogenized problem* (63)*. From the definitions of the effective density* (60) *and of the homogenized elastic tensor* (61) *the effects of the fluid on the structure are the following:*

- *An added mass effect, so that the effective density* (60) *of the homogenized porous medium is also its averaged density,*
- *A mean pressure term, which is nonlocal in the micro–scale cell problems* (54) *and appears in the effective elastic tensor* (61) *as a contribution to the compressibility factor of the material. This is the consequence of the phenomenon described in Remark 2 for finite values of ε: the pressure term in each hole results in an effect which is nonlocal at the microscopic scale, but local at the macroscopic scale. In fact, this is the same effective tensor that was found in [7] in the static case (modulo a different air compressibility factor, because we have used here a different state law for the gas).*

On the whole, the resulting homogenized model (63) *behaves like a linearized elastic material. This is in agreement with the experimental data since low–frequency sound propagates in the lungs without much attenuation [29].*

Let us study the properties of problem (63). The sesquilinear form that appears on the left hand side is not coercive. However, the following ellipticity properties of $\mathcal{A}^*$ show that the homogenized problem keeps much of the operator structure of linearized elasticity.

Proposition 3. *The fourth–order real–valued tensor $\mathcal{A}^*(\mathbf{x})$ defined in* (61) *has the following properties:*

1. *(Symmetry) The coefficients of $\mathcal{A}^*$ satisfy the property:*

$$\mathcal{A}^*_{ijkl} = \mathcal{A}^*_{ijlk} = \mathcal{A}^*_{klij}, \tag{64}$$

2. *(Strong Ellipticity) There exists $\kappa > 0$ depending only on μ_0 and the geometry of the cell $\mathcal{Y}$ such that for any $\mathbf{x} \in \Omega$ and any $d \times d$ real symmetric matrix $\underline{\xi}$,*

$$\mathcal{A}^*(\mathbf{x})\underline{\xi} : \underline{\xi} \ge \kappa \underline{\xi} : \underline{\xi}; \tag{65}$$

3. *(Definite positiveness)*

$$\mathcal{A}^*(\mathbf{x})\underline{\xi} : \underline{\xi} = 0 \Leftrightarrow \underline{\xi} = 0. \tag{66}$$

Proof. The proof of the first point is standard and follows exactly the same lines as in [2, 7]. Consequently, we refer to these works for details. We will focuss only on the second point. Note that the third item follows directly from the ellipticity property.

Let us prove the uniform coercivity. Since $\mathcal{A}^*(\mathbf{x})$ is positive definite in a finite dimensional space, it is known that there exists a scalar $\kappa(\mathbf{x}) > 0$ such that $\mathcal{A}^*(\mathbf{x})\underline{\xi} : \underline{\xi} \geq \kappa(\mathbf{x})\underline{\xi} : \underline{\xi}$. However, $\kappa(\mathbf{x})$ depends both on the geometry and on the Lamé coefficient $\mu(\mathbf{x}, \mathbf{y})$, in a way that is not clear at this point. We are going to prove a uniform lower bound for $\kappa(\mathbf{x})$, independent of $\mathbf{x}$ and of the continuity properties of λ and μ, that makes these dependencies explicit. Let us define the function

$$\boldsymbol{\phi}_\xi = \xi_{ij}\boldsymbol{\phi}^{ij}.$$

We have

$$\mathcal{A}^*(\mathbf{x})\underline{\xi} : \underline{\xi} = a_y^{\#}(\boldsymbol{\phi}_\xi(\mathbf{x}), \boldsymbol{\phi}_\xi(\mathbf{x})) \geq \mu_0 \|e_{\mathbf{y}}(\boldsymbol{\phi}_\xi(\mathbf{x}))\|^2_{L^2(\mathcal{Y}_S)}. \tag{67}$$

