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# Verification and comparison of four numerical schemes for a 1D viscoelastic blood flow model

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## Abstract

A reliable and fast numerical scheme is crucial for the 1D simulation of blood flow. In this paper, a 1D blood flow model is incorporated with a Kelvin-Voigt viscoelastic constitutive relation of wall. This lead to a nonlinear hyperbolic-parabolic system, which is then solved with four numerical schemes, namely: MacCormack, Taylor-Galerkin, second order finite volume and local discontinuous Galerkin. The numerical schemes are tested on an uniform vessel, a simple bifurcation and a network with 55 arteries. In all of the cases, the numerical solutions are checked favorably against analytic, semi-analytic solutions or clinical observations. Among the numerical schemes, comparisons are made in four aspects: the accuracy, the ability to capture shock-like phenomena, the computation speed and the complexity of the implementation. The suitable conditions for the application of each scheme are discussed.

**Keywords:** blood flow; 1D flow modeling; vascular network; numerical simulation

## 1 Introduction

There exist direct 3D simulations of blood flow in arteries, nevertheless they are known to be time and memory consuming and therefore most of them are restricted to local positions (single vessel, confluences) [4, 14]. If we assume an axisymmetric circular velocity profile in the vessel, the 3D problem can be reduced to a 2D problem. If we further take the advantage of long wave length, a 1D model can be obtained. The 1D model is specially interesting for several reasons. First, this model captures well the behaviours of pulse wave, from which one can extract a lot of useful information about the cardiovascular system. For

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example, the Pulse Wave Velocity (PWV) has been recognized by European Society of Hypertension as a very important marker to the diagnosis and treatment of hypertension [24, 5]. Second, it allows fast numerical computations, which allows real-time applications for medical planning. Third, it also provide pertinent boundary conditions for 3D simulations in multi-scale models [9].

The 1D governing equations can be obtained by integrating Navier-Stokes equations across the cross-section of a circular vessel with the assumption of a long-wave perturbation and axisymmetric velocity profile [11, 38, 18]. It results in a system of two partial differential equations (PDEs) for mass and momentum conservation which involve the flow rate  $Q$ , the cross-section area  $A$  and the average pressure  $P$ . To close the system, the constitutive relation of the arterial wall comes in, which relates  $P$  and  $A$ . After the insertion of this relation into the PDEs, a nonlinear hyperbolicity-dominated system is obtained. Depending on the details of the model of the arterial wall, there may be a diffusive term by the viscosity of the wall or/and a dispersive term by the axial tension.

In case of weak nonlinearity (i.e. small perturbation around the equilibrium state [21, 31]), we can linearize the 1D governing equations and find solutions in frequency domain [29, 42]. But for the full nonlinear system, analytic solutions are not available yet. Thus several numerical schemes have been proposed and used to solve the system in the time domain, and roughly we classify them in:

- Finite Difference (FD) [36, 35, 32, 30, 40, 46]
- Finite Volume (FV) [7, 6, 44]
- Finite Element (FE), [23, 1, 37, 10]
- Discontinuous Galerkin (DG) [1, 26, 28, 27, 37]

These schemes have been successfully applied in other communities where researchers have to solve similar hyperbolic problems. For instance, the Mac-Comack scheme (FD) was principally designed for gas dynamics (i.e. 1D compressible Euler equations) and it was then successfully used to compute blood flow in veins [12]. From ideas frequently applied in shallow water equations, Delestre et al. [7] obtained “well balanced” schemes which properly treat the source term induced by a tapered artery. The 1D model and the numerical solutions have been validated by *in vitro* experimental [36, 43, 1] or *in vivo* clinical data [35, 34]. But usually only one particular scheme was chosen in a study and no cross comparisons among the schemes can be found. Thus the advantage/drawback and the suitable conditions for applications of each scheme were not discussed in those literatures.

Our objective in this paper is to make a cross comparison of the four kind of schemes and to suggest the suitable conditions of application for each scheme. In general, we note that FD schemes are not flexible to treat complex computational geometries in high dimensions (2D or 3D). However, in 1D dimension and low order accuracy schemes, FD, FE and FV schemes are in fact completely equivalent for linear problems. But for problems with large nonlinearities, solutions with sharp gradient may appear and the performance of different schemes

could be different. Equally important is the accuracy. For DG scheme it may be tuned either by the degree of the polynomial or by the mesh size. But if a diffusive term is added to the governing equations, it will be hard to treat by an implicit time marching method (e.g. Crank-Nicolson) in the DG setting, thus the time step may be very severely limited for stability. Therefore, the performance of each scheme depend on the main features of the problems at hand. And the problems with different main features actually arise in a wide range of applications. For instance, no shock is observed in arteries in normal physiological conditions but shock-like phenomena may arise in veins [8, 25] or in arteries when the human body suffers from a blunt impact by accident [16]. For another instance, in some conditions diffusive term or dispersive term may arise as a source term [1] and the proper treatment of these terms will pose different levels of difficulty in each numerical framework. Thus to make a cross comparison of the numerical schemes is interesting and useful.

In this paper, Section 2 presents the governing equations and the characteristic structure of the homogeneous part of the nonlinear system. Section 3 presents the numerical solvers. In particular, a large amount of details of computation are given because this kind of information is scattered in literature. In this section, firstly an operator splitting is proposed (in the FD, FV and FE frameworks) to separate the hyperbolic and parabolic part. Then the treatment of the boundary conditions is discussed. Following that, MacCormack, Taylor-Galerkin, and second order finite volume are presented to integrate the hyperbolic subproblem. The parabolic subproblem is treated by Crank-Nicolson method. At the end of this section, a local discontinuous Galerkin method is presented for the hyperbolic-parabolic problem. Section 4 shows the analytic and numerical results. The system is linearized and asymptotic solutions are obtained for cases when there are different source terms in the system. The physical interpretations of the solutions are various behaviours of the wave: propagation, attenuation or diffusion in an uniform tube. The effect of the skin friction and the viscosity of the wall on the pulse wave is clearly observed. Moreover, a case with a larger nonlinearity is computed and the ability of the four schemes to properly capture the shock-like phenomena is tested. After the tests on a single vessel, a simple bifurcation is computed and the numerical reflection and transmission coefficients are compared with analytic ones predicted with linearized equations. Finally, a network with 55 arteries is computed. In all of the cases, the numerical solutions are compared favorably with the analytic, semi-analytic solutions or clinical observations. In the last section, comparisons among the four schemes are made in four aspects: the accuracy, the ability to capture shock-like phenomena, the computation speed and the complexity of the implementation. The suitable conditions for the application of each scheme are discussed.

## 2 The 1D model of arterial blood flow

### 2.1 1D mathematical model

Under the assumption of an axisymmetric velocity profile, the 1D arterial blood flow model can be written as:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad (1a)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \alpha \frac{Q^2}{A} \right) + \frac{A}{\rho} \frac{\partial P}{\partial x} = -C_f \frac{Q}{A}, \quad (1b)$$

where as stated above,  $A$  is the cross-section area of the artery,  $Q$  the flow rate or flux, and  $\rho$  the density of blood. The coefficient  $\alpha$  is the momentum correction factor, and  $C_f$  is the skin friction coefficient. They depend on the shape of the velocity profile. Usually, the profile can be estimated from the Womersley number which is defined as  $R\sqrt{\omega/\nu}$ , with  $R$  the radius of the vessel,  $\omega$  the frequency of the pulse wave and  $\nu$  the kinematic viscosity of the fluid. With a small Womersley number, we can take a Poiseuille (parabolic) profile. In that case  $\alpha = \frac{4}{3}$  and  $C_f = 8\pi\nu$ . This choice is only valid for very viscous flows [18, 19]. In practice, viscosity is not so large, and the profile is more flat. For a completely flat profile  $\alpha$  equals 1. This value is often used since it leads to a considerable simplification in analysis and the loss of relevance of the model is very small in most cases [10]. Thus we assume its value is 1 in this paper. The value of  $C_f$  needs special attention because it has significant influence on the pulse wave. In practical applications, its value has to be determined according to the particular problem at hand (both *in vitro* and *in vivo*). We assume its value is  $8\pi\nu$  according to a Poiseuille profile. We are aware of the limit of this approximation. However, as our purpose is comparison of numerical schemes, we do not discuss any more the value of  $\alpha$  and  $C_f$ .

To close the system, several viscoelastic constitutive relations for arterial wall have been presented in literature, like [2, 1, 33]. We choose the Kelvin-Voigt model for simplicity. We assume that the arterial wall is thin, isotropic, homogeneous, incompressible, and moreover that it deforms axisymmetrically with each circular cross-section independently of the others. We denote the undeformed cross-section area by  $A_0$  and the external pressure of the vessel by  $P_{ext}$ . Then, the relation linking  $A$  and  $P$  is:

$$P = P_{ext} + \beta(\sqrt{A} - \sqrt{A_0}) + \nu_s \frac{\partial A}{\partial t}, \quad (2)$$

with the stiffness coefficient  $\beta$ ,

$$\beta = \frac{\sqrt{\pi} E h}{(1 - \eta^2) A_0},$$

and the viscosity coefficient  $\nu_s$ ,

$$\nu_s = \frac{\sqrt{\pi} \phi h}{2(1 - \eta^2) \sqrt{A_0} A}, \quad (3)$$

where  $\eta$  is the Poisson ratio, which is 0.5 for an incompressible material,  $E$  the Young's modulus,  $h$  the thickness of the wall and  $\phi$  the viscosity of the material. For convenience, we further define  $C_v = \frac{A\nu_s}{\rho}$  for reasons which will be clear very soon in the next section. We also note that in absence of the wall viscosity we retrieve the classical Hooke's law.

