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Grégoire Allaire, Charles Dapogny, Pascal Frey

A MESH EVOLUTION ALGORITHM BASED ON THE LEVEL SET METHOD FOR GEOMETRY AND TOPOLOGY OPTIMIZATION

G. ALLAIRE [1] C. DAPOGNY[2,3], P. FREY [2]

[1] *Centre de Mathématiques Appliquées (UMR 7641), École Polytechnique 91128 Palaiseau, France.*
[3] *UPMC Univ Paris 06, UMR 7598, Laboratoire J.-L. Lions, F-75005 Paris, France.*
[3] *Renault DREAM-DELT'A Guyancourt, France.*

ABSTRACT

We propose an approach for structural optimization which combines the flexibility of the level set method for handling large deformations and topology changes with the accurate description of the geometry provided by an exact mesh of the shape. The key ingredients of our method are efficient algorithms for (i) moving a level set function on an unstructured mesh, (ii) remeshing the surface corresponding to the zero level set and (iii) simultaneously adapting the volumic mesh which fits to this surfacic mesh.

Keywords: Geometry and topology optimization and Level set method and Local mesh modifications.

1. INTRODUCTION

The goal of this brief note is to report on the 3-d extension of our 2-d work [1], where we described a method for topology optimization relying on an exact meshing of the shape at each stage of the iterative process.

We follow the lead of [2] by describing an admissible shape Ω, included in a fixed computational domain D, with a *level set function* $\phi : D \to \mathbb{R}$ such that:

$$(1) \qquad \begin{cases} \phi(x) < 0 & \text{if} & x \in \Omega, \\ \phi(x) = 0 & \text{if} & x \in \partial\Omega, \\ \phi(x) > 0 & \text{if} & x \in D \setminus \overline{\Omega}. \end{cases}$$

The evolution of the shape Ω through the optimization process is then deduced from the solution of a Hamilton-Jacobi equation for ϕ [9]. In [2], as well as in almost all other works on the level set method for structural optimization, a fixed mesh of D is used and the mechanical analysis is performed through an *Ersatz material approach* in the whole domain D instead of Ω which is never meshed.

Here, as in [1], we add an extra ingredient to this process, asking that, at each iteration, the shape is exactly meshed, which enables precise mechanical computations. On the other hand it requires that the mesh of D be *unstructured* and change from one iteration to the next since the shape is *explicitly* discretize as part of the mesh of D. Nevertheless, we still retain the versatility of the level set method when it comes to topology changes.

Such a change in perspectives raises several difficulties: let alone the fact that we can no longer rely on finite difference schemes for, among other things, solving Hamilton-Jacobi equations on D (because the mesh of D is no longer Cartesian and we don't want to use two different meshes), we need efficient tools to switch from the level set representation of shapes to a meshed re presentation, and conversely.

Our method differs from previous works (always in 2-d) on shape optimization using exact meshes of the shapes. In [13], the authors relied on the knowledge of a level set function to move points of the background mesh so that it coincides with a mesh of the associated shape. In [12] and [8], the authors inferred from the knowledge of a level set function a sample set of points associated to the shape, then resorted to the Delaunay algorithm to construct a computational mesh of it. Unfortunately, such methods are difficult to extend to 3-d. On the contrary, as we shall show below, our method extends to three dimensions without additional theoretical difficulties (even though it is considerably more tedious to carry out). Moreover, the

use of unstructured meshes enables the use of mesh adaptation techniques [7], which is a natural direction for future work.

2. Structural optimization in linear elasticity

2.1. A model problem in linear elasticity.

We consider *shapes* which are (smooth) bounded domains $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$), filled with a linear elastic isotropic material with Hooke's law A. Each shape Ω is clamped on a given part $\Gamma_D \subset \partial\Omega$, submitted to external loads g on another part $\Gamma_N \subset \partial\Omega$, with $\Gamma_D \cap \Gamma_N = \emptyset$ (we omit body forces), so that the displacement field u of the structure is the unique solution to the system

$$(2) \qquad \begin{cases} -\mathrm{div}\,(Ae(u)) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ Ae(u).n = g & \text{on } \Gamma_N, \\ Ae(u).n = 0 & \text{on } \Gamma, \end{cases}$$

where $e(u) = (\nabla u + {}^t\nabla u)/2$ is the *linearized strain tensor*, $\Gamma := \partial\Omega \setminus (\Gamma_D \cup \Gamma_N)$ is the *free boundary*, and n is the outer unit normal to $\partial\Omega$.

Our problem is to minimize a functional $J(\Omega)$ of the domain, among a set of *admissible shapes* Ω, such that, among other things $(\Gamma_D \cup \Gamma_N) \subset \partial\Omega$, leaving only Γ subject to optimization. In this note, we limit ourselves with a weighted sum of the *compliance* of the structure and its volume,

$$(3) \qquad J(\Omega) = \int_{\Gamma_N} g.u\, ds + \ell \int_{\Omega} dx,$$

where ℓ is a (positive) Lagrange multiplier associated to the volume constraint.