Now, let $\mathbf{z}_1, \ldots, \mathbf{z}_d$, be a basis of $\mathbf{Z}$ (and $\mathbb{R}^d$) such that for d faces of the unit cell $\mathcal{Y}$, denoted by $F_1, \ldots, F_d$, the translated surfaces $F_1 + \mathbf{z}_1, \ldots, F_d + \mathbf{z}_d$ are also faces of $\mathcal{Y}_F$. Then, for $i = 1, \ldots, d$ and any $\mathbf{y} \in F_i$, by $\mathcal{Y}$–periodicity of $\boldsymbol{\chi}^{kl}$ we have

$$\underline{\xi}\mathbf{z}_i = p_\xi(\mathbf{z}_i) = \boldsymbol{\phi}_\xi(\mathbf{y} + \mathbf{z}_i) - \boldsymbol{\phi}_\xi(\mathbf{y}).$$

Because the trace operator is continuous from $\mathbf{H}^1(\mathcal{Y}_S)$ on F_i and $F_i + \mathbf{z}_i$, there exists a constant C depending on $\mathcal{Y}_S$, $\mathcal{Y}$ only such that

$$\|\underline{\xi}\mathbf{z}_i\| \leq C\|\boldsymbol{\phi}_\xi\|_{\mathbf{H}^1(\mathcal{Y}_S)}.$$

Since the $\mathbf{z}_i$ form a basis of $\mathbb{R}^d$, we have

$$\sqrt{\underline{\xi} : \underline{\xi}} < C \sup_{i=1,\ldots,d} \|\underline{\xi}\mathbf{z}_i\| \leq C\|\boldsymbol{\phi}_\xi\|_{\mathbf{H}^1(\mathcal{Y}_S)}.$$

Here, C depends on $\mathcal{Y}_S$ and $\mathcal{Y}$ only. To conclude, we need to use the special version of Korn inequality for the space on which the $\boldsymbol{\phi}_\xi$ live which is proved in the Annex, Lemma 8. This yields

$$\|\boldsymbol{\phi}_\xi\|_{\mathbf{H}^1(\mathcal{Y}_S)} \leq C\|e(\boldsymbol{\phi}_\xi)\|_{L^2(\mathcal{Y}_S)},$$

where C does not depend on $\mathbf{x}$ and depends only on $\mathcal{Y}_S$ and $\mathcal{Y}$. Combining estimates (67) and (58), we have proved that for some constant $C > 0$ depending only on $\mathcal{Y}_S$ and $\mathcal{Y}$,

$$\mathcal{A}^*(\mathbf{x})\underline{\xi} : \underline{\xi} > C\mu_0\underline{\xi} : \underline{\xi} \qquad \forall \mathbf{x} \in \Omega. \tag{68}$$

□

We are going to apply the Fredholm theory to the homogenized problem to show that there is a discrete set of resonant frequencies ω for this limit problem. We denote by $(\,\cdot\,,\,\cdot\,)_{\mathbf{L}^2}$ the $\mathbf{L}^2$–scalar product in $\mathbf{L}^2(\Omega)$.

Definition 2. *Let B be the unbounded operator $\mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ such that:*

$$\begin{cases} D(B) = \left\{\mathbf{u} \in \mathbf{H}^1_0(\Omega), -\mathbf{div}\,(\mathcal{A}^*(\mathbf{x})e(\mathbf{u})) \in \mathbf{L}^2(\Omega)\right\} \\ B\mathbf{u} = -\mathbf{div}\,(\mathcal{A}^*(\mathbf{x})e(\mathbf{u}))\,, \end{cases} \tag{69}$$

and b be the associated sesquilinear form in $\mathbf{H}^1(\Omega)$, that is

$$b(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathcal{A}^*(\mathbf{x})e(\mathbf{u}) : \overline{e(\mathbf{v})} = (B\mathbf{u}, \mathbf{v})_{\mathbf{L}^2}. \tag{70}$$

Define the family of operators $A_\omega = B - \omega^2 I$ with $D(A_\omega) = D(B)$, and the associated family of sesquilinear forms a_ω appearing on the left–hand side of (63):

$$a_\omega(\mathbf{u}, \mathbf{v}) = \int_\Omega -\rho^* \omega^2 \mathbf{u} \cdot \overline{\mathbf{v}} + \mathcal{A}^*(\mathbf{x}) e(\mathbf{u}) : \overline{e(\mathbf{v})} = (A_\omega \mathbf{u}, \mathbf{v})_{\mathbf{L}^2}. \tag{71}$$

