

ON A FRACTIONAL GINZBURG-LANDAU EQUATION AND 1/2-HARMONIC MAPS INTO SPHERES

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ABSTRACT. This paper is devoted to the asymptotic analysis of a fractional version of the Ginzburg-Landau equation in bounded domains, where the Laplacian is replaced by an integro-differential operator related to the square root Laplacian as defined in Fourier space. In the singular limit $\varepsilon \rightarrow 0$, we show that solutions with uniformly bounded energy converge weakly to sphere valued 1/2-harmonic maps, *i.e.*, the fractional analogues of the usual harmonic maps. In addition, the convergence holds in smooth functions spaces away from a countably \mathcal{H}^{n-1} -rectifiable closed set of finite $(n-1)$ -Hausdorff measure. The proof relies on the representation of the square root Laplacian as a Dirichlet-to-Neumann operator in one more dimension, and on the analysis of a boundary version of the Ginzburg-Landau equation. Besides the analysis of the fractional Ginzburg-Landau equation, we also give a general partial regularity result for stationary 1/2-harmonic maps in arbitrary dimension.

Keywords: Fractional Ginzburg-Landau equations; Fractional harmonic maps; Square root Laplacian; Dirichlet-to-Neuman operator; Nonlinear boundary reactions; Partial regularity; Generalized varifolds.

1. INTRODUCTION

Let $n \geq 1$ and $m \geq 2$ be given integers, and let $\omega \subseteq \mathbb{R}^n$ be a smooth bounded open set. In this paper we investigate the asymptotic behavior, as $\varepsilon \downarrow 0$, of weak solutions $v_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^m$ of the fractional Ginzburg-Landau equation

$$(-\Delta)^{\frac{1}{2}} v_\varepsilon = \frac{1}{\varepsilon} (1 - |v_\varepsilon|^2) v_\varepsilon \quad \text{in } \omega, \quad (1.1)$$

possibly subject to an exterior Dirichlet condition

$$v_\varepsilon = g \quad \text{on } \mathbb{R}^n \setminus \omega, \quad (1.2)$$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a smooth function satisfying $|g| = 1$ in $\mathbb{R}^n \setminus \omega$. The action of the integro-differential operator $(-\Delta)^{\frac{1}{2}}$ on a smooth bounded function $v : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined by

$$(-\Delta)^{\frac{1}{2}} v(x) := \text{p.v.} \left(\gamma_n \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x - y|^{n+1}} dy \right) \quad \text{with } \gamma_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}},$$

and the notation p.v. means that the integral is taken in the *Cauchy principal value* sense. In Fourier space¹, this operator has symbol $2\pi|\xi|$, while $4\pi^2|\xi|^2$ is the symbol of $-\Delta$. In particular, $(-\Delta)^{\frac{1}{2}}$ is the square root of the standard Laplacian when acting on the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

The weak sense for equation (1.1) will be understood variationally using the distributional formulation of the fractional Laplacian in the open set ω . More precisely, we define the action of $(-\Delta)^{\frac{1}{2}} v$ on an element $\varphi \in \mathcal{D}(\omega; \mathbb{R}^m)$ by setting

$$\begin{aligned} \langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle_\omega &:= \frac{\gamma_n}{2} \iint_{\omega \times \omega} \frac{(v(x) - v(y)) \cdot (\varphi(x) - \varphi(y))}{|x - y|^{n+1}} dx dy \\ &\quad + \gamma_n \iint_{\omega \times (\mathbb{R}^n \setminus \omega)} \frac{(v(x) - v(y)) \cdot (\varphi(x) - \varphi(y))}{|x - y|^{n+1}} dx dy. \end{aligned}$$

This formula turns out to define a distribution on ω whenever $v \in L^2_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ satisfies

$$\mathcal{E}(v, \omega) := \frac{\gamma_n}{4} \iint_{\omega \times \omega} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy + \frac{\gamma_n}{2} \iint_{\omega \times (\mathbb{R}^n \setminus \omega)} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy < \infty. \quad (1.3)$$

In such a case we say that v is admissible, and $(-\Delta)^{\frac{1}{2}} v$ belongs to $H^{-1/2}(\omega; \mathbb{R}^m)$, *i.e.*, the topological dual space of $H^{1/2}_0(\omega; \mathbb{R}^m)$ (functions in $H^{1/2}(\mathbb{R}^n; \mathbb{R}^m)$ vanishing outside ω). When prescribing the exterior Dirichlet condition (1.2), the class of admissible functions is given by the affine space $H_g^{1/2}(\mathbb{R}^n; \mathbb{R}^m) :=$

¹We consider here the ordinary frequency Fourier transform $v \mapsto \hat{v}$ given by $\hat{v}(\xi) := \int_{\mathbb{R}^n} v(x) e^{-2i\pi x \cdot \xi} d\xi$.

$g + H_{00}^{1/2}(\omega; \mathbb{R}^m)$. Note that weak and strong definitions of $(-\Delta)^{\frac{1}{2}}$ are consistent, *i.e.*, if v is smooth and bounded, then v is admissible, and the pointwise and distributional definitions of $(-\Delta)^{\frac{1}{2}}v$ agree, see *e.g.* [68]. Then observe that the derivative at v of $\mathcal{E}(\cdot, \omega)$ in the direction $\varphi \in \mathcal{D}(\omega; \mathbb{R}^m)$ is precisely given by $\langle (-\Delta)^{\frac{1}{2}}v, \varphi \rangle_\omega$. Hence the energy $\mathcal{E}(\cdot, \omega)$, which is built on the $H^{1/2}$ -seminorm, can be thought as *Dirichlet 1/2-energy* in ω associated to the operator $(-\Delta)^{\frac{1}{2}}$. Integrating the potential in the right hand side of (1.1) we obtain the *Ginzburg-Landau 1/2-energy* associated to the equation in ω ,

$$\mathcal{E}_\varepsilon(v, \omega) := \frac{1}{2} \int_\omega e(v, \omega) + \frac{1}{2\varepsilon} (1 - |v|^2)^2 dx, \quad (1.4)$$

where we have set $e(v, \omega)$ to be the *nonlocal energy density* in ω given by

$$e(v, \omega) := \frac{\gamma_n}{2} \int_\omega \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dy + \gamma_n \int_{\mathbb{R}^n \setminus \omega} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dy. \quad (1.5)$$

We define weak solutions of (1.1) as critical points of \mathcal{E}_ε with respect to perturbations supported in ω .

If $-\Delta$ is used instead of its square root and $m = 1$, equation (1.1) reduces to the well known Allen-Cahn equation. This scalar equation appears in the van der Waals-Cahn-Hilliard theory of phase transitions in fluids. From the geometrical point of view, the Allen-Cahn equation provides an approximation, as $\varepsilon \rightarrow 0$, of codimension one minimal surfaces, *i.e.*, limits of critical points are functions with values in $\{\pm 1\}$, and the limiting interfaces between the two phases ± 1 are *generalized minimal hypersurfaces*, see *e.g.* [45, 56]. Since the work of ALBERTI, BOUCHITTÉ, & SEPPECHER [1, 2], fractional generalizations of the Allen-Cahn equation involving $(-\Delta)^s$ with $0 < s < 1$ have been investigated by many authors, see *e.g.* [13, 14, 15, 16, 17, 37, 58, 72], specially in the case $s = 1/2$ which is related to models for dislocations in crystals [35, 38] (see also [25, 27] for $s \neq 1/2$), and to thin-film micromagnetics [46, 47]. The convergence as $\varepsilon \rightarrow 0$ of *minimizers* of the fractional Ginzburg-Landau s -energy has been treated very recently in [64, 65]. In these papers, Dirichlet and Ginzburg-Landau s -energies are built on the H^s -seminorm and are defined as in (1.3)-(1.4)-(1.5) with the weight $|x - y|^{-(n+2s)}$ in place of $|x - y|^{-(n+1)}$. It is proved in [64, 65] that limits of minimizers still take values in $\{\pm 1\}$, and the geometrical characterization of the associated interfaces depends on whether $s \geq 1/2$ or $s < 1/2$. For $s \geq 1/2$, limiting interfaces turn out to be (usual generalized) area-minimizing minimal surfaces, while the picture completely changes for $s < 1/2$, where limiting interfaces are the so-called *nonlocal s -minimal surfaces* introduced by CAFFARELLI, ROQUEJOFFRE, & SAVIN [18]. The interface induced by a map v taking values in $\{\pm 1\}$ is said to be a minimizing s -minimal surface if v itself is minimizing the Dirichlet s -energy in the class of maps with values in $\{\pm 1\}$. In a sense, nonlocal s -minimal surfaces can be thought as *fractional s -harmonic maps* with values in $\mathbb{S}^0 \simeq \{\pm 1\}$. The dichotomy in the parameter s essentially comes from the fact that characteristic functions of (smooth) sets do not belong to H^s for $s \geq 1/2$, so that the s -energy density concentrates near a limiting interface very much like in the local case. For $s < 1/2$, the H^s -regularity allows for (some) characteristic functions, and the energy of minimizers remains uniformly bounded as ε tends to 0. In this paper, we focus on the case $s = 1/2$ for some specific geometrical features that we shall explain shortly.

In the vectorial case $m = 2$, the local Ginzburg-Landau equation (*i.e.*, with $-\Delta$ instead of its square root) has been widely studied because it shares many of the relevant features of more elaborated models of superconductivity or superfluidity, see *e.g.* [9, 61, 62]. In the spirit of the classical Landau's theory of phase transitions, fractional Ginzburg-Landau equations have been recently suggested in the physics literature in order to incorporate a long-range dependence posed by a nonlocal ordering, as it might appear in certain high temperature superconducting compounds, see [54, 77, 79] and the references therein. In arbitrary dimensions $n \geq 1$ and $m \geq 2$, the local Ginzburg-Landau equation has a geometrical flavor like in the scalar case, and the limiting system in the asymptotic $\varepsilon \rightarrow 0$ is given by the harmonic map equation into the sphere \mathbb{S}^{m-1} (or into more general manifolds according to the potential well). Harmonic maps can be seen as higher dimensional generalizations of geodesics and are defined as critical points of the Dirichlet energy with respect to perturbations in the target. If the target is a sphere, it corresponds to variations of the Dirichlet energy of the form $t \mapsto \frac{v_* + t\varphi}{|v_* + t\varphi|}$ for bounded test functions φ and $|t| \ll 1$. It leads to the Euler-Lagrange equation (in the sense of distributions)

$$-\Delta v_* = |\nabla v_*|^2 v_* \iff -\Delta v_* \perp T_{v_*} \mathbb{S}^{m-1}. \quad (1.6)$$

The equation in the right hand side of (1.6) arises when using test functions satisfying $\varphi(x) \in T_{v_*(x)}\mathbb{S}^{m-1}$ a.e., while the equation in the left hand side appears when φ is chosen with no constraint. The implication from right to left can be easily deduced (at least formally) writing $\Delta v_* = (\Delta v_* \cdot v_*)v_*$ and using the identity $0 = \Delta(|v_*|^2) = 2\Delta v_* \cdot v_* + 2|\nabla v_*|^2$. A main feature of the harmonic map system is its critical structure concerning regularity. Indeed, the source term in the left hand side equation of (1.6) has *a priori* no better integrability than L^1 . This is precisely the borderline case where linear elliptic regularity does not apply. In dimension $n = 2$, harmonic maps are smooth by the famous result of HÉLEIN [44]. However, equation (1.6) admits highly discontinuous solutions in dimensions $n \geq 3$ [60], and specific assumptions are required to obtain at least partial regularity (*i.e.*, smoothness outside some "small set", see Subsection 4.1 for a more detailed discussion and references). In addition to this regularity issue, equation (1.6) turns out to be invariant under composition with conformal maps in dimension $n = 2$. This conformal invariance implies that sequences of harmonic maps are usually non-compact in the energy space [49], and this lack of compactness is formally transferred to the local Ginzburg-Landau equation in the asymptotic $\varepsilon \rightarrow 0$. The weak convergence of critical points of the local Ginzburg-Landau energy towards (weakly) harmonic maps has been proved by LIN & WANG [50, 51, 52] (see also [20]). In the spirit of the blow-up analysis for harmonic maps developed in [49], the authors have determined in [50, 51, 52] the geometrical nature of the possible defect measure arising in the weak convergence process, as well as the rectifiability of the energy concentration set.

The main object of the present article is to provide a careful analysis of critical points of the Ginzburg-Landau 1/2-energy \mathcal{E}_ε as $\varepsilon \rightarrow 0$ in the spirit of [50, 51, 52], and as a byproduct to extend the previous works on the scalar case to the vectorial setting. By analogy with the local case, we expect solutions of the fractional Ginzburg-Landau equation (1.1) with uniformly bounded energy in ε to converge weakly as $\varepsilon \rightarrow 0$ to critical points of the Dirichlet 1/2-energy under the constraint to be \mathbb{S}^{m-1} -valued. In other words, a weak limit v_* of critical points of the Ginzburg-Landau 1/2-energy should satisfy

$$\left[\frac{d}{dt} \mathcal{E} \left(\frac{v_* + t\varphi}{|v_* + t\varphi|}, \omega \right) \right]_{t=0} = 0 \quad \text{for all } \varphi \in \mathcal{D}(\omega; \mathbb{R}^m). \quad (1.7)$$

Computing the associated Euler-Lagrange equation, we obtain

$$(-\Delta)^{\frac{1}{2}} v_* \perp T_{v_*} \mathbb{S}^{m-1} \quad \text{in } \omega, \quad (1.8)$$

in the distributional sense. In the case where $|v_*| = 1$ in \mathbb{R}^n , the Lagrange multiplier related to the constraint takes a quite simple form and leads to the equation

$$(-\Delta)^{\frac{1}{2}} v_*(x) = \left(\frac{\gamma_n}{2} \int_{\mathbb{R}^n} \frac{|v_*(x) - v_*(y)|^2}{|x - y|^{n+1}} dy \right) v_*(x) \quad \text{in } \omega, \quad (1.9)$$

which is in clear analogy with (1.6). We shall refer to as *weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω* any map taking values into \mathbb{S}^{m-1} in ω and satisfying condition (1.7).

The notion of 1/2-harmonic map into a manifold has been introduced in the case $\omega = \mathbb{R}$ by DA LIO & RIVIÈRE [22, 23]. More precisely, weak 1/2-harmonic maps from \mathbb{R} into \mathbb{S}^{m-1} are defined in [22] as critical points of the *line energy*

$$\mathcal{L}(v) := \frac{1}{2} \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} v|^2 dx,$$

under the \mathbb{S}^{m-1} -constraint as in (1.7). Here $(-\Delta)^{\frac{1}{4}}$ is the operator with symbol $\sqrt{2\pi|\xi|}$. A classical computation in Fourier space gives $\mathcal{L}(v) = \mathcal{E}(v, \mathbb{R})$, so that the two notions of 1/2-harmonic maps indeed coincide. Recently, 1/2-harmonic maps into a manifold have found applications in geometrical problems such as *minimal surfaces with free boundary* (see [24, 33, 76], and Remark 4.28), and they are intimately related to *harmonic maps with partially free boundary* (see Remark 4.5, and [41, 42, 57]). While the main purpose of [22, 23] was to show the regularity of 1/2-harmonic maps for $n = 1$, it was suggested there that they arise as limits of the fractional Ginzburg-Landau equation. Our primary objective here is to prove this assertion in arbitrary dimensions and for arbitrary domains, prescribing eventually an exterior Dirichlet condition like in (1.2). Working on such a program, it is natural to face the regularity problem for 1/2-harmonic maps in arbitrary dimension, and this is our secondary objective. Looking at the 1/2-harmonic map equation (1.8)-(1.9), one immediately realizes that it has precisely the same critical

structure than the one of the local case, for both regularity and compactness. Here, equation (1.8) turns out to be invariant in dimension $n = 1$ under composition with traces on \mathbb{R} of conformal maps from the half plane \mathbb{R}_+^2 into itself. By analogy with the local case again, it suggests that solutions of the fractional Ginzburg-Landau equation (1.1) are not strongly compact as $\varepsilon \rightarrow 0$ in general. Describing the possible defect of strong convergence shall be one of the main concern in our asymptotic analysis of the fractional Ginzburg-Landau equation.

Our first main result, stated in Theorem 1.1 below, deals with the fractional Ginzburg-Landau equation supplemented with the Dirichlet exterior condition (1.2), and provides an answer to the questions above. We assume in this theorem the uniform energy bound with respect to ε which easily holds in most cases whenever the exterior condition g admits an \mathbb{S}^{m-1} -valued extension in ω of finite energy. Note that, if g does not admits a smooth \mathbb{S}^{m-1} -valued extension in ω , singularities have to appear in the limit $\varepsilon \rightarrow 0$ as in usual Ginzburg-Landau problems. If $n = m$, this is the case when $\partial\omega \simeq \mathbb{S}^{n-1}$ and $g : \partial\omega \rightarrow \mathbb{S}^{n-1}$ has a non vanishing topological degree.

Theorem 1.1. *Let $\omega \subseteq \mathbb{R}^n$ be a smooth bounded open set, and let $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a smooth map satisfying $|g| = 1$ in $\mathbb{R}^n \setminus \omega$. Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, and let $\{v_k\}_{k \in \mathbb{N}} \subseteq H_g^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ be such that for each $k \in \mathbb{N}$, v_k weakly solves*

$$\begin{cases} (-\Delta)^{\frac{1}{2}} v_k = \frac{1}{\varepsilon_k} (1 - |v_k|^2) v_k & \text{in } \omega, \\ v_k = g & \text{in } \mathbb{R}^n \setminus \omega. \end{cases}$$

If $\sup_k \mathcal{E}_{\varepsilon_k}(v_k, \omega) < \infty$, then there exist a (not relabeled) subsequence and $v_ \in H_g^{1/2}(\omega; \mathbb{R}^m)$ a weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω such that $v_k - v_* \rightarrow 0$ weakly in $H_{00}^{1/2}(\omega)$. In addition, there exist a finite nonnegative Radon measure μ_{sing} on ω , a countably \mathcal{H}^{n-1} -rectifiable relatively closed set $\Sigma \subseteq \omega$ of finite $(n-1)$ -dimensional Hausdorff measure, and a Borel function $\theta : \Sigma \rightarrow (0, \infty)$ such that*

- (i) $e(v_k, \omega) \mathcal{L}^n \llcorner \omega \xrightarrow{*} e(v_*, \omega) \mathcal{L}^n \llcorner \omega + \mu_{\text{sing}}$ weakly* as Radon measures on ω ;
- (ii) $\frac{(1 - |v_k|^2)^2}{\varepsilon_k} \rightarrow 0$ in $L_{\text{loc}}^1(\omega)$;
- (iii) $\frac{1 - |v_k(x)|^2}{\varepsilon_k} \rightarrow \frac{\gamma_n}{2} \int_{\mathbb{R}^n} \frac{|v_*(x) - v_*(y)|^2}{|x - y|^{n+1}} dy + \mu_{\text{sing}}$ in $\mathcal{D}'(\omega)$;
- (iv) $\mu_{\text{sing}} = \theta \mathcal{H}^{n-1} \llcorner \Sigma$;
- (v) $v_* \in C^\infty(\omega \setminus \Sigma)$ and $v_n \rightarrow v_*$ in $C_{\text{loc}}^\ell(\omega \setminus \Sigma)$ for every $\ell \in \mathbb{N}$;
- (vi) if $n \geq 2$, the limiting 1/2-harmonic map v_* and the defect measure μ_{sing} satisfy the stationarity relation

$$\left[\frac{d}{dt} \mathcal{E}(v_* \circ \phi_t, \omega) \right]_{t=0} = \frac{1}{2} \int_{\Sigma} \text{div}_{\Sigma} X \, d\mu_{\text{sing}} \quad (1.10)$$

for all vector fields $X \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ compactly supported in ω , where $\{\phi_t\}_{t \in \mathbb{R}}$ denotes the flow on \mathbb{R}^n generated by X ;

- (vii) if $n = 1$, the set Σ is finite and $v_* \in C^\infty(\omega)$.

The proof of this theorem rests on the representation of $(-\Delta)^{\frac{1}{2}}$ as *Dirichlet-to-Neumann operator* associated to the harmonic extension to the open half space $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$ given by the convolution product with the Poisson kernel, see (2.9). More precisely, denoting by $v \mapsto v^e$ this harmonic extension, we shall prove that it is well defined on the space of functions of finite Dirichlet 1/2-energy, and that $\partial_\nu v^e = (-\Delta)^{\frac{1}{2}} v$ as distributions on ω . Here, ∂_ν denotes the exterior normal differentiation on $\partial\mathbb{R}_+^{n+1} \simeq \mathbb{R}^n$. When applying the extension procedure to a solution v_ε of the fractional Ginzburg-Landau equation (1.1), we end up with the following *system of Ginzburg-Landau boundary reactions*

$$\begin{cases} \Delta v_\varepsilon^e = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial v_\varepsilon^e}{\partial \nu} = \frac{1}{\varepsilon} (1 - |v_\varepsilon^e|^2) v_\varepsilon^e & \text{on } \omega. \end{cases} \quad (1.11)$$

The asymptotic analysis of this system as $\varepsilon \rightarrow 0$ will lead us to the main conclusions stated in Theorem 1.1. To this purpose, we shall first establish an *epsilon-regularity* type of estimate for (1.11) in the spirit of the regularity theory for harmonic maps [8, 67] or usual Ginzburg-Landau equations [20]. This estimate is the key to derive the convergence result and the rectifiability of the defect measure. Note that Theorem 1.1 actually shows that μ_{sing} is a $(n-1)$ -rectifiable varifold in the sense of ALMGREN, see e.g. [70]. We emphasize that identity (1.10) is precisely the coupling equation between the limiting 1/2-harmonic v_* and the defect measure μ_{sing} . It states that the first inner variation of the Dirichlet 1/2-energy of v_* is equal to $-\frac{1}{2}$ times the first inner variation of the varifold μ_{sing} , see [70, formulas 15.7 and 16.2]. We shall achieve (1.10) in two independent steps. The first step consists in proving an analogous identity when passing to the limit $\varepsilon \rightarrow 0$ in system (1.11). In the spirit of [52], the convenient way to let $\varepsilon \rightarrow 0$ in the first inner variation of the Dirichlet energy of v_ε^e is to use the notion of *generalized varifold* of AMBROSIO & SONER [4], once adapted to the boundary setting. In turn, the second step allows us to return to the original formulation on \mathbb{R}^n . It shows that the first inner variation of the Dirichlet 1/2-energy of an arbitrary map v is equal to the first inner variation up to ω of the Dirichlet energy of its harmonic extension v^e , see Lemma 4.16. Note that this crucial observation also implies that an harmonic extension v^e is stationary up to ω for the Dirichlet energy whenever the original function v is stationary for the Dirichlet 1/2-energy. Here, stationarity means stationarity with respect to inner variations (i.e., variations of the domain).

Concerning the regularity issue for 1/2-harmonic maps, it is also fruitful to rephrase the problem in terms of harmonic extensions. Applying the extension procedure to a weak \mathbb{S}^{m-1} -valued 1/2-harmonic map v_* in ω leads to the system

$$\begin{cases} \Delta v_*^e = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial v_*^e}{\partial \nu} \perp T_{v_*^e} \mathbb{S}^{m-1} & \text{on } \omega. \end{cases} \quad (1.12)$$

This system turns out to be (almost) included in the class of *harmonic maps with free boundary* (see Remark 4.5) for which a regularity theory do exist [29, 30, 43, 66]. This theory provides partial regularity results, and it requires *minimality*, or at least *stationarity* up to the free boundary. As already mentioned in the discussion above, if v_* is assumed to be stationary, then the extension v_*^e is stationary up to ω . Similarly, we shall see that harmonic extensions also inherit minimality up to ω . This is then enough to derive a general regularity result for 1/2-harmonic maps under a minimality or stationarity assumption. We finally point out that the relation between fractional harmonic maps and harmonic maps with a free boundary has been previously noticed by MOSER [57] for a (non-explicit) operator slightly different from the square root Laplacian $(-\Delta)^{\frac{1}{2}}$, and leading to slightly weaker results.

Our regularity results for 1/2-harmonic maps can be summarized in the following theorem. For simplicity we only state it for a sphere target, but we would like to stress that it actually holds for more general target manifolds, see Remark 4.24. Notice that it is an *interior regularity* result. The regularity at the boundary when prescribing an exterior Dirichlet condition (1.2) remains an open question.

Theorem 1.2. *Let $\omega \subseteq \mathbb{R}^n$ be a smooth bounded open set, and let $v_* \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n)$ be a weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω . Then $v_* \in C^\infty(\omega \setminus \text{sing}(v_*))$, where $\text{sing}(v_*)$ denotes the complement of the largest open set on which v_* is continuous. Moreover,*

- (i) *if $n = 1$, then $\text{sing}(v_*) \cap \omega = \emptyset$;*
- (ii) *if $n \geq 2$ and v_* is stationary, then $\mathcal{H}^{n-1}(\text{sing}(v_*) \cap \omega) = 0$;*
- (iii) *if v_* is minimizing, then $\dim_{\mathcal{H}}(\text{sing}(v_*) \cap \omega) \leq n - 2$ for $n \geq 3$, and $\text{sing}(v_*) \cap \omega$ is discrete for $n = 2$.*

Before concluding this introduction, let us briefly comment on some possible extensions of the present results and further open questions. First of all, in the energy (1.4) one could replace the Ginzburg-Landau potential $(1 - |u|^2)^2$ by a more general nonnegative potential $W(u)$ having a zero set $\{W = 0\}$ given by a smooth compact submanifold \mathcal{N} of \mathbb{R}^m without boundary, and then consider the corresponding fractional Ginzburg-Landau equation. In this context, the singular limit $\varepsilon \rightarrow 0$ leads to the 1/2-harmonic map system into \mathcal{N} (see Remark 4.24). If the codimension of \mathcal{N} is equal to 1 (plus some non degeneracy

assumptions on W), the proof of Theorem 1.1 can certainly be reproduced with minor modifications. However, the higher codimension case seems to require additional analysis since our *epsilon-regularity estimate* (Theorem 5.1) strongly uses the codimension 1 structure. It could be interesting to have a proof handling both cases. In another direction, one could consider the fractional Ginzburg-Landau equation in some open subset ω of a more general (smooth) complete Riemannian manifold $(\mathcal{M}, \mathbf{g})$ with a fractional $1/2$ -Laplacian $(-\Delta_{\mathbf{g}})^{\frac{1}{2}}$ given by the square root of the Laplace-Beltrami operator on \mathcal{M} as defined through spectral theory and functional calculus. In the most difficult case of a noncompact manifold, the construction of $(-\Delta_{\mathbf{g}})^{\frac{1}{2}}$ (or any fractional power $s \in (0, 1)$) can be found in [6] (and the references therein). Under some geometrical assumption on \mathcal{M} (e.g. \mathcal{M} has bounded geometry), [6] follows the approach of [75] to construct the Poisson kernel for the harmonic extension in $\mathcal{M} \times \mathbb{R}_+$ (endowed with the product metric $\mathbf{g} + |dt|^2$) of functions defined on \mathcal{M} , and it is shown that the fractional Laplacian $(-\Delta_{\mathbf{g}})^{\frac{1}{2}}$ coincides with the Dirichlet-to-Neumann operator associated to this harmonic extension. This is then enough to rephrase the fractional Ginzburg-Landau equation on \mathcal{M} (and $1/2$ -harmonic maps) as a system of Ginzburg-Landau boundary reactions on $\partial(\mathcal{M} \times \mathbb{R}_+) \simeq \mathcal{M}$ (respectively, harmonic maps with partially free boundary), and we believe that most of our results should extend to this setting. Note that this approach was successfully used in [39] in the scalar case $m = 1$ and for any fractional power $s \in (0, 1)$ to study the so-called *layer solutions* in the hyperbolic space \mathbb{H}^n . As already mentioned, we have considered here only the case $s = 1/2$ for the specific relations with the geometrical problems of harmonic maps with partially free boundary and minimal surfaces with free boundary. However, it could be interesting to investigate in the case $s \neq 1/2$ the analogues of this geometric objects, and then perform the asymptotic analysis as $\varepsilon \rightarrow 0$ of the corresponding Ginzburg-Landau equation. For this question, some aspects of the present analysis can actually be used for an arbitrary $s \in (0, 1)$ by means of the $2s$ -harmonic extension procedure discovered in [19]. In a forthcoming article [53], we shall consider the scalar Ginzburg-Landau (Allen-Cahn) equation in the case $s \in (0, 1/2)$ and the related nonlocal s -minimal surfaces in the spirit of [18, 64]. Interpreting nonlocal s -minimal surfaces as fractional s -harmonic maps with values in $\mathbb{S}^0 \simeq \{\pm 1\}$, and taking advantage of the $2s$ -harmonic extension, we shall prove in [53] the convergence, as $\varepsilon \rightarrow 0$, of arbitrary solutions of the fractional Allen-Cahn equation with equibounded energy towards nonlocal s -minimal surfaces (possibly non-minimizing), thus extending a result in [64] to the non-minimizing case.

The paper is organized as follows. In Section 2, we present all the needed material on the functional aspects of the *square root Laplacian/Dirichlet-to-Neumann operator*, as well as some key estimates related to harmonic extensions. Actually, we believe that this section can be of independent interest. Section 3 is devoted to the qualitative analysis of the fractional Ginzburg-Landau equation for ε fixed, and provides regularity results for this equation. Section 4 is entirely devoted to the analysis of $1/2$ -harmonic maps. In a first part, we give the regularity theory and prove Theorem 1.2. The second part deals with explicit examples of $1/2$ -harmonic maps underlying their geometrical nature, and stressing the analogies with standard harmonic maps. In Section 5, we prove the aforementioned *epsilon-regularity estimate* for the system of Ginzburg-Landau boundary reactions, and its asymptotic analysis as $\varepsilon \rightarrow 0$ follows in Section 6. Then we return to the fractional Ginzburg-Landau equation in Section 7 where we perform its asymptotic analysis with and without exterior Dirichlet condition. The special case of energy minimizers under exterior Dirichlet condition is also treated. Finally, we collect in Appendix A the proofs of several statements from Section 2, and Appendix B provides some elliptic estimates coming into play in the proof of the epsilon-regularity result.

Notations. Throughout the paper, \mathbb{R}^n is identified with $\partial\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \{0\}$. We denote by $B_r(x)$ the open ball in \mathbb{R}^{n+1} of radius r centered at x , while $D_r(x) := B_r(x) \cap \mathbb{R}^n$ is the open disc in \mathbb{R}^n centered at $x \in \mathbb{R}^n$. For a set $A \subseteq \mathbb{R}^{n+1}$, we write $A^+ := A \cap \mathbb{R}_+^{n+1}$, $\partial^+ A := \partial A \cap \mathbb{R}_+^{n+1}$, and \overline{A}^+ the closure of A^+ in \mathbb{R}^{n+1} . If $A \subseteq \mathbb{R}^n$ and no confusion arises, the complement of A in \mathbb{R}^n is simply denoted by A^c . The Froebenius inner product between two matrices $M = (M_{ij})$ and $N = (N_{ij})$ is denoted by $M : N := \sum_{i,j} M_{ij} N_{ij}$.