2.2. Hadamard's boundary variation method.

To define a *shape derivative* of a functional $J(\Omega)$, only variations of the form $(I+\theta)(\Omega)$, where $\theta \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ is a *small* displacement field, are considered. The shape derivative of J at Ω is then the Fréchet differential at 0 of the underlying mapping $\theta \mapsto J((I + \theta)(\Omega))$. The *structure theorem* [6] states that, for a wide class of functionals J, this derivative reads

$$(4) \qquad \forall \theta \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d), \;\; J'(\Omega)(\theta) = \int_{\Gamma} v\, \theta.n\, ds,$$

for a certain scalar function v on Γ. A *descent direction* for J is then revealed as $-v\, n$. For instance, the derivative of (3) is [2]

$$(5) \qquad J'(\Omega)(\theta) = \int_{\Gamma} (\ell - Ae(u) : e(u))\; \theta.n\, ds.$$

3. An overlook of the proposed method

Let $D \subset \mathbb{R}^d$ be a fixed computational domain which encloses all admissible shapes : as described in [1], we rely on two alternative descriptions of shapes $\Omega \subset D$:

- *The level-set description* : Ω is known as the negative subdomain of a scalar function ϕ as in (1). In the numerical context, ϕ is discretized at the vertices of a simplicial mesh of D.
- *The meshed description*: the whole domain D is equipped with a (conformal) simplicial mesh $\mathcal{T}_\Omega$, a part of which is a mesh of Ω, i.e. the entities (edges, faces, etc...) of a mesh of Ω also belong to $\mathcal{T}_\Omega$.

We now describe the operators which allows us to switch from one representation to the other.

3.1. From a meshed domain to a level set description.

Let $\mathcal{T}_\Omega$ be a simplicial mesh of D, in which $\Omega \subset D$ is explicitly discretized. In order to generate a level set function ϕ associated to Ω on $\mathcal{T}_\Omega$, we compute an approximation of the *signed distance function* d_Ω to Ω, which enjoys crucial properties as regards numerical stability [3]. To achieve this, a numerical scheme for computing the signed distance function to a discrete contour on a simplicial mesh based on properties of the *time-dependent Eikonal equation* is used (see [5]).

3.2. **Meshing the negative subdomain of a level set function.**

Let $\mathcal{T}$ be a simplicial mesh of D and ϕ a (piecewise linear) level set function on D, defined at the vertices of $\mathcal{T}$, which accounts for a (polyhedral) domain $\Omega := \{x \in D \mid \phi(x) < 0\}$. Note that Ω is not explicitly discretized in $\mathcal{T}$. The proposed method for modifying $\mathcal{T}$ into a new mesh $\widetilde{\mathcal{T}}$ of D, in which Ω is explicitly discretized, involves two steps (see [10] for another interesting approach).

Step 1 : Rough discretization of Ω into $\mathcal{T}$. Each simplex $K \in \mathcal{T}$ which is crossed by the 0 level set $\partial\Omega$ of ϕ is split into several simplices, in such a way that $K \cap \partial\Omega$ explicitly appears in the resulting mesh. This is a rather easy, purely logical step, which relies on patterns, depending on the relative signs and values of ϕ at the vertices of each such simplex [7]. This produces a mesh $\mathcal{T}_{temp}$ of D in which Ω appears as a submesh. Unfortunately, $\mathcal{T}_{temp}$ is bound to be of very poor *mesh quality* - i.e. to contain very thin, or flat elements - whereas it is well-known that the performances (in terms of accuracy, convergence rate, etc.) of finite element methods strongly depend on the uniformity of the mesh.

Step 2 : Quality-oriented local mesh modifications. This step is the most tedious of the whole process, and is the only one that is fundamentally different from its two-dimensional equivalent. We simply sketch the main ingredients, referring to [4] for details. From the ill-shaped mesh $\mathcal{T}_{temp}$, a well-shaped mesh $\widetilde{\mathcal{T}}$ is obtained by repeatedly applying the following local operations [7]:
  - Split the edges of $\mathcal{T}_{temp}$ that are 'too long', introducing a new vertex at their middle, then updating connectivities accordingly. An edge is said 'too long' when its length is larger than a prescribed size taking into account a user-specified size feature and the local curvature of Ω.
  - Merge the two endpoints of an edge of $\mathcal{T}_{temp}$ whenever it is 'too short' and provided the resulting configuration is not 'too far' from Ω.
  - Swap connectivities, or slightly move vertices of the mesh, whenever it helps improving the overall quality of the mesh and does not jeopardize the accuracy of the description of Ω.