Then, we have the well–known properties, since B is elliptic:

Proposition 4. *1. B is self–adjoint and has compact resolvent,*

2. the eigenvalues of B form a sequence of nonnegative real numbers converging to $+\infty$ $(\lambda_n)_{n\geq 0}$, $0 < \lambda_0 < ... < \lambda_n < ...$

3. A_ω is invertible iff $\omega^2 \notin (\lambda_n)_{n\geq 0}$,

4. If $\omega^2 = \lambda_n$, the solutions of $A_\omega u = 0$ form a subspace $\mathbf{V}_n$ of finite dimension d_n for which there exists an orthonormal basis of eigenvectors of B, $(\boldsymbol{\phi}^k)_{1\leq k \leq d_n}$, and $A_\omega \mathbf{u} = \mathbf{f}$ is solvable iff $(\boldsymbol{\phi}^k, \mathbf{f})_{\mathbf{L}^2} = 0$ for all $1 \leq k \leq d_n$.

Remark 15. *In the case of Neumann boundary conditions, the main difference is that $\lambda_0 = 0$ is an eigenvalue of the problem (with multiplicity $d_0 = 6$) corresponding to the infinitesimal rigid displacements. Except this everything else stands. Indeed, the homogenization process and $\mathcal{A}^*$ do not depend on the boundary conditions.*

3.2 Proof of the *a priori* bounds and Theorem 1

We are now going to prove Theorem 1, making good use of our knowledge of the homogenized system (63) and its eigenvalue set. The idea is to proceed by contradiction. Suppose that the segments of Theorem 1 are false for some ω for which the problem (63) is well–posed. Then, the following alternative holds true:

- The problem (22) is ill–posed for arbitrary small values of ε. In this case, there is a sequence $(\varepsilon_n)_{n\geq 0}$ converging to zero such that for all $n \geq 0$, the problem (22) is ill–posed. We know then by Proposition 5 that the homogeneous problem (22) with vanishing data $(\mathbf{f}_n, g_n) = (\mathbf{0}, 0)$ has a non–zero solution $(\mathbf{u}_n, \phi_n)$. Since the problem is linear, we can require that $(\mathbf{u}_n, \phi_n)$ is normalized meaning that:

$$\|\mathbf{u}_n\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon_n})} + \|\phi_n\|^2_{H^1(\Omega_{F,\varepsilon_n})} = 1.$$

- Or the problem (22) is well–posed for ε small enough, but the solutions do not satisfy *a priori* bounds uniform in ε. Then, there exists a sequence $(\varepsilon_n)_{n\geq 0}$ converging to zero such that for some sequence $(\mathbf{f}_n, g_n) \in \mathbf{L}^2(\Omega) \times H^1(\Omega)$ indexed by $n \geq 0$, the sequence of solutions $(\mathbf{u}_n, \phi_n)$ of (22) satisfies

$$1 = \|\mathbf{u}_n\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon_n})} + \|\phi_n\|^2_{H^1(\Omega_{F,\varepsilon_n})} > n \left(\|\mathbf{f}_n\|^2_{\mathbf{L}^2(\Omega)} + \|g_n\|^2_{H^1(\Omega)} \right).$$

In either case, we have obtained a sequence $(\varepsilon_n)_{n\geq 0}$ converging to zero and a sequence of data $(\mathbf{f}_n, g_n)$ converging strongly to zero, such that the sequence $(\mathbf{u}_n, \phi_n)$ is a sequence of solutions of (22) and is bounded independently of n in $\mathbf{H}^1(\Omega_{S,\varepsilon}) \times H^1(\Omega_{F,\varepsilon})$:

$$\|\mathbf{u}_n\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon_n})} + \|\phi_n\|^2_{H^1(\Omega_{F,\varepsilon_n})} = 1. \tag{72}$$

We are going to show that $\widehat{\mathbf{u}_n}$ and $\widehat{\phi_n}$ converge strongly to zero in $L^2(\Omega)$, and then, using the estimate (35), we will conclude that $\lim_{n\to+\infty} \|\mathbf{u}_n\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon_n})} + \|\phi_n\|^2_{H^1(\Omega_{F,\varepsilon_n})} = 0$ which is absurd considering (72).