## 2.2 Characteristic structure of the system

After presenting the system of equations, we remind its hyperbolic feature by discussing the characteristic structure. The discussion is classical, and can be found in text books [11, 20]. The notations we introduce here will be useful for the discussion of the numerical solvers. We assume  $P_{ext}$  is constant along the axial variable  $x$ , and substitute the constitutive relation (2) into Eq. (1b). We note that  $\frac{\partial A}{\partial t}$  can be replaced by  $-\frac{\partial Q}{\partial x}$  thanks to Eq. 1a. The equation for the balance of momentum turns out

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} + \frac{\beta}{3\rho} A^{\frac{3}{2}} \right) - \frac{A}{\rho} \frac{\partial}{\partial x} \left( \nu_s \frac{\partial Q}{\partial x} \right) = -C_f \frac{Q}{A} + \frac{A}{\rho} \left( \frac{\partial(\beta\sqrt{A_0})}{\partial x} - \frac{2}{3}\sqrt{A} \frac{\partial\beta}{\partial x} \right). \quad (4)$$

Under the assumption of a small perturbation of  $A$ , we approximate the term  $\frac{A}{\rho} \frac{\partial}{\partial x} \left( \nu_s \frac{\partial Q}{\partial x} \right)$  by  $C_v \frac{\partial^2 Q}{\partial x^2}$  with the already defined coefficient  $C_v = \frac{A\nu_s}{\rho} = \frac{\sqrt{\pi}\phi h}{2\rho(1-\eta^2)\sqrt{A_0}}$ , which turns out to be independent of  $A$  or  $Q$ . The governing equations may be written as:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S, \quad (5)$$

where

$$U = \begin{pmatrix} A \\ Q \end{pmatrix}, \quad F = F_c + F_v = \begin{pmatrix} Q \\ \frac{Q^2}{A} + \frac{\beta}{3\rho} A^{\frac{3}{2}} \end{pmatrix} + \begin{pmatrix} 0 \\ -C_v \frac{\partial Q}{\partial x} \end{pmatrix}$$

and

$$S = \begin{pmatrix} 0 \\ -C_f \frac{Q}{A} + \frac{A}{\rho} \left( \frac{\partial(\beta\sqrt{A_0})}{\partial x} - \frac{2}{3}\sqrt{A} \frac{\partial\beta}{\partial x} \right) \end{pmatrix}.$$

In this equation,  $U$  is the conservative variable,  $F$  the corresponding flux and  $S$  the source term. Note that the flux (scaled by constant density) consists of two parts, the convective  $F_c$  and the diffusive  $F_v$ . We recognize  $\frac{Q^2}{A}$  due to the fluid flow,  $\frac{\beta}{3\rho} A^{\frac{3}{2}}$  due to the elasticity, and  $-C_v \frac{\partial Q}{\partial x}$  due to the viscosity of the wall. In general, the suitable numerical techniques for the convective and diffusive flux are different. Thus it is common to separate the diffusive term and put it on the right side. Thus we write now the problem in a convection-diffusion form:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S + D \quad (6)$$

with

$$F = F_c, \quad D = \begin{pmatrix} 0 \\ C_v \frac{\partial^2 Q}{\partial x^2} \end{pmatrix}.$$

We consider firstly the homogeneous part and later the non-homogeneous part. Expanding the derivative of the flux, the homogeneous part can be written in a quasi-linear form

$$\frac{\partial U}{\partial t} + J_c \frac{\partial U}{\partial x} = 0 \quad (7)$$

where  $J_c$  is the Jacobian matrix

$$J_c = \begin{pmatrix} 0 & 1 \\ \frac{Q^2}{A^2} + c^2 & 2\frac{Q}{A} \end{pmatrix}$$

with the Moens-Korteweg celerity

$$c = \sqrt{\frac{\beta}{2\rho}} A^{\frac{1}{2}}. \quad (8)$$

Actually,  $A$  is always positive. Therefore  $c$  is real, which is the speed of the pressure wave with respect to the fluid flow. The matrix  $J_c$  has two different eigenvalues

$$\lambda_{1,2} = \frac{Q}{A} \pm c. \quad (9)$$

Linear algebra shows  $J_c$  must be diagonalizable in the form  $J_c = R\Lambda R^{-1}$ . The columns of  $R$  are the right eigenvectors of  $J_c$ . Left multiplying Eq. (7) by  $R^{-1}$ , one obtains

$$R^{-1} \frac{\partial U}{\partial t} + R^{-1} R \Lambda R^{-1} \frac{\partial U}{\partial x} = 0.$$

By introducing a new unknown variable vector which satisfies  $\partial_U W = R^{-1}$ , the previous equation can be transformed into

$$\frac{\partial W}{\partial t} + \Lambda \frac{\partial W}{\partial x} = 0. \quad (10)$$

$W_{1,2}$  can be readily obtained by integrating  $\partial_U W = R^{-1}$  componentwise

$$W_{1,2} = \frac{Q}{A} \pm 4c. \quad (11)$$

$W = [W_1, W_2]^T$  is called Riemann invariant vector or characteristics. In time-space plane,  $W_{1,2}$  are constants along the lines  $D_t X_{1,2}(t) = \lambda_{1,2}$ . In physiological conditions,  $\lambda_1 > 0 > \lambda_2$ . The two families of characteristic propagate in opposite directions. The homogeneous part is a subcritical hyperbolic system. For further use, we get the expression for  $A$  and  $Q$  by inverting the relation (11),

$$A = \frac{(W_1 - W_2)^4}{1024} \left( \frac{\rho}{\beta} \right)^2, \quad Q = A \frac{W_1 + W_2}{2}. \quad (12)$$

In the non-homogeneous part, the skin friction term dissipates the momentum and the second order derivative of  $Q$  is diffusive. Thus the full system

has hyperbolic-parabolic features. In physiological conditions, the Womersley number is not too big and the artery is almost uniform, thus the source term will be very small and the system is dominated by the hyperbolicity feature. If the properties of the artery have sharp variations, large source terms will be introduced. In this case, we will treat the artery as two different connected segments.

### 3 Numerical solvers

Having defined the problem and notations, in this section we present the numerical solvers. The original problem is split into two subproblems, hyperbolic and parabolic. Three numerical schemes are presented to treat the hyperbolic subproblem. For the parabolic subproblem, Crank-Nicolson method is suitable. Because of the duplication of values at the interface of elements in the DG setting, there are difficulties to apply Crank-Nicolson scheme. A local discontinuous Galerkin method is adopted to treat the unsplit problem.

#### 3.1 Operator splitting

There are explicit high resolution schemes for hyperbolic problems. But for parabolic problem, implicit schemes are necessary in general for a reasonable time step for time integration. Thus we applied a fractional step or operator splitting method. Starting from Eq. (6), the original problem is split into to a hyperbolic subproblem,

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S \quad (13)$$

and a parabolic one,

$$\frac{\partial U}{\partial t} = D. \quad (14)$$

Let us consider the time intervals  $(t^n, t^{n+1})$ , for  $n = 0, 1, \dots$ , with  $t^n = n\Delta t$ . In every time interval, the hyperbolic problem is solved to get a predictor  $U^*$ , which is used as the initial condition (I.C.) of the second problem. The second step can be viewed as a corrector. The original problem is approximated by a sequential application of the two subproblems in a certain order.

From data  $U^n$ , we may make a prediction  $U^*$  by evolving time  $\Delta t$  of the hyperbolic subproblem, and correct it with the evolution over  $\Delta t$  of the parabolic subproblem,

$$U^n \xrightarrow{e^{\Delta t \mathcal{H}}} U^* \xrightarrow{e^{\Delta t \mathcal{P}}} U^{n+1},$$

where  $e^{\Delta t \mathcal{H}}$  ( $e^{\Delta t \mathcal{P}}$ ) means to solve the hyperbolic (parabolic) subproblem over  $\Delta t$ . This method is called Godunov splitting. If the two subproblems are not commutable, the splitting error is  $\mathcal{O}(\Delta t)$ , see Chapter 17 of reference [20].

There is a 3-stage splitting called Strang splitting, which has a leading error term  $\mathcal{O}(\Delta t^2)$ ,

$$U^n \xrightarrow{e^{\frac{1}{2}\Delta t \mathcal{P}}} U^* \xrightarrow{e^{\Delta t \mathcal{H}}} U^{**} \xrightarrow{e^{\frac{1}{2}\Delta t \mathcal{P}}} U^{n+1}.$$



We will see in the section about diffusion that in reality the errors induced by the two splittings are very close. That is because the coefficient of the term  $\mathcal{O}(\Delta t)$  is much smaller than the coefficient of  $\mathcal{O}(\Delta t^2)$ . Thus, usually Godunov splitting is sufficient.

Because the unsplit problem is dominated by the hyperbolicity, the system must be driven mainly by the Boundary Conditions (B.C.) through the first subproblem. Thus we discuss the B.C. of the hyperbolic part in the next subsection and present the treatment of B.C. for the parabolic part in Section 3.6 together with Crank-Nicolson scheme.

## 3.2 Initial and boundary conditions

### 3.2.1 Initial conditions

Assume we are interested in the blood flow in an arterial segment  $(0, L)$  within a time interval  $(0, T)$ . For an evolutionary problem, a proper I.C. is needed. In reality, the information contained in I.C. flows out after a certain interval of time, and it will not have influence on the system thereafter. Thus, the I.C. can be set arbitrarily, say,  $U(t = 0, x) = (A_0, 0)$ , for convenience.

### 3.2.2 Inlet and outlet of the hyperbolic part

Let us look back to the vector Eq. (10) again. The two components of this system are

$$\frac{\partial W_1}{\partial t} + \lambda_1 \frac{\partial W_1}{\partial x}(U) = 0, \quad (15a)$$

$$\frac{\partial W_2}{\partial t} + \lambda_2 \frac{\partial W_2}{\partial x}(U) = 0. \quad (15b)$$

Since the two eigenvalues have opposite signs, there is exactly one incoming characteristic at each end of the computational domain. The incoming characteristic carries information from outside the domain and thus is essential to guarantee the problem to be well-posed. That is to say, the system must be supplemented by B.C. in the form

$$W_1(0, t) = g_1(t), \quad W_2(L, t) = g_2(t), t > 0.$$

The outgoing characteristic carries information from inside the domain, which can be given by the differential equations. Since  $W_{1,2}$  are constants along the lines  $D_t X_{1,2}(t) = \lambda_{1,2}$  in time-space plane, we can get  $W_2^{n+1}(0)$  and  $W_1^{n+1}(L)$  by interpolation in the data of the n-time step:

$$W_2^{n+1}(0) = W_1^n(-\lambda_1^n(0)\Delta t), \quad W_1^{n+1}(L) = W_2^n(L - \lambda_2^n(L)\Delta t). \quad (16)$$

The characteristics are then transformed to physical variables by relation (12) for numerical computation.

In reality, we rarely have the explicit expression of incoming characteristics. Usually, we want to impose B.C. in physical term  $A$  or  $Q$ . At the inlet, if  $A(t)$  is given, one can use the relation (11) to deduce:

$$W_1^{n+1} = W_2^{n+1} + 8\sqrt{\frac{\beta}{2\rho}A(t)^{1/2}}.$$

If  $Q(t)$  is given, similarly we obtain an approximation

$$W_1^{n+1} = -W_2^{n+1} + 2\frac{Q^{n+1}}{A(0)}.$$

If  $P(t)$  is given, from the wall relation (2) simplified with no viscous effect ( $\nu_s = 0$ ), we in fact impose:

$$W_1^{n+1} = W_2^{n+1} + 8\sqrt{\frac{1}{2\rho}(P(t) + \beta A_0^{1/2})}.$$

At the outlet, some part of the perturbation of outgoing characteristic  $W_1$  may be reflected,

$$W_2^{n+1} = W_2^0 - R_t(W_1^{n+1} - W_1^0)$$

where  $R_t$  is the coefficient of reflection. If  $R_t = 0$ , the B.C. is nonreflecting. That means that the outgoing characteristic goes out without leaving any effect and that the incoming characteristic is a constant in time. If  $R_t \neq 0$ , the reflection is due to the resistance in the downstream arteries.