This steps ends with a mesh $\widetilde{\mathcal{T}}$ of D which is suitable for computations (see figure 1 for an example).

4. The proposed algorithm

Let Ω^0 be an initial shape. The tools of section 3 yield the following algorithm for structural optimization.

For $n = 0, \dots$ till convergence, start with a shape Ω^n, given by the data of a mesh $\mathcal{T}_{\Omega^n}$ of D, in which Ω^n is explicitly discretized.
  (1) Retain only the part of $\mathcal{T}_{\Omega^n}$ corresponding to Ω^n, and compute the solution of (2) in Ω^n by a standard finite element method.
  (2) On the whole mesh $\mathcal{T}_{\Omega^n}$ of D, generate the signed distance function d_{Ω^n} to Ω^n.
  (3) Infer from (5) a descent direction θ^n for (3).
  (4) Chose a descent step $\tau^n > 0$ and, on mesh $\mathcal{T}_{\Omega^n}$, solve the *level set advection equation*,

$$(6) \qquad \begin{cases} \frac{\partial\phi}{\partial t} + \theta^n(x).\nabla\phi = 0 & \text{for } x \in D, \ t \in (0, \tau^n), \\ \phi(0, x) = d_{\Omega^n}(x) & \text{for } x \in D, \end{cases}$$

as a linear, implicit-in-time approximation of the true nonlinear Hamilton-Jacobi equation for level set evolution (using a method of characteristics [11]). Notice that, because (6) is solved on the same mesh $\mathcal{T}_{\Omega^n}$ as the one used for the finite element analysis, no projection of the velocity field whatsoever is involved. This yields a level set function $\phi^{n+1} := \phi(\tau^n, .)$ on $\mathcal{T}_{\Omega^n}$ associated to the new shape Ω^{n+1}.
  (5) Discretize the 0 level set of ϕ^{n+1} in the mesh $\mathcal{T}_{\Omega^n}$ along the lines of section 3.2 to obtain a new mesh $\mathcal{T}_{\Omega^{n+1}}$ of D, in which Ω^{n+1} is explicitly discretized.

5. Numerical examples

The proposed method is applied to the *bridge* test case, as depicted on figure 2: a structure, embedded in a box of dimensions $40 \times 200 \times 60$, made of an isotropic elastic material of Young modulus $E = 1$ and
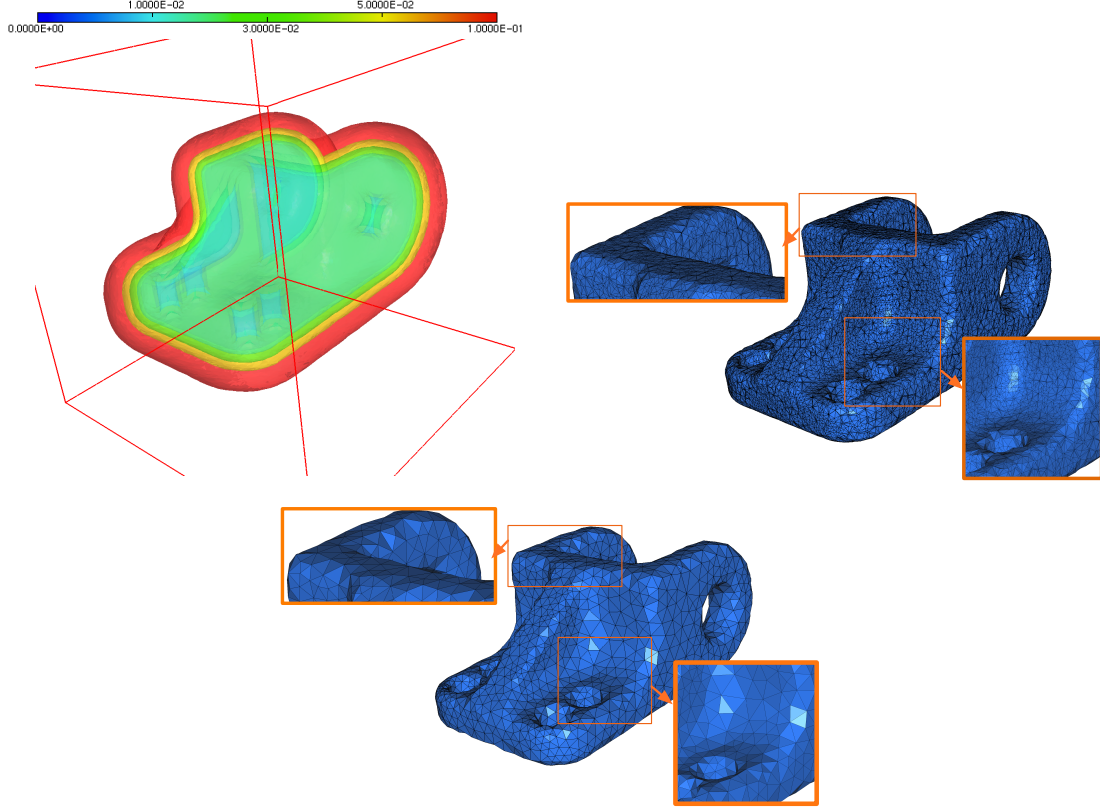
FIGURE 1. *Top-left:* Isovalues of a level set function ϕ defined over the whole box D; *top-right:* resulting ill-shaped mesh $\mathcal{T}_{temp}$ after the rough discretization of the 0 level set of ϕ; *bottom:* final, nicely-shaped mesh $\widetilde{\mathcal{T}}$, enclosing a mesh of Ω as a submesh (only $\partial\Omega$ is represented).