Concerning bounded open sets $\Omega \subseteq \mathbb{R}_+^{n+1}$, we shall say that Ω is an **admissible open set** whenever

- $\partial\Omega$ is Lipschitz regular;

- the open set $\partial^0\Omega \subseteq \mathbb{R}^n$ defined by

$$\partial^0\Omega := \{x \in \partial\Omega \cap \mathbb{R}^n : B_r^+(x) \subseteq \Omega \text{ for some } r > 0\},$$

is non empty and has Lipschitz boundary;

- $\partial\Omega = \partial^+\Omega \cup \overline{\partial^0\Omega}$.

Finally, we shall always denote by C a generic positive constant which only depends on the dimensions n and m , and possibly changing from line to line. If a constant depends on additional given parameters, we shall write those parameters using the subscript notation.

2. ENERGY SPACES AND LOCAL REPRESENTATION OF THE 1/2-LAPLACIAN

2.1. Basics on $H^{1/2}$ -spaces. Given an open set $\omega \subseteq \mathbb{R}^n$, the fractional Sobolev space $H^{1/2}(\omega; \mathbb{R}^m)$ is defined by

$$H^{1/2}(\omega; \mathbb{R}^m) := \{v \in L^2(\omega; \mathbb{R}^m) : [v]_{H^{1/2}(\omega)} < \infty\},$$

where

$$[v]_{H^{1/2}(\omega)}^2 := \frac{\gamma_n}{4} \iint_{\omega \times \omega} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy, \quad \gamma_n := \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}},$$

and we recall that $H^{1/2}(\omega; \mathbb{R}^m)$ is an Hilbert space normed by

$$\|v\|_{H^{1/2}(\omega)} := \|v\|_{L^2(\omega)} + [v]_{H^{1/2}(\omega)}.$$

Then $H_{\text{loc}}^{1/2}(\mathbb{R}^n; \mathbb{R}^m)$ denotes the class of functions which belongs to $H^{1/2}(\omega; \mathbb{R}^m)$ for every bounded open set $\omega \subseteq \mathbb{R}^n$. The linear space $H_{00}^{1/2}(\omega; \mathbb{R}^m)$ is defined by

$$H_{00}^{1/2}(\omega; \mathbb{R}^m) := \left\{v \in H^{1/2}(\mathbb{R}^n; \mathbb{R}^m) : v = 0 \text{ a.e. in } \mathbb{R}^n \setminus \omega\right\} \subseteq H^{1/2}(\mathbb{R}^n; \mathbb{R}^m).$$

Endowed with the norm induced by $H^{1/2}(\mathbb{R}^n; \mathbb{R}^m)$, the space $H_{00}^{1/2}(\omega; \mathbb{R}^m)$ is also an Hilbert space, and for $v \in H_{00}^{1/2}(\omega; \mathbb{R}^m)$,

$$[v]_{H^{1/2}(\mathbb{R}^n)}^2 = \frac{\gamma_n}{4} \iint_{\omega \times \omega} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy + \frac{\gamma_n}{2} \iint_{\omega \times \omega^c} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy. \quad (2.1)$$

The topological dual space of $H_{00}^{1/2}(\omega; \mathbb{R}^m)$ will be denoted by $H^{-1/2}(\omega; \mathbb{R}^m)$. We recall that if the boundary of ω is smooth enough (e.g. if $\partial\omega$ is Lipschitz regular), then

$$H_{00}^{1/2}(\omega; \mathbb{R}^m) = \overline{\mathcal{D}(\omega; \mathbb{R}^m)}^{\|\cdot\|_{H^{1/2}(\mathbb{R}^n)}}, \quad (2.2)$$

see [40, Theorem 1.4.2.2].

Throughout the paper we shall be interested in functions for which the right hand side of (2.1) is finite. We denote this class of functions by

$$\widehat{H}^{1/2}(\omega; \mathbb{R}^m) := \{v \in L_{\text{loc}}^2(\mathbb{R}^n; \mathbb{R}^m) : \mathcal{E}(v, \omega) < \infty\},$$

where $\mathcal{E}(\cdot, \omega)$ is the $1/2$ -Dirichlet energy defined in (1.3). The following properties hold for any bounded open subsets ω and ω' of \mathbb{R}^n :

- $\widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ is a linear space;
- $\widehat{H}^{1/2}(\omega; \mathbb{R}^m) \subseteq \widehat{H}^{1/2}(\omega'; \mathbb{R}^m)$ whenever $\omega' \subseteq \omega$, and $\mathcal{E}(v, \omega') \leq \mathcal{E}(v, \omega)$;
- $\widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap H_{\text{loc}}^{1/2}(\mathbb{R}^n; \mathbb{R}^m) \subseteq \widehat{H}^{1/2}(\omega'; \mathbb{R}^m)$;
- $H_{\text{loc}}^{1/2}(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n) \subseteq \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$.

An elementary property concerning $\widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ is given in Lemma 2.1 below whose proof can be found in Appendix A. Using this lemma, it is then elementary to see that $\widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ is a Hilbert space for the scalar product associated to the norm $v \mapsto \|v\|_{L^2(\omega)} + (\mathcal{E}(v, \omega))^{1/2}$.

Lemma 2.1. *Let $x_0 \in \omega$ and $\rho > 0$ be such that $D_\rho(x_0) \subseteq \omega$. There exists a constant $C_\rho > 0$, independent of x_0 , such that*

$$\int_{\mathbb{R}^n} \frac{|v(x)|^2}{(|x - x_0| + 1)^{n+1}} dx \leq C_\rho (\mathcal{E}(v, D_\rho(x_0)) + \|v\|_{L^2(D_\rho(x_0))}^2)$$

for all $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$.

Remark 2.2 (Exterior Dirichlet condition). We observe that for any $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$,

$$v + H_{00}^{1/2}(\omega; \mathbb{R}^m) \subseteq \widehat{H}^{1/2}(\omega; \mathbb{R}^m). \quad (2.3)$$

Moreover, if $v = g$ a.e. in $\mathbb{R}^n \setminus \omega$ for some $g \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$, then $v - g \in H_{00}^{1/2}(\omega; \mathbb{R}^m)$. As a consequence, the affine subspace

$$H_g^{1/2}(\omega; \mathbb{R}^m) := \left\{ v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) : v = g \text{ a.e. in } \mathbb{R}^n \setminus \omega \right\}$$

is characterized by

$$H_g^{1/2}(\omega; \mathbb{R}^m) = g + H_{00}^{1/2}(\omega; \mathbb{R}^m). \quad (2.4)$$

Finally, we shall keep in mind that

$$H_g^{1/2}(\omega; \mathbb{R}^m) \subseteq H_{\text{loc}}^{1/2}(\mathbb{R}^n; \mathbb{R}^m) \quad (2.5)$$

whenever $g \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap H_{\text{loc}}^{1/2}(\mathbb{R}^n; \mathbb{R}^m)$.

2.2. The fractional Laplacian. Let $\omega \subseteq \mathbb{R}^n$ be a bounded open set. We define the fractional 1/2-Laplacian $(-\Delta)^{\frac{1}{2}} : \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \rightarrow (\widehat{H}^{1/2}(\omega; \mathbb{R}^m))'$ as the continuous linear operator induced by the quadratic form $\mathcal{E}(\cdot, \omega)$. More precisely, for a given map $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$, we define the distribution $(-\Delta)^{\frac{1}{2}}v$ through its action on elements of $\widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ by setting

$$\begin{aligned} \langle (-\Delta)^{\frac{1}{2}}v, \varphi \rangle_\omega &:= \frac{\gamma_n}{2} \iint_{\omega \times \omega} \frac{(v(x) - v(y)) \cdot (\varphi(x) - \varphi(y))}{|x - y|^{n+1}} dx dy \\ &\quad + \gamma_n \iint_{\omega \times \omega^c} \frac{(v(x) - v(y)) \cdot (\varphi(x) - \varphi(y))}{|x - y|^{n+1}} dx dy. \end{aligned} \quad (2.6)$$

Note that for every $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$, the restriction of $(-\Delta)^{\frac{1}{2}}v$ to $H_{00}^{1/2}(\omega; \mathbb{R}^m)$ belongs to $H^{-1/2}(\omega; \mathbb{R}^m)$ with the estimate

$$\|(-\Delta)^{\frac{1}{2}}v\|_{H^{-1/2}(\omega)}^2 \leq 2\mathcal{E}(v, \omega), \quad (2.7)$$

which obviously follows from (2.1) and Cauchy-Schwarz Inequality. In addition, formula (2.6) being precisely twice the *symmetric bilinear form* associated to $\mathcal{E}(\cdot, \omega)$, we observe that the restriction of $(-\Delta)^{\frac{1}{2}}v$ to $H_{00}^{1/2}(\omega; \mathbb{R}^m)$ corresponds to the first variation of the energy $\mathcal{E}(v, \omega)$ with respect to perturbations supported in ω , *i.e.*,

$$\langle (-\Delta)^{\frac{1}{2}}v, \varphi \rangle_\omega = \left[\frac{d}{dt} \mathcal{E}(v + t\varphi, \omega) \right]_{t=0} \quad (2.8)$$

for all $\varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m)$.

Remark 2.3. Consider an open set $\omega' \subseteq \omega$. Since $\widehat{H}^{1/2}(\omega; \mathbb{R}^m) \subseteq \widehat{H}^{1/2}(\omega'; \mathbb{R}^m)$ and $H_{00}^{1/2}(\omega'; \mathbb{R}^m) \subseteq H_{00}^{1/2}(\omega; \mathbb{R}^m)$, direct computations from (2.6) yield

$$\langle (-\Delta)^{\frac{1}{2}}v, \varphi \rangle_\omega = \langle (-\Delta)^{\frac{1}{2}}v, \varphi \rangle_{\omega'}$$

for all $\varphi \in H_{00}^{1/2}(\omega'; \mathbb{R}^m)$.

Remark 2.4. If v is a smooth bounded function, the distribution $(-\Delta)^{\frac{1}{2}}v$ can be rewritten from (2.6) as a pointwise defined function through the formula

$$(-\Delta)^{\frac{1}{2}}v(x) = \text{p.v.} \left(\gamma_n \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x - y|^{n+1}} dy \right) := \lim_{\delta \downarrow 0} \gamma_n \int_{\mathbb{R}^n \setminus D_\delta(x)} \frac{v(x) - v(y)}{|x - y|^{n+1}} dy.$$

In turn, this last formula can be written in a non-singular form, that is

$$(-\Delta)^{\frac{1}{2}}v(x) = \frac{\gamma_n}{2} \int_{\mathbb{R}^n} \frac{v(x) - v(x+h) - \nabla v(x) \cdot h}{|h|^{n+1}} dh.$$

or

$$(-\Delta)^{\frac{1}{2}}v(x) = \frac{\gamma_n}{2} \int_{\mathbb{R}^n} \frac{-v(x+h) + 2v(x) - v(x-h)}{|h|^{n+1}} dh.$$

Finally, as a motivation to the following subsection, we recall that $(-\Delta)^{\frac{1}{2}}$ is the infinitesimal generator of the Poisson Kernel, *i.e.*,

$$e^{-t(-\Delta)^{\frac{1}{2}}}(x, y) = \frac{\gamma_n t}{(|x - y|^2 + t^2)^{\frac{n+1}{2}}}, \quad t > 0,$$

see *e.g.* [48].

2.3. Harmonic extension and the Dirichlet-to-Neumann operator. Throughout the paper, for a measurable function v defined over \mathbb{R}^n , we shall denote by v^e its extension to the half-space \mathbb{R}_+^{n+1} given by the convolution of v with the Poisson kernel, *i.e.*,

$$v^e(x) := \gamma_n \int_{\mathbb{R}^n} \frac{x_{n+1} v(z)}{(|x' - z|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} dz, \quad (2.9)$$

where we write $x = (x', x_{n+1}) \in \mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$. Notice that v^e is well defined if v belongs to the Lebesgue space L^p over \mathbb{R}^n with respect to the finite measure

$$m := (1 + |z|)^{-(n+1)} dz \quad (2.10)$$

for some $1 \leq p \leq \infty$. In particular, v^e can be defined whenever $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ for some open bounded set $\omega \subseteq \mathbb{R}^n$ by Lemma 2.1. Moreover, if $v \in L^\infty(\mathbb{R}^n)$, then $v^e \in L^\infty(\mathbb{R}_+^{n+1})$ and

$$\|v^e\|_{L^\infty(\mathbb{R}_+^{n+1})} \leq \|v\|_{L^\infty(\mathbb{R}^n)}. \quad (2.11)$$

For an admissible function v , the extension v^e is harmonic in \mathbb{R}_+^{n+1} . In addition, it has a pointwise trace on $\partial\mathbb{R}_+^{n+1} = \mathbb{R}^n$ which is equal to v at every Lebesgue point. In other words, v^e solves

$$\begin{cases} \Delta v^e = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ v^e = v & \text{on } \mathbb{R}^n. \end{cases} \quad (2.12)$$

The operator $v \mapsto v^e$ enjoys some useful continuity properties. Among them, we shall use the following lemma which is proven in Appendix A.

Lemma 2.5. *For every $R > 0$, the linear operator $\mathfrak{P}_R : L^2(\mathbb{R}^n, m) \rightarrow L^2(B_R^+)$ defined by*

$$\mathfrak{P}_R(v) := v^e|_{B_R^+}, \quad (2.13)$$

is continuous.

If $v \in \dot{H}^{1/2}(\mathbb{R}^n; \mathbb{R}^m)$, it is well known that the harmonic extension v^e belongs to $\dot{H}^1(\mathbb{R}_+^{n+1}; \mathbb{R}^m)$, and the $H^{1/2}$ -seminorm of v can be computed from the Dirichlet energy of v^e (here $\dot{H}^{1/2}$ and \dot{H}^1 denote the homogeneous Sobolev spaces). Moreover, v^e turns out to be the extension of v of minimal energy. This result, summarized in Lemma 2.6 below, is a classical application of Fourier Transform (see *e.g.* [26]).

Lemma 2.6. *Let $v \in \dot{H}^{1/2}(\mathbb{R}^n; \mathbb{R}^m)$, and let v^e be its harmonic extension to \mathbb{R}_+^{n+1} given by (2.9). Then v^e belongs to $\dot{H}^1(\mathbb{R}_+^{n+1}; \mathbb{R}^m)$ and*

$$[v]_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 = \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} |\nabla v^e|^2 dx = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} |\nabla u|^2 dx : u \in \dot{H}^1(\mathbb{R}_+^{n+1}; \mathbb{R}^m), u = v \text{ on } \mathbb{R}^n \right\}.$$

For a bounded open set $\omega \subseteq \mathbb{R}^n$, we have the following analogous result whose proof is postponed to Appendix A. In the following statement and hereafter, we denote by $H_{\text{loc}}^1(\mathbb{R}_+^{n+1} \cup \omega; \mathbb{R}^m)$ the family of maps which belongs to $H^1(\Omega; \mathbb{R}^m)$ for every bounded open set $\Omega \subseteq \mathbb{R}_+^{n+1}$ such that $\overline{\Omega} \subseteq \mathbb{R}_+^{n+1} \cup \omega$.

Lemma 2.7. *Let $\omega \subseteq \mathbb{R}^n$ be a bounded open set. For every $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$, the harmonic extension v^e given by (2.9) belongs to $H_{\text{loc}}^1(\mathbb{R}_+^{n+1} \cup \omega; \mathbb{R}^m) \cap L_{\text{loc}}^2(\overline{\mathbb{R}_+^{n+1}})$. In addition, for every $x_0 \in \omega$, $R > 0$ and $\rho > 0$ such that $D_{3\rho}(x_0) \subseteq \omega$, there exist constant $C_{R,\rho} > 0$ and $C_R > 0$, independent of v and x_0 , such that*

$$\|v^e\|_{L^2(B_R^+(x_0))}^2 \leq C_{R,\rho} \left(\mathcal{E}(v, D_{2\rho}(x_0)) + \|v\|_{L^2(D_{2\rho}(x_0))}^2 \right),$$

and

$$\|\nabla v^e\|_{L^2(B_\rho^+(x_0))}^2 \leq C_\rho \left(\mathcal{E}(v, D_{2\rho}(x_0)) + \|v\|_{L^2(D_{2\rho}(x_0))}^2 \right).$$

Remark 2.8. By the previous lemma, for any $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap H_{\text{loc}}^1(\mathbb{R}^n; \mathbb{R}^m)$, the harmonic extension v^e belongs to $H_{\text{loc}}^1(\overline{\mathbb{R}_+^{n+1}}; \mathbb{R}^m)$, and for any $R > 0$,

$$\|v^e\|_{H^1(B_R^+)}^2 \leq C_R \left(\mathcal{E}(v, D_{2R}) + \|v\|_{L^2(D_{2R})}^2 \right).$$

If $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ for some bounded open set $\omega \subseteq \mathbb{R}^n$ with Lipschitz boundary, we now observe that v^e admits a distributional (exterior) normal derivative $\partial_\nu v^e$ on ω by its harmonicity in \mathbb{R}_+^{n+1} . More precisely, we define $\partial_\nu v^e$ through its action on $\varphi \in \mathcal{D}(\omega; \mathbb{R}^m)$ by

$$\left\langle \frac{\partial v^e}{\partial \nu}, \varphi \right\rangle := \int_{\mathbb{R}_+^{n+1}} \nabla v^e \cdot \nabla \Phi \, dx, \quad (2.14)$$

where Φ is any smooth extension of φ compactly supported in $\mathbb{R}_+^{n+1} \cup \omega$. Note that the right hand side of (2.14) is well defined by Lemma 2.7. Using the harmonicity of v^e and the divergence theorem, it is routine to check that the integral in (2.14) does not depend on the choice of the extension Φ , and that $\partial_\nu v^e$ coincides with the classical exterior normal derivative of v^e on $\omega \subseteq \partial \mathbb{R}_+^{n+1}$ whenever v is smooth. It turns out that $\partial_\nu v^e$ coincides with the distribution $(-\Delta)^{\frac{1}{2}} v$ defined in the previous subsection. Here again the proof is left to Appendix A.

Lemma 2.9. *Let $\omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. For every $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ we have*

$$\langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle_\omega = \left\langle \frac{\partial v^e}{\partial \nu}, \varphi \right\rangle \quad \text{for all } \varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m),$$

where v^e is the harmonic extension of v to \mathbb{R}_+^{n+1} given by (2.9).

Up to now, we have not said anything about the local counterpart of Lemma 2.6 concerning the minimality of v^e for a function v in $\widehat{H}^{1/2}(\omega; \mathbb{R}^m)$. This is the purpose of the following lemma inspired from [18, Lemma 7.2].

Corollary 2.10. *Let $\omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $\Omega \subseteq \mathbb{R}_+^{n+1}$ be an admissible bounded open set such that $\overline{\partial^0 \Omega} \subseteq \omega$. Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$, and let v^e be its harmonic extension to \mathbb{R}_+^{n+1} given by (2.9). Then,*

$$\frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2} \int_\Omega |\nabla v^e|^2 \, dx \geq \mathcal{E}(u, \omega) - \mathcal{E}(v, \omega) \quad (2.15)$$

for all $u \in H^1(\Omega; \mathbb{R}^m)$ such that $u - v^e$ is compactly supported in $\Omega \cup \partial^0 \Omega$. In the right hand side of (2.15), the trace of u on $\partial^0 \Omega$ is extended by v outside $\partial^0 \Omega$.

Proof. Let $u \in H^1(\Omega; \mathbb{R}^m)$ such that $u - v^e$ is compactly supported in $\Omega \cup \partial^0 \Omega$. We extend u by v^e outside Ω . Then $h := u - v^e \in H^1(\mathbb{R}_+^{n+1}; \mathbb{R}^m)$ and h is compactly supported in $\Omega \cup \partial^0 \Omega$. Hence $h|_{\mathbb{R}^n} \in H_{00}^{1/2}(\partial^0 \Omega; \mathbb{R}^m)$. Since $v \in \widehat{H}^{1/2}(\partial^0 \Omega; \mathbb{R}^m)$ we deduce from (2.3) that u admits a trace on \mathbb{R}^n which belongs to $\widehat{H}^{1/2}(\partial^0 \Omega; \mathbb{R}^m)$.

Using Lemma 2.6 and Lemma 2.9, we now estimate

$$\begin{aligned} \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2} \int_\Omega |\nabla v^e|^2 \, dx &= \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} |\nabla h|^2 \, dx + \int_{\mathbb{R}_+^{n+1}} \nabla v^e \cdot \nabla h \, dx \\ &= \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} |\nabla h|^2 \, dx + \langle (-\Delta)^{\frac{1}{2}} v, h|_{\mathbb{R}^n} \rangle_{\partial^0 \Omega} \\ &\geq [h|_{\mathbb{R}^n}]_{H^{1/2}(\mathbb{R}^n)}^2 + \langle (-\Delta)^{\frac{1}{2}} v, h|_{\mathbb{R}^n} \rangle_{\partial^0 \Omega} \\ &= \mathcal{E}(h|_{\mathbb{R}^n}, \partial^0 \Omega) + \langle (-\Delta)^{\frac{1}{2}} v, h|_{\mathbb{R}^n} \rangle_{\partial^0 \Omega}. \end{aligned} \quad (2.16)$$

Using the fact that $u|_{\mathbb{R}^n}, v \in \widehat{H}^{1/2}(\partial^0 \Omega; \mathbb{R}^m)$, we derive

$$\mathcal{E}(h|_{\mathbb{R}^n}, \partial^0 \Omega) = \mathcal{E}(u|_{\mathbb{R}^n}, \partial^0 \Omega) + \mathcal{E}(v, \partial^0 \Omega) - \langle (-\Delta)^{\frac{1}{2}} v, u|_{\mathbb{R}^n} \rangle_{\partial^0 \Omega}, \quad (2.17)$$

and

$$\langle (-\Delta)^{\frac{1}{2}} v, h|_{\mathbb{R}^n} \rangle_{\partial^0 \Omega} = \langle (-\Delta)^{\frac{1}{2}} v, u|_{\mathbb{R}^n} \rangle_{\partial^0 \Omega} - 2\mathcal{E}(v, \partial^0 \Omega). \quad (2.18)$$

Gathering (2.16)-(2.17)-(2.18) yields

$$\frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2} \int_\Omega |\nabla v^e|^2 \, dx \geq \mathcal{E}(u|_{\mathbb{R}^n}, \partial^0 \Omega) - \mathcal{E}(v, \partial^0 \Omega).$$

Since $u|_{\mathbb{R}^n} = v$ outside $\partial^0 \Omega$, we infer that

$$\mathcal{E}(u|_{\mathbb{R}^n}, \partial^0 \Omega) - \mathcal{E}(v, \partial^0 \Omega) = \mathcal{E}(u|_{\mathbb{R}^n}, \omega) - \mathcal{E}(v, \omega),$$

and the conclusion follows. \square

3. THE FRACTIONAL GINZBURG-LANDAU EQUATION

We consider for the entire section a bounded open set $\omega \subseteq \mathbb{R}^n$ with Lipschitz boundary. We are interested in weak solutions $v_\varepsilon \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ of the fractional Ginzburg-Landau equation

$$(-\Delta)^{\frac{1}{2}} v_\varepsilon = \frac{1}{\varepsilon} (1 - |v_\varepsilon|^2) v_\varepsilon \quad \text{in } \omega. \quad (3.1)$$

Here the notion of weak solution is understood in the duality sense according to the formulation (2.6) of the fractional Laplacian, *i.e.*,

$$\langle (-\Delta)^{\frac{1}{2}} v_\varepsilon, \varphi \rangle_\omega = \frac{1}{\varepsilon} \int_\omega (1 - |v_\varepsilon|^2) v_\varepsilon \cdot \varphi \, dx \quad \text{for all } \varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$$

(or equivalently for all $\varphi \in \mathcal{D}(\omega; \mathbb{R}^m)$). By (2.8), such solutions correspond to critical points in ω of the Ginzburg-Landau 1/2-energy

$$\mathcal{E}_\varepsilon(v, \omega) := \mathcal{E}(v, \omega) + \frac{1}{4\varepsilon} \int_\omega (1 - |v|^2)^2 \, dx.$$

In other words, we are interested in maps $v_\varepsilon \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ satisfying

$$\left[\frac{d}{dt} \mathcal{E}_\varepsilon(v_\varepsilon + t\varphi, \omega) \right]_{t=0} = 0 \quad \text{for all } \varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega).$$

Among all kinds of critical points are the *minimizers*. We say that $v_\varepsilon \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ is a minimizer of \mathcal{E}_ε in ω if

$$\mathcal{E}_\varepsilon(v_\varepsilon, \omega) \leq \mathcal{E}_\varepsilon(v_\varepsilon + \varphi, \omega) \quad \text{for all } \varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$$

(or equivalently for all $\varphi \in \mathcal{D}(\omega; \mathbb{R}^m)$). The most standard way to construct minimizing solutions (and in particular solutions of (3.1)) is to minimize $\mathcal{E}_\varepsilon(\cdot, \omega)$ under an exterior Dirichlet condition. More precisely, given a map $g \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$, one consider the minimization problem

$$\min \left\{ \mathcal{E}_\varepsilon(v, \omega) : v \in H_g^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega) \right\}, \quad (3.2)$$

whose resolution follows directly from the Direct Method of Calculus of Variations.

To investigate the qualitative properties of solutions of (3.1), we rely on the harmonic extension to the half space \mathbb{R}_+^{n+1} introduced in the previous section. According to Lemma 2.9, if $v_\varepsilon \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ is a weak solution of (3.1), then its harmonic extension v_ε^e given by (2.9) weakly solves

$$\begin{cases} \Delta v_\varepsilon^e = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial v_\varepsilon^e}{\partial \nu} = \frac{1}{\varepsilon} (1 - |v_\varepsilon^e|^2) v_\varepsilon^e & \text{on } \omega. \end{cases} \quad (3.3)$$

In view of Lemma 2.7 and (2.14), the weak sense for this system corresponds to

$$\int_{\mathbb{R}_+^{n+1}} \nabla v_\varepsilon^e \cdot \nabla \Phi \, dx = \frac{1}{\varepsilon} \int_\omega (1 - |v_\varepsilon^e|^2) v_\varepsilon^e \cdot \Phi \, d\mathcal{H}^n \quad (3.4)$$

for all $\Phi \in H^1(\mathbb{R}_+^{n+1}; \mathbb{R}^m) \cap L^4(\omega)$ compactly supported in $\mathbb{R}_+^{n+1} \cup \omega$. System (3.3) also has a variational structure. Indeed, considering an admissible bounded open set $\Omega \subseteq \mathbb{R}_+^{n+1}$ such that $\partial^0 \Omega \subseteq \omega$, and setting for $u \in H^1(\Omega; \mathbb{R}^m) \cap L^4(\partial^0 \Omega)$,

$$E_\varepsilon(u, \Omega) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{4\varepsilon} \int_{\partial^0 \Omega} (1 - |u|^2)^2 \, d\mathcal{H}^n, \quad (3.5)$$

the weak formulation (3.4) can be rephrased as

$$\left[\frac{d}{dt} E_\varepsilon(v_\varepsilon^e + t\Phi, \Omega) \right]_{t=0} = 0$$

for all $\Phi \in H^1(\Omega; \mathbb{R}^m) \cap L^4(\partial^0 \Omega)$ compactly supported in $\Omega \cup \partial^0 \Omega$. Hence v_ε^e is a critical point of E_ε in Ω for all admissible bounded open set $\Omega \subseteq \mathbb{R}_+^{n+1}$ such that $\partial^0 \Omega \subseteq \omega$. The energy E_ε is what we may refer to as *Ginzburg-Landau boundary energy*.

If v_ε turns out to be minimizing, we can transfer the minimality of v_ε to v_ε^e with the help of Corollary 2.10. In the following proposition, we say that $u \in H^1(\Omega; \mathbb{R}^m) \cap L^4(\partial^0 \Omega)$ is minimizer of E_ε in Ω if

$$E_\varepsilon(u, \Omega) \leq E_\varepsilon(u + \Phi, \Omega)$$

for all $\Phi \in H^1(\Omega; \mathbb{R}^m) \cap L^4(\partial^0\Omega)$ compactly supported in $\Omega \cup \partial^0\Omega$.

Proposition 3.1 (Minimality transfer). *Let $v_\varepsilon \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ be a minimizer of \mathcal{E}_ε in ω , and let v_ε^e be its harmonic extension to \mathbb{R}_+^{n+1} given by (2.9). Then v_ε^e is a minimizer of E_ε in every admissible bounded open set $\Omega \subseteq \mathbb{R}_+^{n+1}$ such that $\overline{\partial^0\Omega} \subseteq \omega$.*

Concerning regularity for system (3.1), it suffices to investigate the boundary regularity of the harmonic extension which satisfies (3.3). For this issue we can rely on the well known regularity theory for linear Neumann boundary value problems, see e.g. [36]. The following theorem deals with regularity in the interior of the free boundary, and a simple proof can be found in [17, Lemma 2.2] (which actually deals with scalar valued equations, but a quick inspection of the proof shows that the result still holds in the vectorial case).

Theorem 3.2. *Let $R > 0$, and let $u_\varepsilon \in H^1(B_{2R}^+; \mathbb{R}^m) \cap L^\infty(B_{2R}^+)$ be a weak solution of*

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } B_{2R}^+, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{1}{\varepsilon}(1 - |u_\varepsilon|^2)u_\varepsilon & \text{on } D_{2R}. \end{cases}$$

Then $u \in C^\infty(B_R^+ \cup D_R)$.

Applying Theorem 3.2 to system (3.3) yields interior regularity for *bounded* variational solutions of the fractional Ginzburg-Landau system (3.1).

Corollary 3.3. *Let $v_\varepsilon \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n)$ be a weak solution of (3.1). Then $v_\varepsilon \in C^\infty(\omega)$.*

Dealing with arbitrary critical points of $\mathcal{E}_\varepsilon(\cdot, \omega)$ in $H_g^{1/2}(\omega; \mathbb{R}^m)$ for a smooth domain ω and a smooth (exterior) Dirichlet data g , one may wonder if regularity holds across the boundary $\partial\omega$. If v_ε is such a critical point and v_ε^e is its harmonic extension, we end up with the mixed boundary value problem

$$\begin{cases} \Delta v_\varepsilon^e = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial v_\varepsilon^e}{\partial \nu} = \frac{1}{\varepsilon}(1 - |v_\varepsilon^e|^2)v_\varepsilon^e & \text{on } \omega, \\ v_\varepsilon^e = g & \text{on } \mathbb{R}^n \setminus \omega. \end{cases} \quad (3.6)$$

Even if we obviously expect regularity in the interior of ω , it might not hold across the edge $\partial\omega$, since solutions of linear mixed boundary value problems are usually not better than Hölder continuous (and $1/2$ is in general the best possible Hölder exponent, see e.g. [63] and the references therein). Hölder continuity for variational solutions of (non homogeneous) linear mixed boundary value problems follows from the general results of STAMPACCHIA [73, 74] (see also [21]), and an estimate on the order of regularity is given by the classical result of SHAMIR [69]. In our case, it is enough to prove an *a priori* global L^∞ -bound on v_ε to derive from [69] the regularity stated in the following theorem.