Thanks to (72), we can apply our analysis from Section 3.1 directly. In particular, $\widehat{\mathbf{u}_n}$ and $\widehat{\phi_n}$ two–scale converge: there exists functions $\mathbf{u} \in \mathbf{H}^1_D(\Omega)$, $\mathbf{u}^1 \in \mathbf{L}^2(\Omega, H^1_\#(\mathcal{Y})/\mathbb{C})$ and $\phi \in L^2(\Omega, H^1_\#(\mathcal{Y}))$ such that

$$\widehat{\mathbf{u}_n} \twoheadrightarrow \mathbf{u}, \qquad \nabla\widehat{\mathbf{u}_n} \twoheadrightarrow \nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{y}}\mathbf{u}^1,$$
$$\widehat{\phi_n} \twoheadrightarrow 0, \qquad \nabla\widehat{\phi_n} \twoheadrightarrow \nabla_{\mathbf{y}}\phi.$$

Moreover, $(\mathbf{u}, \mathbf{u}^1|_{\Omega\times\mathcal{Y}_S}, \phi|_{\Omega\times\mathcal{Y}_F})$ satisfy the homogenized problem (73) with zero right hand side. We have supposed that the variational problem (73) is well–posed for our choice of ω. As a consequence, $\mathbf{u}$ is equal to zero and the respective restrictions of $\mathbf{u}^1$ and ϕ to $\Omega\times\mathcal{Y}_S$ and $\Omega\times\mathcal{Y}_F$ are also zero. Let us now show that this implies that $\widehat{\mathbf{u}_n}$ and $\widehat{\phi_n}$ converge to 0 strongly in $H^1(\Omega)$. The first difficulty is that we do not control $\mathbf{u}^1$ and ϕ on the whole domain $\Omega\times\mathcal{Y}$, and thus the two–scale limits of the gradients $\nabla\widehat{\mathbf{u}_n}$ and $\nabla\widehat{\phi_n}$ are not *a priori* uniquely defined. We prove that this is not the case for the weak H^1–limits. We know that $\nabla\widehat{\mathbf{u}_n}$ converges weakly to $\int_{\mathcal{Y}} \nabla_{\mathbf{x}}\mathbf{u} + \nabla_y\mathbf{u}^1$ so, since, $\mathbf{u} = \mathbf{0}$ and $\mathbf{u}^1|_{\mathcal{Y}_S} = \mathbf{0}$,

$$\nabla\hat{\mathbf{u}} \rightharpoonup \int_{\mathcal{Y}_F} \nabla_y\mathbf{u}^1 \quad \text{weakly in } \mathbf{L}^2(\Omega).$$

Then, for almost every $\mathbf{x}\in\Omega$, $\mathbf{u}^1(\mathbf{x},\cdot)$ is zero on $\mathcal{Y}_S$ and belongs to $\mathbf{H}^1_\#(\mathcal{Y})$, so clearly $\mathbf{u}^1 = 0$ on $\Gamma_F = \partial\mathcal{Y}_F$. Integrating by parts, we obtain for any $i, j \in \{1,\dots,d\}$,

$$\int_{\mathcal{Y}_F} \partial_i u^1_j(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} = \int_{\Gamma_F} u^1_j(\mathbf{x},\mathbf{y}) n_{S,i}(\mathbf{y}) = 0.$$

In the same way, we know that $\nabla\widehat{\phi_n}$ converges weakly to $\int_{\mathcal{Y}_S} \nabla_{\mathbf{y}}\phi$ in $\mathbf{H}^1(\Omega)$ and for any $i \in \{1,\dots,d\}$,

$$\int_{\mathcal{Y}_S} \partial_i\phi(\mathbf{y})\mathrm{d}\mathbf{y} = \int_{\Gamma_F} \phi(\mathbf{x},\mathbf{y}) n_{S,i}(\mathbf{y}) = 0.$$