Poisson ratio $\nu = 0.3$, is clamped on each side of its base, and submitted to a unit vertical load $g = -e_z$ on the middle of its base. The initial shape is the lower part of the box (without any holes). We minimize the objective (3) with a Lagrange multiplier $\ell = 20$ for the volume. We run 70 iterations of the above algorithm; each mesh $\mathcal{T}_{\Omega^n}$ has about $12,000$ vertices, and the entire computation takes roughly one hour on a laptop computer (which is a reasonable overhead compared to a similar approach on a fixed mesh). Note that our algorithm has been able to change dramatically the topology and yet shapes are exactly meshed at each iteration. Figure 3 shows the behavior of the objective function in the course of the iterative process.

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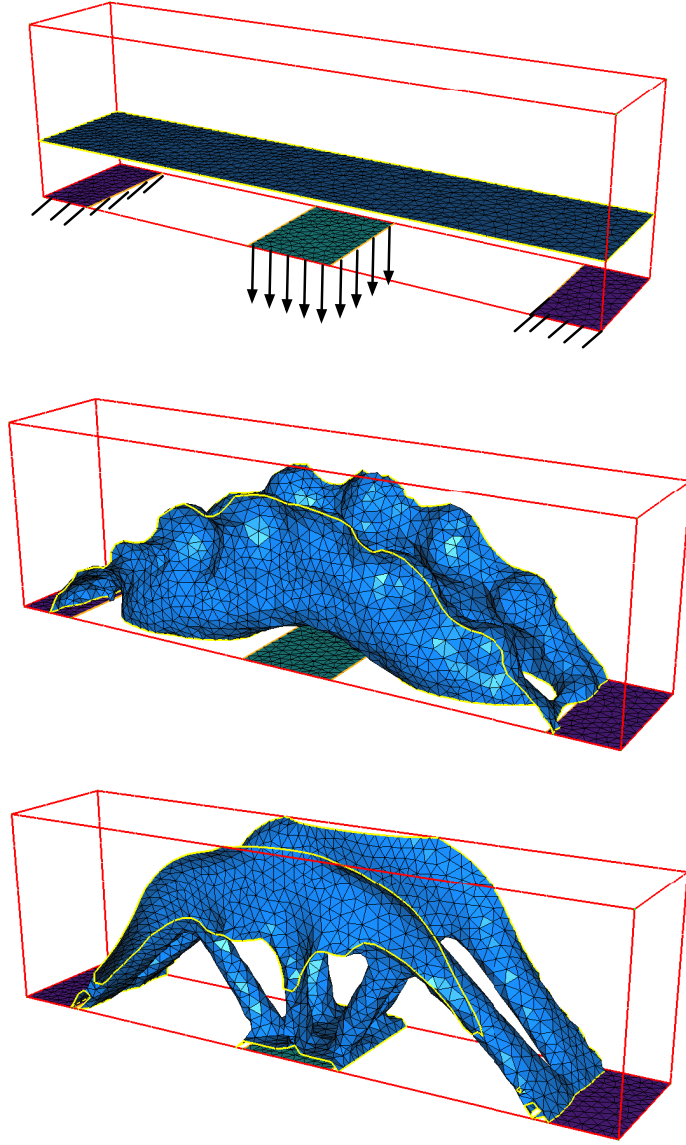
FIGURE 2. *From top to bottom :* Initial, 20th and final iterations of the bridge test-case. Only the boundary $\partial\Omega$ of each shape Ω is displayed.
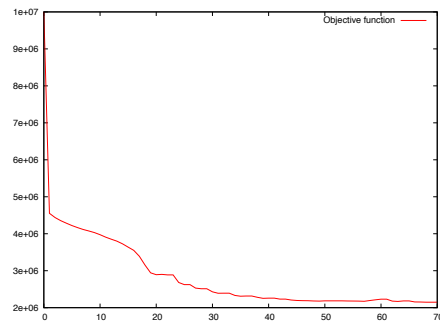

FIGURE 3. Objective function versus iteration number for the bridge test-case.

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