Theorem 3.4. *Assume that $\partial\omega$ is smooth. Let $g \in C^1(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n)$, and let $v_\varepsilon \in H_g^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ be a weak solution of (3.1). Then $v_\varepsilon \in C^\infty(\omega)$ and v_ε is α -Hölder continuous near $\partial\omega$ for every $\alpha \in (0, 1/2)$.*

As mentioned in the few lines above, the proof of Theorem 3.4 rely on the boundedness of v_ε . Indeed, once the L^∞ -bound on v_ε is obtained, the right hand side in the Neumann boundary equation of (3.6) remains bounded. This is enough to apply the regularity result of [69], and then deduce the Hölder continuity of v_ε^e (and thus of v_ε). The higher order regularity away from $\partial\omega$ follows from Theorem 3.2.

Lemma 3.5. *Let ω and g be as in Theorem 3.4. Let $v_\varepsilon \in H_g^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ be a weak solution of (3.1). Then $v_\varepsilon \in L^\infty(\mathbb{R}^n)$.*

Proof. Let us fix for the whole proof an admissible bounded open set $\Omega \subseteq \mathbb{R}_+^{n+1}$ with smooth boundary such that $\overline{\omega} \subseteq \partial^0\Omega$. We shall use an argument in the spirit of the proof by BREZIS & KATO for the standard Laplacian.

Step 1. By (2.5) and Remark 2.8, $v_\varepsilon^e \in H_{\text{loc}}^1(\overline{\mathbb{R}_+^{n+1}}; \mathbb{R}^m)$ and v_ε^e weakly solves (3.6). By standard elliptic regularity for the Dirichlet problem, we have $v_\varepsilon^e \in C^1(\overline{\mathbb{R}_+^{n+1}} \setminus \overline{\omega})$. Since $\text{dist}(\partial^+ \Omega, \overline{\omega}) > 0$, we have

$$M := \|v_\varepsilon^e\|_{L^\infty(\partial^+ \Omega)} + \|\nabla v_\varepsilon^e\|_{L^\infty(\partial^+ \Omega)} < \infty.$$

Let us now consider the scalar function

$$\eta := \sqrt{|v_\varepsilon^e|^2 + \lambda} \in H^1(\Omega) \cap L^4(\omega)$$

with $\lambda := \max(1, \|g\|_{L^\infty(\mathbb{R}^n \setminus \omega)}^2)$, and fix an arbitrary nonnegative function $\Phi \in C^\infty(\overline{\Omega})$ with compact support in $\Omega \cup \omega \cup \partial^+ \Omega$. By the chain-rule formula for Sobolev functions we have

$$\int_{\Omega} \nabla \eta \cdot \nabla \Phi \, dx = \sum_{j=1}^{n+1} \int_{\Omega} \frac{(v_\varepsilon^e \cdot \partial_j v_\varepsilon^e) \partial_j \Phi}{\sqrt{|v_\varepsilon^e|^2 + \lambda}} \, dx.$$

Since

$$\frac{v_\varepsilon^e}{\sqrt{|v_\varepsilon^e|^2 + \lambda}} \in H^1(\Omega; \mathbb{R}^m),$$

we deduce that

$$\int_{\Omega} \nabla \eta \cdot \nabla \Phi \, dx = \int_{\Omega} \nabla v_\varepsilon^e \cdot \nabla \left(\frac{\Phi v_\varepsilon^e}{\sqrt{|v_\varepsilon^e|^2 + \lambda}} \right) \, dx - \int_{\Omega} \left(\frac{|\nabla v_\varepsilon^e|^2}{\sqrt{|v_\varepsilon^e|^2 + \lambda}} - \sum_{j=1}^{n+1} \frac{|v_\varepsilon^e \cdot \partial_j v_\varepsilon^e|^2}{(|v_\varepsilon^e|^2 + \lambda)^{3/2}} \right) \Phi \, dx.$$

On the other hand $\Phi \geq 0$, so that

$$\int_{\Omega} \nabla \eta \cdot \nabla \Phi \, dx \leq \int_{\Omega} \nabla v_\varepsilon^e \cdot \nabla \left(\frac{\Phi v_\varepsilon^e}{\sqrt{|v_\varepsilon^e|^2 + \lambda}} \right) \, dx.$$

Using equation (3.6), we infer that

$$\int_{\Omega} \nabla \eta \cdot \nabla \Phi \, dx \leq \int_{\partial^+ \Omega} \left(v_\varepsilon \cdot \frac{\partial v_\varepsilon^e}{\partial \nu} \right) \frac{\Phi}{\sqrt{|v_\varepsilon^e|^2 + \lambda}} \, d\mathcal{H}^n + \frac{1}{\varepsilon} \int_{\omega} (1 - |v_\varepsilon^e|^2) \frac{|v_\varepsilon^e|^2 \Phi}{\sqrt{|v_\varepsilon^e|^2 + \lambda}} \, d\mathcal{H}^n,$$

whence

$$\int_{\Omega} \nabla \eta \cdot \nabla \Phi \, dx \leq \int_{\partial^+ \Omega} \left(v_\varepsilon \cdot \frac{\partial v_\varepsilon^e}{\partial \nu} \right) \frac{\Phi}{\eta} \, d\mathcal{H}^n - \frac{1}{\varepsilon} \int_{\omega} \frac{|v_\varepsilon^e|^2 (\eta + \sqrt{1 + \lambda})}{\eta} (\eta - \sqrt{1 + \lambda}) \Phi \, d\mathcal{H}^n. \quad (3.7)$$

Then we conclude by approximation that (3.7) actually holds for any nonnegative $\Phi \in H^1(\Omega) \cap L^4(\omega)$ satisfying $\Phi = 0$ \mathcal{H}^n -a.e. on $\partial^0 \Omega \setminus \omega$.

Step 2. Given $T > 0$ and $\beta > 0$, we define

$$\rho := \max\{\eta - \sqrt{2\lambda}, 0\}, \quad \rho_T := \min(\rho, T), \quad \Psi_{T,\beta} := \rho_T^\beta \rho, \quad \Phi_{T,\beta} := \rho_T^{2\beta} \rho.$$

Those functions belong to $H^1(\Omega) \cap L^4(\omega)$ and vanish on $\partial^0 \Omega \setminus \omega$. Setting $\Omega_T := \{0 < \rho < T\} \cap \Omega$, straightforward computations yield

$$\int_{\Omega} |\nabla \Psi_{T,\beta}|^2 \, dx = \int_{\Omega} \rho_T^{2\beta} |\nabla \eta|^2 \, dx + (\beta^2 + 2\beta) \int_{\Omega_T} \rho^{2\beta} |\nabla \eta|^2 \, dx,$$

and

$$\int_{\Omega} \nabla \eta \cdot \nabla \Phi_{T,\beta} \, dx = \int_{\Omega} \rho_T^{2\beta} |\nabla \eta|^2 \, dx + 2\beta \int_{\Omega_T} \rho^{2\beta} |\nabla \eta|^2 \, dx.$$

From this two equalities we now infer that

$$\int_{\Omega} |\nabla \Psi_{T,\beta}|^2 \, dx \leq (\beta + 1) \int_{\Omega} \nabla \eta \cdot \nabla \Phi_{T,\beta} \, dx.$$

Using $\Phi_{T,\beta}$ as a test function in (3.7), we deduce that

$$\int_{\Omega} |\nabla \Psi_{T,\beta}|^2 \, dx \leq (\beta + 1) \int_{\partial^+ \Omega} \left(v_\varepsilon \cdot \frac{\partial v_\varepsilon^e}{\partial \nu} \right) \frac{\rho_T^{2\beta} \rho}{\eta} \, d\mathcal{H}^n - \frac{\beta + 1}{\varepsilon} \int_{\omega} \frac{|v_\varepsilon^e|^2 (\eta + \sqrt{1 + \lambda})}{\eta} \rho_T^{2\beta} \rho^2 \, d\mathcal{H}^n,$$

which leads to

$$\int_{\Omega} \left| \nabla \left(\rho_T^\beta \rho \right) \right|^2 \, dx \leq (\beta + 1) \mathcal{H}^n(\partial^+ \Omega) M^{2\beta+2}.$$

Applying the Poincaré Inequality in [80, Corollary 4.5.2] to $\rho_T^\beta \rho$ yields

$$\int_{\partial \Omega} |\rho_T^\beta \rho|^2 \, dx \leq C_{\Omega,\omega} (\beta + 1) M^{2\beta+2},$$

for a constant $C_{\Omega, \omega} > 0$ which only depends on Ω and ω . Next we let $T \rightarrow \infty$ in this last inequality to obtain

$$\|\rho\|_{L^{2(\beta+1)}(\partial\Omega)} \leq C_{\Omega, \omega}^{1/(2\beta+2)} (\beta+1)^{1/(2\beta+2)} M.$$

Letting now $\beta \rightarrow \infty$ leads to $\|\rho\|_{L^\infty(\partial\Omega)} \leq M$, which in turn implies $v_\varepsilon \in L^\infty(\omega)$. Since $v_\varepsilon = g$ outside ω , we have thus proved that $v_\varepsilon \in L^\infty(\mathbb{R}^n)$. \square

We now deduce from the maximum principle the following upper bound on the modulus of a critical point v_ε satisfying an exterior Dirichlet condition.

Corollary 3.6. *Let ω and g be as in Theorem 3.4. Let $v_\varepsilon \in H_g^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ be a weak solution of (3.1). Then $\|v_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leq \max(1, \|g\|_{L^\infty(\mathbb{R}^n \setminus \omega)})$.*

Proof. By Theorem 3.4, v_ε^e is smooth in $\mathbb{R}^{n+1} \cup \omega$ and continuous in $\overline{\mathbb{R}_+^{n+1}}$. We consider the function $m_\varepsilon := \lambda^2 - |v_\varepsilon^e|^2$ with $\lambda := \max(1, \|g\|_{L^\infty(\mathbb{R}^n \setminus \omega)})$. Then m_ε satisfies

$$\begin{cases} -\Delta m_\varepsilon = 2|\nabla v_\varepsilon^e|^2 \geq 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial m_\varepsilon}{\partial \nu} = -\frac{2}{\varepsilon}|v_\varepsilon^e|^2(m_\varepsilon + 1 - \lambda^2) & \text{on } \omega, \\ m_\varepsilon \geq 0 & \text{on } \mathbb{R}^n \setminus \omega. \end{cases}$$

Assume that m_ε achieves its minimum over \mathbb{R}^n at a point $x_0 \in \omega$. Then x_0 is a point of maximum of $|v_\varepsilon|$, and hence x_0 is an absolute minima of m_ε over the whole half space $\overline{\mathbb{R}_+^{n+1}}$ by (2.11). If $m_\varepsilon(x_0) < 0$ we obtain $\partial_\nu m_\varepsilon(x_0) > 0$. On the other hand, by the strong maximum principle and the Hopf boundary lemma, we have $\partial_\nu m_\varepsilon(x_0) < 0$ which leads to contradiction. \square

Remark 3.7. In the case where the exterior condition g satisfies $\|g\|_{L^\infty(\mathbb{R}^n \setminus \omega)} \leq 1$, Corollary 3.6 provides the estimate $|v_\varepsilon| \leq 1$ which is very standard for the usual (local) Ginzburg-Landau equation. We emphasize that, in this case, we have $|v_\varepsilon^e| \leq 1$.

4. 1/2-HARMONIC MAPS: DEFINITIONS, REGULARITY, AND EXAMPLES

4.1. Definitions and regularity theory. In this subsection we assume that $\omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary. We start introducing the concept of *1/2-harmonic map in ω with values into the sphere \mathbb{S}^{m-1}* as well as the related notion of *boundary harmonic map*. We shall then discuss their regularity in view of the existing theory for classical harmonic maps. To simplify slightly the presentation, we made the choice to focus on the case where the target manifold is a sphere. However definitions and results extend to more general compact manifolds without boundary, see Remark 4.24 at the end of this subsection.

Definition 4.1. Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ be such that $|v| = 1$ a.e. in ω . We say that v is *weakly 1/2-harmonic into \mathbb{S}^{m-1} in ω* if

$$\left[\frac{d}{dt} \mathcal{E} \left(\frac{v + t\varphi}{|v + t\varphi|}, \omega \right) \right]_{t=0} = 0 \quad (4.1)$$

for all $\varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m) \cap L^\infty(\omega)$ compactly supported in ω .

Writing explicitly the criticality condition (4.1), we find the variational formulation of the Euler-Lagrange equation for 1/2-harmonic into \mathbb{S}^{m-1} as stated in the following proposition.

Proposition 4.2. *Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ be such that $|v| = 1$ a.e. in ω . Then v is a weak 1/2-harmonic into \mathbb{S}^{m-1} in ω if and only if*

$$\langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle_\omega = 0 \quad (4.2)$$

for all $\varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m)$ satisfying

$$\varphi(x) \in T_{v(x)}\mathbb{S}^{m-1} \quad \text{a.e. in } \omega.$$

Proof. Step 1. We start proving that (4.2) implies (4.1). By the density of compactly supported smooth functions stated in (2.2), it is enough to consider the case where $\varphi \in \mathcal{D}(\omega; \mathbb{R}^m)$. Then we observe that

$(\varphi \cdot v)v \in H_{00}^{1/2}(\omega; \mathbb{R}^m)$. Hence $\psi := \varphi - (\varphi \cdot v)v$ belongs to $H_{00}^{1/2}(\omega; \mathbb{R}^m)$, and it satisfies $\psi \cdot v = 0$ a.e. in ω . Noticing that

$$\frac{v(x) + t\varphi(x)}{|v(x) + t\varphi(x)|} = v(x) + t\psi(x) + O(t^2) \quad \text{as } t \rightarrow 0,$$

we obtain by dominated convergence and assuming (4.2) that

$$\left[\frac{d}{dt} \mathcal{E} \left(\frac{v + t\varphi}{|v + t\varphi|}, \omega \right) \right]_{t=0} = \langle (-\Delta)^{\frac{1}{2}} v, \psi \rangle_{\omega} = 0,$$

and this first step is complete.

Step 2. We now prove the reverse implication. Let $\varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m)$ be such that $\varphi \cdot v = 0$ a.e. in ω . By standard truncations and cut-off arguments, we may assume without loss of generality that φ is compactly supported in ω and that $\varphi \in L^{\infty}(\omega)$. Then,

$$\frac{v(x) + t\varphi(x)}{|v(x) + t\varphi(x)|} = \frac{v(x) + t\varphi(x)}{\sqrt{1 + t^2|\varphi(x)|^2}} = v(x) + t\varphi(x) + O(t^2) \quad \text{as } t \rightarrow 0,$$

and we deduce again by dominated convergence and assumption (4.1) that

$$0 = \left[\frac{d}{dt} \mathcal{E} \left(\frac{v + t\varphi}{|v + t\varphi|}, \omega \right) \right]_{t=0} = \langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle_{\omega},$$

which ends the proof. \square

Remark 4.3. Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ be a weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω . In view of Proposition 4.2, the Euler-Lagrange equation can be written in the form

$$(-\Delta)^{\frac{1}{2}} v \perp T_v \mathbb{S}^{m-1} \quad \text{in } H^{-1/2}(\omega), \quad (4.3)$$

which is the clear analogue of the classical (weak) harmonic map system. Of course this equation is not completely explicit, but one can rewrite it computing the Lagrange multiplier associated to the constraint. In the case where $|v| = 1$ a.e. in all of \mathbb{R}^n , it takes a quite simple form. Indeed, for $\varphi \in \mathcal{D}(\omega; \mathbb{R}^m)$, Proposition 4.2 yields

$$\langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle_{\omega} = \langle (-\Delta)^{\frac{1}{2}} v, (\varphi \cdot v)v \rangle_{\omega}.$$

Using the representation (2.6) and the pointwise identity (based on the fact that $|v| = 1$),

$$(v(x) - v(y)) \cdot ((\varphi(x) \cdot v(x))v(x) - (\varphi(y) \cdot v(y))v(y)) = \frac{1}{2} |v(x) - v(y)|^2 (\varphi(x) \cdot v(x) + \varphi(y) \cdot v(y)),$$

we obtain

$$\langle (-\Delta)^{\frac{1}{2}} v, (\varphi \cdot v)v \rangle_{\omega} = \frac{\gamma_n}{2} \iint_{\omega \times \mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} v(x) \cdot \varphi(x) \, dx dy.$$

The identity above then leads to

$$(-\Delta)^{\frac{1}{2}} v(x) = \left(\frac{\gamma_n}{2} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} \, dy \right) v(x) \quad \text{in } \mathcal{D}'(\omega),$$

which is again in clear analogy with the standard harmonic map system into a sphere.

We now introduce what we call \mathbb{S}^{m-1} -boundary harmonic maps. Those maps correspond to critical points of the Dirichlet energy under the constraint to be \mathbb{S}^{m-1} -valued on a prescribed portion of the boundary.

Definition 4.4. Let $\Omega \subseteq \mathbb{R}_+^{n+1}$ be an admissible bounded open set, and let $u \in H^1(\Omega; \mathbb{R}^m)$ be such that $|u| = 1$ \mathcal{H}^n -a.e. on $\partial^0 \Omega$. We say that u is weakly harmonic in Ω with respect to the (partially) free boundary condition $u(\partial^0 \Omega) \subseteq \mathbb{S}^{m-1}$ if

$$\int_{\Omega} \nabla u \cdot \nabla \Phi \, dx = 0 \quad (4.4)$$

for all $\Phi \in H^1(\Omega; \mathbb{R}^m) \cap L^{\infty}(\Omega)$ compactly supported $\Omega \cup \partial^0 \Omega$ and satisfying

$$\Phi(x) \in T_{u(x)} \mathbb{S}^{m-1} \quad \mathcal{H}^n\text{-a.e. on } \partial^0 \Omega. \quad (4.5)$$

In short, we shall say that u is a weak $(\mathbb{S}^{m-1}, \partial^0 \Omega)$ -boundary harmonic map in Ω .

Defintion 4.4 can be motivated by the fact that

$$\left[\frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} |\nabla u_t|^2 dx \right) \right]_{t=0} = \int_{\Omega} \nabla u \cdot \nabla \Phi dx$$

for variations u_t of the form

$$u_t := \frac{u + t\Phi}{\sqrt{1 + t^2|\Phi|^2}},$$

with Φ satisfying (4.5). Such variations u_t belong to $H^1(\Omega; \mathbb{R}^m)$, and they satisfy the constraint $|u_t| = 1$ \mathcal{H}^n -a.e. on $\partial^0\Omega$ by (4.5). Actually, if $u \in H^1(\Omega; \mathbb{R}^m)$ is a weak $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic map in Ω , a standard truncation argument shows that (4.4) holds for any $\Phi \in H^1(\Omega; \mathbb{R}^m)$ with compact support support in $\Omega \cup \partial^0\Omega$ and satisfying (4.5). Moreover, choosing Φ with compact support in Ω in formula (4.4) shows that $\Delta u = 0$ in Ω . Integrating by parts in (4.4) then yields

$$\int_{\partial^0\Omega} \frac{\partial u}{\partial \nu} \cdot \Phi d\mathcal{H}^n = 0$$

for all smooth functions Φ compactly supported in $\Omega \cup \partial^0\Omega$ and satisfying (4.5). In general the integral above has of course to be understood in the $H^{-1/2} - H_{00}^{1/2}$ duality sense. Therefore, the weak $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic map system can be reformulated as

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} \perp T_u \mathbb{S}^{m-1} & \text{in } H^{-1/2}(\partial^0\Omega). \end{cases} \quad (4.6)$$

Remark 4.5 (Harmonic maps with partially free boundary). In view of the discussion above, let us mention that $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic maps belong to a larger class of harmonic maps known in the literature as *harmonic maps with partially free boundary*, see [5, 28, 29, 30, 41, 42, 43, 66] and references therein. In most studies, one considers a smooth compact Riemannian manifold \mathcal{M} without boundary (that we can assume to be isometrically embedded in some Euclidean space by the Nash embedding Theorem), and \mathcal{N} a smooth closed submanifold of \mathcal{M} . The boundary portion $\partial^0\Omega$ is called *the partially free boundary*, and \mathcal{N} is *the supporting manifold*. Then \mathcal{M} -valued (weak) harmonic maps in Ω with the partially free boundary condition $u(\partial^0\Omega) \subseteq \mathcal{N}$ are defined as critical points of the Dirichlet energy under the constraints $u(x) \in \mathcal{M}$ for a.e. $x \in \Omega$ and $u(x) \in \mathcal{N}$ for \mathcal{H}^n -a.e. $x \in \partial^0\Omega$. For $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic maps, we may consider the submanifold $\mathcal{N} = \mathbb{S}^{m-1}$ of the target $\mathcal{M} = \mathbb{R}^m$. However, to apply known results on harmonic maps, the compactness of \mathcal{M} is usually required. Similarly to [57, Sec. 3], a way to avoid this problem is to consider *bounded* $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic maps, noticing that \mathbb{S}^{m-1} can be viewed as a submanifold of a flat torus $\mathcal{M} = \mathbb{R}^m/r\mathbb{Z}^m$ with factor $r > 2$. Indeed, if u is a $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic map in Ω satisfying $\|u\|_{L^\infty(\Omega)} < r/2$, then it can be considered as a \mathcal{M} -valued harmonic map. We finally point out that such L^∞ -bound is not too restrictive in many applications. For instance, if we consider an additional Dirichlet boundary condition $u = g$ on $\partial^+\Omega$ with $g : \partial^+\Omega \rightarrow \mathbb{S}^{m-1}$ smooth, then we usually obtain $\|u\|_{L^\infty(\Omega)} \leq 1$ through some maximum principle.

As a consequence of (2.14), Lemma 2.9, and Proposition 4.2, we can relate 1/2-harmonic maps into \mathbb{S}^{m-1} to $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic maps, as we already did for the fractional Ginzburg-Landau equation.

Proposition 4.6 (Criticality transfer). *Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ be a weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω , and let v^e be its harmonic extension to \mathbb{R}_+^{n+1} given by (2.9). Then v^e is a weak $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic map in Ω for every admissible bounded open set $\Omega \subseteq \mathbb{R}_+^{n+1}$ such that $\partial^0\Omega \subseteq \omega$.*

For what concerns regularity, let us emphasize that a full regularity result cannot hold in general neither for weakly 1/2-harmonic maps into \mathbb{S}^{m-1} nor for weak $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic maps due to topological constraints. In the case of boundary harmonic maps, the regularity issue is of course the regularity at the free boundary $\partial^0\Omega$, since the system is the usual Laplace equation in Ω (so that boundary harmonic maps are smooth in the interior of the domain). At the end of the next subsection, we shall give examples of weakly 1/2-harmonic maps from \mathbb{R}^2 into \mathbb{S}^1 which are not continuous at the origin (see Proposition 4.30), the prototypical one being $v(x) = \frac{x}{|x|}$. Obviously, such examples do not exclude

some partial regularity to hold, but dealing with general weak solutions we actually do not expect any reasonable kind of regularity by analogy with the classical harmonic map equation into a manifold and the famous counterexample of RIVIÈRE [60] (of a weak harmonic map from the three dimensional ball into \mathbb{S}^2 which is everywhere discontinuous). In the context of 1/2-harmonic maps into spheres, we are not aware of an analogous counterexample to regularity, but we believe that it should exist. As it is the case for harmonic maps into a manifold, it is then reasonable to ask for an extra assumption on a weak 1/2-harmonic map to derive at least partial regularity. The (usual) assumption we make is either *energy minimality* or *stationarity*, i.e., criticality under inner variations. We now recall these notions for 1/2-harmonic maps and for boundary harmonic maps.

Definition 4.7. Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ be such that $|v| = 1$ a.e. in ω . We say that v is a *minimizing 1/2-harmonic map into \mathbb{S}^{m-1}* in ω if

$$\mathcal{E}(v, \omega) \leq \mathcal{E}(\tilde{v}, \omega)$$

for all $\tilde{v} \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ such that $|\tilde{v}| = 1$ a.e. in ω , and $\tilde{v} - v$ is compactly supported in ω .

Definition 4.8. Let $\Omega \subseteq \mathbb{R}_+^{n+1}$ be an admissible bounded open set, and let $u \in H^1(\Omega; \mathbb{R}^m)$ be such that $|u| = 1$ \mathcal{H}^n -a.e. on $\partial^0\Omega$. We say that u is a *minimizing harmonic map in Ω with respect to the (partially) free boundary condition $u(\partial^0\Omega) \subseteq \mathbb{S}^{m-1}$* if

$$\frac{1}{2} \int_{\Omega} |\nabla u|^2 dx \leq \frac{1}{2} \int_{\Omega} |\nabla \tilde{u}|^2 dx$$

for all $\tilde{u} \in H^1(\Omega; \mathbb{R}^m)$ such that $|\tilde{u}| = 1$ \mathcal{H}^n -a.e. on $\partial^0\Omega$, and $\tilde{u} - u$ is compactly supported in $\Omega \cup \partial^0\Omega$. In short, we shall say that u is a *minimizing $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic map* in Ω .

Obviously, if $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ is a minimizing 1/2-harmonic map into \mathbb{S}^{m-1} in ω , then v is a weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω . In the same way, minimizing $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic maps are weak $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic maps. As already pursued for the fractional Ginzburg-Landau equation, the minimality of a 1/2-harmonic map can be transferred to its harmonic extension with the help of Corollary 2.10.

Proposition 4.9 (Minimality transfer). Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ be a minimizing 1/2-harmonic map into \mathbb{S}^{m-1} in ω . Let v^e be the harmonic extension of v in \mathbb{R}_+^{n+1} given by (2.9). Then v^e is a minimizing $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic map in Ω for every admissible bounded open set $\Omega \subseteq \mathbb{R}_+^{n+1}$ such that $\overline{\partial^0\Omega} \subseteq \omega$.

As mentioned above, the regularity theory for harmonic maps also deals with stationary harmonic maps. For boundary harmonic maps, the stationarity criteria has to allow inner variations up to the partially free boundary $\partial^0\Omega$, and it leads to the following definition.

Definition 4.10. Let $\Omega \subseteq \mathbb{R}_+^{n+1}$ be an admissible bounded open set, and let $u \in H^1(\Omega; \mathbb{R}^m)$ be a weak $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic map in Ω . We say that u is *stationary* if

$$\left[\frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} |\nabla(u \circ \Phi_t)|^2 dx \right) \right]_{t=0} = 0 \quad (4.7)$$

for any differentiable 1-parameter group of smooth diffeomorphisms $\Phi_t : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ satisfying

- $\Phi_0 = \text{id}_{\mathbb{R}^{n+1}}$;
- $\Phi_t(\mathbb{R}^n) \subseteq \mathbb{R}^n$;
- $\Phi_t - \text{id}_{\mathbb{R}^{n+1}}$ is compactly supported, and $\text{supp}(\Phi_t - \text{id}_{\mathbb{R}^{n+1}}) \cap \overline{\mathbb{R}_+^{n+1}} \subseteq \Omega \cup \partial^0\Omega$.

Remark 4.11. It is well known that the usual stationarity definition can be recast in terms of the infinitesimal generator \mathbf{X} of the 1-parameter group $\{\Phi_t\}_{t \in \mathbb{R}}$ of diffeomorphisms (see e.g. [71, Chapter 2.2]). In the free boundary context, a map $u \in H^1(\Omega; \mathbb{R}^m)$ is stationary in Ω up to the free boundary $\partial^0\Omega$ if and only if

$$\int_{\Omega} \left(|\nabla u|^2 \text{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i u \cdot \partial_j u) \partial_j \mathbf{X}_i \right) dx = 0 \quad (4.8)$$

for all vector fields $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_{n+1}) \in C^1(\overline{\Omega}; \mathbb{R}^{n+1})$ compactly supported in $\Omega \cup \partial^0\Omega$ and satisfying $\mathbf{X}_{n+1} = 0$ on $\partial^0\Omega$ (the left hand side of (4.8) being exactly minus twice the value of the derivative in (4.7)).

Moreover, using the so-called *stress-energy tensor* $T = (T_i)_{i=1}^{n+1}$ given by

$$(T_i)_j(u) := |\nabla u|^2 \delta_{ij} - 2(\partial_i u \cdot \partial_j u), \quad 1 \leq j \leq n+1,$$

identity (4.8) can be rewritten as the following system of conservation laws

$$\begin{cases} \operatorname{div} T_i = 0 & \text{in } \Omega \text{ for each } i \in \{1, \dots, n+1\}, \\ T_i \cdot \nu = 0 & \text{on } \partial^0 \Omega \text{ for each } i \in \{1, \dots, n\}, \end{cases} \quad (4.9)$$

in the sense of distributions.

Remark 4.12. For any $u \in H^1(\Omega; \mathbb{R}^m)$ satisfying $\Delta u = 0$ in Ω , it turns out that the integral in the left hand side of (4.8) only depends on the trace of $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ on $\partial^0 \Omega$. Indeed, if u is harmonic in Ω , then we have $\operatorname{div} T = 0$ in Ω . Therefore, if \mathbf{X} vanishes on $\partial^0 \Omega$, integrating by parts this integral shows that it is equal to zero (no matter what the trace of u on $\partial^0 \Omega$ is).

Remark 4.13. Note that minimizing boundary harmonic maps are always stationary, but a stationary boundary harmonic map might not be minimizing. On the other hand, if a boundary harmonic map is smooth enough up to the free boundary $\partial^0 \Omega$, then it is stationary. Indeed, under such a smoothness assumption a simple computation shows that a solution of (4.6) satisfies (4.9).

Concerning the 1/2-Dirichlet energy, we define the notion of stationarity in the usual way through inner variations.

Definition 4.14. Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ be a weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω . We say that v is *stationary* in ω if

$$\left[\frac{d}{dt} \mathcal{E}(v \circ \phi_t, \omega) \right]_{t=0} = 0$$

for any differentiable 1-parameter group of smooth diffeomorphisms $\phi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying

- $\phi_0 = \operatorname{id}_{\mathbb{R}^n}$;
- $\phi_t - \operatorname{id}_{\mathbb{R}^n}$ is compactly supported, and $\operatorname{supp}(\phi_t - \operatorname{id}_{\mathbb{R}^n}) \subseteq \omega$.

Another consequence of Corollary 2.10 is that stationarity for 1/2-harmonic maps implies stationarity for harmonic extensions as stated in the following proposition.

Proposition 4.15 (Stationarity transfer). *Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ be a stationary weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω . Let v^e be the harmonic extension of v in \mathbb{R}_+^{n+1} given by (2.9). Then v^e is a stationary weak $(\mathbb{S}^{m-1}, \partial^0 \Omega)$ -boundary harmonic map in Ω for every admissible bounded open set $\Omega \subseteq \mathbb{R}_+^{n+1}$ such that $\overline{\partial^0 \Omega} \subseteq \omega$.*

As matter of fact, Proposition 4.15 is directly implied by the following lemma together with Remarks 4.11 & 4.12.