### 3.2.3 Conjunction points

There are many cases when conjunction points need to be considered: when there are changes of topology, sharp changes in geometry or mechanic properties. Topological change corresponds to the large amount of bifurcations and some trifurcations in the arterial network. Sharp changes correspond to the sharp variation of the properties of the vessel wall, e.g. sharp increase of stiffness  $K$  due to stenting or  $A_0$  due to aneurysm. As the derivatives of the corresponding variables in the source term lead to a singularity or very near a singularity, the vessel can be treated as two segments conjoined together.

Since all of the conjunctions can be treated with the same method, we consider a branching point as a sample problem: a main vessel with two daughter arteries. At the branching point, there are then six boundary conditions,  $A_p^{n+1}$  and  $Q_p^{n+1}$  for the outlet of the parent artery and  $A_{d_1}^{n+1}$ ,  $Q_{d_1}^{n+1}$ ,  $A_{d_2}^{n+1}$  and  $Q_{d_2}^{n+1}$  for the inlets of the two daughter arteries. From the physical point of view, the scheme has to preserve the conservation of mass flux

$$Q_p^{n+1} - Q_{d_1}^{n+1} - Q_{d_2}^{n+1} = 0 \quad (17a)$$

and conservation of momentum flux

$$\frac{1}{2}\rho\left(\frac{Q_p^{n+1}}{A_p^{n+1}}\right)^2 + P_p^{n+1} - \frac{1}{2}\rho\left(\frac{Q_{d_i}^{n+1}}{A_{d_i}^{n+1}}\right)^2 - P_{d_i}^{n+1} = 0 \quad i = 1, 2. \quad (17b)$$

$P_p^{n+1}$  and  $P_{d_i}^{n+1}$  are given by the constitutive relation (2). Moreover, the outgoing characteristics in each artery can be determined by the interpolation formula (16).  $(W_1)_p^{n+1}$  is given by the interpolation on the  $n$ -th step data of the parent vessel and it must be equal to  $W_1(U_p^{n+1})$ , which is given by relation (11)

$$(W_1)_p^{n+1} - W_1(U_p^{n+1}) = 0. \quad (17c)$$

The same holds for  $W_2$  on the two daughter arteries.

$$(W_2)_{d_i}^{n+1} - W_2(U_{d_i}^{n+1}) = 0 \quad i = 1, 2. \quad (17d)$$

Combining Eqs. (17a), (17b), (17c) and (17d), there are 6 Eqs. with 6 unknowns. This nonlinear algebraic system can be readily solved by Newton iterative method or some other nonlinear algebraic solvers with  $U^n$  as the initial guess.

### 3.3 MacCormack scheme

In FD framework, MacCormack method [22] is very suitable for nonlinear hyperbolic systems of conservation laws. It is equivalent to the Lax-Wendroff scheme for linear systems. It has the following characteristics: conservative, three-point spatial stencil and two time levels (predictor and corrector), second-order accurate in time and space.

For the conservative system (13), an approximate solution  $U^*$  is obtained from  $U^n$  in the first step and then corrected in the second to give the solution  $U^{n+1}$  at the next time step  $t + \Delta t$ . The numerical solution is performed in a mesh with  $N + 1$  points which defines the spatial resolution  $\Delta x = \frac{L}{N}$ , see Figure 1. The finite difference equations (at the interior grid points) are then :

1. predictor step

$$U_i^* = U_i^n - \frac{\Delta t}{\Delta x}(F_{i+1}^n - F_i^n) + \Delta t S_i^n, \quad i = 2, \dots, N$$

2. corrector step

$$U_i^{n+1} = \frac{1}{2}(U_i^n + U_i^*) - \frac{\Delta t}{2\Delta x}(F_i^* - F_{i-1}^*) + \frac{\Delta t}{2}S_i^*, \quad i = 2, \dots, N$$

where  $F^*$  and  $S^*$  are evaluated as functions of the predicted solution  $U^*$ . Note that the predictor step applies a forward differencing and the corrector step a backward differencing. The order of the two kind of differencing can be reversed. The grid points  $i = 1$  and  $i = N + 1$  represent the boundary conditions.

### 3.4 Taylor-Galerkin scheme

In this section, we follow the presentations of Formaggia et al. [10, 11] and Sherwin et al. [37] for the Taylor-Galerkin scheme. Other forms are also possible,

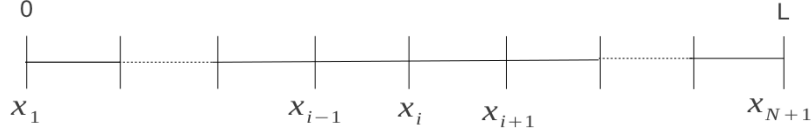


Figure 1: Mesh for FD and FE

see, for example [41]. The Taylor series for  $U^{n+1}$  up to second order is

$$U^{n+1} = U^n + \Delta t \frac{\partial U^n}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 U^n}{\partial t^2} + \mathcal{O}(\Delta t^3). \quad (18)$$

From Eq. (13), one obtains,

$$\frac{\partial U^n}{\partial t} = S^n - \frac{\partial F^n}{\partial x}. \quad (19)$$

Differentiation both sides with respect to  $t$  and exchange of the order of spatial and temporal differentiation in the second term gives,

$$\frac{\partial^2 U^n}{\partial t^2} = \left( S_U \frac{\partial U}{\partial t} \right)^n - \frac{\partial}{\partial x} \left( H \frac{\partial U}{\partial t} \right)^n \quad (20)$$

where  $S_U = \frac{\partial S}{\partial U}$  and  $H = \frac{\partial F}{\partial U}$ . Substituting Eq. (19) into Eq. (20) and then both of them into Eq. (18), one gets,

$$U^{n+1} = U^n - \Delta t \frac{\partial}{\partial x} \left[ F^n + \frac{\Delta t}{2} H^n S^n \right] - \frac{\Delta t^2}{2} \left[ S_U^n \frac{\partial F^n}{\partial x} - \frac{\partial}{\partial x} \left( H^n \frac{\partial F^n}{\partial x} \right) \right] + \Delta t \left( S^n + \frac{\Delta t}{2} S_U^n S^n \right). \quad (21)$$

For convenience, we adopt the notation

$$F_{LW}(U) = F(U) + \frac{\Delta t}{2} H(U) S(U),$$

$$S_{LW}(U) = S(U) + \frac{\Delta t}{2} S_U(U) S(U).$$

The piecewise linear function space associated with the mesh (Figure 1) is given as,

$$V_h^0 = \{ [v_h]^2 | v_h \in \mathcal{C}^0, v_h|_{[x_i, x_{i+1}]} \in \mathcal{C}^1, v_h(0) = v_h(L) = 0, i = 1 \dots N \}.$$

The shape function in this space has the property  $\psi_i(x_j) = \delta_{ij}$ , where  $\delta_{ij}$  is Kronecker delta. This is both the trial function space and test function space in Galerkin framework.  $U$  is approximated by  $U_h \in V_h^0$ . We further define the inner product

$$(u, v) = \int_0^L u v dx.$$

Replace  $U$  by  $U_h$  in Eq. (21), multiply both sides by basis test functions, and integrate over the domain  $[0, L]$ , finally we get

$$(U_h^{n+1}, \psi_i) = (U_h^n, \psi_i) + \Delta t (F_{LW}(U_h^n), \frac{d\psi_i}{dx}) - \frac{\Delta t^2}{2} (S_U(U_h^n) \frac{\partial F(U_h^n)}{\partial x}, \psi_i) - \frac{\Delta t^2}{2} (H(U_h^n) \frac{\partial F(U_h^n)}{\partial x}, \frac{d\psi_i}{dx}) + \Delta t (S_{LW}(U_h^n), \psi_i) \quad i = 2, \dots, N. \quad (22)$$

$U_h^{n+1}$  and  $U_h^n$  are expanded as  $U_h = \sum_{j=2}^{j=N} U_j \psi_j$ . Instead of evaluated directly as nonlinear functions of  $U_h^n$ , the terms  $F(U_h^n)$ ,  $F_{LW}(U_h^n)$ ,  $S_{LW}(U_h^n)$ ,  $S_U(U_h^n)$  and  $H(U_h^n)$  are projected onto the trial function space and expanded by a group finite element method. That is, for example,  $F(U_h^n) = \sum_{j=2}^{j=N} F_j^n \psi_j$ , with  $F_j^n = F(U_j^n)$ . Finally, the matrix form of the FE scheme writes

$$\mathcal{M}U^{n+1} = \mathcal{M}U^n + \Delta t \mathcal{K}^T F_{LW}^n + \Delta t \mathcal{M} S_{LW}^n - \frac{\Delta t^2}{2} \tilde{\mathcal{M}} F^n - \frac{\Delta t^2}{2} \tilde{\mathcal{K}} F^n, \quad (23)$$

where

$$\mathcal{M}_{ij} = (\psi_i, \psi_j), \quad \mathcal{K}_{ij} = (\psi_i, \frac{\partial \psi_j}{\partial x})$$

and

$$\tilde{\mathcal{M}}_{ij}(S_u) = \left( \sum_k (S_u)_k \psi_k \frac{\partial \psi_i}{\partial x}, \psi_j \right), \quad \tilde{\mathcal{K}}_{ij}(H) = \left( \sum_k H_k \psi_k \frac{\partial \psi_i}{\partial x}, \frac{\psi_j}{\partial x} \right).$$

Please note our abuse of notation that in this equation  $U^{n+1}$ ,  $F^n$  etc. stand for discretized vectors whose values are associated with the grid points. Also note that  $\tilde{\mathcal{M}}$  and  $\tilde{\mathcal{K}}$  are functions of  $S_u$  and  $H$ , therefore they must be updated in every time step.