This proves that $\widehat{\mathbf{u}_n}$ and $\widehat{\phi_n}$ converge weakly to zero in $\mathbf{H}^1(\Omega)$ and $H^1(\Omega)$ respectively. By compactness of the injection of $H^1_0(\Omega)$ in $L^2(\Omega)$, there exists a subsequence (still denoted by n) such that $\widehat{\mathbf{u}_n}$ and $\widehat{\phi_n}$ converge strongly to 0 in $L^2(\Omega)$. We now use the estimate (35). Since $(\mathbf{u}_n, \phi_n)$ are solutions of (22), we get:

$$\|\mathbf{u}_n\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon_n})} + \|\phi_n\|^2_{H^1(\Omega_{F,\varepsilon_n})} \\ \leq C(\omega)\left(\|\mathbf{u}_n\|^2_{\mathbf{L}^2(\Omega_{S,\varepsilon_n})} + \|\phi_n\|^2_{L^2(\Omega_{F,\varepsilon_n})} + \|\mathbf{f}_n\|^2_{\mathbf{L}^2(\Omega)} + \|g_n\|^2_{H^1(\Omega)}\right).$$

Hence since $\widehat{\mathbf{u}_n}$ and $\widehat{\phi_n}$ converge strongly to zero in $L^2(\Omega)$, we obtain

$$\lim_{n\to\infty} \|\mathbf{u}_n\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon_n})} + \|\phi_n\|^2_{H^1(\Omega_{F,\varepsilon_n})} = 0.$$

But this is in contradiction with the construction of the sequence, since

$$\|\mathbf{u}_n\|^2_{\mathbf{H}^1(\Omega_{S,\varepsilon_n})} + \|\phi_n\|^2_{H^1(\Omega_{F,\varepsilon_n})} = 1.$$

This ends the proof of Theorem 1. □

Finally, we have the following Proposition, which completes Theorem 1:

Proposition 5. *Let $0 < \lambda_0 \leq \cdots < \lambda_n \leq \ldots$ be the ordered sequence of eigenvalues of the homogeneous variational problem on $\mathbf{H}_0^1(\Omega)$*

$$-\rho^* \lambda^2 \mathbf{u} - \mathbf{div}\,(\sigma^*(\mathbf{u})) = 0.$$

then, for any $\omega \in \mathbb{R} \setminus \{\lambda_n\}_{n\in\mathbb{N}}$, there exists $\varepsilon_0(\omega)$ and $C(\omega)$ in $\mathbb{R}_+^*$ such that for $0 < \varepsilon < \varepsilon_0(\omega)$, the problem (22) is well posed and for any data $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $g \in H^1(\Omega)$, the solution $(\mathbf{u}_\varepsilon, \phi_\varepsilon)$ satisfies the a priori estimate:

$$\|\mathbf{u}_\varepsilon\|^2_{H^1(\Omega_{S,\varepsilon})} + \|\phi_\varepsilon\|^2_{H^1(\Omega_{F,\varepsilon})} \leq C(\omega)\left(\|\mathbf{f}\|^2_{L^2(\Omega)} + \|g\|^2_{H^1(\Omega)}\right).$$

3.3 Convergence Theorem and homogenized problem

We can now sum up the asymptotic behavior of the solutions in the following theorem.

Theorem 2. *Two–scale homogenization of problem (22)*
Let the frequency $\omega \geq 0$ be such that ω^2 is in the resolvent set of B, then for ε small enough the problem (22) is well posed.
Moreover, let the data $(\mathbf{f}_\varepsilon, g_\varepsilon)_{\varepsilon>0} \subset \mathbf{L}^2(\Omega) \times H^1(\Omega)$ be a sequence such that $\mathbf{f}_\varepsilon$ and g_ε converge strongly to $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $g \in H^1(\Omega)$, then the solutions $(\mathbf{u}_\varepsilon, \phi_\varepsilon)$ of the problem (22) two–scale converge in the sense that:

$$\chi_{S,\varepsilon}\widehat{\mathbf{u}_\varepsilon} \twoheadrightarrow \mathbf{u}\chi_S,$$
$$\widehat{\phi}_\varepsilon \twoheadrightarrow 0, \qquad \chi_{F,\varepsilon}\nabla\widehat{\phi}_\varepsilon \twoheadrightarrow \mathbf{u}\chi_F,$$

where $\mathbf{u}$ is the solution of the homogenized problem:

$$\begin{aligned} -\rho^*\omega^2\mathbf{u} - \mathbf{div}(\sigma^*(\mathbf{u})) &= |\mathcal{Y}_S|\mathbf{f} + |\mathcal{Y}_F|\nabla g && \text{on } \Omega, \\ \mathbf{u} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{73}$$

and the coefficients σ^* and ρ^* can be explicitly computed using formulas (60), (61), (62).

Proof. The only result of this theorem which we have not yet proved is the two–scale convergence of the whole sequences $\chi_{S,\varepsilon}\widehat{\mathbf{u}_\varepsilon}$, $\chi_{F,\varepsilon}\widehat{\phi}_\varepsilon$, and not only subsequences. This is a consequence of the uniqueness of the solution of the homogenized problem (73), since every subsequence then converges to the same limit. □

4 Conclusion

We have obtained an homogenized system of equations for the modeling of sound propagation in a foam like material such as the lung tissue. Starting from a model coupling elastic and acoustic equations, we obtain at the limit a linearized elastic-like medium. In particular, we have shown that the resonances of the material do not change the homogenized model: in fact, the resonances of the real material, for a given $\varepsilon > 0$, are shown to be close to the resonances of the homogenized material.

Obviously, this model is limited in its physical description of the lung tissue, but is nevertheless valid for the low–frequency range since we recover the model introduced by Rice [29]. However, for higher frequencies some of the phenomena we have neglected, may become more important, in particular viscous attenuation or scattering by the alveoli as the wavelength becomes smaller [23]. Indeed, it is well-known that sounds of a frequency above 1kHz are quickly attenuated when propagating through the parenchyma [28, 30]. We refer to [14] for the numerical study of an other model showing some memory effects due to a viscoelastic micro–structure.

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Annex

We provide here the proof of Lemma 8. We recall the definition of the functions $\mathbf{p}^{kl} \in \mathbf{H}^1(Y_S)$:

$$\mathbf{p}^{kl}(\mathbf{y}) = \frac{1}{2}\left(y_k \mathbf{e}^l + y_l \mathbf{e}^k\right) \qquad \text{for } 1 \leq k, l \leq d, \tag{74}$$

where the vectors $\mathbf{e}^k$ for $1 \leq k \leq d$ are the unit vectors of $\mathbb{R}^d$ whose components are $\mathbf{e}^k_l = \delta_{kl}$ for $1 \leq k,\ l \leq d$.

Lemma 9. *Consider the space of real–valued functions on Y_S defined as follows:*

$$\mathbf{V} = \mathit{Span}\left\{(\mathbf{p}^{kl})_{1\leq k,l\leq d}\right\} + \mathbf{H}^1_\#(Y_S, \mathbb{R})/\mathbb{R}^d \subset \mathbf{H}^1(Y_S, \mathbb{R})/\mathbb{R}^d, \tag{75}$$

where the family $(\mathbf{p}_{kl})_{1\leq k,l\leq d}$ is defined by (74). Then the following Korn's inequality holds in $\mathbf{V}$: there exists $C > 0$ depending only on the geometry of Y_S such that

$$\|\boldsymbol{\phi}\|_{H^1_\#(Y_S)} \leq C\|e(\boldsymbol{\phi})\|_{L^2(Y_S)} \qquad \forall \boldsymbol{\phi} \in \mathbf{V}. \tag{76}$$

Proof. We follow the proof of Theorem 6.3–4 in [15].