Lemma 4.16. *Let $\Omega \subseteq \mathbb{R}_+^{n+1}$ be an admissible bounded open set such that $\overline{\partial^0 \Omega} \subseteq \omega$. Given $X \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ compactly supported in $\partial^0 \Omega$, let $\{\phi_t\}_{t \in \mathbb{R}}$ be the flow generated by X . For each $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ we have*

$$\left[\frac{d}{dt} \mathcal{E}(v \circ \phi_t, \omega) \right]_{t=0} = -\frac{1}{2} \int_{\Omega} \left(|\nabla v^e|^2 \operatorname{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i v^e \cdot \partial_j v^e) \partial_j \mathbf{X}_i \right) dx,$$

where $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_{n+1}) \in C^1(\overline{\Omega}; \mathbb{R}^{n+1})$ is any vector field compactly supported in $\Omega \cup \partial^0 \Omega$ satisfying $\mathbf{X} = (X, 0)$ on $\partial^0 \Omega$.

Proof. Let $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_{n+1}) \in C^1(\overline{\Omega}; \mathbb{R}^{n+1})$ be an arbitrary vector field compactly supported in $\Omega \cup \partial^0 \Omega$ and satisfying $\mathbf{X} = (X, 0)$ on $\partial^0 \Omega$. Then consider a compactly supported C^1 -extension of \mathbf{X} to the whole space \mathbb{R}^{n+1} , still denoted by \mathbf{X} , such that $\mathbf{X} = (X, 0)$ on \mathbb{R}^n . We define $\{\Phi_t\}_{t \in \mathbb{R}}$ to be the flow generated by \mathbf{X} , i.e., for each $x \in \mathbb{R}^{n+1}$ the map $t \mapsto \Phi_t(x)$ is defined as the solution of

$$\begin{cases} \frac{d}{dt} \Phi_t(x) = \mathbf{X}(\Phi_t(x)), \\ \Phi_0(x) = x. \end{cases}$$

One can easily check that the family $\{\Phi_t\}_{t \in \mathbb{R}}$ is admissible for Definition 4.10. Noticing that $\Phi_t = (\phi_t, 0)$ on \mathbb{R}^n , we now infer from Corollary 2.10 that

$$\frac{1}{2} \int_{\Omega} |\nabla(v^e \circ \Phi_t)|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla v^e|^2 dx \geq \mathcal{E}(v \circ \phi_t, \omega) - \mathcal{E}(v, \omega).$$

Dividing both sides of this inequality by $t > 0$, and then letting $t \rightarrow 0$, we obtain

$$-\frac{1}{2} \int_{\Omega} \left(|\nabla v^e|^2 \operatorname{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i v^e \cdot \partial_j v^e) \partial_j \mathbf{X}_i \right) dx \geq \left[\frac{d}{dt} \mathcal{E}(v \circ \phi_t, \omega) \right]_{t=0}. \quad (4.10)$$

Here, straightforward computations yield

$$\begin{aligned} \left[\frac{d}{dt} \mathcal{E}(v \circ \phi_t, \omega) \right]_{t=0} &= \frac{(n+1)\gamma_n}{4} \iint_{\omega \times \omega} \frac{|v(x) - v(y)|^2 (x-y) \cdot (X(x) - X(y))}{|x-y|^{n+1} |x-y|^2} dx dy \\ &+ \frac{(n+1)\gamma_n}{2} \iint_{\omega \times \omega^c} \frac{|v(x) - v(y)|^2 (x-y) \cdot X(x)}{|x-y|^{n+1} |x-y|^2} dx dy \\ &- \frac{\gamma_n}{2} \iint_{\omega \times \mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+1}} \operatorname{div} X(x) dx dy. \end{aligned} \quad (4.11)$$

Since X was chosen arbitrary, inequality (4.10) holds with $(-X)$ and $(-\mathbf{X})$ instead of X and \mathbf{X} respectively, and we conclude that inequality (4.10) is in fact an equality. \square

Remark 4.17. From formula (4.11), one can actually deduce that any sufficiently smooth 1/2-harmonic map v in ω is stationary.

Proof. To see this, we write the first two terms in the right hand side of (4.11) as A and B , respectively. We have

$$A = \lim_{\delta \downarrow 0} \frac{\gamma_n}{2} \int_{\omega} \left(\int_{\omega \setminus D_{\delta}(y)} |v(x) - v(y)|^2 X(x) \cdot \nabla_x \left(\frac{-1}{|x-y|^{n+1}} \right) dx \right) dy =: \lim_{\delta \downarrow 0} A_{\delta}.$$

Since X is compactly supported in ω , we can write for $\delta > 0$ small enough

$$B = \frac{\gamma_n}{2} \int_{\omega^c} \left(\int_{\omega \setminus D_{\delta}(y)} |v(x) - v(y)|^2 X(x) \cdot \nabla_x \left(\frac{-1}{|x-y|^{n+1}} \right) dx \right) dy.$$

Given $y \in \mathbb{R}^n \setminus \partial\omega$ and $\delta > 0$ small enough, we integrate by parts to find that

$$\begin{aligned} &\int_{\omega \setminus D_{\delta}(y)} |v(x) - v(y)|^2 X(x) \cdot \nabla_x \left(\frac{-1}{|x-y|^{n+1}} \right) dx = \int_{\omega \setminus D_{\delta}(y)} \frac{|v(x) - v(y)|^2}{|x-y|^{n+1}} \operatorname{div} X(x) dx \\ &+ 2 \int_{\omega \setminus D_{\delta}(y)} \frac{v(x) - v(y)}{|x-y|^{n+1}} \cdot (X(x) \cdot \nabla v(x)) dx + \frac{\chi_{\omega}(y)}{\delta^{n+1}} \int_{\partial D_{\delta}(y)} |v(x) - v(y)|^2 \frac{(x-y) \cdot X(x)}{|x-y|} d\mathcal{H}_x^{n-1}, \end{aligned}$$

where χ_{ω} denotes the characteristic function of the set ω . Therefore, for δ small we have

$$\begin{aligned} A_{\delta} + B &= \frac{\gamma_n}{2} \int_{\mathbb{R}^n} \int_{\omega \setminus D_{\delta}(y)} \frac{|v(x) - v(y)|^2}{|x-y|^{n+1}} \operatorname{div} X(x) dx dy \\ &+ \gamma_n \int_{\mathbb{R}^n} \int_{\omega \setminus D_{\delta}(y)} \frac{v(x) - v(y)}{|x-y|^{n+1}} \cdot (X(x) \cdot \nabla v(x)) dx dy \\ &+ \frac{\gamma_n}{2\delta^{n+1}} \int_{\omega} \int_{\partial D_{\delta}(y)} |v(x) - v(y)|^2 \frac{(x-y) \cdot X(x)}{|x-y|} d\mathcal{H}_x^{n-1} dy =: I_{\delta} + II_{\delta} + III_{\delta}. \end{aligned} \quad (4.12)$$

Obviously,

$$\lim_{\delta \downarrow 0} I_{\delta} = \frac{\gamma_n}{2} \iint_{\omega \times \mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+1}} \operatorname{div} X(x) dx dy.$$

Then we rewrite

$$II_{\delta} = \gamma_n \int_{\omega} \left(\int_{\mathbb{R}^n \setminus D_{\delta}(x)} \frac{v(x) - v(y)}{|x-y|^{n+1}} dy \right) \cdot (X(x) \cdot \nabla v(x)) dx,$$

so that

$$\lim_{\delta \downarrow 0} II_{\delta} = \int_{\omega} (-\Delta)^{\frac{1}{2}} v(x) \cdot (X(x) \cdot \nabla v(x)) dx.$$

Finally, expanding v around y , we easily get that

$$\lim_{\delta \downarrow 0} III_{\delta} = \frac{\gamma_n}{2} \int_{\omega} \left(\int_{\mathbb{S}^{n-1}} |\sigma \cdot \nabla v(y)|^2 X(y) \cdot \sigma d\mathcal{H}_{\sigma}^{n-1} \right).$$

We claim that the integral above vanishes, *i.e.*, $\lim_{\delta} III_{\delta} = 0$. Indeed, write for $y \in \omega$ fixed,

$$\int_{\mathbb{S}^{n-1}} |\sigma \cdot \nabla v(y)|^2 X(y) \cdot \sigma \, d\mathcal{H}_{\sigma}^{n-1} = \sum_{i=1}^m \int_{\mathbb{S}^{n-1}} |\vec{a}_i \cdot \sigma|^2 \vec{b} \cdot \sigma \, d\mathcal{H}_{\sigma}^{n-1},$$

where we have set $\vec{a}_i := \nabla v_i(y)$ and $\vec{b} := X(y)$. Given $i \in \{1, \dots, m\}$, we can assume that \vec{a}_i and \vec{b} belongs to $\mathbb{R}^2 \times \{(0, \dots, 0)\}$ by invariance under rotation. Then,

$$\int_{\mathbb{S}^{n-1}} |\vec{a}_i \cdot \sigma|^2 \vec{b} \cdot \sigma \, d\mathcal{H}_{\sigma}^{n-1} = C_n \int_{\mathbb{S}^1} |\vec{a}_i \cdot \sigma|^2 \vec{b} \cdot \sigma \, d\mathcal{H}_{\sigma}^1,$$

for a dimensional constant C_n . Now observe that the function in the right hand side is a homogeneous polynomial in σ of degree 3. Hence its integral over \mathbb{S}^1 vanishes, and the claim is proved.

Gathering (4.11) with (4.12), we have thus shown that

$$\left[\frac{d}{dt} \mathcal{E}(v \circ \phi_t, \omega) \right]_{t=0} = \int_{\omega} (-\Delta)^{\frac{1}{2}} v(x) \cdot (X(x) \cdot \nabla v(x)) \, dx. \quad (4.13)$$

Finally, since $v(x) \in \mathbb{S}^{n-1}$ for $x \in \omega$, we have $\partial_j v(x) \in T_{v(x)} \mathbb{S}^{n-1}$ for $x \in \omega$ and $j = 1, \dots, n$. Then the Euler-Lagrange equation (4.3) yields $(-\Delta)^{\frac{1}{2}} v \cdot \partial_j v = 0$ in ω for $j = 1, \dots, n$. Therefore the function under the integral in (4.13) vanishes in ω , which shows that v is stationary. \square

Going back to our discussion on the regularity of harmonic maps, we now mention that there is only one case where full (interior) regularity holds for general critical points. It is the case where the starting dimension equals two (the conformal dimension), and this is a well known result due to HÉLEIN [44]. In higher dimensions, the (optimal) partial regularity result for minimizing harmonic maps has been obtained in the pioneering work of SCHOEN & UHLENBECK [67]. It has then been extended to stationary harmonic maps by BETHUEL [8] (see also [31]). All this results have an analogue in the context of harmonic maps with partially free boundary, where the new issue is of course to determine the partial regularity at the relative interior of the free boundary $\partial^0 \Omega$. Higher order regularity starting from continuous solutions has been studied and proved by GULLIVER & JOST [41], see also [5, 42]. The minimizing case has been handled independently by HARDT & LIN [43] and DUZAAR & STEFFEN [29, 30], while full regularity in the conformal dimension and partial regularity under the stationarity condition has been more recently proved by SCHEVEN [66, Theorem 2.2, Theorem 4.1]. All these results on partial regularity essentially deals with an estimate on the Hausdorff dimension of the so-called *singular set*. The singular set $\text{sing}(u)$ of a map u is usually defined as the complement of the largest (relative) open set on which u is continuous. It is therefore a relatively closed subset of the domain where u is defined. For a harmonic map with partially free boundary it is then a relatively closed subset of $\Omega \cup \partial^0 \Omega$, and in case of a $(\mathbb{S}^{m-1}, \partial^0 \Omega)$ -boundary harmonic map it is a relatively closed subset of $\partial^0 \Omega$ (since regularity holds in Ω). In view of Remark 4.5, we can obtain a partial regularity theory for *bounded* \mathbb{S}^{m-1} -boundary harmonic maps from the known results mentioned above about harmonic maps with partially free boundary. In the statements below, $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

Theorem 4.18 ([29, 30, 41, 43, 66]). *Let $\Omega \subseteq \mathbb{R}_+^{n+1}$ be an admissible bounded open set. If $u \in H^1(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega)$ is a weak $(\mathbb{S}^{m-1}, \partial^0 \Omega)$ -boundary harmonic map in Ω , then $u \in C^\infty((\Omega \cup \partial^0 \Omega) \setminus \text{sing}(u))$. Moreover,*

- (i) *if $n = 1$, then $\text{sing}(u) = \emptyset$;*
- (ii) *if $n \geq 2$ and u is stationary, then $\mathcal{H}^{n-1}(\text{sing}(u)) = 0$;*
- (iii) *if u is minimizing, then $\dim_{\mathcal{H}}(\text{sing}(u)) \leq n - 2$ for $n \geq 3$, and $\text{sing}(u)$ is discrete for $n = 2$.*

Proof. As indicated in Remark 4.5, the sphere \mathbb{S}^{m-1} can be considered as a submanifold of a flat torus $\mathcal{M} := \mathbb{R}^m / r\mathbb{Z}^m$ with $r > 2\|u\|_{L^\infty(\Omega)}$. Then u is a weakly harmonic map in Ω into the compact (boundaryless) manifold \mathcal{M} satisfying the partially free boundary condition $u(\partial^0 \Omega) \subseteq \mathbb{S}^{m-1}$. Items (i) and (ii) follow from [66, Theorem 2.2, Theorem 4.1] (note that for $n = 1$, stationarity is not required since the conclusion of [66, Corollary 1.3] trivially holds in this case), while item (iii) is proved in [29, 30, 43]. Finally, the higher order regularity away from the singular set is obtained from [41, Section 4]. \square

Thanks to Theorem 4.18, we can now deduce the partial regularity theory for *bounded* 1/2-harmonic maps into \mathbb{S}^{m-1} .

Proof of Theorem 1.2. Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n)$ be a weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω , and let $\Omega \subseteq \mathbb{R}_+^{n+1}$ be an arbitrary admissible bounded open set such that $\overline{\partial^0\Omega} \subseteq \omega$. By Proposition 4.6, v^e is a weak $(\mathbb{S}^{m-1}, \partial^0\Omega)$ -boundary harmonic map in Ω . If $n = 1$, then Theorem 4.18 shows that $v^e \in C^\infty(\Omega \cup \partial^0\Omega)$, whence $v \in C^\infty(\omega)$ by arbitrariness of Ω . Let us now assume that $n \geq 2$, and that v is stationary. By Proposition 4.15, v^e is stationary. Since $\text{sing}(v) \cap \partial^0\Omega = \text{sing}(v^e) \cap \partial^0\Omega$, Theorem 4.18 yields $\mathcal{H}^{n-1}(\text{sing}(v) \cap \partial^0\Omega) = 0$, and the conclusion follows by monotone convergence letting $\partial^0\Omega \uparrow \omega$. In the case where v is minimizing, the argument follows the same lines using Proposition 4.9. \square

Remark 4.19. In recent papers, DA LIO & RIVIÈRE have obtained a direct proof for the full regularity of weak 1/2-harmonic maps in the case $\omega = \mathbb{R}$, first for a sphere target in [22], and then for more general target manifolds in [23]. A main point of their proof is to rewrite the Euler-Lagrange equation (4.3) in a form that exhibits a special algebraic structure which allows to use some compensated compactness arguments (somehow in the spirit of [44]). Even if we have deduced here the regularity theory for 1/2-harmonic maps from the existing literature, we believe that the analysis in [22, 23] could be of first importance for further investigations.

Remark 4.20. The relation between 1/2-harmonic maps and harmonic maps with partially free boundary was first noticed by MOSER [57] for a different, non-explicit, integro-differential operator (which coincides with $(-\Delta)^{\frac{1}{2}}$ only in the case $\omega = \mathbb{R}^n$). Roughly speaking, the Dirichlet-to-Neumann operator considered in [57] is associated to a ‘‘homogeneous Neumann type condition’’ on the exterior of ω , while we are dealing with Dirichlet exterior conditions. In this Neumann framework and under an additional technical condition, [57] provides a similar partial regularity result from the regularity theory for harmonic maps with partially free boundary.

Remark 4.21. In the context of harmonic map with partially free boundary, DUZAAR & GROTHOWSKI [28] have studied regularity for the mixed boundary value problem which consists in prescribing a Dirichlet data on the remaining part $\partial^+\Omega$ of the boundary of Ω . Under suitable smoothness conditions on the Dirichlet data, $\partial^+\Omega$ and $\partial^0\Omega$, and assuming that $\partial^+\Omega$ and $\partial^0\Omega$ meet orthogonally, they have proved that minimizing solutions are continuous across the edge $\overline{\partial^+\Omega} \cap \overline{\partial^0\Omega}$. Unfortunately, this result cannot be used for 1/2-harmonic maps with exterior Dirichlet condition (even for minimizing maps) since the two parts of the boundary would have to meet *tangentially*. Up to our knowledge, there is no regularity result at the boundary $\partial\omega$ for the 1/2-harmonic map problem with a prescribed exterior Dirichlet condition.

Remark 4.22 (Energy Monotonicity). For usual harmonic maps, it is well known that the stationarity property yields a monotonic control on the energy on balls with respect to the radius. For a stationary boundary harmonic map $u \in H^1(\Omega; \mathbb{R}^m)$, the relation (4.8) leads to the monotonicity of the function

$$r \in (0, \text{dist}(x_0, \partial^+\Omega)) \mapsto \frac{1}{2r^{n-1}} \int_{B_r^+(x_0)} |\nabla u|^2 dx,$$

for every $x_0 \in \partial^0\Omega$.

Remark 4.23 (Liouville property). As a consequence of the energy monotonicity above, if $n \geq 2$ and $u \in \dot{H}^1(\mathbb{R}_+^{n+1}; \mathbb{R}^m)$ is an entire stationary boundary harmonic map (i.e., a stationary boundary harmonic map in B_r^+ for every $r > 0$), then u is constant. As a byproduct, if $n \geq 2$ and $v \in \dot{H}^{1/2}(\mathbb{R}^n; \mathbb{R}^m)$ is an entire stationary 1/2-harmonic map (i.e., a stationary 1/2-harmonic map in D_r for every $r > 0$), then v is constant.

Remark 4.24 (General target manifold). The definition of 1/2-harmonic maps extends to a more general target submanifold $\mathcal{N} \subseteq \mathbb{R}^m$ in the following way. Assuming that \mathcal{N} is smooth, compact, and without boundary, the nearest point retraction $\pi_{\mathcal{N}}$ on \mathcal{N} is smooth in a small tubular neighborhood of \mathcal{N} . We then say that a map $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ satisfying $v(x) \in \mathcal{N}$ for a.e. $x \in \omega$, is weakly 1/2-harmonic into \mathcal{N} in ω if

$$\left[\frac{d}{dt} \mathcal{E}(\pi_{\mathcal{N}}(v + t\varphi), \omega) \right]_{t=0} = 0$$

for all $\varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m) \cap L^\infty(\omega)$ compactly supported in ω . The associated Euler-Lagrange equation reads

$$(-\Delta)^{\frac{1}{2}} v \perp T_v \mathcal{N} \quad \text{in } H_{00}^{-1/2}(\omega),$$

i.e., $\langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle_\omega = 0$ for all $\varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m)$ satisfying $\varphi(x) \in T_{v(x)} \mathcal{N}$ for a.e. $x \in \omega$.

Concerning the definition of $(\mathcal{N}, \partial^0 \Omega)$ -valued boundary harmonic maps, we simply reproduce Definition 4.4 replacing \mathbb{S}^{m-1} by the manifold \mathcal{N} . The notions of minimality and stationarity for both boundary harmonic and 1/2-harmonic maps remain unchanged. With these definitions, all the results of this subsection do hold. In particular, we have the same partial regularity theory for $(\mathcal{N}, \partial^0 \Omega)$ -boundary harmonic maps as stated in Theorem 4.18 by the general regularity results in [43, 29, 30, 66]. As a consequence, we have the same partial regularity theory for 1/2-harmonic maps into \mathcal{N} as stated in Theorem 1.2.

4.2. 1/2-harmonic lines into \mathbb{S}^1 . We provide in this subsection some explicit examples of 1/2-harmonic maps into \mathbb{S}^1 enlightening the geometric flavour of the 1/2-harmonic map equation, as well as its analogy with usual harmonic maps. We start with an explicit representation formula for *all entire 1/2-harmonic maps* from the line \mathbb{R} into \mathbb{S}^1 with finite energy. In the sequel, we identify \mathbb{R}^2 with the complex plane \mathbb{C} writing $z = x_1 + ix_2$. The open unit disc of \mathbb{C} is denoted by \mathbb{D} , and $\partial \mathbb{D}$ is identified with \mathbb{S}^1 .

Theorem 4.25. *Let $v \in \dot{H}^{1/2}(\mathbb{R}; \mathbb{S}^1)$ be a non-constant entire 1/2-harmonic map into \mathbb{S}^1 , and let v^e be its harmonic extension to \mathbb{R}_+^2 given by (2.9). There exist some $d \in \mathbb{N}$, $\theta \in \mathbb{R}$, $\{\lambda_k\}_{k=1}^d \subseteq (0, \infty)$, and $\{a_k\}_{k=1}^d \subseteq \mathbb{R}$ such that $v^e(z)$ or its complex conjugate equals*

$$e^{i\theta} \prod_{k=1}^d \frac{\lambda_k(z - a_k) - i}{\lambda_k(z - a_k) + i}. \quad (4.14)$$

In addition,

$$\mathcal{E}(v, \mathbb{R}) = [v]_{H^{1/2}(\mathbb{R})}^2 = \frac{1}{2} \int_{\mathbb{R}_+^2} |\nabla v^e|^2 dz = \pi d. \quad (4.15)$$

Remark 4.26. Theorem 4.25 shows that the map

$$x \in \mathbb{R} \mapsto \left(\frac{x^2 - 1}{x^2 + 1}, \frac{-2x}{x^2 + 1} \right) \in \mathbb{S}^1$$

is a 1/2-harmonic map. Indeed, it corresponds to the case $\theta = 0$, $d = 1$, $\lambda_1 = 1$, and $a_1 = 0$ in (4.14). Note that this map is precisely the inverse of the stereographic projection with pole at $(1, 0)$ going from the circle into the line.

The proof of Theorem 4.25 is based on an observation due to MIRONESCU & PISANTE [55] together with the following preliminary lemma.

Lemma 4.27. *Let $v \in \dot{H}^{1/2}(\mathbb{R}; \mathbb{R}^2)$ be a nontrivial entire 1/2-harmonic map into \mathbb{S}^1 , and let v^e be its harmonic extension to \mathbb{R}_+^2 given by (2.9). Then v^e is either a conformal or an anti-conformal transformation of $\overline{\mathbb{R}_+^2}$ into the closed unit disc $\overline{\mathbb{D}}$.*

Proof. First recall that v^e is smooth in $\overline{\mathbb{R}_+^2}$ by Theorem 4.18, and v^e takes values in $\overline{\mathbb{D}}$ by (2.11). Let us now consider the Hopf differential \mathcal{H} of v^e defined by

$$\mathcal{H}(z) := (|\partial_1 v^e|^2 - |\partial_2 v^e|^2) - 2i(\partial_1 v^e \cdot \partial_2 v^e).$$

It is well-known that v^e is conformal or anti-conformal if and only if $\mathcal{H} \equiv 0$. We thus have to prove that \mathcal{H} vanishes identically.

First notice that \mathcal{H} is holomorphic since v^e is harmonic. Since $v(x) \in \mathbb{S}^1$ we find that $\partial_1 v^e(x) \in T_{v(x)} \mathbb{S}^1$ for every $x \in \mathbb{R}$. By the boundary equation in (4.6), it implies that $\partial_1 v^e \cdot \partial_2 v^e$ vanishes on \mathbb{R} . Setting g to be the imaginary part of \mathcal{H} , the function g is harmonic in \mathbb{R}_+^2 by holomorphicity of \mathcal{H} . Since g vanishes on \mathbb{R} , we can extend g to the whole plane by odd reflection (i.e., setting $g(z) := -g(\bar{z})$ if $\text{Im}(z) < 0$), and the resulting function (still denoted by g) is harmonic in \mathbb{R}^2 . On the other hand, $v^e \in \dot{H}^1(\mathbb{R}_+^2; \mathbb{R}^2)$ by Lemma 2.6. Hence $g \in L^1(\mathbb{R}^2)$, which leads to $g \equiv 0$ by the mean value property of harmonic functions. Therefore \mathcal{H} takes real values, and it must be constant by holomorphicity. We then conclude that $\mathcal{H} \equiv 0$ since $\mathcal{H} \in L^1(\mathbb{R}_+^2)$. \square

Remark 4.28 (Minimal surfaces). According to Remark 4.24, Lemma 4.27 still holds if \mathbb{S}^1 is replaced by a smooth compact manifold $\mathcal{N} \subseteq \mathbb{R}^m$ without boundary, *i.e.*, if $v \in \dot{H}^{1/2}(\mathbb{R}; \mathbb{R}^m)$ is a nontrivial entire 1/2-harmonic map into \mathcal{N} , then v° is conformal or anti-conformal. If $m = 3$ and \mathcal{N} is a surface, it shows that the image of \mathbb{R}_+^2 by v° is a minimal surface in \mathbb{R}^3 whose boundary lies in \mathcal{N} , and meets \mathcal{N} orthogonally, see [76]. If \mathcal{N} is a closed boundary curve, then the image of \mathbb{R}_+^2 by v° is a minimal surface spanned by \mathcal{N} .

Proof of Theorem 4.25. By Lemma 4.27 we may assume that v° is conformal (otherwise we simply consider the complex conjugate of v). We introduce the so-called *Caley transform* $\mathfrak{C} : \mathbb{R}_+^2 \rightarrow \mathbb{D}$ defined by

$$\mathfrak{C}(z) := \frac{z - i}{z + i}.$$

It is well known that \mathfrak{C} is a conformal map which is one-to-one from \mathbb{R}_+^2 into \mathbb{D} , and one-to-one from \mathbb{R} into $\mathbb{S}^1 \setminus \{(1, 0)\}$. Setting

$$w := v^\circ \circ \mathfrak{C}^{-1},$$

we find that w is conformal in \mathbb{D} , and thus holomorphic in \mathbb{D} . Moreover, by conformality of \mathfrak{C} , we have $w \in H^1(\mathbb{D}; \mathbb{C})$, and more precisely

$$\int_{\mathbb{D}} |\nabla w|^2 dz = \int_{\mathbb{R}_+^2} |\nabla v^\circ|^2 dz. \quad (4.16)$$

In particular $g := w|_{\mathbb{S}^1} \in H^{1/2}(\mathbb{S}^1; \mathbb{C})$. But g is smooth on $\mathbb{S}^1 \setminus \{(0, 1)\}$ and $g(z) = v \circ \mathfrak{C}^{-1}(z)$ for every $z \in \mathbb{S}^1 \setminus \{(0, 1)\}$, so that $g \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$. Now, w being holomorphic, it is also harmonic in \mathbb{D} . It is therefore the harmonic extension to the disc of the map g . By a result of BREZIS & NIRENBERG [11, 12], it implies that $|w(z)| \rightarrow 1$ uniformly as $|z| \rightarrow 1$. For a nonconstant holomorphic map this latter property implies that it is a (finite) Blaschke product, see [55] and the references therein. In other words, we can find a positive $d \in \mathbb{N}$, $\tilde{\theta} \in \mathbb{R}$, and $\alpha_1, \dots, \alpha_d \in \mathbb{D}$ such that

$$w(z) = e^{i\tilde{\theta}} \prod_{k=1}^d \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}. \quad (4.17)$$

Then (4.14) follows from the previous formula with

$$\lambda_k := \frac{|1 - \alpha_k|^2}{1 - |\alpha_k|^2}, \quad a_k := -\frac{2\text{Im}(\alpha_k)}{|1 - \alpha_k|^2}, \quad \theta := \tilde{\theta} + \sum_{k=1}^d \theta_k \text{ where } \theta_k \in \mathbb{R} \text{ and } e^{i\theta_k} = \frac{1 - \alpha_k}{1 - \bar{\alpha}_k}.$$

On the other hand, it is easy to check from (4.17) (see *e.g.* [55]) that

$$\int_{\mathbb{D}} |\nabla w|^2 dz = 2\pi d,$$

which combined with (4.16) implies (4.15) by Lemma 2.6. \square

Remark 4.29 (1/2-harmonic circles). By analogy with our definition of \mathcal{E} in (1.3), we can consider on $H^{1/2}(\mathbb{S}^1; \mathbb{C})$ the energy

$$\mathcal{E}(g, \mathbb{S}^1) := \frac{\gamma_1}{4} \iint_{\mathbb{S}^1 \times \mathbb{S}^1} \frac{|g(x) - g(y)|^2}{|x - y|^2} dx dy.$$

It is well known that for every $g \in H^{1/2}(\mathbb{S}^1; \mathbb{C})$,

$$\mathcal{E}(g, \mathbb{S}^1) = \frac{1}{2} \int_{\mathbb{D}} |\nabla w_g|^2 dx,$$

where $w_g \in H^1(\mathbb{D}; \mathbb{C})$ denotes the harmonic extension of g to the disc. In view of identity (4.16) and Lemma 2.6, we have

$$\mathcal{E}(g, \mathbb{S}^1) = \mathcal{E}(g \circ \mathfrak{C}, \mathbb{R}) \quad \text{for all } g \in H^{1/2}(\mathbb{S}^1; \mathbb{C}),$$

where \mathfrak{C} is the Caley transform. Defining (weak) 1/2-harmonic maps from \mathbb{S}^1 into \mathbb{S}^1 as critical points of $\mathcal{E}(\cdot, \mathbb{S}^1)$ with respect to perturbations in the target (as in Definition 4.1, see (4.19) below), we deduce that g is such a 1/2-harmonic map if and only if $g \circ \mathfrak{C}$ is a (finite energy) 1/2-harmonic map from \mathbb{R} into \mathbb{S}^1 . In terms of the harmonic extension w_g , the Euler-Lagrange equation for 1/2-harmonic maps from \mathbb{S}^1 into \mathbb{S}^1 writes

$$\frac{\partial w_g}{\partial \nu} \wedge g = 0 \quad \text{on } \mathbb{S}^1, \quad (4.18)$$

where $\frac{\partial}{\partial \nu}$ is the exterior normal derivative on $\partial \mathbb{D} \simeq \mathbb{S}^1$. As a consequence of Theorem 4.25, we find that g is a 1/2-harmonic map from \mathbb{S}^1 into \mathbb{S}^1 if and only if g is the restriction to \mathbb{S}^1 of a (finite) Blaschke

product (i.e., g is of the form (4.17)). This fact has also been discovered independently by BERLYAND, MIRONESCU, RYBALKO, & SANDIER [7] (with essentially the same proof), and by DA LIO & RIVIÈRE [24]. Note that the explicit formula (4.17) shows that the energy is quantized by the topological degree, i.e.,

$$\mathcal{E}(g, \mathbb{S}^1) = \pi |\deg(g)|,$$

for any 1/2-harmonic map g from \mathbb{S}^1 into \mathbb{S}^1 . By the result of MIRONESCU & PISANTE [55], it shows in particular that every 1/2-harmonic map from \mathbb{S}^1 into \mathbb{S}^1 is minimizing in its own homotopy class. Here the homotopy classes are classified by the topological degree for $H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ -maps as defined in [11, 12]. These properties of 1/2-harmonic circles into \mathbb{S}^1 are in clear analogy with the theory of usual harmonic maps where it is well known that harmonic 2-spheres into \mathbb{S}^2 are minimizing in their own homotopy class and have an energy quantized by the degree.