### 3.5 Second order finite volume scheme

For finite volume method, the domain is decomposed into finite volumes or cells with vertex  $x_i$  as the center of cell  $[x_{i-1/2}, x_{i+1/2}]$ , see Figure 2. For every cell, the conservation law must hold,

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial U}{\partial t} dx + \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial F}{\partial x} dx = \int_{x_{i-1/2}}^{x_{i+1/2}} S dx.$$

Application of Gauss's law on the second term gives

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial U}{\partial t} dx + F|_{x_{i+1/2}} - F|_{x_{i-1/2}} = \int_{x_{i-1/2}}^{x_{i+1/2}} S dx. \quad (24)$$

In the cells, average values are considered,

$$U_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} U(x) dx, \quad S_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} S(x) dx.$$

Thus Eq. 24 turns into an ordinary differential equation

$$\frac{dU_i}{dt} = -\frac{(F|_{x_{i+1/2}} - F|_{x_{i-1/2}})}{\Delta x} + S_i. \quad (25)$$

We have a local Riemann problem at each interface of neighboring cells, since the values of  $U$  at the two sides of the interface,  $U_L$  to the left and  $U_R$  to the right, are not equal in general. By solving the Riemann problem, a numerical flux  $F^*$  can be obtained. Depending on the approximate approaches on solving the Riemann problem, different numerical fluxes are possible. Among them, Rusanov (or called local Lax-Friedrichs) flux is widely used. It writes

$$F^*(U_L, U_R) = \frac{F(U_L) + F(U_R)}{2} - c \frac{U_R - U_L}{2},$$

with

$$c = \sup_{U=U_L, U_R} (\sup_{j \in \{1, 2\}} |\lambda_j(U)|)$$

where  $\lambda_1(U)$  and  $\lambda_2(U)$  are the eigenvalues of  $J_c$ . HLL flux is another option and it has less numerical diffusivity. Since Rusanov flux is more simple and robust, it is adopted in this paper. If  $U_L$  and  $U_R$  equal the average values at the cell, the numerical flux will be of first order accuracy. Linear reconstructions of  $U$  within the cells are necessary for a second order numerical flux. To this end, let us consider the slope of a scalar  $s$  within the  $i$ -th cell  $Ds_i$ , which can be approximated by  $(s_i - s_{i-1})/\Delta x$ ,  $(s_{i+1} - s_i)/\Delta x$  or  $(s_{i+1} - s_{i-1})/2\Delta x$ . Then the values of  $s$  at the interfaces associated with this cell can be recovered as

$$s_{i-1/2+} = s_i - \frac{\Delta x}{2} Ds_i \text{ and } s_{i+1/2-} = s_i + \frac{\Delta x}{2} Ds_i.$$

The discretization of derivative in space can achieve a second order accuracy by this method. But the solution will have nonphysical oscillations. Some examples of oscillations induced by these methods can be found in Chapter 6 of reference [20]. Slope or flux limiter and non-oscillatory solution are integral characteristics of FV schemes. MUSCL (monotonic upwind scheme for conservation law) is one popular slope limited linear reconstruction technique. To present MUSCL, we first define a slope limiter,

$$\text{minmod}(x, y) = \begin{cases} \min(x, y) & \text{if } x, y \geq 0, \\ \max(x, y) & \text{if } x, y \leq 0, \\ 0 & \text{else} \end{cases}$$

Then the slope  $Ds_i$  is modified as

$$Ds_i = \text{minmod}\left(\frac{s_i - s_{i-1}}{\Delta x}, \frac{s_{i+1} - s_i}{\Delta x}\right).$$

The values of  $A$  and  $Q$  at the interface can be obtained as

$$A_{i-1/2+} = A_i - \frac{\Delta x}{2} DA_i, \quad A_{i+1/2-} = A_i + \frac{\Delta x}{2} DA_i$$

and

$$Q_{i-1/2+} = Q_i - \frac{\Delta x}{2} DQ_i, \quad Q_{i+1/2-} = Q_i + \frac{\Delta x}{2} DQ_i.$$

It is easy to verify that the variables are conserved by this reconstruction

$$\frac{R_{i-1/2+} + R_{i+1/2-}}{2} = R_i, \quad \frac{Q_{i-1/2+} + Q_{i+1/2-}}{2} = Q_i.$$

The adopted numerical integration in time is also of second order accuracy. Let us rewrite Eq. (25) as

$$\frac{dU_i}{dt} = \Phi(U_i)$$

where

$$\Phi(U_i) = -\frac{(F_{i+1/2}^* - F_{i-1/2}^*)}{\Delta x} + S_i.$$

Note that the fluxes have been replaced by numerical ones. A 2-step second order Adams-Bashforth (A-B) scheme is applied for the temporal integration,

$$U_i^{n+1} = U_i^n + \Delta t \left( \frac{3}{2} \Phi(U_i^n) - \frac{1}{2} \Phi(U_i^{n-1}) \right).$$

This scheme can be initiated by a forward Euler method. A second order Runge-Kutta (R-K) method is also possible. But the R-K method requires once more resolution of  $\Phi(U)$  at every step. This may be offset by a larger time step allowed by the R-K method. But note also that the boundary conditions are determined dynamically. The A-B method allows less resolutions of the nonlinear algebraic equations at conjunction points. Thus we choose the A-B method for the temporal integration.

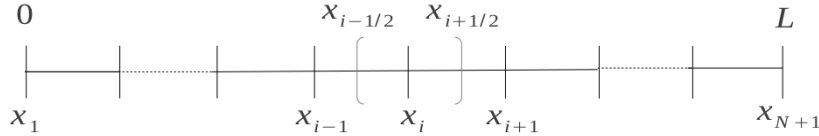


Figure 2: Mesh for FV

### 3.6 Treatment of the parabolic subproblem

For the previous 3 schemes, only the hyperbolic subproblem resulted from splitting is solved. For the parabolic subproblem, Crank-Nicolson method is very suitable. The temporal and spatial discretization has the form,

$$\frac{U_i^{n+1} - U_i^*}{\Delta t} = \frac{C_v}{2} \left( \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2} + \frac{U_{i+1}^* - 2U_i^* + U_{i-1}^*}{\Delta x^2} \right),$$

where  $U^*$  is the solution of the first hyperbolic subproblem. The matrix of the resulting algebraic system is tridiagonal, which is quite cheap to invert.

This scheme is second order accurate both on time and space. Moreover, it is unconditionally stable. It is natural to set a homogeneous Neumann B.C. for the parabolic subproblem,  $\partial_x U_p(0, t) = \partial_x U_p(L, t) = 0$ . The subscript  $p$  stands for parabolic.

### 3.7 Local Discontinuous Galerkin scheme

In FV framework, the recovery of  $U_L$  and  $U_R$  of higher accuracy requires bigger stencil. In higher dimensions, this kind of reconstruction leads to difficulties. On the other hand, it is quite straightforward to increase the order of approximation polynomials in one finite element. Unlike the global FE, the neighboring elements do not share the same values at the interface. Numerical flux are obtained from these values, where the dynamics of the system can be considered. We present a nodal DG scheme, following Hesthaven and Warburton's book [15]. The domain is decomposed into  $K$  non-overlapping elements, see Figure 3. At each element, the local approximation to the solution is a polynomial of order  $N = N_p - 1$ . The global approximation to  $U$  is direct summation of these local solutions:

$$U_h = \bigoplus_{k=1}^{k=K} U_h^k. \quad (26)$$

Similarly, the flux  $F$  and the source term  $S$  can also be approximated by direct summation of piecewise  $N$ -th degree polynomials. The local form of the conservation law on the  $k$ -th element has the form,

$$\frac{\partial U_h^k}{\partial t} + \frac{\partial F_h^k}{\partial x} = S_h^k. \quad (27)$$

Multiplication by a test function  $\phi^k$  at both sides of Eq. (27), and integration over one element gives

$$\left( \frac{\partial U_h^k}{\partial t}, \phi^k \right)_{D_k} + \left( \frac{\partial F_h^k}{\partial x}, \phi^k \right)_{D_k} = \left( S_h^k, \phi^k \right)_{D_k}. \quad (28)$$

Applying integration by part on the second term, we have:

$$\left( \frac{\partial U_h^k}{\partial t}, \phi^k \right)_{D_k} - \left( F_h^k, \frac{\partial \phi^k}{\partial x} \right)_{D_k} + F_h^k \phi^k \Big|_{x_k}^{x_{k+1}} = \left( S_h^k, \phi^k \right)_{D_k}. \quad (29)$$

Again, since the value of  $U_h$  at the two sides of the interface,  $U_h^k(x_k)$  and  $U_h^{k+1}(x_k)$ , are not equal, a numerical flux  $F^*$  is introduced here. Through the numerical flux, information is communicated between elements. In practice, the second term is integrated by part again for convenience of computation. Thus we have

$$\left( \frac{\partial U_h^k}{\partial t}, \phi^k \right)_{D_k} + \left( \frac{\partial F_h^k}{\partial x}, \phi^k \right)_{D_k} + (-F_h^k \phi^k + F^* \phi^k) \Big|_{x_k}^{x_{k+1}} = \left( S_h^k, \phi^k \right)_{D_k}. \quad (30)$$



If we introduce  $N_p$  nodes within the element  $D_k$ , the local solution can be expanded as

$$U_h^k(x, t) = \sum_{i=1}^{N_p} U_h^k(x_i^k, t) l_i^k(x), \quad (31)$$

where  $l_i^k(x)$  is the Lagrange interpolant associated with the  $i$ -th point. For the Galerkin scheme, Eq. (30) must hold for every test function  $l_i^k(x)$ . Thus we have  $N_p$  equations for  $N_p$  unknowns. In matrix form, the system can be written as,

$$\mathcal{M}^k \frac{dU_k}{dt} + \mathcal{K}^k F^k + (-F^k l^k + F^{*k} l^k) \Big|_{x_k}^{x_{k+1}} = \mathcal{M}^k S^k, \quad (32)$$

where

$$\mathcal{M}^k = (l_i^k, l_j^k)_{D_k}, \quad \mathcal{K}^k = \left( \frac{dl_i^k}{dx}, l_j^k \right)_{D_k},$$

and  $l^k$  is the vector of equations  $(l_1^k, l_2^k, \dots, l_{N_p}^k)^T$ . The system of equations can be turned into a semi-discrete form,

$$\frac{dU_k}{dt} = -\mathcal{D}^k F^k - (\mathcal{M}^k)^{-1} (-F^k l^k + F^{*k} l^k) \Big|_{x_k}^{x_{k+1}} + S^k. \quad (33)$$

where

$$\mathcal{D}_{(i,j)}^k = (\mathcal{M}^k)^{-1} \mathcal{K}_{(i,j)}^k = \frac{dl_j^k}{dr} \Big|_{r_i}$$

is the local differentiation operator [15]. The computation of  $\mathcal{M}^k$  and  $\mathcal{D}^k$  is crucial. We define an affine mapping from a reference element  $(-1, 1)$  to  $D_k$ ,

$$x(r) = x_k + \frac{1+r}{2}(x_{k+1} - x_k).$$

The local operators can be readily computed as,

$$\mathcal{M}^k = \mathcal{J}_k \int_{-1}^1 l_i l_j dr, \quad \mathcal{D}^k = \mathcal{J}_k \frac{dl_j}{dr} \Big|_{r_i},$$

where  $\mathcal{J}_k = (x_{k+1} - x_k)/2$  and  $l_i$  is the Lagrange interpolant at the reference element. Note that the terms  $\int_{-1}^1 l_i l_j dr$  and  $\frac{dl_j}{dr} \Big|_{r_i}$  can be precomputed and stored. Legendre-Gauss-Lobatto points has to be chosen as the interpolation points to minimize the computation error. For more details, we refer to Chapter 3 of reference [15]. For the temporal integration, a second order A-B scheme is applied for reasons as discussed in Section 3.5.

The scheme previously presented can treat a hyperbolic problem. But in this setting Crank-Nicolson method is hard to apply, because the values at the interfaces are duplicated. We consider the problem formulation of Eq. (5), where the flux contains convective part  $F_c$  and diffusive part  $F_v$ . For the convective part, Godunov flux is applicable. For the diffusive flux, a straight idea is to use

the central flux,  $(F_v(U_L) + F_v(U_R))/2$ . But as pointed out by Shu et al. [39], this choice is inconsistent.