Step 1. We begin by showing that $\mathbf{V}$ is a closed subspace of $\mathbf{H}^1(Y_S)/\mathbb{R}^d$. $\mathbf{H}^1_\#(Y_S)$ is closed in $\mathbf{H}^1(Y_S)$ since it is the closure of $C^\infty_\#(Y_S)^d$ in $\mathbf{H}^1(Y_S)$.
Since the space of constant functions, noted $\mathbb{R}^d$ for simplicity, is a subspace of $\mathbf{H}^1_\#(Y_S)$ with finite dimension, it is closed both in $\mathbf{H}^1(Y_S)$ and in $\mathbf{H}^1_\#(Y_S)$. Identifying the quotient spaces $\mathbf{H}^1(Y_S)/\mathbb{R}^d$ and $\mathbf{H}^1_\#(Y_S)/\mathbb{R}^d$ with the orthogonal complement of $\mathbb{R}^d$ in each space, it is clear that $\mathbf{H}^1_\#(Y_S)/\mathbb{R}^d$ is a closed subspace of $\mathbf{H}^1(Y_S)/\mathbb{R}^d$.

Step 2. Let $\mathbf{M}$ be the orthogonal complement of $\mathbf{H}^1_\#(Y_S)/\mathbb{R}^d$ in $\mathbf{H}^1(Y_S)/\mathbb{R}^d$. For each choice of k, l, $1 \leq k, l \leq d$, we can decompose each $\mathbf{p}^{kl}$ according to the direct sum $\mathbf{H}^1(Y_S)/\mathbb{R}^d = \mathbf{M} \oplus \mathbf{H}^1_\#(Y_S)/\mathbb{R}^d$:

$$\mathbf{p}^{kl} = \mathbf{p}^{kl}_0 + \boldsymbol{\psi}^{kl} \qquad \mathbf{p}^{kl}_0 \in \mathbf{M}, \quad \boldsymbol{\psi}^{kl} \in \mathbf{H}^1_\#(Y_S)/\mathbb{R}^d.$$

Let $(\boldsymbol{\phi}^n)$ be a sequence of elements in $\mathbf{V}$, such that $\boldsymbol{\phi}^n \to \phi$ in $\mathbf{H}^1(Y_S)/\mathbb{R}^d$. We have a unique decomposition

$$\boldsymbol{\phi}^n = \alpha^n_{kl}\mathbf{p}^{kl}_0 + \boldsymbol{\psi}^n, \qquad \underline{\alpha}^n \in \mathbb{R}^{d\times d}, \quad \boldsymbol{\psi}^n \in \mathbf{H}^1_\#(Y_S)/\mathbb{R}^d,$$

and $\|\boldsymbol{\phi}^n\|^2_{\mathbf{H}^1(Y_S)} = \|\sum_{kl} \alpha^n_{kl}\mathbf{p}^{kl}_0\|^2_{\mathbf{H}^1(Y_S)} + \|\boldsymbol{\psi}^n\|^2_{\mathbf{H}^1(Y_S)}$, so $(\alpha^n_{kl}\mathbf{p}^{kl}_0)$ is bounded. Since the space $\mathit{Span}\left\{(\mathbf{p}^{kl}_0)_{1\leq k,l\leq d}\right\}$ has a finite dimension, there exists $\mathbf{p} \in \mathit{Span}\left\{(\mathbf{p}^{kl}_0)_{1\leq k,l\leq d}\right\}$ such that up to a subsequence

$$\alpha^n_{kl}\mathbf{p}^{kl}_0 \to \mathbf{p}.$$

Then, $\boldsymbol{\psi}_n$ converges to $\boldsymbol{\psi}$ in $\mathbf{H}^1(Y_S)/\mathbb{R}^d$, so since $\mathbf{H}^1_\#(Y_S)/\mathbb{R}^d$ is closed in $\mathbf{H}^1(Y_S)/\mathbb{R}^d$,

$$\boldsymbol{\psi}_n \to \boldsymbol{\psi} \in \mathbf{H}^1_\#(Y_S)/\mathbb{R}^d.$$

Finally, $\boldsymbol{\phi} = \mathbf{p} + \boldsymbol{\psi} \in \mathbf{V}$ and $\mathbf{V}$ is closed as a subspace of $\mathbf{H}^1(Y_S)/\mathbb{R}^d$.