With the help of Theorem 4.25 we can now give an explicit representation of 0-homogeneous maps which are weakly 1/2-harmonic from the unit disc of \mathbb{R}^2 into \mathbb{S}^1 . Those maps provide examples of singular weak 1/2-harmonic maps. In particular, it shows that $\frac{x}{|x|}$ is a weak 1/2-harmonic map into \mathbb{S}^1 . It would be interesting to determine which maps are minimizing or stationary, and then to compare the result with [10] (which deals with 0-homogeneous harmonic maps from the unit ball in \mathbb{R}^3 into \mathbb{S}^2).

Proposition 4.30. *For $n = 2$, let $v \in \widehat{H}^{1/2}(D_1; \mathbb{R}^2)$ be a 0-homogeneous map in all of \mathbb{R}^2 such that $|v| = 1$ a.e. in \mathbb{R}^2 . Then v is a weak 1/2-harmonic map into \mathbb{S}^1 in D_1 if and only if $v(x) = g(\frac{x}{|x|})$ for a 1/2-harmonic circle g into \mathbb{S}^1 in the sense of Remark 4.29.*

Proof. Step 1. Let us consider an arbitrary map $v \in \widehat{H}^{1/2}(D_1; \mathbb{R}^2)$ which is 0-homogeneous. By a rescaling argument we first deduce that $v \in H_{loc}^{1/2}(\mathbb{R}^2; \mathbb{R}^2)$, and in view of Remark 2.8, $v^e \in H_{loc}^1(\overline{\mathbb{R}}^3; \mathbb{R}^2)$. Since v^e clearly inherits the 0-homogeneity of v , we infer that the map $\tilde{v}^e := v^e|_{\mathbb{S}_+^2}$ belongs to $H^1(\mathbb{S}_+^2; \mathbb{R}^2)$ where $\mathbb{S}_+^2 := \mathbb{S}^2 \cap \mathbb{R}_+^3$. Therefore $g := \tilde{v}^e|_{\partial\mathbb{S}_+^2} \in H^{1/2}(\mathbb{S}^1; \mathbb{R}^2)$ identifying $\partial\mathbb{S}_+^2$ with \mathbb{S}^1 . Obviously, we have $v = g(\frac{x}{|x|})$ and $v^e(x) = \tilde{v}^e(\frac{x}{|x|})$. From the harmonicity of v^e , we deduce that \tilde{v}^e is the unique solution of

$$\begin{cases} \Delta_{\mathbb{S}^2} \tilde{v}^e = 0 & \text{on } \mathbb{S}_+^2, \\ \tilde{v}^e = g & \text{on } \partial\mathbb{S}_+^2, \end{cases}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator on \mathbb{S}^2 .

We now introduce the one-to-one conformal mapping $\mathfrak{G} : \mathbb{D} \rightarrow \overline{\mathbb{S}_+^2}$ defined by

$$\mathfrak{G}(x) := \left(\frac{2x_1}{|x|^2 + 1}, \frac{2x_2}{|x|^2 + 1}, \frac{1 - |x|^2}{|x|^2 + 1} \right).$$

Note that \mathfrak{G} is the identity on \mathbb{S}^1 . By conformality of \mathfrak{G} the map $w_g := \tilde{v}^e \circ \mathfrak{G} \in H^1(\mathbb{D}; \mathbb{R}^2)$ is harmonic in \mathbb{D} , and $w_g = g$ on \mathbb{S}^1 in the trace sense. It is therefore the harmonic extension of g to the disc \mathbb{D} , and we infer from Remark 4.29 and the conformality of \mathfrak{G} that

$$\frac{1}{2} \int_{\mathbb{S}_+^2} |\nabla_T \tilde{v}^e|^2 d\mathcal{H}^2 = \frac{1}{2} \int_{\mathbb{D}} |\nabla w_g|^2 dx = \mathcal{E}(g, \mathbb{S}^1),$$

where ∇_T denotes the tangential gradient.

Step 2. Assume that $v \in \widehat{H}^{1/2}(D_1; \mathbb{R}^2)$ is a 0-homogeneous 1/2-harmonic map into \mathbb{S}^1 in D_1 . We claim that the associated map g as defined in Step 1 is a 1/2-harmonic circle into \mathbb{S}^1 , i.e.,

$$\left[\frac{d}{dt} \mathcal{E} \left(\frac{g + t\varphi}{|g + t\varphi|}, \mathbb{S}^1 \right) \right]_{t=0} = 0 \quad (4.19)$$

for all $\varphi \in H^{1/2}(\mathbb{S}^1; \mathbb{R}^2) \cap L^\infty(\mathbb{S}^1)$. Arguing as in the proof of Proposition 4.2, we infer that it is enough to prove (4.19) for all $\varphi \in H^{1/2}(\mathbb{S}^1; \mathbb{R}^2) \cap L^\infty(\mathbb{S}^1)$ such that $\varphi \cdot g = 0$ a.e. on \mathbb{S}^1 . For such a function φ , we have

$$\left[\frac{d}{dt} \mathcal{E} \left(\frac{g + t\varphi}{|g + t\varphi|}, \mathbb{S}^1 \right) \right]_{t=0} = \left[\frac{d}{dt} \mathcal{E}(g + t\varphi, \mathbb{S}^1) \right]_{t=0} = \left[\frac{d}{dt} \left(\frac{1}{2} \int_{\mathbb{D}} |\nabla(w_g + tw_\varphi)|^2 dx \right) \right]_{t=0},$$

where w_φ is the harmonic extension of φ to the disc \mathbb{D} (notice that $w_\varphi \in H^1(\mathbb{D}; \mathbb{R}^2) \cap L^\infty(\mathbb{D})$). Setting $\Phi := w_\varphi \circ \mathfrak{S}^{-1} \in H^1(\mathbb{S}_+^2; \mathbb{R}^2) \cap L^\infty(\mathbb{S}_+^2)$, we can argue as in Step 1 to obtain

$$\frac{1}{2} \int_{\mathbb{D}} |\nabla(w_g + tw_\varphi)|^2 dx = \frac{1}{2} \int_{\mathbb{S}_+^2} |\nabla_T(\tilde{v}^e + t\Phi)|^2 d\mathcal{H}^2.$$

Consequently,

$$\left[\frac{d}{dt} \mathcal{E} \left(\frac{g + t\varphi}{|g + t\varphi|}, \mathbb{S}^1 \right) \right]_{t=0} = \left[\frac{d}{dt} \left(\frac{1}{2} \int_{\mathbb{S}_+^2} |\nabla_T(\tilde{v}^e + t\Phi)|^2 d\mathcal{H}^2 \right) \right]_{t=0} = \int_{\mathbb{S}_+^2} \nabla_T \tilde{v}^e \cdot \nabla_T \Phi d\mathcal{H}^2. \quad (4.20)$$

Let us now consider a function $\eta \in C^1((0, +\infty))$ with compact support in the interval $(0, 1)$ satisfying $\int_0^1 \eta(r)r^2 dr = 1$. Define for $x \in \mathbb{R}_+^3$, $\tilde{\Phi} := \eta(|x|)\Phi(\frac{x}{|x|})$. Then $\tilde{\Phi} \in H^1(B_1^+; \mathbb{R}^2) \cap L^\infty(B_1^+)$, $\tilde{\Phi}$ has compact support in $B_1^+ \cup D_1$, and $\tilde{\Phi} \cdot v^e = 0$ \mathcal{H}^2 -a.e. on D_1 . Since v is assumed to be 1/2-harmonic, we deduce from Proposition 4.6 that

$$\int_{B_1^+} \nabla v^e \cdot \nabla \tilde{\Phi} dx = 0. \quad (4.21)$$

On the other hand, by homogeneity of v^e , we have

$$\int_{B_1^+} \nabla v^e \cdot \nabla \tilde{\Phi} dx = \left(\int_0^1 \eta(r)r^2 dr \right) \int_{\mathbb{S}_+^2} \nabla_T \tilde{v}^e \cdot \nabla_T \Phi d\mathcal{H}^2 = \int_{\mathbb{S}_+^2} \nabla_T \tilde{v}^e \cdot \nabla_T \Phi d\mathcal{H}^2. \quad (4.22)$$

Gathering (4.20)-(4.21)-(4.22) leads to (4.19), and the claim is proved.

Step 3. Consider a 1/2-harmonic circle $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, and set $v := g(\frac{x}{|x|})$. From the smoothness of g , we infer that $v \in H_{\text{loc}}^{1/2}(\mathbb{R}^2; \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \subseteq \hat{H}^{1/2}(D_1; \mathbb{R}^2)$, and v is smooth away from the origin. Moreover, we deduce from Step 1 that

$$v^e(x) = w_g \circ \mathfrak{S}^{-1} \left(\frac{x}{|x|} \right).$$

As a consequence,

$$\frac{\partial v^e}{\partial \nu}(x) = \frac{1}{|x|} \frac{\partial w_g}{\partial \nu_p}(p) \quad \text{on } \mathbb{R}^2 \setminus \{0\},$$

where $p := \mathfrak{S}^{-1} \left(\frac{x}{|x|} \right) \in \mathbb{S}^1$ and ν_p denotes the exterior normal to $\partial \mathbb{D} \simeq \mathbb{S}^1$ at the point p . In view of (4.18) we have

$$\frac{\partial v^e}{\partial \nu}(x) \wedge v(x) = \frac{1}{|x|} \frac{\partial w_g}{\partial \nu_p}(p) \wedge g(p) = 0 \quad \text{on } \mathbb{R}^2 \setminus \{0\}.$$

From Lemma 2.9 we then infer that $(-\Delta)^{\frac{1}{2}} v \wedge v = 0$ in $\mathbb{R}^2 \setminus \{0\}$, and thus v satisfies (4.2) in $D_1 \setminus \overline{D}_\rho$ for every $0 < \rho < 1$. Since v belongs to $\hat{H}^{1/2}(D_1; \mathbb{R}^2)$, a standard capacity argument shows that (4.2) actually holds in D_1 . Hence v is a weak 1/2-harmonic map in D_1 by Proposition 4.2. \square

5. SMALL ENERGY ESTIMATE FOR GINZBURG-LANDAU BOUNDARY REACTIONS

In this section we perform a preliminary analysis on the asymptotic, as $\varepsilon \rightarrow 0$, of critical points of the Ginzburg-Landau boundary energy E_ε defined in (3.5). The first step we make here is to prove an *epsilon-regularity* type of estimate which allows to control the regularity of solutions under the assumption that the energy, suitably renormalized, is small. This is the purpose of the following theorem which is a corner stone of the present paper. The entire section is devoted to its proof.

Theorem 5.1. *Let $\varepsilon > 0$ and $R > 0$ such that $\varepsilon \leq R$. There exist constants $\eta_0 > 0$ and $C_0 > 0$ independent of ε and R such that for each map $u_\varepsilon \in C^2(B_R^+ \cup D_R; \mathbb{R}^m)$ satisfying $|u_\varepsilon| \leq 1$ and solving*

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } B_R^+, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{1}{\varepsilon} (1 - |u_\varepsilon|^2) u_\varepsilon & \text{on } D_R, \end{cases} \quad (5.1)$$

the condition $E_\varepsilon(u_\varepsilon, B_R^+) \leq \eta_0 R^{n-1}$ implies

$$\sup_{B_{R/4}^+} |\nabla u_\varepsilon|^2 + \sup_{D_{R/4}} \frac{(1 - |u_\varepsilon|^2)^2}{\varepsilon^2} \leq \frac{C_0}{R^2} \eta_0. \quad (5.2)$$

The proof of Theorem 5.1 is divided into three main parts according to the following subsections. For the local Ginzburg-Landau equation, the argument leading to the analogous estimate is essentially based on the so-called *Bochner inequality* satisfied by the energy density, see e.g. [20]. In our case, such inequality does not seem to be available, and to prove (5.2) we better use a compactness approach in the spirit of [78].

5.1. Energy monotonicity and clearing-out lemma. As usual in Ginzburg-Landau or harmonic map problems, the first main ingredient to derive regularity estimates is a useful monotonicity formula. This is the purpose of the following lemma.

Lemma 5.2 (Monotonicity formula). *Let $R > 0$ and $u_\varepsilon \in C^2(B_R^+ \cup D_R; \mathbb{R}^m)$ satisfying (5.1). Then, for every $x_0 \in D_R$ and every $0 < \rho < r < \text{dist}(x_0, \partial D_R)$,*

$$\begin{aligned} & \frac{1}{r^{n-1}} E_\varepsilon(u_\varepsilon, B_r^+(x_0)) - \frac{1}{\rho^{n-1}} E_\varepsilon(u_\varepsilon, B_\rho^+(x_0)) \\ &= \int_\rho^r \frac{1}{t^{n-1}} \left(\int_{\partial^+ B_t(x_0)} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 d\mathcal{H}^n \right) dt \\ & \quad + \frac{1}{4\varepsilon} \int_\rho^r \frac{1}{t^n} \left(\int_{D_t(x_0)} (1 - |u_\varepsilon|^2)^2 d\mathcal{H}^n \right) dt. \end{aligned}$$

Proof. Step 1. Consider a smooth vector field $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_{n+1}) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ compactly supported in B_R and satisfying $\mathbf{X}_{n+1} = 0$ on \mathbb{R}^n . For $t \in \mathbb{R}$ small, the map $\Phi_t(x) := x - t\mathbf{X}(x)$ defines a smooth diffeomorphism from B_R into B_R satisfying $\Phi_t(D_R) \subseteq D_R$, and $\Phi_t(B_R^+) \subseteq B_R^+$. Setting $u_t := u_\varepsilon \circ \Phi_t$, standard computations (see e.g. [71, Chapter 2.2]) yield

$$\left[\frac{d}{dt} \left(\frac{1}{2} \int_{B_R^+} |\nabla u_t|^2 dx \right) \right]_{t=0} = \frac{1}{2} \int_{B_R^+} \left(|\nabla u_\varepsilon|^2 \text{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i u_\varepsilon \cdot \partial_j u_\varepsilon) \partial_j \mathbf{X}_i \right) dx, \quad (5.3)$$

and

$$\left[\frac{d}{dt} \left(\frac{1}{4\varepsilon} \int_{D_R} (1 - |u_t|^2)^2 d\mathcal{H}^n \right) \right]_{t=0} = \frac{1}{4\varepsilon} \int_{D_R} (1 - |u_\varepsilon|^2)^2 \text{div}_{\mathbb{R}^n} \mathbf{X} d\mathcal{H}^n.$$

Using (5.1) we integrate by parts in (5.3) to find

$$\left[\frac{d}{dt} \left(\frac{1}{2} \int_{B_R^+} |\nabla u_t|^2 dx \right) \right]_{t=0} = -\frac{1}{4\varepsilon} \int_{D_R} (1 - |u_\varepsilon|^2)^2 \text{div}_{\mathbb{R}^n} \mathbf{X} d\mathcal{H}^n,$$

whence

$$\left[\frac{d}{dt} E_\varepsilon(u_t, B_R^+) \right]_{t=0} = 0. \quad (5.4)$$

Step 2. Let $x_0 \in D_R$, and $0 < r < \text{dist}(x_0, \partial D_R)$. Without loss of generality we may assume that $x_0 = 0$ (to simplify the notation). Let $\eta \in C^\infty(\mathbb{R}; [0, 1])$ be an even function such that $\eta(t) = 0$ for $|t| \geq r$. Using the vector field $\mathbf{X}(x) = \eta(|x|)x$ in Step 1, we find

$$\begin{aligned} & \frac{n-1}{2} \int_{B_R^+} |\nabla u_\varepsilon|^2 \eta(|x|) dx + \frac{1}{2} \int_{B_R^+} |\nabla u_\varepsilon|^2 \eta'(|x|) |x| dx \\ & \quad - \int_{B_R^+} \left| \frac{\partial u_\varepsilon}{\partial |x|} \right|^2 \eta'(|x|) |x| dx + \frac{n}{4\varepsilon} \int_{D_R} (1 - |u_\varepsilon|^2)^2 \eta(|x|) d\mathcal{H}^n \\ & \quad + \frac{1}{4\varepsilon} \int_{D_R} (1 - |u_\varepsilon|^2)^2 \eta'(|x|) |x| d\mathcal{H}^n = 0. \end{aligned} \quad (5.5)$$

Then, given an arbitrary $t \in (0, r)$, we consider a sequence $\{\eta_k\}$ of functions as above such that η_k converges weakly* in BV as $k \rightarrow \infty$ to the characteristic function of the interval $[-t, t]$. Using η_k as a test function in (5.5) and letting $k \rightarrow \infty$ leads to

$$\begin{aligned} & \frac{n-1}{2} \int_{B_t^+} |\nabla u_\varepsilon|^2 dx - \frac{t}{2} \int_{\partial^+ B_t} |\nabla u_\varepsilon|^2 d\mathcal{H}^n + t \int_{\partial^+ B_t} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 d\mathcal{H}^n \\ & \quad + \frac{n}{4\varepsilon} \int_{D_t} (1 - |u_\varepsilon|^2)^2 d\mathcal{H}^n - \frac{t}{4\varepsilon} \int_{\partial D_t} (1 - |u_\varepsilon|^2)^2 d\mathcal{H}^{n-1} = 0. \end{aligned}$$

Dividing by t^n and rearranging terms, we end up with

$$\frac{d}{dt} \left(\frac{1}{t^{n-1}} E_\varepsilon(u_\varepsilon, B_t^+) \right) = \frac{1}{t^{n-1}} \int_{\partial^+ B_t} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 d\mathcal{H}^n + \frac{1}{4\varepsilon t^n} \int_{D_t} (1 - |u_\varepsilon|^2)^2 d\mathcal{H}^n.$$

Integrating this equality between $\rho > 0$ and r yields to the announced result. \square

Remark 5.3 (Liouville property). As a consequence of Lemma 5.2 and the stationarity equation (5.4), any entire (smooth) solution $u_\varepsilon : \overline{\mathbb{R}}_+^{n+1} \rightarrow \mathbb{R}^m$ of the Ginzburg-Landau boundary equation (*i.e.*, u_ε solves (5.1) for every radius $R > 0$) satisfying $E_\varepsilon(u_\varepsilon, \mathbb{R}_+^{n+1}) < \infty$, has to be *constant*. In view of (3.3) and Lemma 2.6, the same Liouville property holds for entire solutions of the fractional Ginzburg-Landau equation having finite energy. More precisely, if $v_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a (smooth) solution of (3.1) in \mathbb{R}^n satisfying $\mathcal{E}_\varepsilon(v_\varepsilon, \mathbb{R}^n) < \infty$, then v_ε is constant. The argument goes as follows.

Proof. If $n \geq 2$, Lemma 5.2 implies $E_\varepsilon(u_\varepsilon, B_\rho^+) \leq C(r/\rho)^{1-n}$ for every $0 < \rho \leq r$, and the constancy of u_ε follows letting $r \rightarrow \infty$. The case $n = 1$ is slightly more involved, and relies on (5.4). For $n = 1$ we consider a nonnegative cut-off function $\zeta \in C^\infty(\mathbb{R}^2)$ such that $\zeta = 1$ on B_1 and $\zeta = 0$ on $\mathbb{R}^2 \setminus B_2$. Setting $\zeta_k(x) := \zeta(x/k)$, we use the vector field $\mathbf{X}(x) = \zeta_k(x)x$ in (5.4) to obtain

$$\begin{aligned} \frac{1}{2} \int_{B_{2k}^+ \setminus B_k^+} \left(|\nabla u_\varepsilon|^2 (x \cdot \nabla \zeta_k) - 2 \sum_{i,j=1}^2 (\partial_i u_\varepsilon \cdot \partial_j u_\varepsilon) x_i \partial_j \zeta_k \right) dx \\ + \frac{1}{4\varepsilon} \int_{D_{2k}} (1 - |u_\varepsilon|^2)^2 \zeta_k d\mathcal{H}^1 + \frac{1}{4\varepsilon} \int_{D_{2k} \setminus D_k} (1 - |u_\varepsilon|^2)^2 x_1 \partial_1 \zeta_k d\mathcal{H}^1 = 0. \end{aligned}$$

Since $|\nabla \zeta_k| \leq C/k$, this identity yields

$$\frac{1}{\varepsilon} \int_{D_k} (1 - |u_\varepsilon|^2)^2 d\mathcal{H}^1 \leq C E_\varepsilon(u_\varepsilon, B_{2k}^+ \setminus B_k^+) \xrightarrow{k \rightarrow \infty} 0,$$

and we deduce that $|u_\varepsilon| \equiv 1$. We then infer that u_ε is a bounded solution of

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \mathbb{R}^n, \end{cases}$$

and the constancy of u_ε follows from Liouville Theorem on bounded entire harmonic functions through the usual reflection argument across $\partial \mathbb{R}_+^{n+1} \simeq \mathbb{R}^n$. \square

The second key ingredient in Ginzburg-Landau problems is the so-called *clearing-out property* which essentially asserts that a solution must take values in a small neighborhood of the potential-well whenever the energy is small enough. This is precisely the object of the following lemma. The proof uses some ingredients from [66, Lemma 3.1] suitably modified to fit the Ginzburg-Landau setting.

Lemma 5.4 (Clearing-out). *Let $\varepsilon \in (0, 1]$. There exists a constant $\eta_1 > 0$ independent of ε such that for each map $u_\varepsilon \in C^2(B_1^+ \cup D_1; \mathbb{R}^m)$ satisfying $|u_\varepsilon| \leq 1$ and (5.1) with $R = 1$, the condition $E_\varepsilon(u_\varepsilon, B_1^+) \leq \eta_1$ implies*

$$|u_\varepsilon| \geq \frac{1}{2} \quad \text{in } \overline{B}_{1/2}^+. \quad (5.6)$$

Proof. Step 1. We assume in this first step that $\varepsilon \geq 1/2$. We claim that we can find $\eta_2 > 0$ only depending on n and m such that if $E_\varepsilon(u_\varepsilon, B_1^+) \leq \eta_2$, then $|u_\varepsilon| \geq 1/2$ in $\overline{B}_{1/2}^+$. Indeed, for an arbitrary map u_ε satisfying the statement of the theorem, since $|u_\varepsilon| \leq 1$ and $\varepsilon \geq 1/2$, we deduce from [17, Lemma 2.2] that $\|u_\varepsilon\|_{C^{2,\beta}(B_{1/2}^+)} \leq C_\beta$ for a constant C_β only depending on n , m , and $\beta \in (0, 1)$. We now proceed by contradiction assuming that there exist sequences $\{\varepsilon_k\} \subseteq [1/2, 1]$, $\{x_k\} \subseteq \overline{B}_{1/2}^+$, and $\{u_k\} \subseteq C^2(B_1^+ \cup D_1; \mathbb{R}^m)$ satisfying $|u_k| \leq 1$ and (5.1) with $R = 1$, such that $E_{\varepsilon_k}(u_k, B_1^+) \rightarrow 0$ and $|u_k(x_k)| < 1/2$. Then we can find a (not relabeled) subsequence such that u_k converges uniformly on $\overline{B}_{1/2}^+$. Since $E_{\varepsilon_k}(u_k, B_1^+) \rightarrow 0$ the limit has to be a constant of modulus one. As a consequence, $|u_k| \rightarrow 1$ uniformly on $\overline{B}_{1/2}^+$, which contradicts the assumption $|u_k(x_k)| < 1/2$.

Step 2. We now consider the case $\varepsilon < 1/2$. Let us fix an arbitrary point $x_0 = (x'_0, (x_0)_{n+1})$ in $B_{1/2}^+$, and set $x_1 := (x'_0, 0) \in D_{1/2}$. By the Monotonicity Formula in Lemma 5.2 we have

$$\frac{1}{r^{n-1}} E_\varepsilon(u_\varepsilon, B_r^+(x_1)) \leq (1 - |x_1|)^{1-n} E_\varepsilon(u_\varepsilon, B_{1-|x_1|}^+(x_1)) \leq 2^{n-1} E_\varepsilon(u_\varepsilon, B_1^+) \leq 2^{n-1} \eta_1 \quad (5.7)$$

for every $0 < r < 1 - |x_1|$. We set

$$R := \frac{1}{3}|x_1 - x_0|.$$

Case 1) Let us first assume that $R \leq \varepsilon/7$. Since $\varepsilon < 1/2 < 1 - |x_1|$ we have

$$\frac{1}{\varepsilon^{N-1}} E_\varepsilon(u_\varepsilon, B_\varepsilon^+(x_1)) \leq 2^{n-1} \eta_1.$$

Next we define for $y \in B_1^+ \cup D_1$,

$$v_\varepsilon(x) := u_\varepsilon(x_1 + \varepsilon y),$$

so that

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{in } B_1^+, \\ \frac{\partial v_\varepsilon}{\partial \nu} = (1 - |v_\varepsilon|^2)v_\varepsilon & \text{on } D_1, \end{cases}$$

and

$$E_1(v_\varepsilon, B_1^+) = \frac{1}{\varepsilon^{n-1}} E_\varepsilon(u_\varepsilon, B_\varepsilon^+(x_1)) \leq 2^{n-1} \eta_1.$$

Choosing η_1 such that $2^{n-1} \eta_1 \leq \eta_2$, we infer from Step 1 that $|v_\varepsilon| \geq 1/2$ in $B_{1/2}^+$. Hence $|u_\varepsilon| \geq 1/2$ in $B_{\varepsilon/2}^+(x_1)$, and thus $|u_\varepsilon(x_0)| \geq 1/2$.

Case 2) We assume that $R > \varepsilon/7$, so that $\varepsilon/R < 7$. Noticing that $5R < 1 - |x_1|$, we have $B_{5R}^+(x_1) \subseteq B_1^+$. Let us set

$$\bar{u}_\varepsilon := \frac{1}{|B_{5R}^+(x_1)|} \int_{B_{5R}^+(x_1)} u_\varepsilon(y) dy.$$

We consider for $x \in B_{2R}(x_0) \subseteq B_{5R}^+(x_1)$,

$$v_\varepsilon(x) := u_\varepsilon(x) - \bar{u}_\varepsilon.$$

Denote by G_{x_0} the fundamental solution in \mathbb{R}^{n+1} of the Laplace equation with pole at x_0 . We recall that

$$|\nabla G_{x_0}(x)| \leq \frac{C}{|x - x_0|^n} \quad \text{for all } x \in \mathbb{R}^{n+1} \setminus \{x_0\}. \quad (5.8)$$

Then consider a smooth cut-off function $\zeta \in C_c^\infty(B_{2R}(x_0); [0, 1])$ such that $\zeta \equiv 1$ in $B_R(x_0)$, $\zeta \equiv 0$ in $B_{2R}(x_0) \setminus B_{3R/2}(x_0)$, and $|\nabla \zeta| \leq C/R$.

We estimate

$$\begin{aligned} |v_\varepsilon(x_0)|^2 &= \int_{B_{2R}(x_0)} G_{x_0} \Delta(|v_\varepsilon|^2 \zeta^2) dx \\ &= - \int_{B_{2R}(x_0)} \nabla G_{x_0} \cdot \nabla(|v_\varepsilon|^2 \zeta^2) dx \\ &= -2 \int_{B_{3R/2}(x_0)} \zeta^2 \left(\sum_{j=1}^n \partial_j G_{x_0} \partial_j v_\varepsilon \cdot v_\varepsilon \right) dx - 2 \int_{A_R(x_0)} |v_\varepsilon|^2 \zeta \nabla G_{x_0} \cdot \nabla \zeta dx \\ &\leq \|v_\varepsilon\|_{L^\infty(B_{3R/2}(x_0))} \int_{B_{3R/2}(x_0)} |\nabla G_{x_0}| |\nabla u_\varepsilon| dx + \frac{C}{R^{n+1}} \int_{A_R(x_0)} |v_\varepsilon|^2 dx, \end{aligned} \quad (5.9)$$

where $A_R(x_0) := B_{2R}(x_0) \setminus B_R(x_0)$.