To solve this problem, we rewrite the original equations as

$$\begin{aligned} \frac{\partial U}{\partial t} + \frac{\partial(F_c - C_v q)}{\partial x} &= S \\ q - \frac{\partial Q}{\partial x} &= 0 \end{aligned}$$

In semi-discrete form, the equations for one element are

$$\begin{aligned} \frac{dU_k}{dt} &= -\mathcal{D}^k F^k - (\mathcal{M}^k)^{-1}(-F^k l^k + F^* l^k) \Big|_{x_k}^{x_{k+1}} + S^k \\ q^k &= \mathcal{D}^k Q^k - (\mathcal{M}^k)^{-1}(-Q^k l^k + Q^* l^k) \Big|_{x_k}^{x_{k+1}} \end{aligned}$$

The flux in these equations have to be modified accordingly:  $F^k = F_c^k - C_v q^k$ ,  $F^* = F_c^* - (C_v q)^*$ .  $F_c^*$  is defined by Godunov flux. The numerical flux  $(C_v q)^*$  and  $Q^*$  are defined by the central flux. The introduction of an auxiliary variable  $q$  stabilizes the scheme. Note that the auxiliary equation does not involve time evolution, and thus  $q^k$  can be eliminated in every time step. The addition of storage or computational cost is very limited. This method is called local discontinuous Galerkin scheme.

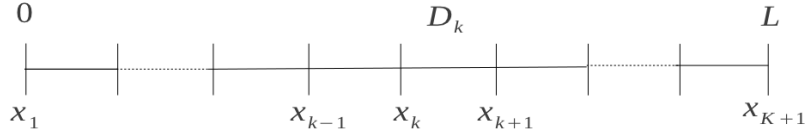


Figure 3: Mesh for DG

## 4 Results and discussion

In this section, the numerical solutions in various cases are verified against analytic, semi-analytic solutions or clinical observations. At first, the computations are done on a single uniform vessel. In case of small perturbations, a linearized system is obtained. If this system is homogeneous, it allows pure wave solution. If the source terms due to the skin friction and the viscosity of the wall are added respectively, asymptotic solutions are obtained. In case of larger perturbations, the full nonlinear system allows shocks. The shock-capturing property of each scheme is tested in this case. After the test on a single vessel, a simple bifurcation is computed and the reflection and transmission coefficients are compared with analytic ones predicted by linearized system. At the final of this section, a network with 55 arteries is computed and the numerical solutions are checked against clinical observations reported in literature.

## 4.1 Propagation in an uniform tube

In this subsection, we compare the numerical results with analytic ones for a pulse wave on a single uniform vessel ( $\partial_x(\beta\sqrt{A_0}) = \partial_x\beta = 0$ ). To avoid reflections, non reflective B.C. is set at the outflow to mimic a semi-infinite tube. Adding a small perturbation ( $(\epsilon\tilde{A}, \epsilon\tilde{Q})$ ) to the equilibrium solution ( $U = (A_0, 0)$ ), substituting it into the governing equations and dropping the terms with quadratics of  $\epsilon$ , we obtain the equations for the perturbations in a linear form:

$$\frac{\partial\tilde{A}}{\partial t} + \frac{\partial\tilde{Q}}{\partial x} = 0, \quad \frac{\partial\tilde{Q}}{\partial t} + c_0^2 \frac{\partial\tilde{A}}{\partial x} = -\frac{C_f}{A_0}\tilde{Q} + C_v \frac{\partial^2\tilde{Q}}{\partial x^2} \quad (34)$$

with  $c_0 = \sqrt{\frac{\beta}{2\rho}\sqrt{A_0}}$ , the Moens-Korteweg celerity. To investigate the propagation phenomena at first, we drop the non-homogeneous part ( $C_f = 0$  and  $C_v = 0$ ). Then Eqs. (34) become d'Alembert equations, which admit the pure wave solution. If we assume that the initial condition is at equilibrium ( $A = A_0, Q = 0$ ), and the inflow is prescribed as  $Q(0, t) = Q_{in}(t)$  with

$$Q_{in}(t) = Q_c \sin\left(\frac{2\pi}{T_c}t\right)H\left(-t + \frac{T_c}{2}\right), \quad t > 0,$$

where  $H(t)$  is the Heaviside function,  $T_c$  the period of the sinusoidal wave and  $Q_c$  the amplitude. The solution is  $c_0\tilde{A} = \tilde{Q} = Q_{in}(x - c_0t)$ , which means that the waveform propagates to the right with a speed of  $c_0$ .

We propose a numerical test with parameters of the tube inspired by [37]:  $L = 250\text{cm}$ ,  $A_0 = 3.2168\text{cm}^2$ ,  $\beta = 1.8734 \times 10^6\text{Pa/m}$  and  $\rho = 1.050 \times 10^3\text{kg/m}^3$ , and accordingly,  $c_0 = 400\text{cm/s}$ . To impose a small perturbation, we choose  $Q_c = 1\text{ml/s}$  and  $T_c = 0.4\text{s}$ . In this case the change ratio of the radius is  $\Delta R/R_0 = Q_0/(2A_0c_0) = 0.04\%$ , thus the perturbation is assured small enough. We take the linearized analytic solution at time  $t = 0.4\text{s}$  as reference to compute the errors of the numerical solutions. The norm of the error vector is defined by

$$\|E\| = \frac{1}{N} \sum_{i=1}^N \left| \frac{Q_{numerical} - Q_{analytic}}{Q_c} \right|.$$

To specify the time step, we define the formula,  $\Delta t = C_t \frac{L}{Nc_0}$ . For the DG scheme, the time step formula is modified accordingly as  $\Delta t = \frac{C_t}{\mathcal{P}} \frac{L}{Nc_0}$ , with  $\mathcal{P}$  the degree of the polynomial. The value of  $C_f$  must satisfy the CFL (Courant-Friedrichs-Lewy) condition.

To test the spatial convergence, we fix  $C_t = 0.1$ , and vary the number of mesh nodes  $N$ . The log-log plot of  $\|E\|$  against  $\Delta x$  can be seen in Figure 4(a). We have two main observations. First, all of the schemes converge with an order between 1 and 2 and the DG schemes converges faster (see Figure 4(a)). Second, as shown by Figure 4(b) the difference between the analytic solutions and all of the numerical solutions are hardly discernible with a moderate number of mesh points ( $N_{TG} = N_{FV} = N_{FD} = 800, N_{DG-\mathcal{P}_1} = N_{DG-\mathcal{P}_2} = 100$ ).

To test the temporal convergence, we fix the mesh ( $N_{TG} = N_{FV} = N_{FD} = 800$ ,  $N_{DG-\mathcal{P}_1} = N_{DG-\mathcal{P}_2} = 100$ ) and increase (or decrease) the time steps, the error varies slightly for all of the schemes except FV (see Figure 4(c)). If  $C_t > 0.6$ , Taylor-Galerkin, FV and MacCormack become unstable. For the DG schemes,  $C_t$  can not be greater than 0.1 (see Figure 4(d)). For the convergence of the temporal integration, the FV scheme has to choose a smaller time step than the value prescribed by the CFL condition.

To compare the actual speed and accuracy of the four schemes, we set  $N$  and  $C_t$  such that the errors achieve the same order of magnitude (see Table 1 and Figure 4(f)). Except the Taylor-Galerkin scheme, all of the schemes have the similar accuracy with very close running time (see Figure 4(e) and 4(f)). At this point, the Taylor-Galerkin scheme shows the worst accuracy and needs to run the longest time. We note that large global matrix arise in Taylor-Galerkin scheme while the operators in other schemes are local and have small size. That explains the relative poor performance of Taylor-Galerkin although a larger time step is allowed by this scheme. we will see that in case of a network of real size, the largest number of  $N$  is about 100 and Taylor-Galerkin shows a good balanced property between accuracy and speed (Section 4.6).

scheme	N	$C_t$
Taylor-Gakerkin	800	0.5
FV	800	0.3
MacComack	1600	0.5
$DG - \mathcal{P}_1$	200	0.1
$DG - \mathcal{P}_2$	100	0.1

Table 1: Number of elements and coefficient of time step

## 4.2 Attenuation due to the viscosity of blood

We now consider the same linearized system Eq. (34) with the small term due to skin friction ( $C_f \neq 0$  and  $C_v = 0$ ). The main dynamics of the system will be grossly the same traveling wave but attenuated by viscosity of blood. This behaviour can be predicted by asymptotic analysis. We have a small non-dimensional parameter  $\epsilon_f = T_c C_f / A_0$ , which is the ratio of the characteristic time of pulse  $T_c$  to the characteristic time of attenuation  $A_0 / C_f$ . In order to see how the waveform slowly evolves when it propagates to, say right, we make a change of variables to  $\tau = \epsilon_f t$  and  $\xi = x - c_0 t$  (slow time, moving frame). The two differential operators  $\partial_t$  and  $\partial_x$  expand as

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} = \epsilon_f \frac{\partial}{\partial \tau} - c_0 \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi}. \end{aligned}$$