Step 3. Let us show that $\mathbf{V}$ contains no infinitesimal rigid displacement of a solid body. Suppose we have two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ such that

$$\mathbf{V} \ni \mathbf{a} + \mathbf{b} \times \mathbf{y} = \underline{B}_{kl}\mathbf{p}^{kl} + \boldsymbol{\psi}, \qquad \underline{B} \in \mathbb{R}^{d\times d},\ \boldsymbol{\psi} \in \mathbf{H}^1_\#(Y_S).$$

Recall that $\mathbf{p}^{kl}$ is defined by (74). Since $\mathbf{p}^{kl}$, $\mathbf{b} \times \mathbf{y}$ and $\mathbf{a}$ are all polynomial functions in the variable $\mathbf{y}$, $\boldsymbol{\psi}$ is one too. Then $\boldsymbol{\psi}$ is a periodic polynomial function, therefore it has to be equal to a constant $\mathbf{c}$. Then $\mathbf{a} = \mathbf{c}$ because $\mathbf{p}^{kl}(0) = 0$, see definition (74). Now, we have

$$\mathbf{b} \times \mathbf{y} = \frac{1}{2}B_{kl}y_k\mathbf{e}^l + \frac{1}{2}B_{kl}y_l\mathbf{e}^k = \frac{1}{2}(\underline{B} + \underline{B}^T)\mathbf{y}.$$

Observe that the cross product on the left can be represented only by a skew–symmetric matrix, while we have a symmetric matrix on the right of the identity. Therefore both matrices are in fact zero. This means that $\mathbf{b} = \mathbf{0}$ and since we have taken the quotient by the constants in definition (75), $\mathbf{V}$ contains no infinitesimal rigid displacement of a solid body aside from $\{\mathbf{0}\}$.

Step 4. Now, suppose assertion (76) is wrong. Then, there exists $(\boldsymbol{\phi}^n)$ a sequence of elements of $\mathbf{V}$ such that:

$$\|\boldsymbol{\phi}^n\|_{\mathbf{H}^1(Y_S)} = 1 \text{ for all } n \in \mathbb{N}, \text{ and } \lim_{n\to\infty} \|e(\boldsymbol{\phi}^n)\|_{L^2(Y_S)} = 0.$$

Using the Rellich–Kondrasov theorem, there exists a subsequence (still denoted by n) such that $\boldsymbol{\phi}^n$ converges strongly in $\mathbf{L}^2(\Omega)$. Since $e(\boldsymbol{\phi}^n)$ also converges strongly in $L^2(\Omega)$, we deduce that $\boldsymbol{\phi}^n$ is a Cauchy sequence with respect to the norm

$$\boldsymbol{\phi} \mapsto \sqrt{\|\boldsymbol{\phi}\|^2_{\mathbf{L}^2(Y_S)} + \|e(\boldsymbol{\phi})\|^2_{L^2(Y_S)}}.$$

By the standard Korn's inequality in $\mathbf{H}^1(Y_S)$, this norm is equivalent to the norm $\|\cdot\|_{\mathbf{H}^1(Y_S)}$ on $\mathbf{H}^1(Y_S)$. Hence, since $\mathbf{V}$ is closed and therefore complete, there exists $\boldsymbol{\phi} \in \mathbf{V}$ such that $\boldsymbol{\phi}^n$ converges to $\boldsymbol{\phi}$ strongly. Moreover, the limit $\boldsymbol{\phi}$ satisfies

$$\|e(\boldsymbol{\phi})\|_{L^2(Y_S)} = \lim_{n\to\infty} \|e(\boldsymbol{\phi}^n)\|_{L^2(Y_S)} = 0.$$

Now $\boldsymbol{\phi}$ is an infinitesimal rigid dislacement of a solid body and belongs to $\mathbf{V}$, hence $\boldsymbol{\phi} = \mathbf{0}$. This is a contradiction, since $\|\boldsymbol{\phi}^n\|_{\mathbf{H}^1(Y_S)} = 1$ for all $n \in \mathbb{N}$.

□