By harmonicity of u_ε , ∇u_ε is harmonic in B_1^+ . Since $B_{R/4}(x) \subseteq B_{5R}^+(x_1)$ for all $x \in B_{3R/2}(x_0)$, we deduce that

$$\nabla u_\varepsilon(x) = \frac{1}{|B_{R/4}(x)|} \int_{B_{R/4}(x)} \nabla u_\varepsilon(y) dy \quad \text{for all } x \in B_{3R/2}(x_0).$$

From Jensen's inequality and (5.7), we then infer that

$$|\nabla u_\varepsilon(x)|^2 \leq \frac{1}{|B_{R/4}(x)|} \int_{B_{R/4}(x)} |\nabla u_\varepsilon(y)|^2 dy \leq \frac{C}{R^{n+1}} E_\varepsilon(u_\varepsilon, B_{5R}^+(x_1)) \leq \frac{C}{R^2} \eta_1$$

for all $x \in B_{3R/2}(x_0)$. Consequently,

$$|\nabla u_\varepsilon(x)| \leq \frac{C}{R} \sqrt{\eta_1} \quad \text{for all } x \in B_{3R/2}(x_0). \quad (5.10)$$

Using (5.8) together with (5.10), we obtain that

$$\int_{B_{3R/2}(x_0)} |\nabla G_{x_0}| |\nabla u_\varepsilon| dx \leq C \sqrt{\eta_1}. \quad (5.11)$$

Estimate (5.10) also yields

$$\|v_\varepsilon\|_{L^\infty(B_{3R/2}(x_0))} \leq |v_\varepsilon(x_0)| + \frac{3R}{2} \|\nabla u_\varepsilon\|_{L^\infty(B_{3R/2}(x_0))} \leq |v_\varepsilon(x_0)| + C\sqrt{\eta_1}. \quad (5.12)$$

Next we infer from Poincaré Inequality that

$$\int_{B_{2R}(x_0) \setminus B_R(x_0)} |v_\varepsilon|^2 dx \leq \int_{B_{5R}^+(x_1)} |u_\varepsilon - \bar{u}_\varepsilon|^2 dx \leq CR^2 \int_{B_{5R}^+(x_1)} |\nabla u_\varepsilon|^2 dx,$$

so that

$$\frac{1}{R^{n+1}} \int_{B_{2R}(x_0) \setminus B_R(x_0)} |v_\varepsilon|^2 dx \leq \frac{C}{R^{n-1}} E_\varepsilon(u_\varepsilon, B_{5R}^+(x_1)) \leq C\eta_1. \quad (5.13)$$

Gathering (5.9), (5.11), (5.12), and (5.13), we conclude that

$$|v_\varepsilon(x_0)|^2 \leq C\sqrt{\eta_1}(|v_\varepsilon(x_0)| + \sqrt{\eta_1}) \leq \frac{1}{2}|v_\varepsilon(x_0)|^2 + C\eta_1,$$

and thus

$$|u_\varepsilon(x_0) - \bar{u}_\varepsilon| \leq C\sqrt{\eta_1}.$$

In particular,

$$|1 - |u_\varepsilon(x_0)|| \leq |1 - |\bar{u}_\varepsilon|| + C\sqrt{\eta_1}. \quad (5.14)$$

Now we claim that

$$|1 - |\bar{u}_\varepsilon|| \leq C\sqrt{\eta_1}. \quad (5.15)$$

Setting for $y \in \mathbb{R}^m$, $d(y) := |1 - |y||$, the function d is 1-Lipschitz. Since $d(\bar{u}_\varepsilon) \leq d(u_\varepsilon(x)) + |u_\varepsilon(x) - \bar{u}_\varepsilon|$, we have

$$d(\bar{u}_\varepsilon) \leq \frac{1}{|B_{5R}^+(x_1)|} \int_{B_{5R}^+(x_1)} d(u_\varepsilon(x)) dx + \frac{1}{|B_{5R}^+(x_1)|} \int_{B_{5R}^+(x_1)} |u_\varepsilon(x) - \bar{u}_\varepsilon| dx. \quad (5.16)$$

Then we use Jensen and Poincaré inequalities together with (5.7), to derive

$$\begin{aligned} \left(\frac{1}{|B_{5R}^+(x_1)|} \int_{B_{5R}^+(x_1)} |u_\varepsilon(x) - \bar{u}_\varepsilon| dx \right)^2 &\leq \frac{1}{|B_{5R}^+(x_1)|} \int_{B_{5R}^+(x_1)} |u_\varepsilon(x) - \bar{u}_\varepsilon|^2 dx \\ &\leq \frac{C}{R^{n-1}} \int_{B_{5R}^+(x_1)} |\nabla u_\varepsilon|^2 dx \leq C\eta_1. \end{aligned} \quad (5.17)$$

Using again Jensen and Poincaré inequalities for the function $d \circ u_\varepsilon$, estimate (5.7), and the facts that $|\nabla d| \leq 1$ and $\varepsilon/R < 7$, we obtain

$$\begin{aligned} \left(\frac{1}{|B_{5R}^+(x_1)|} \int_{B_{5R}^+(x_1)} d \circ u_\varepsilon dx \right)^2 &\leq \frac{1}{|B_{5R}^+(x_1)|} \int_{B_{5R}^+(x_1)} |d \circ u_\varepsilon|^2 dx \\ &\leq C \left(\frac{1}{R^n} \int_{D_{5R}(x_1)} |d \circ u_\varepsilon|^2 d\mathcal{H}^n + \frac{1}{R^{n-1}} \int_{B_{5R}^+(x_1)} |\nabla(d \circ u_\varepsilon)|^2 dx \right) \\ &\leq \frac{C}{R^{n-1}} \left(\frac{1}{\varepsilon} \int_{D_{5R}(x_1)} (1 - |u_\varepsilon|^2)^2 d\mathcal{H}^n + \int_{B_{5R}^+(x_1)} |\nabla u_\varepsilon|^2 dx \right) \\ &\leq C\eta_1. \end{aligned} \quad (5.18)$$

Combining (5.16), (5.17), and (5.18), yields (5.15)

Now (5.14) and (5.15) imply that $|1 - |u_\varepsilon(x_0)|| \leq C\sqrt{\eta_1}$, and thus $|u_\varepsilon(x_0)| \geq 1/2$ whenever η_1 is chosen small enough, smallness depending only on the dimensions n and m . \square

5.2. Proof of Theorem 5.1, Step 1. By rescaling variables, it suffices to consider the case $R = 1$ and $\varepsilon \in (0, 1]$. We shall choose $\eta_0 \leq \eta_1$ where η_1 is given by Lemma 5.4, so that

$$|u_\varepsilon| \geq \frac{1}{2} \quad \text{in } \bar{B}_{1/2}^+. \quad (5.19)$$

Arguing as in the proof of Lemma 5.4, we infer that

$$\frac{1}{r^{n-1}} E_\varepsilon(u_\varepsilon, B_r^+(x)) \leq 2^{n-1} \eta_0 \quad \text{for all } x \in D_{1/2} \text{ and } 0 < r < 1 - |x|. \quad (5.20)$$

We are going to prove that if η_0 is small enough, smallness depending only on n and m , then

$$|\nabla u_\varepsilon|^2 \leq C\eta_0 \quad \text{in } \bar{B}_{1/4}^+.$$

In view of the Neumann boundary condition, it clearly implies the full estimate (5.2). We shall achieve it considering two different cases. In the spirit of [20], we start with the localization of an interior region where $|\nabla u_\varepsilon|^2$ is large.

Since u_ε is smooth, we can find $\sigma_\varepsilon \in (0, 1/2)$ such that

$$(1/2 - \sigma_\varepsilon)^2 \sup_{B_{\sigma_\varepsilon}^+} |\nabla u_\varepsilon|^2 = \max_{0 \leq \sigma \leq 1/2} \left((1/2 - \sigma)^2 \sup_{B_\sigma^+} |\nabla u_\varepsilon|^2 \right), \quad (5.21)$$

and $x_\varepsilon \in B_{\sigma_\varepsilon}^+$ such that

$$2|\nabla u_\varepsilon(x_\varepsilon)|^2 \geq \sup_{x \in B_{\sigma_\varepsilon}^+} |\nabla u_\varepsilon|^2 =: e_\varepsilon. \quad (5.22)$$

Denote by $\bar{x}_\varepsilon \in D_{\sigma_\varepsilon}$ the orthogonal projection of x_ε on \mathbb{R}^n , and set

$$\rho_\varepsilon := \frac{1}{2}(1/2 - \sigma_\varepsilon) \quad \text{and} \quad R_\varepsilon := \frac{1}{3}|x_\varepsilon - \bar{x}_\varepsilon|.$$

We now distinguish two cases.

Case 1). We first assume that $R_\varepsilon > \rho_\varepsilon/7$. Since ∇u_ε is harmonic in B_1^+ , we have

$$\nabla u_\varepsilon(x_\varepsilon) = \frac{1}{|B_{R_\varepsilon}(x_\varepsilon)|} \int_{B_{R_\varepsilon}(x_\varepsilon)} \nabla u_\varepsilon \, dx.$$

Then we deduce from the harmonicity of u_ε , Jensen's inequality, (5.20), and (5.22) that

$$e_\varepsilon \leq \frac{2}{|B_{R_\varepsilon}(x_\varepsilon)|} \int_{B_{R_\varepsilon}(x_\varepsilon)} |\nabla u_\varepsilon|^2 \, dx \leq \frac{C}{R_\varepsilon^{n+1}} \int_{B_{5R_\varepsilon}^+(\bar{x}_\varepsilon)} |\nabla u_\varepsilon|^2 \, dx \leq \frac{C}{R_\varepsilon^2} \eta_0 \leq \frac{C}{\rho_\varepsilon^2} \eta_0.$$

Here we have also used that $B_{R_\varepsilon}(x_\varepsilon) \subseteq B_{5R_\varepsilon}^+(\bar{x}_\varepsilon) \subseteq B_1^+$. Consequently,

$$\rho_\varepsilon^2 e_\varepsilon = \frac{1}{4}(1/2 - \sigma_\varepsilon)^2 e_\varepsilon \leq C\eta_0.$$

Choosing $\sigma = 1/4$ in (5.21) yields

$$\sup_{B_{1/4}^+} |\nabla u_\varepsilon|^2 \leq C\eta_0,$$

which is the announced estimate.

Case 2). We now assume that $R_\varepsilon \leq \rho_\varepsilon/7$, and we claim that if η_0 is small enough, only depending on the dimensions n and m , then

$$R_\varepsilon \leq \frac{1}{7\sqrt{e_\varepsilon}}. \quad (5.23)$$

Indeed, assume that (5.23) does not hold. Then, arguing as in Case 1), we obtain

$$e_\varepsilon \leq \frac{C}{R_\varepsilon^2} \eta_0 \leq C e_\varepsilon \eta_0.$$

Hence $1 \leq C\eta_0$, which is impossible whenever η_0 is small enough.

Next we assume that η_0 is sufficiently small so that (5.23) holds. Then,

$$|x_\varepsilon - \bar{x}_\varepsilon| \leq \min \left\{ \frac{1}{2\sqrt{e_\varepsilon}}, \frac{\rho_\varepsilon}{2} \right\}.$$

Noticing that $B_{\rho_\varepsilon}^+(\bar{x}_\varepsilon) \subseteq B_{\sigma_\varepsilon + \rho_\varepsilon}^+$, we deduce that

$$\sup_{B_{\rho_\varepsilon}^+(\bar{x}_\varepsilon)} |\nabla u_\varepsilon|^2 \leq \sup_{B_{\sigma_\varepsilon + \rho_\varepsilon}^+} |\nabla u_\varepsilon|^2. \quad (5.24)$$

Since $\sigma_\varepsilon + \rho_\varepsilon \in (0, 1/2)$, we infer from (5.21) and (5.24) that

$$\sup_{B_{\rho_\varepsilon}^+(\bar{x}_\varepsilon)} |\nabla u_\varepsilon|^2 \leq 4e_\varepsilon. \quad (5.25)$$

Let us now introduce the quantity

$$r_\varepsilon := \rho_\varepsilon \sqrt{e_\varepsilon}.$$

We claim that

$$r_\varepsilon \leq 1. \quad (5.26)$$

The proof of (5.26) is postponed to the next subsection, and we complete now the argument. Then, taking (5.26) for granted, (5.21) yields

$$\max_{0 \leq \sigma \leq 1/2} \left((1/2 - \sigma)^2 \sup_{B_\sigma^+} |\nabla u_\varepsilon|^2 \right) = (1/2 - \sigma_\varepsilon)^2 \sup_{B_{\sigma_\varepsilon}^+} |\nabla u_\varepsilon|^2 = 4\rho_\varepsilon^2 e_\varepsilon \leq 4.$$

Choosing $\sigma = 1/4$ in the inequality above, we obtain

$$\sup_{B_{1/4}^+} |\nabla u_\varepsilon|^2 \leq 64 \leq C\eta_0,$$

as desired. \square

5.3. Proof of Theorem 5.1, Step 2. We are now going to prove (5.26) by contradiction, assuming that $r_\varepsilon > 1$. The following proposition is the key point of the argument.

Proposition 5.5. *If $\eta_0 > 0$ is small enough (depending only on n and m), then there exists a constant $\varsigma_0 > 0$ (depending only on n and m) such that for each $\varepsilon > 0$ and each map $v_\varepsilon \in C^2(\overline{B_1^+}; \mathbb{R}^m)$ solving*

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{in } B_1^+, \\ \frac{\partial v_\varepsilon}{\partial \nu} = \frac{1}{\varepsilon}(1 - |v_\varepsilon|^2)v_\varepsilon & \text{on } D_1, \end{cases} \quad (5.27)$$

with the estimates

$$1/2 \leq |v_\varepsilon| \leq 1 \quad \text{and} \quad |\nabla v_\varepsilon| \leq 2 \quad \text{in } \overline{B_1^+}, \quad (5.28)$$

and such that

$$|\nabla v_\varepsilon(z_\varepsilon)|^2 \geq \frac{1}{2} \quad (5.29)$$

for some point $z_\varepsilon \in B_{1/2}^+$, the condition

$$E_\varepsilon(v_\varepsilon, B_1^+) \leq 2^{n-1}\eta_0 \quad (5.30)$$

implies $\varepsilon \geq \varsigma_0$.

Proof. We argue by contradiction assuming that there exist a sequence $\varepsilon_k \downarrow 0$, and corresponding solutions $\{v_{\varepsilon_k}\}_{k \in \mathbb{N}}$ of (5.27) satisfying (5.28), (5.29), and (5.30). To simplify the notation, we shall write $v_k := v_{\varepsilon_k}$ and $z_k := z_{\varepsilon_k}$. Note that v_k is smooth in $B_1^+ \cup D_1$ by Theorem 3.2.

Thanks to (5.28) we can consider the smooth functions

$$a_k := |v_k| \quad \text{and} \quad w_k := \frac{v_k}{|v_k|}.$$

Noticing that

$$|\nabla v_k|^2 = |\nabla a_k|^2 + a_k^2 |\nabla w_k|^2, \quad (5.31)$$

we deduce from (5.28) that

$$1/2 \leq a_k \leq 1, \quad |\nabla a_k| \leq 2, \quad |w_k| = 1, \quad |\nabla w_k| \leq 4. \quad (5.32)$$

In addition, system (5.27) yields

$$\begin{cases} -\Delta a_k + |\nabla w_k|^2 a_k = 0 & \text{in } B_1^+, \\ \frac{\partial a_k}{\partial \nu} = \frac{1}{\varepsilon_k}(1 - a_k^2)a_k & \text{on } D_1, \end{cases} \quad (5.33)$$

and

$$\begin{cases} -\operatorname{div}(a_k^2 \nabla w_k) = a_k^2 |\nabla w_k|^2 w_k & \text{in } B_1^+, \\ \frac{\partial w_k}{\partial \nu} = 0 & \text{on } D_1. \end{cases} \quad (5.34)$$

In view of the boundary condition in (5.34), we can extend a_k and w_k to B_1 by even reflection across D_1 , i.e.,

$$a_k(x) = a_k(x', -x_{n+1}) \quad \text{and} \quad w_k(x) = w_k(x', -x_{n+1}) \quad \text{for } x \in B_1 \setminus B_1^+, \quad (5.35)$$

and then derive from (5.34),

$$-\operatorname{div}(a_k^2 \nabla w_k) = a_k^2 |\nabla w_k|^2 w_k \quad \text{in } B_1. \quad (5.36)$$

From (5.32) and the boundary condition in (5.33), we infer that

$$a_k \rightarrow a_* \text{ in } C^{0,\alpha}(\overline{B_1}) \text{ for every } 0 < \alpha < 1, \quad (5.37)$$

for a (not relabeled) subsequence and a function a_* satisfying

$$\frac{1}{2} \leq a_* \leq 1, \quad |\nabla a_*| \leq 2, \quad a_* = 1 \text{ on } D_1. \quad (5.38)$$

From (5.32), (5.36), and standard elliptic regularity, we can find a further subsequence (not relabeled) such that

$$w_k \rightarrow w_* \text{ in } C_{\text{loc}}^{1,\alpha}(B_1) \cap C^{0,\alpha}(\overline{B_1}) \text{ for every } 0 < \alpha < 1, \quad (5.39)$$

for a map w_* satisfying $|w_*| = 1$, $|\nabla w_*| \leq 4$, and solving

$$\begin{cases} -\operatorname{div}(a_*^2 \nabla w_*) = a_*^2 |\nabla w_*|^2 w_* & \text{in } B_1^+, \\ \frac{\partial w_*}{\partial \nu} = 0 & \text{on } D_1. \end{cases} \quad (5.40)$$

In view of (5.38), we infer from [36, Theorem 8.32] that

$$\|w_*\|_{C^{1,\alpha}(B_r)} \leq C_{\alpha,r} \quad (5.41)$$

for every $0 < \alpha < 1$ and every $0 < r < 1$, where $C_{\alpha,r}$ denotes a constant which only depends on n, m, α , and r .

In turn, we infer from the $C_{\text{loc}}^{1,\alpha}$ -convergence of w_k and (5.33) that

$$a_k \rightarrow a_* \text{ in } C_{\text{loc}}^{1,\alpha}(B_1^+) \text{ for every } 0 < \alpha < 1, \quad (5.42)$$

again by elliptic regularity. As a consequence, a_* solves

$$\begin{cases} -\Delta a_* + |\nabla w_*|^2 a_* = 0 & \text{in } B_1^+, \\ a_* = 1 & \text{on } D_1. \end{cases} \quad (5.43)$$

Hence $a_* \in C_{\text{loc}}^{1,\alpha}(B_1^+ \cup D_1)$ for every $0 < \alpha < 1$, and more precisely [36, Corollary 8.36] yields

$$\|a_*\|_{C^{1,\alpha}(B_r^+)} \leq C_{\alpha,r} \quad (5.44)$$

for every $0 < \alpha < 1$ and every $0 < r < 1$.

Finally the convergence of a_k and w_k implies that $v_k \rightarrow v_* := a_* w_*$ in $C_{\text{loc}}^{1,\alpha}(B_1^+) \cap C^{0,\alpha}(B_1^+)$. In addition $v_* \in C_{\text{loc}}^{1,\alpha}(B_1^+ \cup D_1)$ satisfies for every $0 < \alpha < 1$ and every $0 < r < 1$,

$$\|v_*\|_{C^{1,\alpha}(B_r^+)} \leq C_{\alpha,r}, \quad (5.45)$$

thanks to (5.41) and (5.44).

We now claim that v_k actually converges to v_* in the $C^{1,\alpha}$ -topology locally up to the boundary D_1 . Then we shall see that the estimates leading to the $C^{1,\alpha}$ -convergence can be iterated to obtain a $C_{\text{loc}}^\ell(B_1^+ \cup D_1)$ -convergence for every $\ell \in \mathbb{N}$. We shall complete the proof of Proposition 5.5 right after proving the two following lemmas.

Lemma 5.6. *We have $v_k \rightarrow v_*$ in $C_{\text{loc}}^{1,\alpha}(B_1^+ \cup D_1)$ for every $0 < \alpha < 1$.*

Proof. By (5.39) we have $w_k \rightarrow w_*$ in $C_{\text{loc}}^{1,\alpha}(B_1^+ \cup D_1)$, and it is enough to prove that $a_k \rightarrow a_*$ in $C_{\text{loc}}^{1,\alpha}(B_1^+ \cup D_1)$ for every $0 < \alpha < 1$. In view of (5.42), it remains to show the desired convergence near D_1 . We fix $x_0 \in D_1$, $0 < \alpha < 1$, and $0 < r < \operatorname{dist}(x_0, \partial D_1)$ arbitrary. Without loss of generality we may assume that $x_0 = 0$.

We introduce the function

$$h_k := 1 - a_k,$$

and we notice that (5.32) and the boundary condition in (5.33) yield

$$0 \leq h_k \leq 3\varepsilon_k \quad \text{on } D_1. \quad (5.46)$$

Then we define for $y \in D_{r/4}$ fixed,

$$\vartheta_{k,y}(x) := h_k(x+y) - h_k(x) \quad \text{for } x \in \overline{B_{r/2}}^+. \quad (5.47)$$

By (5.32) the Lipschitz constant of h_k is bounded by 2, and hence

$$\|\vartheta_{k,y}\|_{L^\infty(B_{r/2}^+)} \leq 2|y|. \quad (5.48)$$

In addition $\vartheta_{k,y}$ satisfies

$$\begin{cases} -\Delta \vartheta_{k,y} = f_{k,y} & \text{in } B_{r/2}^+, \\ \frac{\varepsilon_k}{a_k(1+a_k)} \frac{\partial \vartheta_{k,y}}{\partial \nu} + \vartheta_{k,y} = \varepsilon_k g_{k,y} & \text{on } D_{r/2}, \end{cases}$$

with

$$f_{k,y}(x) := |\nabla w_k(x+y)|^2 a_k(x+y) - |\nabla w_k(x)|^2 a_k(x),$$

and

$$g_{k,y}(x) := \left(\frac{1}{a_k(x)(1+a_k(x))} - \frac{1}{a_k(x+y)(1+a_k(x+y))} \right) \frac{\partial h_k(x+y)}{\partial \nu}.$$

From (5.32), (5.39), and (5.41), we infer that

$$\|f_{k,y}\|_{L^\infty(B_{r/2}^+)} \leq C_{\alpha,r}|y|^\alpha, \quad (5.49)$$

and

$$\|g_{k,y}\|_{L^\infty(D_{r/2})} \leq C_{\alpha,r}|y|^\alpha. \quad (5.50)$$

Next we consider the unique (variational) solution $\zeta_k \in H^1(B_{r/2}^+)$ of

$$\begin{cases} -\Delta \zeta_k = 1 & \text{in } B_{r/2}^+, \\ \zeta_k = 1 & \text{on } \partial^+ B_{r/2}, \\ \frac{4\varepsilon_k}{3} \frac{\partial \zeta_k}{\partial \nu} + \zeta_k = 0 & \text{on } D_{r/2}, \end{cases}$$

i.e., ζ_k is the unique critical point of the strictly convex and coercive functional

$$J_k(\zeta) := \frac{1}{2} \int_{B_{r/2}^+} |\nabla \zeta|^2 dx - \int_{B_{r/2}^+} \zeta dx + \frac{3}{8\varepsilon_k} \int_{D_{r/2}} \zeta^2 d\mathcal{H}^n$$

defined over the affine space $\{\zeta \in H^1(B_{r/2}^+) : \zeta = 1 \text{ on } \partial^+ B_{r/2}\}$. By Lemma 9.1 in Appendix B, $\zeta_k \in C^{0,\beta}(\overline{B_{r/2}^+}) \cap C^\infty(\overline{B_{r/2}^+} \setminus \partial D_{r/2})$ for some $\beta \in (0, 1)$, and

$$0 \leq \zeta_k \leq C_r \varepsilon_k \quad \text{on } D_{r/4}, \quad (5.51)$$

for a constant C_r which only depends on n and r .

Let us now define

$$\kappa_k(r, y) := \max \left\{ \|\vartheta_{k,y}\|_{L^\infty(B_{r/2}^+)}, \|f_{k,y}\|_{L^\infty(B_{r/2}^+)} \right\} \leq C_{\alpha,r}|y|^\alpha \quad (5.52)$$

(recall (5.48), (5.49), and that $|y| \leq r/4$), and consider the functions

$$H_k^+(x) := \kappa_k(r, y) \zeta_k(x) - \vartheta_{k,y}(x) + \varepsilon_k \|g_{k,y}\|_{L^\infty(D_{r/2})},$$

and

$$H_k^-(x) := \kappa_k(r, y) \zeta_k(x) + \vartheta_{k,y}(x) + \varepsilon_k \|g_{k,y}\|_{L^\infty(D_{r/2})}.$$

By construction H_k^\pm satisfies

$$\begin{cases} -\Delta H_k^\pm \geq 0 & \text{in } B_{r/2}^+, \\ H_k^\pm \geq 0 & \text{on } \partial^+ B_{r/2}, \\ \frac{\varepsilon_n}{a_k(1+a_k)} \frac{\partial H_k^\pm}{\partial \nu} + H_k^\pm \geq 0 & \text{on } D_{r/2}. \end{cases}$$

By the maximum principle and the Hopf boundary lemma, it follows that $H_k^\pm \geq 0$ in $B_{r/2}^+$ (see e.g. the proof of Lemma 9.1). Therefore,

$$|\vartheta_{k,y}(x)| \leq \kappa_k(r, y) \zeta_k(x) + \varepsilon_k \|g_{k,y}\|_{L^\infty(D_{r/2})}. \quad (5.53)$$

Gathering (5.50), (5.51), and (5.52), together this last estimate, we deduce that

$$\left| \frac{h_k(x+y) - h_k(x)}{\varepsilon_k} \right| \leq C_{\alpha,r} |y|^\alpha \quad \text{for every } x, y \in D_{r/4}.$$

In view of (5.46), we have thus proved that

$$\left\| \frac{1 - a_k}{\varepsilon_k} \right\|_{C^{0,\alpha}(D_{r/4})} \leq C_{\alpha,r}. \quad (5.54)$$

From of (5.32) we conclude that

$$\left\| \frac{1}{\varepsilon_k} (1 - a_k^2) a_k \right\|_{C^{0,\alpha}(D_{r/4})} \leq C_{\alpha,r}. \quad (5.55)$$

Applying Lemma 9.2 in Appendix B to equation (5.33) together with estimates (5.39), (5.41), and (5.55), we finally deduce that

$$\|a_k\|_{C^{1,\alpha}(B_{r/8}^+)} \leq C_{\alpha,r},$$

and thus $a_k \rightarrow a_*$ in $C^{1,\alpha'}(B_{r/8}^+)$ for every $0 < \alpha' < \alpha$. \square

Lemma 5.7. *We have $v_k \rightarrow v_*$ in $C_{\text{loc}}^\ell(B_1^+ \cup D_1)$ for every $\ell \in \mathbb{N}$.*

Proof. In view of Lemma 5.6 we can argue by induction assuming that for an integer $\ell \geq 1$, $w_k \rightarrow w_*$ and $a_k \rightarrow a_*$ in $C_{\text{loc}}^{\ell,\alpha}(B_1^+ \cup D_1)$ for every $\alpha \in (0, 1)$. We aim to prove that $w_k \rightarrow w_*$ and $a_k \rightarrow a_*$ in $C_{\text{loc}}^{\ell+1,\alpha}(B_1^+ \cup D_1)$ for every $\alpha \in (0, 1)$. We proceed with the notations used in the proof of Lemma 5.6.

Step 1: Improved convergence of $\{w_k\}$. Using the even reflection across D_1 as stated in (5.35), we first rewrite equation (5.36) as

$$-\Delta w_k = \frac{2}{a_k} \nabla a_k \cdot \nabla w_k + |\nabla w_k|^2 w_k \quad \text{in } B_1. \quad (5.56)$$

(Recall that $a_k \geq 1/2$.) Let us now fix a multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ of length $|\beta| := \sum_i \beta_i = \ell - 1$, and set

$$\partial^\beta := \partial_1^{\beta_1} \partial_2^{\beta_2} \dots \partial_n^{\beta_n}.$$

Using the boundary condition $\partial_\nu w_k = 0$ on D_1 , we notice that the term $\frac{2}{a_k} \nabla a_k \cdot \nabla w_k$ is actually continuous across D_1 . As a consequence, $\{\partial^\beta (\frac{2}{a_k} \nabla a_k \cdot \nabla w_k + |\nabla w_k|^2 w_k)\}$ is bounded in $C_{\text{loc}}^{0,\alpha}(B_1)$ for every $0 < \alpha < 1$, by the induction hypothesis. By elliptic regularity, it implies that $\{\partial^\beta w_k\}$ is bounded in $C_{\text{loc}}^{2,\alpha}(B_1)$ for every $0 < \alpha < 1$.

If $\ell = 1$ the required improved convergence readily follows, and we may now assume that $\ell \geq 2$. In view of the arbitrariness of the multi-index β above, we have showed that

- (A0) $\{\partial^\beta w_k\}$ is bounded in $C_{\text{loc}}^{0,\alpha}(B_1^+ \cup D_1)$ for $\beta \in \mathbb{N}^n$ such that $|\beta| = \ell + 1$ and $0 < \alpha < 1$;
- (B0) $\{\partial_{n+1} \partial^\beta w_k\}$ is bounded in $C_{\text{loc}}^{0,\alpha}(B_1^+ \cup D_1)$ for $\beta \in \mathbb{N}^n$ such that $|\beta| = \ell$ and $0 < \alpha < 1$;
- (C0) $\{\partial_{n+1}^2 \partial^\beta w_k\}$ is bounded in $C_{\text{loc}}^{0,\alpha}(B_1^+ \cup D_1)$ for $\beta \in \mathbb{N}^n$ such that $|\beta| = \ell - 1$ and $0 < \alpha < 1$.

Next we want to estimate the remaining derivatives of w_k of order $\ell + 1$. We proceed by induction assuming that for an integer $0 \leq \gamma \leq \ell - 2$,

- (A γ) $\{\partial_{n+1}^\gamma \partial^\beta w_k\}$ is bounded in $C_{\text{loc}}^{0,\alpha}(B_1^+ \cup D_1)$ for $\beta \in \mathbb{N}^n$ such that $\gamma + |\beta| = \ell + 1$ and $0 < \alpha < 1$;
- (B γ) $\{\partial_{n+1}^{\gamma+1} \partial^\beta w_k\}$ is bounded in $C_{\text{loc}}^{0,\alpha}(B_1^+ \cup D_1)$ for $\beta \in \mathbb{N}^n$ such that $\gamma + |\beta| = \ell$ and $0 < \alpha < 1$;
- (C γ) $\{\partial_{n+1}^{\gamma+2} \partial^\beta w_k\}$ is bounded in $C_{\text{loc}}^{0,\alpha}(B_1^+ \cup D_1)$ for $\beta \in \mathbb{N}^n$ such that $\gamma + |\beta| = \ell - 1$ and $0 < \alpha < 1$,

and we aim to show that $\{\partial_{n+1}^{\gamma+3} \partial^\beta w_k\}$ is bounded in $C_{\text{loc}}^{0,\alpha}(B_1^+ \cup D_1)$ for $\beta \in \mathbb{N}^n$ such that $\gamma + |\beta| = \ell - 2$ and $0 < \alpha < 1$. Using equation (5.56), we have

$$-\partial_{n+1}^{\gamma+3} \partial^\beta w_k = \Delta' (\partial_{n+1}^{\gamma+1} \partial^\beta w_k) + \partial_{n+1}^{\gamma+1} \partial^\beta \left(\frac{2}{a_k} \nabla a_k \cdot \nabla w_k + |\nabla w_k|^2 w_k \right) \quad \text{in } B_1^+,$$

where $\Delta' := \sum_{j=1}^n \partial_j^2$. By (B γ) and the induction hypothesis, the right hand side of the above equation remains bounded in $C_{\text{loc}}^{0,\alpha}(B_1^+ \cup D_1)$, and thus $\{\partial_{n+1}^{\gamma+3} \partial^\beta w_k\}$ is indeed bounded in $C_{\text{loc}}^{0,\alpha}(B_1^+ \cup D_1)$.

In conclusion, we have thus proved that $\{w_k\}$ remains bounded in $C_{\text{loc}}^{\ell+1,\alpha}(B_1^+ \cup D_1)$ for every exponent $0 < \alpha < 1$, and the improved convergence follows.