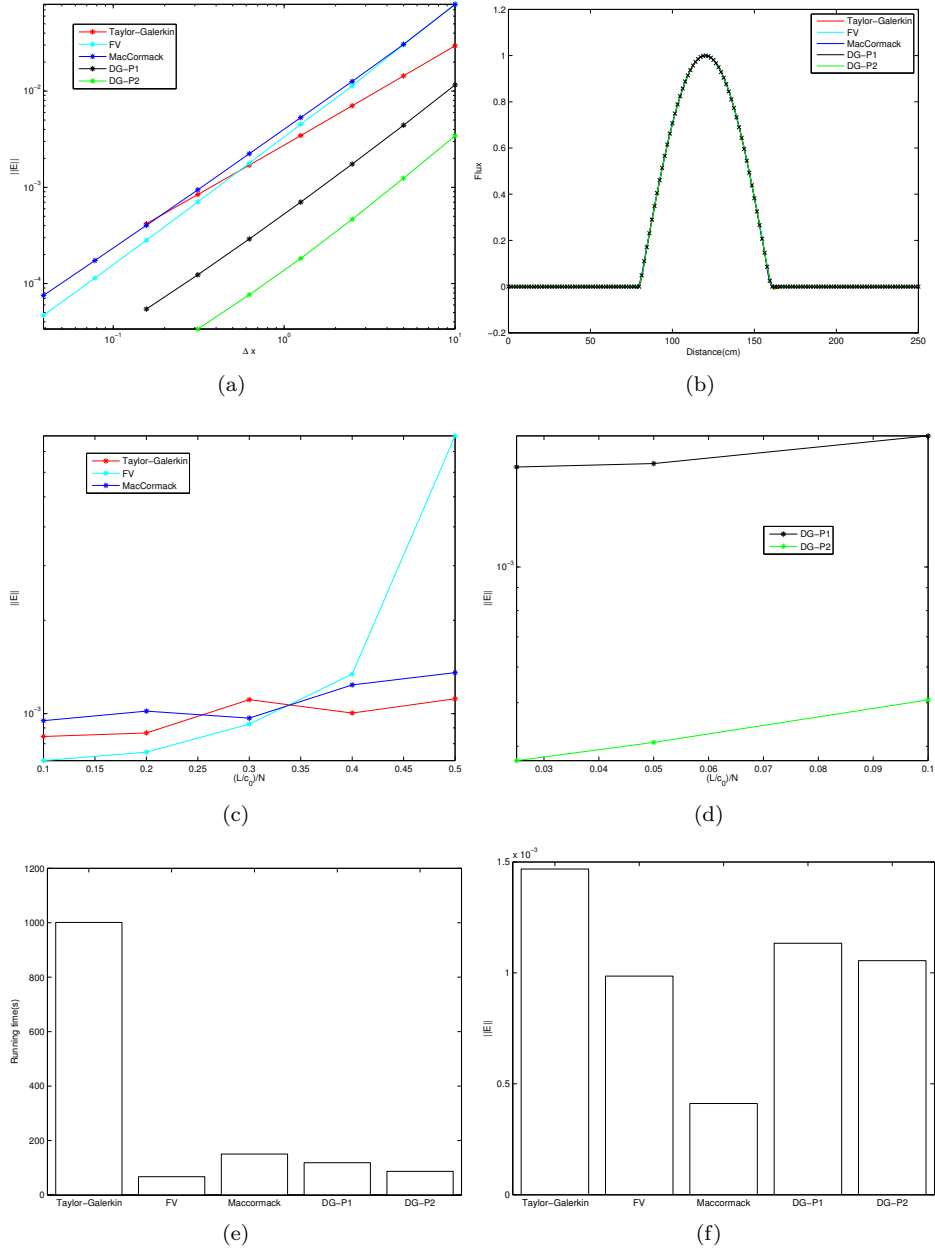


Figure 4: Test on a uniform tube. Top left, errors with different size of elements (cells). Top right, all the solutions for the pulse wave at time 0.4s are overlapped and the analytic solution is indicated by cross signs. Middle left and right, with a fixed mesh ( $N_{TG} = N_{FV} = N_{FD} = 800$ ,  $N_{DG-P1} = 200$  and  $N_{DG-P2} = 100$ ), errors as functions of  $C_t$  coefficient. Bottom left and right, running time and errors for the configuration shown in Table 1.

The solution has the asymptotic expansion

$$\tilde{A} = \tilde{A}_0 + \epsilon_f \tilde{A}_1 + \dots, \quad \tilde{Q} = \tilde{Q}_0 + \epsilon_f \tilde{Q}_1 + \dots$$

Substituting these into the governing equations expressed in new variables and collecting the terms with the same order of  $\epsilon_f$ , one has

$$\begin{aligned} & (-c_0 \frac{\partial \tilde{A}_0}{\partial \xi} + \frac{\partial \tilde{Q}_0}{\partial \xi}) + \epsilon_f (\frac{\partial \tilde{A}_0}{\partial \tau} - c_0 \frac{\partial \tilde{A}_1}{\partial \xi} + \frac{\partial \tilde{Q}_1}{\partial \xi}) + \dots = 0 \\ & (-c_0 \frac{\partial \tilde{Q}_0}{\partial \xi} + c_0^2 \frac{\partial \tilde{A}_0}{\partial \xi}) + \epsilon_f (\frac{\partial \tilde{Q}_0}{\partial \tau} - c_0 \frac{\partial \tilde{Q}_1}{\partial \xi} + c_0^2 \frac{\partial \tilde{A}_1}{\partial \xi} + \frac{\tilde{Q}_0}{T_c}) + \dots = 0. \end{aligned}$$

We take the first order term in  $\epsilon_f$  in the first equation, substitute it in the first order term in  $\epsilon_f$  in the second equation, finally we obtain

$$(\frac{\partial \tilde{Q}_0}{\partial \tau} + c_0 \frac{\partial \tilde{A}_0}{\partial \tau} + \frac{\tilde{Q}_0}{T_c}) = 0.$$

From the terms of the zeroth order in  $\epsilon_f$ , which involve derivative in  $\xi$  only, the solution must have the form  $\tilde{Q}_0 = c_0 \tilde{A}_0(\tau, \xi) + \phi(\tau)$ . Substituting it into the previous equation implies terms  $\frac{\partial \phi}{\partial \tau}$  and  $\phi(\tau)$ . These are secular terms and thus can be set null. So we have  $c_0 \tilde{A}_0 = \tilde{Q}_0$  and  $\frac{\partial \tilde{Q}_0}{\partial \tau} = -\frac{1}{2T_c} \tilde{Q}_0$ , or

$$\tilde{Q}_0 = \tilde{Q}_0(0, \xi) e^{-\tau/(2T_c)} = \tilde{Q}_0(0, x - c_0 t) e^{-\epsilon_f t/(2T_c)}.$$

For more on asymptotic analysis of blood flow in large blood vessels, we refer to reference [45].

In Figure 5, we plot the snapshots of the waveform at time 0.2s, 0.4s, 0.6s and 0.8s. In the computation, the initial and boundary conditions are the same as in the previous subsection. The mesh and the time steps in Table 1 are adopted. The damping rate of the amplitude of the waveform agrees very well with the analytic prediction,  $\exp(-\frac{C_f x}{2A_0 c_0})$ , which is indicated by the dashed line. Also note that the errors of different schemes are not the same. The FV scheme causes the peak of the wave to slightly flatten, while all of the other schemes are dispersive: we have small oscillations at the foot of the signal.

### 4.3 Diffusion due to the viscosity of the arterial wall

We now consider the same linearized system Eq. (34) but with the Kelvin-Voigt effect and no viscous fluid effect ( $C_f = 0$  and  $C_v \neq 0$ ). The small parameter is now  $\epsilon_v = C_v/(c_0^2 T_c)$ . If we apply the same technique as described in the previous subsection, we can readily obtain the diffusive behaviour of the pulse wave in the moving frame:

$$\frac{\partial \tilde{Q}_0}{\partial \tau} = \frac{c_0^2 T_c}{2} \frac{\partial^2 \tilde{Q}_0}{\partial \xi^2}. \quad (35)$$

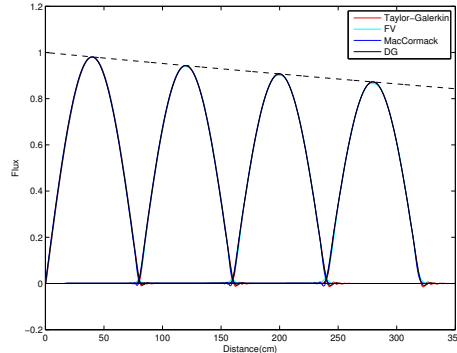


Figure 5: Attenuation due to the skin friction. The snapshots are at time 0.2s, 0.4s, 0.6s and 0.8s. The dashed line is  $\exp(-\frac{C_f x}{2A_0 c_0})$  with  $2A_0 c_0 / C_f \simeq 2000\text{cm}$ . The flux is normalized with respect to  $Q_c$ .

The solution of this equation can be given by the convolution

$$\tilde{Q}_0(\tau, \xi) = \int_{-\infty}^{+\infty} \tilde{Q}_0(0, \xi) G(\tau, \xi - \zeta) d\zeta$$

where  $G$  is the fundamental solution of the heat equation (35)

$$G(\tau, \xi) = \frac{1}{\sqrt{2\pi\tau c_0^2 T_c}} e^{-\xi^2 / (2\tau c_0^2 T_c)}$$

and  $\tilde{Q}_0(\tau, \xi)$  is the initial state. In the test vessel, the parameters are kept the same as in the case of attenuation. The coefficient  $C_v$  is  $0.6275\text{m}^2/\text{s}$  and  $\epsilon_v \simeq 0.1$ . This corresponds to  $\phi=5000\text{Pa} \cdot \text{s}$ , which is in the range of observed values on animals [2]. To facilitate the calculation of the analytic solution, non-reflecting B.C.s are imposed at the two ends of the vessel and I.C. is a half sinusoidal waveform for flux  $Q$  (dashed line in Figure 6) and a constant cross section  $A_0$ . It is clear that half of the initial wave propagates to right and at the same time the waveform is spread out due to the diffusive effect. The analytic solution at time 0.4s (indicated by cross signs) agrees well with the corresponding numerical solution.

Another point worthy noticing is the operator splitting errors. In the DG scheme, no operator splitting error is induced. All of the other numerical schemes adopt operator splitting method. They produce very accurate solutions as well as DG. Thus it verifies the *a priori* judgement that Godunov splitting is sufficient.

#### 4.4 Shock-like phenomena due to the nonlinearity

We now consider the full nonlinear system, but without any source terms ( $C_f = 0$  and  $C_v = 0$ ). The small parameter is now  $\epsilon_2 = Q_c / (c_0 A_0)$ . If we apply the

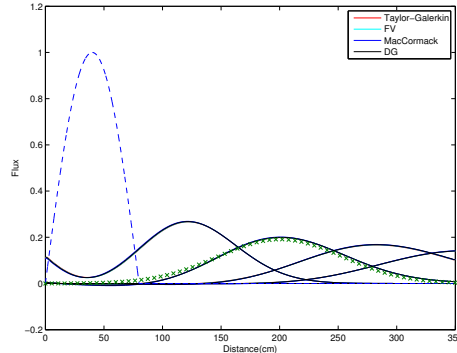


Figure 6: Diffusion due to the viscosity of the wall. The dashed line is the initial condition. One half of the original waveform propagates to right. The snapshots are at time 0.2s, 0.4s, 0.6s and 0.8s. The analytic prediction from the convolution at time 0.4s is indicated by cross signs. The difference between the different numerical solutions is not discernible. The flux is normalized with respect to  $Q_c$ .

same technique as described in the previous subsection, we can readily obtain the equation for the non linear behaviour of the pulse wave in the moving frame (inviscid Burgers equation):

$$\frac{\partial \tilde{Q}_0}{\partial \tau} = \frac{1}{2A_0} \tilde{Q}_0 \frac{\partial \tilde{Q}_0}{\partial \xi} \quad (36)$$

One important consequence of nonlinear hyperbolic system is that shocks may arise even if the initial condition is very smooth. In normal physiological conditions, shocks are not observed in arterial systems. But in venous system, shock-like phenomena may occur on muscular veins during walking and running. The intramuscular pressure (equivalent to  $P_{ext}$  in our model) can rise to 20 – 40 kPa in a few ms [3]. In such situation, experiment and numerical simulations [8, 25] have shown this critical behaviour. For another example, the traumatic rupture of the aorta is responsible for a significant percentage of traffic death and the rupture may be well accounted for by the shock-like transition resulted from the blunt impact to the thorax [16]. Thus we test all of the schemes in such case. To generate a shock, only two parameters are modified:  $L = 800\text{cm}$  and  $Q_c = 200\text{ml/s}$ . The change ratio of the radius is about 7.78%. The number of elements for Taylor-Galerkin, FV and MacComack schemes is 800. The DG scheme uses 200 elements and the order of polynomial is 2. Figure 7 shows that a shock starts to appear near the point 300cm. Strong oscillations are generated at the front foot of the waveform by Taylor-Galerkin scheme. On the other hand, strong oscillations are induced at the back of the large gradients. For the DG scheme, there are some smaller oscillations both in front and back. That is because the characteristic structures are taken into account in the numerical flux. Limiters may be introduced to eliminate the oscillations.



This remedy will be necessary for DG to be applicable on problems with shocks. For the FV scheme, the shock is well captured without nonphysical oscillations. That verifies the total-variation-diminishing (TVD) property of the MUSCL FV scheme.

On Figure 7(b) we plot a case with some viscosity of the wall. The added moderate physical diffusive term smoothes the oscillations and all of the schemes give almost the same result.

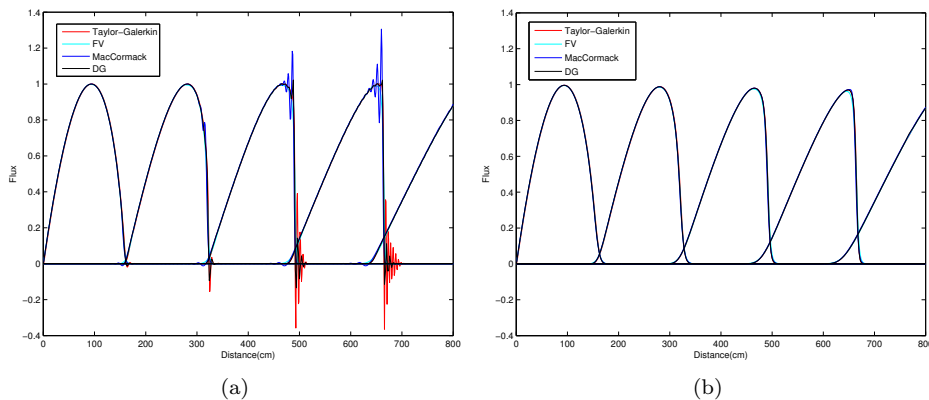


Figure 7: Shock due to the nonlinearity of the system. The left figure (a) shows that sharp gradient arises in a nonlinear hyperbolic system from smooth initial condition. Numerical schemes may cause spurious oscillations. FV scheme with a flux limiter captures the shock without nonphysical oscillations. The right figure (b) shows that all of the schemes give almost the same result for a system with a moderate physical diffusive term.

#### 4.5 Reflection and transmission at a branching point

Up to now, we focused on the various behaviours of wave within a single vessel: propagation, attenuation, diffusion, etc. Now, we look at the boundaries of each artery. Indeed, pressure waves are reflected and transmitted at the conjunction points of a network. For a linearized system, given the impedance  $Z = \frac{\rho c_0}{A_0}$ , the reflection and transmission coefficients at a branching point can be calculated by the formula:

$$\mathcal{R} = \frac{Z_p^{-1} - (Z_{d_1}^{-1} + Z_{d_2}^{-1})}{Z_p^{-1} + (Z_{d_1}^{-1} + Z_{d_2}^{-1})}, \quad \mathcal{T} = \frac{2Z_p^{-1}}{Z_p^{-1} + (Z_{d_1}^{-1} + Z_{d_2}^{-1})},$$

where  $Z_p$  and  $Z_d$  are the characteristic impedance of the parent and daughter vessels [13, 31]. In Figure 8, for sake of illustration, the configuration of the branching and the time profile of pressure at two locations are shown. The amplitude is normalized with respect to  $Q_c = 1 \times 10^{-6} \text{m}^3/\text{s} = 1 \text{ml/s}$ . For the

parent vessel:  $\beta = 2.3633 \times 10^6 \text{Pa/m}$ ,  $A_0 = 4 \text{cm}^2$  and for each of the daughter vessels:  $\beta = 6.3021 \times 10^6 \text{Pa/m}$ ,  $A_0 = 1.5 \text{cm}^2$ . According to the formula,  $\mathcal{R} = 0.2603$  and  $\mathcal{T} = 1.2603$ . The pressure profiles at the points A and B agree very well with the analytic predictions. All of the numerical schemes are compatible with this treatment of conjunction.

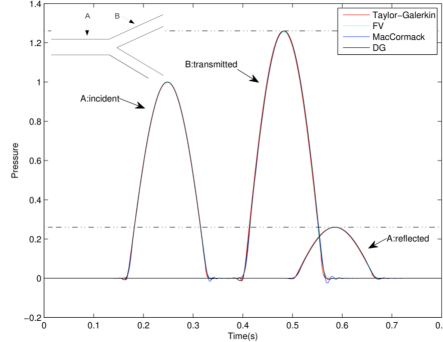


Figure 8: Reflection and transmission of pressure wave at a branching point. The time profiles of the pressure at points A and B are plotted. The analytical reflection and transmission coefficients are 0.2603 and 1.2603 (indicated by the dashed line).

## 4.6 Application on a full systematic arterial network

As already mentioned in the introduction, a relatively realistic description of arterial system has been done in 1D simulations, with different numerical solvers by different teams. For example, in [28, 37], Galerkin approach is used. In these papers, wall viscosity is not included. Note that [35] gives a survey of literature on the details the model, and adopted a viscoelastic model of the wall. But, in all of those papers, usually only one numerical scheme is adopted and cross comparisons among them are not available. In this subsection, we compute a network of 55 arteries with the viscoelastic model we presented above and make a cross comparison among the numerical schemes. To this end, the topology and properties value of the arterial network are adapted from [37]. But the viscosity coefficient of the Kelvin-Voigt model on human body is not given in this paper. In reference [2], the viscosity of aortic wall of dogs was modeled by a Kelvin-Voigt model and it shows that the value of  $\phi$  is in the range of  $3.8 \pm 1.3 \times 10^3 \text{Pa} \cdot \text{s}$  to  $7.8 \pm 1.1 \times 10^3 \text{Pa} \cdot \text{s}$ . Hence, we assume  $\phi = 5 \times 10^3 \text{Pa} \cdot \text{s}$  to calculate the coefficient  $C_v$ . The final parameters of the network we use are shown in Table 3. We note that there may be differences between arteries in human and dog and the arteries in different locations may cause a considerable variation. Nevertheless the inclusion of viscosity term makes it possible to test the numerical schemes in a more realistic condition.

The peak value of the input flux  $Q_c$  is 500 ml/s. This value is very close to

the peak flow rate at the root of aortic artery [35]. We choose  $\min_{i=1}^{i=55} (L^i/c_0^i)$  as a reference length, with  $L^i$  the length of the  $i$ -th artery and  $c_0^i$  the wave speed of the linearized system. For a coarsest possible mesh, the number of elements (cells) of each artery is  $N_{base}^i = \lfloor \frac{L^i}{\min_{i=1}^{i=55} (L^i/c_0^i)} \rfloor$ , where  $\lfloor \cdot \rfloor$  is floor function. We computed the relative change of solutions when the number of the elements (cells) are doubled. Figure 9 shows the relative change of the solutions when the number of the elements (cells) is changed from  $2N_{base}$  to  $4N_{base}$ . The relative change of a quantity (for example flux  $Q$ ) with two meshes  $N_1$  and  $N_2$  is defined as  $\|Q_{N_1} - Q_{N_2}\|_1 / (Q_{max} - Q_{min})$ , where  $\|\cdot\|_1$  stands for 1-norm,  $Q_{max}$  and  $Q_{min}$  are the maximum and minimum values within one heart beat. Figure 9 shows that the change of flux and pressure are less than 1.5% for all of the schemes except DG. Thus we plotted in Figure 10 the results computed with mesh  $2N_{base}$ . The DG scheme is not tested in this manner because it is already converged: result in Figure 10 shows that the error for DG is already very small with the coarsest possible mesh. Time step is prescribed by  $\Delta t = C_t \min_{i=1}^{i=55} (\frac{L^i}{N^i c_0})$ . The coefficient  $C_t$  and the corresponding real time step in the computation is shown in Table 2.

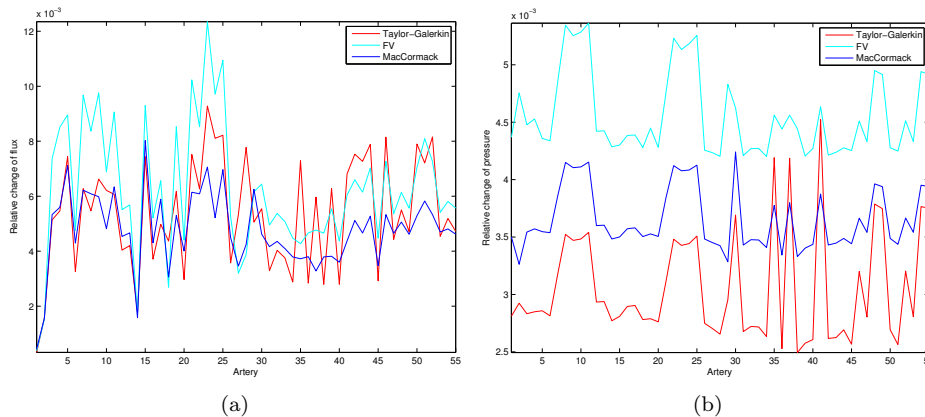


Figure 9: Relative change of the solutions when the mesh is doubled from  $2N_{base}$  to  $4N_{base}$ . The left figure shows that the relative change of all of the flux is less than 1.3%. The right figure shows that the relative change of all of the pressure is less than 0.6%.

In Figure 10 we plot the history profile of flux and pressure at the middle of four representative arteries. All of the numerical solutions agree very well. The main features of the pressure and flux profiles reported in literature [37, 35] are observed. The peak value of pressure waveform increases as we travel down the system. We can also see the dicrotic notch at artery 1. At artery 37, a reverse flow is observed (see Figure 10(f)), which agrees with clinical measurement [35]. The result in this paper is smoother than the result on the corresponding arteries in [37]. The result with viscosities is closer to the clinical observations [35]. We

scheme	$C_t$	$\Delta t$ ( $10^{-6}$ s)	running time (min)
Taylor-Galerkin	0.4	222	22.0
FV	0.25	139	31.9
MacCormack	0.1	55.5	91.2
DG	0.006	6.66	576

Table 2: Time steps and running time for one heart beat using one processor on a standard Linux work station.

realize that it is very important to consider the wall viscosities to give more realistic result. This agrees with the conclusion drawn by the comparison of numerical results with data from *in vitro* experiments [1, 36].