Step 2: Improved convergence of $\{a_k\}$. By elliptic regularity (applied to (5.33)), the improved convergence of $\{w_k\}$ implies the convergence of a_k in $C_{\text{loc}}^{\ell+1,\alpha}(B_1^+)$ for every $0 < \alpha < 1$. It then remains to improve

the convergence of a_k (or equivalently h_k) up to the boundary D_1 . As in the proof of Lemma 5.6, we consider an arbitrary $x_0 \in D_1$, a radius $0 < 2r < \text{dist}(x_0, \partial D_1)$, and $\alpha \in (0, 1)$. Again we can assume that $x_0 = 0$ without loss of generality. Let us now fix a multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ of length $|\beta| := \sum_i \beta_i = \ell$. Recalling that $h_k := 1 - a_k$, we notice that $\partial^\beta h_k$ solves

$$\begin{cases} -\Delta(\partial^\beta h_k) = \partial^\beta (|\nabla w_k|^2 a_k) & \text{in } B_{2r}^+, \\ \frac{\varepsilon_k}{a_k(1+a_k)} \frac{\partial(\partial^\beta h_k)}{\partial \nu} + \partial^\beta h_k = \varepsilon_k R_{\beta,k} & \text{on } D_{2r}, \end{cases} \quad (5.57)$$

where

$$R_{\beta,k} := \frac{1}{a_k(1+a_k)} \frac{\partial(\partial^\beta h_k)}{\partial \nu} - \partial^\beta \left(\frac{1}{a_k(1+a_k)} \frac{\partial h_k}{\partial \nu} \right)$$

involves only derivatives of a_k of order less than or equal to ℓ . Recalling that $a_k \geq 1/2$, we deduce from the induction hypothesis and the boundedness of $\{w_k\}$ in $C^{\ell+1,\alpha}(B_{2r}^+)$ that

$$m_\beta := \sup_k \left(\|\partial^\beta h_k\|_{L^\infty(B_{2r}^+)} + \|R_{\beta,k}\|_{C^{0,\alpha}(D_{2r})} + \|\partial^\beta (|\nabla w_k|^2 a_k)\|_{C^{0,\alpha}(B_{2r}^+)} \right) < \infty. \quad (5.58)$$

Arguing as in the proof of Lemma 5.6 to derive (5.53), we infer that $|\partial^\beta h_k| \leq C_r m_\beta \varepsilon_k$ on D_r , which then yields

$$\left\| \frac{\partial(\partial^\beta h_k)}{\partial \nu} \right\|_{L^\infty(D_r)} \leq C_r m_\beta. \quad (5.59)$$

Next we consider for $y \in D_{r/4}$ the function $\vartheta_{k,y}$ defined in (5.47). By the induction hypothesis, we have $\|\partial^\beta \vartheta_{k,y}\|_{L^\infty(B_{r/2}^+)} \leq c_1 |y|^\alpha$ for some constant c_1 independent of k . On the other hand, $\partial^\beta \vartheta_{k,y}$ solves

$$\begin{cases} -\Delta(\partial^\beta \vartheta_{k,y}) = f_{k,y}^\beta & \text{in } B_{r/2}^+, \\ \frac{\varepsilon_k}{a_k(1+a_k)} \frac{\partial(\partial^\beta \vartheta_{k,y})}{\partial \nu} + \partial^\beta \vartheta_{k,y} = \varepsilon_k g_{k,y}^\beta & \text{on } D_{r/2}, \end{cases}$$

with

$$f_{k,y}^\beta(x) := \partial^\beta (|\nabla w_k|^2 a_k)(x+y) - \partial^\beta (|\nabla w_k|^2 a_k)(x),$$

and

$$g_{k,y}^\beta(x) := R_{\beta,k}(x+y) - R_{\beta,k}(x) + \left(\frac{1}{a_k(x)(1+a_k(x))} - \frac{1}{a_k(x+y)(1+a_k(x+y))} \right) \frac{\partial(\partial^\beta h_k)}{\partial \nu}(x+y).$$

By the induction hypothesis and (5.59), we have

$$\|f_{k,y}^\beta\|_{L^\infty(B_{r/2}^+)} + \|g_{k,y}^\beta\|_{L^\infty(D_{r/2})} \leq c_2 |y|^\alpha$$

for a constant c_2 independent of k . Arguing precisely as in the proof of Lemma 5.6, we deduce that

$$\left| \frac{\partial^\beta h_k(x+y) - \partial^\beta h_k(x)}{\varepsilon_k} \right| \leq c_3 |y|^\alpha \quad \text{for every } x, y \in D_{r/4},$$

and a constant c_3 independent of k . Whence,

$$\sup_k \left\| \frac{\partial^\beta h_k}{\varepsilon_k} \right\|_{C^{0,\alpha}(D_{r/4})} < \infty,$$

and inserting this estimate together with (5.58) in equation (5.57), we conclude from Lemma 9.2 that $\{\partial^\beta h_k\}$ remains bounded in $C^{1,\alpha}(B_{r/8}^+)$.

In view of the arbitrariness of the multi-index β above, we have showed that

(D0) $\{\partial^\beta h_k\}$ is bounded in $C^{0,\alpha}(B_{r/8}^+)$ for every $\beta \in \mathbb{N}^n$ such that $|\beta| = \ell + 1$;

(E0) $\{\partial_{n+1} \partial^\beta h_k\}$ is bounded in $C^{0,\alpha}(B_{r/8}^+)$ for every $\beta \in \mathbb{N}^n$ such that $|\beta| = \ell$.

To complete the proof we need to estimate the remaining derivatives of h_k of order $\ell + 1$. Again, we proceed by induction assuming that for an integer $0 \leq \gamma \leq \ell - 1$,

(D γ) $\{\partial_{n+1}^\gamma \partial^\beta h_k\}$ is bounded in $C^{0,\alpha}(B_{r/8}^+)$ for every $\beta \in \mathbb{N}^n$ such that $\gamma + |\beta| = \ell + 1$;

(E γ) $\{\partial_{n+1}^{\gamma+1} \partial^\beta h_k\}$ is bounded in $C^{0,\alpha}(B_{r/8}^+)$ for every $\beta \in \mathbb{N}^n$ such that $\gamma + |\beta| = \ell$,

and we aim to show that $\{\partial_{n+1}^{\gamma+2}\partial^\beta h_k\}$ is bounded in $C^{0,\alpha}(B_{r/8}^+)$ for every multi-index $\beta \in \mathbb{N}^n$ such that $\gamma + |\beta| = \ell - 1$. Using equation (5.57), we notice that

$$\partial_{n+1}^{\gamma+2}\partial^\beta h_k = -\partial_{n+1}^\gamma \Delta' \partial^\beta h_k - \partial_{n+1}^\gamma \partial^\beta (|\nabla w_k|^2 a_k) \quad \text{in } B_{r/8}^+. \quad (5.60)$$

Clearly the right hand side of (5.60) is bounded in $C^{0,\alpha}(B_{r/8}^+)$ by (D γ) and the induction hypothesis, and thus $\{\partial_{n+1}^{\gamma+2}\partial^\beta h_k\}$ is indeed bounded in $C^{0,\alpha}(B_{r/8}^+)$.

In conclusion, the sequence $\{a_k\}$ is bounded in $C^{\ell+1,\alpha}(B_{r/8}^+)$, and hence $a_k \rightarrow a_*$ in $C^{\ell+1,\alpha'}(B_{r/8}^+)$ for every $0 < \alpha' < \alpha$. \square

Proof of Proposition 5.5 completed. Up to a subsequence, we have $z_k \rightarrow z_*$ for some point $z_* \in \overline{B}_{1/2}^+$. Thanks to Lemma 5.6 we have $\nabla v_k(z_k) \rightarrow \nabla v_*(z_*)$. Then (5.29) leads to $|\nabla v_*(z_*)|^2 \geq 1/2$. From (5.45) we infer that

$$|\nabla v_*|^2 \geq 1/4 \quad \text{in } B_1^+ \cap B_{\varrho_0}(z_*)$$

for some radius $0 < \varrho_0 \leq 1/4$ which only depends on n and m . Using assumption (5.30) we now estimate

$$\frac{\varrho_0^{n+1}|B_1^+|}{4} \leq \int_{B_1^+} |\nabla v_*|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{B_1^+} |\nabla v_k|^2 dx \leq 2^n \eta_0. \quad (5.61)$$

We then find a contradiction if η_0 is small enough, and the proposition is proved. \square

Proof of (5.26) completed. We argue by contradiction assuming that $r_\varepsilon > 1$. We consider the rescaled function

$$\tilde{u}_\varepsilon(x) := u_\varepsilon \left(\frac{x}{\sqrt{e_\varepsilon}} + \bar{x}_\varepsilon \right) \quad \text{for } x \in \overline{B}_1^+,$$

so that $\tilde{u}_\varepsilon \in C^2(\overline{B}_1^+; \mathbb{R}^m)$ solves

$$\begin{cases} \Delta \tilde{u}_\varepsilon = 0 & \text{in } B_1^+, \\ \frac{\partial \tilde{u}_\varepsilon}{\partial \nu} = \frac{1}{\tilde{\varepsilon}}(1 - |\tilde{u}_\varepsilon|^2)\tilde{u}_\varepsilon & \text{on } D_1, \end{cases}$$

with $\tilde{\varepsilon} := \varepsilon\sqrt{e_\varepsilon}$. Moreover, (5.19) and (5.25) imply that $1/2 \leq |\tilde{u}_\varepsilon| \leq 1$ and $|\nabla \tilde{u}_\varepsilon| \leq 2$ in \overline{B}_1^+ . Considering the point $z_\varepsilon := \sqrt{e_\varepsilon}(x_\varepsilon - \bar{x}_\varepsilon) \in B_{1/2}^+$ (which actually belongs to the half axis $\{0\} \times \mathbb{R}_+$ and satisfies $|z_\varepsilon| = 3R_\varepsilon\sqrt{e_\varepsilon} \leq \frac{3}{7}$), we observe that (5.22) yields

$$|\nabla \tilde{u}_\varepsilon(z_\varepsilon)|^2 \geq \frac{1}{2}. \quad (5.62)$$

On the other hand, (5.20) leads to

$$E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon, B_1) = (\sqrt{e_\varepsilon})^{n-1} E_\varepsilon(u_\varepsilon, B_{(\sqrt{e_\varepsilon})^{-1}}(\bar{x}_\varepsilon)) \leq 2^{n-1} \eta_0. \quad (5.63)$$

If η_0 is small enough, we conclude from Proposition 5.5 that $\tilde{\varepsilon} \geq \varsigma_0$ where $\varsigma_0 > 0$ only depends on n and m . Applying [17, Lemma 2.2] we infer that $\|\tilde{u}_\varepsilon\|_{C^{2,\alpha}(B_{3/4}^+)} \leq C_\alpha$ for every $0 < \alpha < 1$, where C_α only depends on α , n , and m . Arguing as in (5.61), we then find a contradiction between (5.62) and (5.63) whenever η_0 is sufficiently small. \square

6. ASYMPTOTICS FOR GINZBURG-LANDAU BOUNDARY REACTIONS

6.1. Convergence to boundary harmonic maps and defect measures. With Theorem 5.1 in hands, we are now able to give a preliminary description of both weak limits as $\varepsilon \downarrow 0$ of critical points of E_ε , and the possible defect measure arising in the weak convergence process. This is the object of the following theorem.

Theorem 6.1. *Let $\Omega \subseteq \mathbb{R}_+^{n+1}$ be an admissible bounded open set. Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, and let $\{u_k\}_{k \in \mathbb{N}} \subseteq H^1(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega)$ be such that for every $k \in \mathbb{N}$, $|u_k| \leq 1$, and u_k weakly solves*

$$\begin{cases} \Delta u_k = 0 & \text{in } \Omega, \\ \frac{\partial u_k}{\partial \nu} = \frac{1}{\varepsilon_k}(1 - |u_k|^2)u_k & \text{on } \partial^0 \Omega. \end{cases}$$

If $\sup_k E_{\varepsilon_k}(u_k, \Omega) < \infty$, then there exist a (not relabeled) subsequence and a bounded weak $(\mathbb{S}^{m-1}, \partial^0 \Omega)$ -boundary harmonic map u_ in Ω such that $u_k \rightharpoonup u_*$ weakly in $H^1(\Omega)$ as $k \rightarrow \infty$. In addition, there exist a*

finite nonnegative Radon measure μ_{sing} on $\partial^0\Omega$ and a relatively closed set $\Sigma \subseteq \partial^0\Omega$ of finite $(n-1)$ -dimensional Hausdorff measure such that

- (i) $|\nabla u_k|^2 \mathcal{L}^{n+1} \llcorner \Omega \rightharpoonup |\nabla u_*|^2 \mathcal{L}^{n+1} \llcorner \Omega + \mu_{\text{sing}}$ weakly* as Radon measures on $\Omega \cup \partial^0\Omega$;
- (ii) $\frac{(1-|u_k|^2)^2}{\varepsilon_k} \rightarrow 0$ in $L^1_{\text{loc}}(\partial^0\Omega)$;
- (iii) $\Sigma = \text{supp}(\mu_{\text{sing}}) \cup \text{sing}(u_*)$;
- (iv) $u_k \rightarrow u_*$ in $C^{\ell}_{\text{loc}}((\Omega \cup \partial^0\Omega) \setminus \Sigma)$ for every $\ell \in \mathbb{N}$;
- (v) if $n = 1$ the set Σ is finite and $u_* \in C^{\infty}(\Omega \cup \partial^0\Omega)$.

Proof. Step 1. First notice that Theorem 3.2 yields $u_k \in C^{\infty}(\Omega \cup \partial^0\Omega)$. By the uniform energy bound and the assumption $|u_k| \leq 1$, we can find a (not relabeled) subsequence such that $u_k \rightharpoonup u_*$ weakly in $H^1(\Omega)$ for some map $u_* \in H^1(\Omega; \mathbb{R}^m)$. Since $|u_k| \leq 1$ and u_k is harmonic in Ω , we deduce that $u_k \rightarrow u_*$ in $C^{\ell}_{\text{loc}}(\Omega)$ for every $\ell \in \mathbb{N}$, u_* is harmonic in Ω , and $|u_*| \leq 1$ in Ω . On the other hand, $|u_k| \rightarrow 1$ in $L^2(\partial^0\Omega)$, and we infer from the compact imbedding $H^1(\Omega) \hookrightarrow L^2(\partial^0\Omega)$ that $|u_*| = 1$ \mathcal{H}^n -a.e. on $\partial^0\Omega$.

It then remains to analyse the asymptotic behavior of u_k near $\partial^0\Omega$. Setting

$$\mu_k := |\nabla u_k|^2 \mathcal{L}^{n+1} \llcorner \Omega + \frac{1}{2\varepsilon_k} (1 - |u_k|^2)^2 \mathcal{H}^n \llcorner \partial^0\Omega,$$

we have $\sup_k \mu_k(\Omega \cup \partial^0\Omega) < \infty$. Hence we can find a further subsequence such that

$$\mu_k \rightharpoonup \mu := |\nabla u_*|^2 \mathcal{L}^{n+1} \llcorner \Omega + \mu_{\text{sing}}, \quad (6.1)$$

weakly* as Radon measures on $\Omega \cup \partial^0\Omega$ for some nonnegative μ_{sing} on $\Omega \cup \partial^0\Omega$. Notice that the local smooth convergence of u_k to u_* in Ω implies that

$$\text{supp}(\mu_{\text{sing}}) \subseteq \partial^0\Omega \quad (6.2)$$

(here $\text{supp}(\mu_{\text{sing}})$ denotes the relative support of μ_{sing} in $\Omega \cup \partial^0\Omega$). By Lemma 5.2, we have

$$\rho^{1-n} \mu_k(B_{\rho}(x)) \leq r^{1-n} \mu_k(B_r(x)) \quad (6.3)$$

for every $x \in \partial^0\Omega$ and every $0 < \rho < r < \text{dist}(x, \partial^+\Omega)$. Therefore,

$$\rho^{1-n} \mu(B_{\rho}(x)) \leq r^{1-n} \mu(B_r(x)) \quad (6.4)$$

for every $x \in \partial^0\Omega$ and every $0 < \rho < r < \text{dist}(x, \partial^+\Omega)$. As a consequence, the $(n-1)$ -dimensional density

$$\Theta^{n-1}(\mu, x) := \lim_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_{n-1} r^{n-1}} \quad (6.5)$$

exists and is finite at every point $x \in \partial^0\Omega$. Here ω_{n-1} denotes the volume of the unit ball in \mathbb{R}^{n-1} if $n \geq 2$, and $\omega_0 = 1$. Note that (6.1) and (6.3) yield

$$\Theta^{n-1}(\mu, x) \leq \frac{C}{(\text{dist}(x, \partial^+\Omega))^{n-1}} \sup_{k \in \mathbb{N}} E_{\varepsilon_k}(u_k, \Omega) < \infty \quad \text{for all } x \in \partial^0\Omega. \quad (6.6)$$

On the other hand, by the smooth convergence of u_k toward u_* in Ω ,

$$\Theta^{n-1}(\mu, x) = 0 \quad \text{for all } x \in \Omega.$$

In addition, we observe that $x \in \partial^0\Omega \mapsto \Theta^{n-1}(\mu, x)$ is upper semicontinuous.

Next we define the concentration set

$$\Sigma := \left\{ x \in \partial^0\Omega : \inf_r \left\{ \liminf_{k \rightarrow \infty} r^{1-n} \mu_k(B_r(x)) : 0 < r < \text{dist}(x, \partial^+\Omega) \right\} \geq \eta_0 \right\},$$

where $\eta_0 > 0$ is the constant given by Theorem 5.1. From (6.3) and (6.4) we infer that

$$\Sigma = \left\{ x \in \partial^0\Omega : \lim_{r \downarrow 0} \liminf_{k \rightarrow \infty} r^{1-n} \mu_k(B_r(x)) \geq \eta_0 \right\} = \left\{ x \in \partial^0\Omega : \lim_{r \downarrow 0} r^{1-n} \mu(B_r(x)) \geq \eta_0 \right\},$$

and consequently,

$$\Sigma = \left\{ x \in \partial^0\Omega : \Theta^{n-1}(\mu, x) \geq \frac{\eta_0}{\omega_{n-1}} \right\}. \quad (6.7)$$

In particular, Σ is a relatively closed subset of $\partial^0\Omega$ (since $\Theta(\mu, \cdot)$ is upper semicontinuous). Moreover, by a well known property of upper densities (see e.g. [3, Theorem 2.56]), we have

$$\frac{\eta_0}{\omega_{n-1}} \mathcal{H}^{n-1}(\Sigma) \leq \mu(\Sigma) < \infty. \quad (6.8)$$

If $n = 1$, it obviously implies that Σ is finite.

Step 2. Let us now show that $u_k \rightarrow u_*$ in $C_{\text{loc}}^\ell((\Omega \cup \partial^0\Omega) \setminus \Sigma)$ for every $\ell \in \mathbb{N}$. In view of the local smooth convergence of u_k in Ω , it suffices to prove the claim near $\partial^0\Omega$. To this purpose, let us fix $x_0 \in \partial^0\Omega \setminus \Sigma$ and $0 < r < \text{dist}(x_0, \partial^+\Omega \cup \Sigma)$ such that $r^{1-n}\mu(B_r(x_0)) < \eta_0$. By the monotonicity in (6.4), we may assume without loss of generality that $\mu(\partial B_r(x_0)) = 0$. Then $\lim_k \mu_k(B_r(x_0)) = \mu(B_r(x_0))$, which in turn implies that $r^{1-n}\mu_k(B_r(x_0)) \leq \eta_0$ for k sufficiently large. Taking k even larger we have $\varepsilon_k \leq r$, and we can then apply Theorem 5.1 and Lemma 5.4 to deduce that $|\nabla u_k| \leq C_r$ and $1/2 \leq |u_k| \leq 1$ in $B_{r/4}^+(x_0)$. This is now enough to reproduce the convergence proof in Proposition 5.5, Lemma 5.6, and Lemma 5.7. It shows that $u_k \rightarrow u_*$ in $C_{\text{loc}}^\ell(B_{r/4}^+(x_0) \cup D_{r/4}(x_0))$ for every $\ell \in \mathbb{N}$. We finally notice that Theorem 5.1 also provides the estimate $(1 - |u_k|^2)^2 \leq C_r \varepsilon_k^2$ in $D_{r/4}(x_0)$. As a consequence,

$$\lim_{k \rightarrow \infty} \frac{1}{\varepsilon_k} \int_{D_{r/4}(x_0)} (1 - |u_k|^2)^2 d\mathcal{H}^n = 0, \quad (6.9)$$

a fact that we shall use later on.

Step 3. Let us now prove that u_* is a weak \mathbb{S}^{m-1} -boundary harmonic map in $\Omega \cup \partial^0\Omega$. We distinguish the two cases $n = 1$ and $n \geq 2$.

Case 1, $n = 1$. Let $\Phi \in H^1(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega)$ with compact support in $\Omega \cup \partial^0\Omega$ such that $\Phi(x) \in T_{u_*(x)}\mathbb{S}^{m-1}$ for \mathcal{H}^n -a.e. $x \in \partial^0\Omega$. By Step 1 the set $\text{supp } \Phi \cap \Sigma$ contains finitely many points b_1, \dots, b_L . Then, fix an arbitrary cut-off function $\zeta \in C^\infty(\mathbb{R}^{n+1}; [0, 1])$ such that $\zeta = 0$ in a small neighborhood of each b_l . We set $\tilde{\Phi} := \zeta\Phi$, so that $\tilde{\Phi}$ has compact support in $(\Omega \cup \partial^0\Omega) \setminus \Sigma$. From the convergence of u_k established in Step 2, we have $|(1 - |u_k|^2)|\tilde{\Phi}| \leq C\varepsilon_k$ on $\partial^0\Omega$, and thus

$$\lim_{k \rightarrow \infty} \frac{1}{\varepsilon_k} \int_{\partial^0\Omega} (1 - |u_k|^2) u_k \cdot \tilde{\Phi} d\mathcal{H}^n = \lim_{k \rightarrow \infty} \frac{1}{\varepsilon_k} \int_{\partial^0\Omega} (1 - |u_k|^2) (u_k - u_*) \cdot \tilde{\Phi} d\mathcal{H}^n = 0,$$

by dominated convergence. On the other hand,

$$\lim_{k \rightarrow \infty} \int_{\Omega} \nabla u_k \cdot \nabla \tilde{\Phi} dx = \int_{\Omega} \nabla u_* \cdot \nabla \tilde{\Phi} dx,$$

and we deduce that

$$\int_{\Omega} \nabla u_* \cdot \nabla \tilde{\Phi} dx = 0.$$

Given an arbitrary $\delta > 0$, we now choose the cut-off function ζ of the form $\zeta(x) = \chi_\delta(x)\hat{\zeta}(x)$ where $\hat{\zeta} \in C^\infty(\mathbb{R}^{n+1}; [0, 1])$ satisfies $\hat{\zeta} = 0$ in a small neighborhood of b_l only for $l \geq 2$, and $\chi_\delta \in C_c^\infty(\mathbb{R}^{n+1}; [0, 1])$ satisfies $\chi_\delta = 0$ in $B_\delta(b_1)$, $\chi_\delta = 1$ outside $B_{2\delta}(b_1)$, and $|\nabla \chi_\delta| \leq C/\delta$. Setting $\hat{\Phi} = \hat{\zeta}\tilde{\Phi}$, we have

$$\int_{\Omega} \chi_\delta \nabla u_* \cdot \nabla \hat{\Phi} dx + \int_{\Omega \cap B_{2\delta}^+(b_1)} \hat{\Phi} \cdot (\nabla \chi_\delta \cdot \nabla u_*) dx = 0. \quad (6.10)$$

Using Cauchy-Schwarz inequality we estimate

$$\left| \int_{\Omega \cap B_{2\delta}^+(b_1)} \hat{\Phi} \cdot (\nabla \chi_\delta \cdot \nabla u_*) dx \right| \leq C \|\hat{\Phi}\|_{L^\infty(\Omega)} \left(\int_{\Omega \cap B_{2\delta}^+(b_1)} |\nabla u_*|^2 dx \right)^{1/2} \xrightarrow{\delta \downarrow 0} 0.$$

Therefore, letting $\delta \downarrow 0$ in (6.10) leads to

$$\int_{\Omega} \nabla u_* \cdot \nabla \hat{\Phi} dx = 0.$$

We then repeat the argument for each point b_l to reach the conclusion

$$\int_{\Omega} \nabla u_* \cdot \nabla \Phi dx = 0. \quad (6.11)$$

Hence u_* is a weak \mathbb{S}^{m-1} -boundary harmonic map in $\Omega \cup \partial^0\Omega$, and $u_* \in C^\infty(\Omega \cup \partial^0\Omega)$ by Theorem 4.18.

Case 2, $n \geq 2$. Consider an arbitrary Φ as in Case 1, and write $K := \text{supp } \Phi$. From Step 1 we know that $\mathcal{H}^{n-1}(\Sigma \cap K) < \infty$. By [32, Theorem 3, p.154] it implies that $\text{cap}_2(\Sigma \cap K) = 0$ where cap_2 denotes the Newtonian capacity. Moreover, the proof of [32, Theorem 3, p.154] provides a sequence of functions

$\{\chi_l\}_{l \in \mathbb{N}}$ such that $\chi_l \in \dot{H}^1(\mathbb{R}^{n+1})$, $0 \leq \chi_l \leq 1$, $\chi_l = 0$ in a neighborhood of $\Sigma \cap K$, $\chi_l \rightarrow 1$ a.e. as $l \rightarrow \infty$, and

$$\lim_{l \rightarrow \infty} \int_{\mathbb{R}^{n+1}} |\nabla \chi_l|^2 dx = 0.$$

Arguing as in Case 1, we obtain that

$$0 = \int_{\Omega} \nabla u_* \cdot \nabla(\chi_l \Phi) dx = \int_{\Omega} \chi_l \nabla u_* \cdot \nabla \Phi dx + \int_{\Omega} \Phi \cdot (\nabla \chi_l \cdot \nabla u_*) dx, \quad (6.12)$$

and we estimate

$$\left| \int_{\Omega} \Phi \cdot (\nabla \chi_l \cdot \nabla u_*) dx \right| \leq C \|\Phi\|_{L^\infty(\Omega)} \|\nabla u_*\|_{L^2(\Omega)} \left(\int_{\mathbb{R}^{n+1}} |\nabla \chi_l|^2 dx \right)^{1/2} \xrightarrow{l \rightarrow \infty} 0.$$

Letting $l \rightarrow \infty$ in (6.12) then shows that (6.11) holds, whence u_* is a weak \mathbb{S}^{m-1} -boundary harmonic map in $\Omega \cup \partial^0 \Omega$.

Step 4. We conclude the proof by showing that $\Sigma = \text{supp}(\mu_{\text{sing}}) \cup \text{sing}(u_*)$. If x_0 does not belong to $\text{supp}(\mu_{\text{sing}}) \cup \text{sing}(u_*)$, we can find $r_0 > 0$ such that $\mu_{\text{sing}}(B_r(x_0)) = 0$ for all $r < r_0$. Hence,

$$r^{1-n} \mu(B_r(x_0)) = \lim_{n \rightarrow \infty} r^{1-n} \mu_k(B_r(x_0)) = \frac{r^{1-n}}{2} \int_{B_r(x_0) \cap \Omega} |\nabla u_*|^2 dx$$

for all $r < r_0$. Since u_* is smooth in a neighborhood of x_0 by Theorem 4.18, we deduce that $\Theta^{n-1}(\mu, x_0) = 0$, and thus $x_0 \notin \Sigma$ by (6.7).

Let us now assume that $x_0 \notin \Sigma$. If $x_0 \in \Omega$ then $x_0 \notin \text{supp}(\mu_{\text{sing}}) \cup \text{sing}(u_*)$ by (6.2) and the smoothness of u_* in Ω . If $x_0 \in \partial^0 \Omega$ we deduce from (6.9) and the convergence established in Step 2 that $x_0 \notin \text{sing}(u_*)$, and

$$\mu(B_r(x_0)) = \lim_{k \rightarrow \infty} \mu_k(B_r(x_0)) = \frac{1}{2} \int_{B_r(x_0) \cap \Omega} |\nabla u_*|^2 dx$$

for a radius $r > 0$ sufficiently small. Therefore, $\mu_{\text{sing}}(B_r(x_0)) = 0$, and thus x_0 does not belong to $\text{supp}(\mu_{\text{sing}})$.

Step 5. In view of (6.1), it only remains to prove (ii). Let K be a compact subset of $\partial^0 \Omega$, and set $\delta_0 := \frac{1}{2} \text{dist}(K, \partial^+ \Omega)$. For $\delta \in (0, \delta_0/2)$, we define $\Sigma_\delta := \{x \in \partial^0 \Omega : \text{dist}(x, \Sigma) \leq \delta\}$. Then, $K \cap \Sigma_\delta$ is a compact set. From Step 4 and the local smooth convergence of u_k toward u_* outside Σ , we deduce that $\varepsilon_k^{-1} \int_{K \setminus \Sigma_\delta} (1 - |u_k|^2)^2 d\mathcal{H}^n \rightarrow 0$. On the other hand, for any $x_0 \in K \cap \Sigma_\delta$ we have $B_\delta^+(x_0) \subseteq \Omega$, and we infer from the monotonicity formula in Lemma 5.2 that

$$\frac{1}{4\varepsilon_k} \int_\delta^{2\delta} \frac{1}{t^n} \int_{D_t(x_0)} (1 - |u_k|^2)^2 d\mathcal{H}^n dt \leq \frac{1}{\delta_0^{n-1}} E_{\varepsilon_k}(u_k, B_{\delta_0}^+(x_0)).$$

Hence,

$$\limsup_{k \rightarrow \infty} \frac{1}{4\varepsilon_k} \int_{D_\delta(x_0)} (1 - |u_k|^2)^2 d\mathcal{H}^n \leq \frac{2^n \delta^n}{\delta_0^{k-1}} \sup_{n \in \mathbb{N}} E_{\varepsilon_k}(u_k, \Omega).$$

Then, by a standard covering argument, we deduce that

$$\lim_{k \rightarrow \infty} \frac{1}{\varepsilon_k} \int_{\Sigma_\delta} (1 - |u_k|^2)^2 d\mathcal{H}^n = 0,$$

and thus $\varepsilon_k^{-1} \int_K (1 - |u_k|^2)^2 d\mathcal{H}^n \rightarrow 0$ as $k \rightarrow \infty$. \square

To complete this subsection, we now prove the $(n-1)$ -rectifiability of the defect measure μ_{sing} through the celebrated PREISS criteria [59].

Proposition 6.2. *Assume that $n \geq 2$. In Theorem 6.1 the set Σ is countably \mathcal{H}^{n-1} -rectifiable, and the defect measure μ_{sing} satisfies*

$$\mu_{\text{sing}} = \theta \mathcal{H}^{n-1} \llcorner \Sigma \quad (6.13)$$

for some positive Borel function $\theta : \Sigma \rightarrow (0, \infty)$.

Proof. By a well known property of Sobolev functions (see e.g. [80, (3.3.28)]), we have

$$\lim_{r \downarrow 0} \frac{1}{r^{n-1}} \int_{\Omega \cap B_r^+(x)} |\nabla u_*|^2 dx = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma. \quad (6.14)$$

Therefore (6.5) yields

$$\Theta^{n-1}(\mu_{\text{sing}}, x) := \lim_{r \downarrow 0} \frac{\mu_{\text{sing}}(B_r(x))}{\omega_{n-1} r^{n-1}} = \Theta^{n-1}(\mu, x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma. \quad (6.15)$$

On the other hand, we derive from (6.6) that

$$\Theta^{*,n-1}(\mu_{\text{sing}}, x) := \limsup_{r \downarrow 0} \frac{\mu_{\text{sing}}(B_r(x))}{\omega_{n-1} r^{n-1}} \leq \frac{C}{(\text{dist}(x, \partial^+ \Omega))^{n-1}} \sup_{k \in \mathbb{N}} E_{\varepsilon_k}(u_k, \Omega) < \infty$$

for all $x \in \Sigma$. Since $\text{supp}(\mu_{\text{sing}}) \subseteq \Sigma$, we infer from [3, Theorem 2.56] that μ_{sing} is absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner \Sigma$. Then we deduce from (6.15) and (6.5)-(6.6)-(6.7) that

$$\Theta^{n-1}(\mu_{\text{sing}}, \cdot) = \Theta^{n-1}(\mu, \cdot) \in (0, \infty) \quad \mu_{\text{sing}}\text{-a.e.},$$

and according to PREISS rectifiability criteria [59], it implies that μ_{sing} is a $(n-1)$ -rectifiable measure (see e.g. [3, Definition 2.59]). Next we infer from [3, Theorem 2.83], (6.15), and (6.6)-(6.7) that μ_{sing} is of the form (6.13) with $\theta(x) := \Theta^{n-1}(\mu, x) \in (0, \infty)$. As a consequence, Σ is a countably \mathcal{H}^{n-1} -rectifiable set. \square

Remark 6.3 (Approximate tangent space). As a consequence of Proposition 6.2 and the fact that $\Sigma \subseteq \partial^0 \Omega \subseteq \mathbb{R}^n$, the measure μ_{sing} admits an $(n-1)$ -dimensional approximate tangent space $T_x \Sigma \subseteq \mathbb{R}^n$ at x with multiplicity $\theta(x)$ for \mathcal{H}^{n-1} -a.e. $x \in \Sigma$ (see e.g. [3, Theorem 2.83]). More precisely, if we denote by $\mathbf{G}_{n,n-1}$ the Grassmann manifold of unoriented $(n-1)$ -dimensional planes in \mathbb{R}^n , then for \mathcal{H}^{n-1} -a.e. $x \in \Sigma$ there exists a plane $T_x \Sigma \in \mathbf{G}_{n,n-1}$ such that

$$\lim_{r \downarrow 0} \frac{1}{r^{n-1}} \int_{\partial^0 \Omega} \phi \left(\frac{y-x}{r} \right) d\mu_{\text{sing}}(y) = \theta(x) \int_{T_x \Sigma} \phi(y) d\mathcal{H}^{n-1}(y)$$

for all $\phi \in C_c^0(\mathbb{R}^n)$.