## 5 Conclusions

In this paper, we incorporated a Kelvin-Voigt viscoelastic constitutive relation of arterial wall with a 1D blood flow model. This led to a hyperbolic-parabolic system which was then solved by four numerical schemes: MacCormack, Taylor-Galerkin, second order finite volume and discontinuous Galerkin. The implementations were verified with analytic, semi-analytic or clinical observations in many cases. At first, a single uniform tube was considered. Under the assumption of small nonlinearities, we obtained asymptotic solutions of linearized systems with different source terms. The propagation, attenuation and diffusion of the waveform were illustrated by both the numerical and analytic solutions. Moreover, in case of a larger nonlinearity, the shock capturing property of each scheme was tested. After the test on a single vessel, a simple bifurcation was computed to check the numerical coupling of different arteries. Finally, we computed a relatively realistic network with 55 arteries. The check of the numerical solutions in all cases was very favorable for all of the schemes. The schemes can be compared in four aspects: the accuracy, the ability to capture shock phenomena, the computation speed and the complexity of the implementation.

1. MacCormack and Taylor-Galerkin schemes generate small oscillations. FV scheme has slight arbitrary steepening effect. Both diffusion and dispersion errors are very small for DG. Nevertheless all of the schemes converge with a moderate fine mesh and precisely capture the various phenomena of this hyperbolicity-dominated hyperbolic-parabolic system.
2. MacCormack and Taylor-Galerkin perform very poorly when there is a steep gradient. Both of them present strong oscillations at one side of the jumping location. DG scheme has smaller oscillations at both sides of the jump. Numerical flux limiters are possible to filter out the oscillations. That will further complicate the schemes and the theory and technique is still under research [17, 25]. On the other hand, there are very mature techniques to impose a slope limiter in the FV scheme. Shock capturing property is unique for FV among the four schemes presented in this paper.

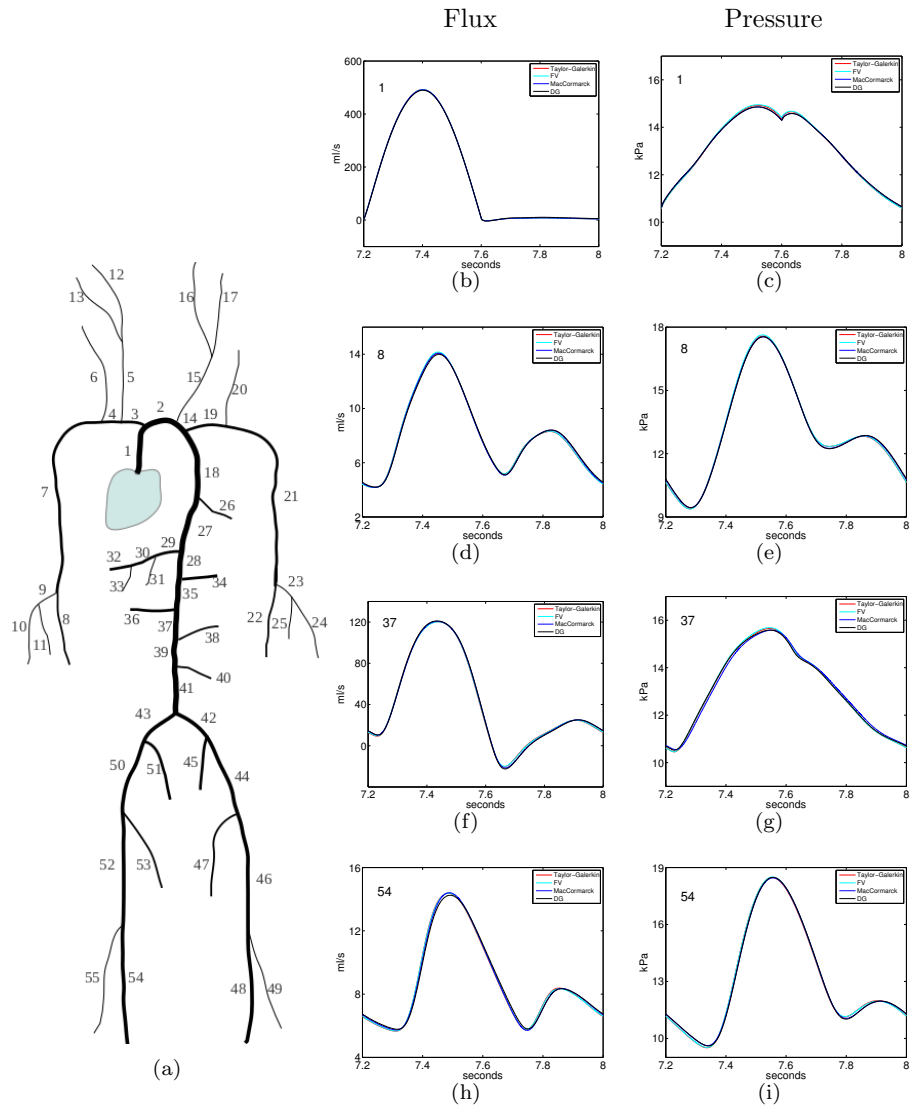


Figure 10: The history profiles of flux and pressure at four locations. Ten heart beats are computed to secure steady state is achieved and the tenth heart beat is plotted. The differences between the four numerical schemes are very small. See Table 3 for time steps and running time of each scheme.

3. For a network of human size, the speed of computation can be ordered from fast to slow as: Taylor-Galerkin, second order FV, MacCormack and DG. The temporal integration in the Taylor-Galerkin scheme is more efficient than Adams-Bashforth 2-step method. Thus it allows a larger time step with a comparable accuracy. But if the number of elements for one artery is too large (larger than 500), the speed of Taylor-Galerkin becomes slower because the size of the global matrix increases quadratically and thus the storing and inverting of matrix become expansive. The DG scheme prevents the application of Crank-Nicolson method on the diffusive term. An explicit method called local DG scheme was adopted in this paper. Even with a moderate diffusion coefficient (within the range observed in physiological condition), a very small time step is necessary for stability. That makes the computation of 1 heart beat cost about 9 hours while all of the other schemes cost only 20 to 90 minutes (using one processor on a standard Linux work station).
4. From easiest to hardest, the implementation of the schemes can be ordered: MacCormack, second order FV, Taylor-Galerkin and local DG.

As a final conclusion from the point of view of practical application, we recommend MacCormack in case of small nonlinearities as it is very simple and robust; second order FV will be a very good option if there maybe shock-like phenomena in the systems; Taylor-Galerkin has quite balanced properties between speed and accuracy if no shock-like phenomena may present in the system; DG is suitable for systems with very small physical diffusive term since both the numerical diffusion and dispersion are very small in this scheme.

Our future work may include these directions: including a physical dispersive term to account for the axial tension of vessels; parallelizing the scheme to speedup the computation and coupling the arterial system with the venous system.

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Table 3: Arterial network

ID	Name	$l$ (cm)	$A_0$ (cm <sup>2</sup> )	$\beta$ (10 <sup>6</sup> Pa/cm)	$C_v$ (10 <sup>4</sup> cm <sup>2</sup> /s)	$R_t$
1	Ascending aorta	4.0	6.789	0.023	0.352	–
2	Aortic arch I	2.0	5.011	0.024	0.317	–
3	Brachiocephalic	3.4	1.535	0.049	0.363	–
4	R.subclavian I	3.4	0.919	0.069	0.393	–
5	R.carotid	17.7	0.703	0.085	0.423	–
6	R.vertebral	14.8	0.181	0.470	0.595	0.906
7	R. subclavian II	42.2	0.833	0.076	0.413	–
8	R.radius	23.5	0.423	0.192	0.372	0.82
9	R.ulnar I	6.7	0.648	0.134	0.322	–
10	R.interosseous	7.9	0.118	0.895	0.458	0.956
11	R.ulnar II	17.1	0.589	0.148	0.337	0.893
12	R.int.carotid	17.6	0.458	0.186	0.374	0.784
13	R. ext. carotid	17.7	0.458	0.173	0.349	0.79
14	Aortic arch II	3.9	4.486	0.024	0.306	–
15	L. carotid	20.8	0.536	0.111	0.484	–
16	L. int. carotid	17.6	0.350	0.243	0.428	0.784
17	L. ext. carotid	17.7	0.350	0.227	0.399	0.791
18	Thoracic aorta I	5.2	3.941	0.026	0.312	–
19	L. subclavian I	3.4	0.706	0.088	0.442	–
20	L. vertebral	14.8	0.129	0.657	0.704	0.906
21	L. subclavian II	42.2	0.650	0.097	0.467	–
22	L. radius	23.5	0.330	0.247	0.421	0.821
23	L. ulnar I	6.7	0.505	0.172	0.364	–
24	L. interosseous	7.9	0.093	1.139	0.517	0.956
25	L. ulnar II	17.1	0.461	0.189	0.381	0.893
26	intercoastals	8.0	0.316	0.147	0.491	0.627
27	Thoracic aorta II	10.4	3.604	0.026	0.296	–
28	Abdominal aorta I	5.3	2.659	0.032	0.311	–
29	Celiac I	2.0	1.086	0.056	0.346	–
30	Celiac II	1.0	0.126	0.481	1.016	–
31	Hepatic	6.6	0.659	0.070	0.340	0.925
32	Gastric	7.1	0.442	0.096	0.381	0.921
33	Splenic	6.3	0.468	0.109	0.444	0.93
34	Sup. mesenteric	5.9	0.782	0.083	0.439	0.934
35	Abdominal aorta II	1.0	2.233	0.034	0.301	–
36	L. renal	3.2	0.385	0.130	0.481	0.861
37	Abdominal aorta III	1.0	1.981	0.038	0.320	–
38	R. renal	3.2	0.385	0.130	0.481	0.861
39	Abdominal aorta IV	10.6	1.389	0.051	0.358	–
40	Inf. mesenteric	5.0	0.118	0.344	0.704	0.918
41	Abdominal aorta V	1.0	1.251	0.049	0.327	–
42	R. com. iliac	5.9	0.694	0.082	0.405	–
43	L. com. iliac	5.8	0.694	0.082	0.405	–
44	L. ext. iliac	14.4	0.730	0.137	0.349	–
45	L. int. iliac	5.0	0.285	0.531	0.422	0.925
46	L. femoral	44.3	0.409	0.231	0.440	–
47	L. deep femoral	12.6	0.398	0.223	0.419	0.885
48	L. post. tibial	32.1	0.444	0.383	0.380	0.724
49	L. ant. tibial	34.3	0.123	1.197	0.625	0.716
50	L. ext. iliac	14.5	0.730	0.137	0.349	–
51	R. int. iliac	5.0	0.285	0.531	0.422	0.925
52	R. femoral	44.4	0.409	0.231	0.440	–
53	R. deep femoral	12.7	0.398	0.223	0.419	0.888
54	R. post. tibial	32.2	0.442	0.385	0.381	0.724
55	R. ant. tibial	34.4	0.122	1.210	0.628	0.716

Data adapted from [37] and [2].