6.2. Stationarity defect and generalized varifolds. In this subsection we assume that $n \geq 2$, and our discussion starts from Theorem 6.1. In this theorem, we point out that the limiting $(\mathbb{S}^{m-1}, \partial^0 \Omega)$ -boundary harmonic map u_* is *a priori* only weakly harmonic, and might not be stationary, i.e., it might not satisfy (4.8). On the other hand, u_* arises as a weak limit of (smooth) critical points of the boundary Ginzburg-Landau energy E_ε , so that the lack of stationarity of u_* should be quantified. For the classical Ginzburg-Landau system, the analogous question is treated by LIN & WANG in [52] where they show that the possible stationarity defect is related to the defect measure through an explicit formula. The main objective in this subsection is to prove that a similar formula holds in the Ginzburg-Landau boundary context, and this is the object of the following theorem.

Theorem 6.4. *Assume that $n \geq 2$. In Theorem 6.1, the limiting $(\mathbb{S}^{m-1}, \partial^0 \Omega)$ -boundary harmonic map u_* and the defect measure μ_{sing} represented in (6.13) satisfy*

$$\int_{\Omega} \left(|\nabla u_*|^2 \text{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i u_* \cdot \partial_j u_*) \partial_j \mathbf{X}_i \right) dx + \int_{\Sigma} \theta \text{div}_{\Sigma} \mathbf{X} d\mathcal{H}^{n-1} = 0$$

for all vector fields $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_{n+1}) \in C^1(\bar{\Omega}; \mathbb{R}^{n+1})$ compactly supported in $\Omega \cup \partial^0 \Omega$ and satisfying $\mathbf{X}_{n+1} = 0$ on $\partial^0 \Omega$.

In the spirit of [52], the proof of Theorem 6.4 relies on the notion of *generalized varifold* introduced by AMBROSIO-SONER [4]. To simplify the proof of Theorem 6.4, we shall assume that the admissible open set $\Omega \subseteq \mathbb{R}_+^{n+1}$ satisfies $\Omega = \tilde{\Omega} \cap \mathbb{R}_+^{n+1}$ for some Lipschitz bounded open set $\tilde{\Omega} \subseteq \mathbb{R}^{n+1}$ which is symmetric with respect to the hyperplane $\mathbb{R}^n = \{x_{n+1} = 0\}$. Since all the arguments are local in nature, the general case can be handled in a similar way with minor modifications. According to this symmetry assumption, we introduce some notations. For $B \subseteq \mathbb{R}^{n+1}$, we write $B^+ := B \cap \mathbb{R}_+^{n+1}$ and $B^- := B \cap \mathbb{R}_-^{n+1}$, where $\mathbb{R}_-^{n+1} := \mathbb{R}^n \times (-\infty, 0)$. For a map $u \in H^1(\Omega; \mathbb{R}^m)$, we denote by $\tilde{u} \in H^1(\tilde{\Omega}; \mathbb{R}^m)$ the extension of u to $\tilde{\Omega}$ obtained by even reflection across $\partial^0 \Omega$, i.e.,

$$\tilde{u}(x', x_{n+1}) := \begin{cases} u(x', x_{n+1}) & \text{for } x = (x', x_{n+1}) \in \Omega, \\ u(x', -x_{n+1}) & \text{for } x = (x', x_{n+1}) \in \Omega^- := \tilde{\Omega}^-, \end{cases}$$

and we write $u^- := \tilde{u}|_{\Omega^-}$.

To recall from [4] the concept of generalized varifolds, we need to introduce the following (compact and convex) set of matrices

$$\mathbf{A}_{n-1} := \{A \in \mathbb{R}^{(n+1) \times (n+1)} : A \text{ is symmetric, } \text{trace}(A) = n-1, -(n+1)I_{n+1} \leq A \leq I_{n+1}\},$$

where I_{n+1} denotes the identity square matrix of size $n+1$.

Definition 6.5. A $(n-1)$ -dimensional generalized varifold V on $\tilde{\Omega}$ is a nonnegative Radon measure on $\tilde{\Omega} \times \mathbf{A}_{n-1}$. The class of all generalized $(n-1)$ -varifolds on $\tilde{\Omega}$ is denoted by $\mathbf{V}_{n-1}^*(\tilde{\Omega})$. For $V \in \mathbf{V}_{n-1}^*(\tilde{\Omega})$ we denote by $\|V\|$ the weight of V defined as the first marginal of V , i.e., $\|V\| := \pi_{\#} V$ where $\pi : \tilde{\Omega} \times \mathbf{A}_{n-1} \rightarrow \tilde{\Omega}$ is the canonical projection. (Notice that $\|V\|$ is a Radon measure on $\tilde{\Omega}$.) The first variation δV of a generalized varifold $V \in \mathbf{V}_{n-1}^*(\tilde{\Omega})$ is the element of $(C_c^1(\tilde{\Omega}; \mathbb{R}^{n+1}))^*$ defined by

$$\langle \delta V, \mathbf{X} \rangle := - \int_{\tilde{\Omega} \times \mathbf{A}_{n-1}} A : \nabla \mathbf{X} \, dV \quad \text{for all } \mathbf{X} \in C_c^1(\tilde{\Omega}; \mathbb{R}^{n+1}).$$

If $\delta V = 0$, then V is said to be *stationary*.

Remark 6.6 (Weak convergence of varifolds). The convergence in $\mathbf{V}_{n-1}^*(\tilde{\Omega})$ is understood as weak* convergence of Radon measures on $\tilde{\Omega} \times \mathbf{A}_{n-1}$. In particular, if $V_k \rightharpoonup V$ in $\mathbf{V}_{n-1}^*(\tilde{\Omega})$, then $\delta V_k \rightharpoonup \delta V$ weakly* in $(C_c^1(\tilde{\Omega}; \mathbb{R}^{n+1}))^*$.

Remark 6.7 (Disintegration and barycenter). Given $V \in \mathbf{V}_{n-1}^*(\tilde{\Omega})$, we denote by $\{V_x\}_{x \in \tilde{\Omega}}$ a *disintegration* of V , i.e., $\{V_x\}_{x \in \tilde{\Omega}}$ is a family of probability measures on \mathbf{A}_{n-1} such that $x \mapsto V_x$ is $\|V\|$ -measurable, and

$$\int_{\tilde{\Omega} \times \mathbf{A}_{n-1}} f(x, A) \, dV = \int_{\tilde{\Omega}} \left(\int_{\mathbf{A}_{n-1}} f(x, A) \, dV_x \right) \, d\|V\| \quad (6.16)$$

for any bounded Borel function $f : \tilde{\Omega} \times \mathbf{A}_{n-1} \rightarrow \mathbb{R}$. The measurability condition on $\{V_x\}_{x \in \tilde{\Omega}}$ means that $x \mapsto V_x(B)$ is $\|V\|$ -measurable for every Borel set $B \subseteq \mathbf{A}_{n-1}$. This fact ensures that the inner integral in the right hand side of (6.16) is $\|V\|$ -measurable, so that its integral is well defined. We refer to [3] for the existence and the uniqueness of $\{V_x\}_{x \in \tilde{\Omega}}$ modulo $\|V\|$ -null sets. Throughout the subsection, we may use the disintegrated notation $V = V_x \|V\|$. We also denote by $\bar{A}_V(x)$ the barycenter of V at x defined by

$$\bar{A}_V(x) := \int_{\mathbf{A}_{n-1}} A \, dV_x.$$

Then $x \mapsto \bar{A}_V(x)$ is $\|V\|$ -measurable, and $\bar{A}_V(x) \in \mathbf{A}_{n-1}$ since V_x is a probability measure. Moreover, we can rewrite the action of the first variation δV as

$$\langle \delta V, \mathbf{X} \rangle := - \int_{\tilde{\Omega}} \bar{A}_V(x) : \nabla \mathbf{X} \, d\|V\|$$

for all $\mathbf{X} \in C_c^1(\tilde{\Omega}; \mathbb{R}^{n+1})$.

We may now present the way to relate our problem to generalized varifolds. We start with the construction of a generalized varifold starting from a Sobolev map. For $u \in H^1(\tilde{\Omega}; \mathbb{R}^m)$, we denote by $V_u \in \mathbf{V}_{n-1}^*(\tilde{\Omega})$ the generalized varifold given by

$$V_u := \frac{1}{2} \delta_{A_u} |\nabla u|^2 \mathcal{L}^{n+1} \llcorner \tilde{\Omega},$$

where $\delta_{A_u(x)}$ is the Dirac mass concentrated at the matrix $A_u(x)$ defined by

$$A_u(x) := \begin{cases} I_{n+1} - 2 \frac{(\nabla u(x))^T (\nabla u(x))}{|\nabla u(x)|^2} & \text{if } |\nabla u(x)| \neq 0, \\ I_{n-1} & \text{otherwise,} \end{cases}$$

and I_{n-1} is the matrix of the orthogonal projection on $\mathbb{R}^{n-1} \simeq \mathbb{R}^{n+1} \times \{(0, 0)\}$. One may easily check that $A_u(x) \in \mathbf{A}_{n-1}$, so that V_u indeed belongs to $\mathbf{V}_{n-1}^*(\tilde{\Omega})$. Next we add the Ginzburg-Landau potential in the construction above to build generalized varifolds based on the Ginzburg-Landau boundary energy. More precisely, for $\varepsilon > 0$ and an arbitrary map $u \in H^1(\tilde{\Omega}; \mathbb{R}^m) \cap L^4(\partial^0 \Omega)$, we set

$$V_u^\varepsilon := V_u + \frac{1}{2\varepsilon} \delta_{I_{n-1}} (1 - |u|^2)^2 \mathcal{H}^n \llcorner \partial^0 \Omega \in \mathbf{V}_{n-1}^*(\tilde{\Omega}),$$

so that

$$\|V_u^\varepsilon\|(B) = \frac{1}{2} \int_B |\nabla u|^2 dx + \frac{1}{2\varepsilon} \int_{B \cap \partial^0 \Omega} (1 - |u|^2)^2 d\mathcal{H}^n$$

for any open subset $B \subseteq \tilde{\Omega}$. The first variation of V_u^ε is then given by

$$\langle \delta V_u^\varepsilon, \mathbf{X} \rangle = -\frac{1}{2} \int_{\tilde{\Omega}} \left(|\nabla u|^2 \operatorname{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i u \cdot \partial_j u) \partial_j \mathbf{X}_i \right) dx - \frac{1}{2\varepsilon} \int_{\partial^0 \Omega} (1 - |u|^2)^2 \operatorname{div}_{\mathbb{R}^{n-1}} \mathbf{X} d\mathcal{H}^n$$

for all $\mathbf{X} \in C_c^1(\tilde{\Omega}; \mathbb{R}^{n+1})$. In the case where $u = \tilde{u}_\varepsilon$ and u_ε is a critical point of the Ginzburg-Landau boundary energy E_ε , the first variation reduces to the following expression.

Lemma 6.8. *Given $\varepsilon > 0$, if $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega)$ is a critical point of E_ε in Ω , then*

$$\langle \delta V_{\tilde{u}_\varepsilon}^\varepsilon, \mathbf{X} \rangle = \frac{1}{2\varepsilon} \int_{\partial^0 \Omega} (1 - |u_\varepsilon|^2)^2 \partial_n \mathbf{X}_n d\mathcal{H}^n$$

for all $\mathbf{X} \in C_c^1(\tilde{\Omega}; \mathbb{R}^{n+1})$.

Proof. If $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega)$ is a critical point of E_ε in Ω , then $u_\varepsilon \in C^\infty(\Omega \cup \partial^0 \Omega)$ by Theorem 3.2. It obviously implies $u_\varepsilon^- \in C^\infty(\Omega^- \cup \partial^0 \Omega)$, and

$$\begin{cases} \Delta u_\varepsilon^- = 0 & \text{in } \Omega^-, \\ \frac{\partial u_\varepsilon^-}{\partial \nu} = \frac{1}{\varepsilon} (1 - |u_\varepsilon|^2) u_\varepsilon & \text{on } \partial^0 \Omega. \end{cases} \quad (6.17)$$

Let us now consider a vector field $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_{n+1}) \in C_c^1(\tilde{\Omega}; \mathbb{R}^{n+1})$. Arguing as in the proof of Lemma 5.2, we integrate by parts to find

$$\begin{aligned} & \int_{\Omega} \left(|\nabla u_\varepsilon|^2 \operatorname{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i u_\varepsilon \cdot \partial_j u_\varepsilon) \partial_j \mathbf{X}_i \right) dx \\ &= -\frac{1}{2\varepsilon} \int_{\partial^0 \Omega} (1 - |u_\varepsilon|^2)^2 \operatorname{div}_{\mathbb{R}^n} \mathbf{X} d\mathcal{H}^n - \int_{\partial^0 \Omega} |\nabla u_\varepsilon|^2 \mathbf{X}_{n+1} d\mathcal{H}^n \\ & \quad + \frac{2}{\varepsilon^2} \int_{\partial^0 \Omega} (1 - |u_\varepsilon|^2)^2 |u_\varepsilon|^2 \mathbf{X}_{n+1} d\mathcal{H}^n. \end{aligned}$$

Similarly, (6.17) yields

$$\begin{aligned} & \int_{\Omega^-} \left(|\nabla u_\varepsilon^-|^2 \operatorname{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i u_\varepsilon^- \cdot \partial_j u_\varepsilon^-) \partial_j \mathbf{X}_i \right) dx \\ &= -\frac{1}{2\varepsilon} \int_{\partial^0 \Omega} (1 - |u_\varepsilon|^2)^2 \operatorname{div}_{\mathbb{R}^n} \mathbf{X} d\mathcal{H}^n + \int_{\partial^0 \Omega} |\nabla u_\varepsilon^-|^2 \mathbf{X}_{n+1} d\mathcal{H}^n \\ & \quad - \frac{2}{\varepsilon^2} \int_{\partial^0 \Omega} (1 - |u_\varepsilon|^2)^2 |u_\varepsilon|^2 \mathbf{X}_{n+1} d\mathcal{H}^n. \end{aligned}$$

Therefore,

$$\langle \delta V_{\tilde{u}_\varepsilon}^\varepsilon, \mathbf{X} \rangle = \frac{1}{2\varepsilon} \int_{\partial^0 \Omega} (1 - |u_\varepsilon|^2)^2 \partial_n \mathbf{X}_n d\mathcal{H}^n + \frac{1}{2} \int_{\partial^0 \Omega} (|\partial_{n+1} u_\varepsilon|^2 - |\partial_{n+1} u_\varepsilon^-|^2) \mathbf{X}_{n+1} d\mathcal{H}^n.$$

Since $\partial_{n+1} u_\varepsilon^- = -\partial_{n+1} u_\varepsilon$ by (6.17), the conclusion follows from this last equality. \square

We now relate a weak limit u_* and its defect measure μ_{sing} to the weak limit of the generalized vari-folds $V_{\tilde{u}_\varepsilon}^\varepsilon$, which has to be stationary by Lemma 6.8 and the vanishing property of the Ginzburg-Landau potential as $\varepsilon \rightarrow 0$.

Corollary 6.9. *Let $\{u_k\}_{k \in \mathbb{N}}$ be the subsequence given by Theorem 6.1, and let μ_{sing} be the defect measure represented in (6.13). Up to a further subsequence (not relabeled), $V_{\tilde{u}_k}^{\varepsilon_k} \rightharpoonup V_*$ for some stationary $V_* \in \mathbf{V}_{n-1}^*(\tilde{\Omega})$. In addition,*

$$V_* = V_{\tilde{u}_*} + V_{\text{sing}}, \quad (6.18)$$

where $V_{\text{sing}} \in \mathbf{V}_{n-1}^*(\tilde{\Omega})$ is supported by $\Sigma \times \mathbf{A}_{n-1}$, and $\|V_{\text{sing}}\| = \mu_{\text{sing}}$.

Proof. Step 1. By symmetry with respect to $\{x_{n+1} = 0\}$, we have $\|V_{\tilde{u}_k}^{\varepsilon_k}\|(\tilde{\Omega}) = 2E_{\varepsilon_k}(u_k, \Omega)$, and thus

$$\sup_{k \in \mathbb{N}} \|V_{\tilde{u}_k}^{\varepsilon_k}\|(\tilde{\Omega}) < \infty.$$

Hence we can find a (not relabeled) further subsequence such that $V_{\tilde{u}_k}^{\varepsilon_k} \rightharpoonup V_*$ for some $V_* \in \mathbf{V}_{n-1}^*(\tilde{\Omega})$. Then $\delta V_{\tilde{u}_k}^{\varepsilon_k} \rightharpoonup \delta V_*$ as distributions. We claim that $\delta V_* = 0$. Indeed, for an arbitrary $\mathbf{X} \in C_c^1(\tilde{\Omega}; \mathbb{R}^{n+1})$, we deduce from item (ii) in Theorem 6.1 and Lemma 6.8 that

$$\langle \delta V_*, \mathbf{X} \rangle = \lim_{k \rightarrow \infty} \frac{1}{2\varepsilon_k} \int_{\partial^0 \Omega} (1 - |u_k|^2)^2 \partial_n \mathbf{X}_n \, d\mathcal{H}^n = 0.$$

Step 2. For $\Phi \in C_c^0(\tilde{\Omega})$, we denote by $\hat{\Phi}$ the reflection of Φ with respect to the hyperplane $\{x_{n+1} = 0\}$, i.e., $\hat{\Phi}(x', x_{n+1}) := \Phi(x', -x_{n+1})$ for $x \in \tilde{\Omega}$. Noticing that

$$\int_{\tilde{\Omega}} \Phi \, d\|V_{\tilde{u}_k}^{\varepsilon_k}\| = \frac{1}{2} \int_{\tilde{\Omega}} |\nabla u_k|^2 (\Phi + \hat{\Phi}) \, dx + \frac{1}{2\varepsilon_k} \int_{\partial^0 \Omega} (1 - |u_k|^2)^2 \Phi \, d\mathcal{H}^n,$$

we infer from Theorem 6.1 that

$$\lim_{k \rightarrow \infty} \int_{\tilde{\Omega}} \Phi \, d\|V_{\tilde{u}_k}^{\varepsilon_k}\| = \frac{1}{2} \int_{\tilde{\Omega}} |\nabla u_*|^2 (\Phi + \hat{\Phi}) \, dx + \int_{\Sigma} \Phi \, d\mu_{\text{sing}} = \frac{1}{2} \int_{\tilde{\Omega}} |\nabla \tilde{u}_*|^2 \Phi \, dx + \int_{\Sigma} \Phi \, d\mu_{\text{sing}}.$$

On the other hand, $\|V_{\tilde{u}_k}^{\varepsilon_k}\| \rightharpoonup \|V_*\|$ weakly* as Radon measures on $\tilde{\Omega}$, and hence

$$\|V_*\| = \|V_{\tilde{u}_*}\| + \mu_{\text{sing}}. \quad (6.19)$$

Next consider $\Psi \in C_c^0(\tilde{\Omega} \times \mathbf{A}_{n-1})$ with compact support in $(\tilde{\Omega} \setminus \Sigma) \times \mathbf{A}_{n-1}$. Then,

$$\int_{\tilde{\Omega} \times \mathbf{A}_{n-1}} \Psi \, dV_{\tilde{u}_k}^{\varepsilon_k} = \frac{1}{2} \int_{\tilde{\Omega}} \Psi(x, A_{\tilde{u}_k}(x)) |\nabla \tilde{u}_k|^2 \, dx + \frac{1}{4\varepsilon_k} \int_{\partial^0 \Omega} \Psi(x, I_{n-1}) (1 - |u_k|^2)^2 \, d\mathcal{H}^n.$$

From the convergences established in items (ii) & (iv) of Theorem 6.1, we deduce that

$$\begin{aligned} \int_{\tilde{\Omega} \times \mathbf{A}_{n-1}} \Psi \, dV_* &= \lim_{k \rightarrow \infty} \int_{\tilde{\Omega} \times \mathbf{A}_{n-1}} \Psi \, dV_{\tilde{u}_k}^{\varepsilon_k} \\ &= \lim_{k \rightarrow \infty} \frac{1}{2} \int_{\tilde{\Omega} \cap \{|\nabla \tilde{u}_*| \neq 0\}} \Psi(x, A_{\tilde{u}_k}(x)) |\nabla \tilde{u}_k|^2 \, dx \\ &= \frac{1}{2} \int_{\tilde{\Omega}} \Psi(x, A_{\tilde{u}_*}(x)) |\nabla \tilde{u}_*|^2 \, dx = \int_{\tilde{\Omega} \times \mathbf{A}_{n-1}} \Psi \, dV_{\tilde{u}_*}. \end{aligned}$$

As a consequence, $V_*(\mathcal{K}) = V_{\tilde{u}_*}(\mathcal{K})$ for every compact set $\mathcal{K} \subseteq (\tilde{\Omega} \setminus \Sigma) \times \mathbf{A}_{n-1}$. By inner regularity, it implies

$$V_*(\mathcal{B} \setminus (\Sigma \times \mathbf{A}_{n-1})) = V_{\tilde{u}_*}(\mathcal{B} \setminus (\Sigma \times \mathbf{A}_{n-1})) = V_{\tilde{u}_*}(\mathcal{B})$$

for all Borel sets $\mathcal{B} \subseteq \tilde{\Omega} \times \mathbf{A}_{n-1}$ (in the last equality, we have used the fact that $\|V_{\tilde{u}_*}\|$ is absolutely continuous with respect to \mathcal{L}^{n+1}). Therefore,

$$V_* \llcorner ((\tilde{\Omega} \setminus \Sigma) \times \mathbf{A}_{n-1}) = V_{\tilde{u}_*}.$$

Finally, setting

$$V_{\text{sing}} := V_* \llcorner (\Sigma \times \mathbf{A}_{n-1}),$$

we clearly have (6.18), and $\|V_{\text{sing}}\| = \mu_{\text{sing}}$ holds by (6.19). \square

The last main step to establish Theorem 6.4 is the following geometrical property on the singular part of the limiting varifold.

Lemma 6.10. *In Corollary 6.9, the barycenter of generalized varifold V_{sing} satisfies*

$$\bar{A}_{V_{\text{sing}}}(x) = A_{T_x \Sigma} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma,$$

where $A_{T_x \Sigma}$ is the matrix of the orthogonal projection on the approximate tangent plane $T_x \Sigma$ of Σ at x according to Remark 6.3.

Proof. Let $x_0 \in \Sigma$ be such that Σ admits an approximate tangent plane $T_{x_0}\Sigma$ at x_0 with multiplicity $\theta(x_0)$, x_0 is a Lebesgue point of $A_{V_{\text{sing}}}$ with respect to μ_{sing} , and (6.14) holds at x_0 . Those properties are satisfied \mathcal{H}^{n-1} -a.e. on Σ .

Next consider an arbitrary $\mathbf{X} \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, and set $\mathbf{X}_r(x) := r\mathbf{X}\left(\frac{x-x_0}{r}\right)$. From Corollary 6.9 and the choice of x_0 , we infer that

$$\begin{aligned} 0 &= \lim_{r \downarrow 0} \frac{1}{r^{n-1}} \langle \delta V_*, \mathbf{X}_r \rangle = \lim_{r \downarrow 0} \frac{1}{r^{n-1}} \langle \delta V_{\text{sing}}, \mathbf{X}_r \rangle \\ &= \lim_{r \downarrow 0} \frac{1}{r^{n-1}} \int_{\partial^0 \Omega} \bar{A}_{V_{\text{sing}}}(x) : \nabla \mathbf{X} \left(\frac{x-x_0}{r} \right) d\mu_{\text{sing}} \\ &= \theta(x_0) \int_{T_{x_0}\Sigma} \bar{A}_{V_{\text{sing}}}(x_0) : \nabla \mathbf{X} d\mathcal{H}^{n-1}. \end{aligned}$$

Since $\theta(x_0) > 0$ and \mathbf{X} is arbitrary, we deduce that

$$\bar{A}_{V_{\text{sing}}}(x_0) \int_{T_{x_0}\Sigma} \nabla \phi d\mathcal{H}^{n-1} = 0 \quad \text{for all } \phi \in C_c^1(\mathbb{R}^{n+1}). \quad (6.20)$$

We claim that

$$\mathcal{F} := \left\{ \int_{T_{x_0}\Sigma} \nabla \phi d\mathcal{H}^{n-1} : \phi \in C_c^1(\mathbb{R}^{n+1}) \right\} = \text{Ker } A_{T_{x_0}\Sigma}. \quad (6.21)$$

Before proving (6.21) we complete the argument. From the last two identities we derive that at least two eigenvalues of $\bar{A}_{V_{\text{sing}}}(x_0)$ vanish. On the other hand, $\bar{A}_{V_{\text{sing}}}(x_0) \in \mathbf{A}_{n-1}$ so that $\text{trace}(\bar{A}_{V_{\text{sing}}}(x_0)) = n-1$ and $\bar{A}_{V_{\text{sing}}}(x_0) \leq I_{n+1}$. It implies that the remaining eigenvalues are actually equal to one. Hence $\bar{A}_{V_{\text{sing}}}(x_0)$ is a matrix of an orthogonal projection over an $(n-1)$ -dimensional plane. Now (6.20) and (6.21) show that $\bar{A}_{V_{\text{sing}}}(x_0)$ and $A_{T_{x_0}\Sigma}$ have the same kernel, whence $\bar{A}_{V_{\text{sing}}}(x_0) = A_{T_{x_0}\Sigma}$.

To prove (6.21), we argue as follows. First notice that we may assume without loss of generality that $T_{x_0}\Sigma = \mathbb{R}^{n-1} \times \{(0, 0)\}$. Since the admissible ϕ 's are compactly supported, we obtain

$$\int_{T_{x_0}\Sigma} \partial_j \phi d\mathcal{H}^{n-1} = 0 \quad \text{for all } j \in \{1, \dots, n-1\},$$

and the inclusion $\mathcal{F} \subseteq \text{Ker } A_{T_{x_0}\Sigma}$ follows. To prove the reverse inclusion it suffices to use admissible functions of the form $\phi(x) = \chi(x_n)\psi(x'')$ or $\phi(x) = \chi(x_{n+1})\psi(x'')$ where we write $x = (x'', x_n, x_{n+1})$, $\chi \in C_c^1(\mathbb{R})$ satisfies $\chi'(0) = 1$, and $\psi \in C_c^1(\mathbb{R}^{n-1})$ is such that $\int_{\mathbb{R}^{n-1}} \psi = 1$. \square

Remark 6.11. In view of Proposition 6.2 and Lemma 6.10, the generalized varifold

$$\bar{V}_{\text{sing}} := \delta_{\bar{A}_{V_{\text{sing}}}} \mu_{\text{sing}}$$

is actually a real $(n-1)$ -rectifiable varifold in the classical sense, see [71].

Proof of Theorem 6.4. We first infer from Corollary 6.9, Lemma 6.10, and (6.13) that

$$\begin{aligned} \langle \delta V_*, \mathbf{X} \rangle &= -\frac{1}{2} \int_{\tilde{\Omega}} \left(|\nabla \tilde{u}_*|^2 \text{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i \tilde{u}_* \cdot \partial_j \tilde{u}_*) \partial_j \mathbf{X}_i \right) dx - \int_{\Sigma} A_{T_x \Sigma} : \nabla \mathbf{X} d\mu_{\text{sing}} \\ &= -\frac{1}{2} \int_{\tilde{\Omega}} \left(|\nabla \tilde{u}_*|^2 \text{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i \tilde{u}_* \cdot \partial_j \tilde{u}_*) \partial_j \mathbf{X}_i \right) dx - \int_{\Sigma} \theta \text{div}_{\Sigma} \mathbf{X} d\mathcal{H}^{n-1} = 0, \end{aligned} \quad (6.22)$$

for all $\mathbf{X} \in C_c^1(\tilde{\Omega}; \mathbb{R}^{n+1})$ (we recall that $\text{div}_{\Sigma} \mathbf{X}(x) := A_{T_x \Sigma} : \nabla \mathbf{X}(x)$).

Let us now consider an arbitrary vector field $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_{n+1}) \in C^1(\tilde{\Omega}; \mathbb{R}^{n+1})$ compactly supported in $\Omega \cup \partial^0 \Omega$ and satisfying $\mathbf{X}_{n+1} = 0$ on $\partial^0 \Omega$. We then extend \mathbf{X} to $\tilde{\Omega}$ by setting

$$\hat{\mathbf{X}}(x) := (\tilde{\mathbf{X}}_1(x), \dots, \tilde{\mathbf{X}}_n(x), -\tilde{\mathbf{X}}_{n+1}(x)),$$

where we recall that $\tilde{\mathbf{X}}_j$ is the extension of \mathbf{X}_j to $\tilde{\Omega}$ obtained by even reflection across $\partial^0 \Omega$. Then $\hat{\mathbf{X}}$ is Lipschitz continuous and compactly supported in $\tilde{\Omega}$. Notice also that $\partial_i \hat{\mathbf{X}}_j$ is continuous in $\tilde{\Omega}$ for all indices $i, j \in \{1, \dots, n\}$. Then, by a standard mollification argument, we can find a sequence $\{\mathbf{X}^k\}_{k \in \mathbb{N}} \subseteq C_c^1(\tilde{\Omega}; \mathbb{R}^{n+1})$ such that $\mathbf{X}^k \rightarrow \hat{\mathbf{X}}$ uniformly on $\tilde{\Omega}$, $\partial_i \mathbf{X}_j^k \rightarrow \partial_i \hat{\mathbf{X}}_j$ uniformly on $\tilde{\Omega}$ for all

$i, j \in \{1, \dots, n\}$, $\nabla \mathbf{X}^k \rightarrow \nabla \widehat{\mathbf{X}}$ a.e. on $\widetilde{\Omega}$ with $\|\nabla \mathbf{X}^k\|_{L^\infty(\widetilde{\Omega})} \leq \|\nabla \widehat{\mathbf{X}}\|_{L^\infty(\widetilde{\Omega})}$. Applying (6.22) to each \mathbf{X}^k and letting $k \rightarrow \infty$, we derive by dominated convergence that

$$\frac{1}{2} \int_{\widetilde{\Omega}} \left(|\nabla \tilde{u}_*|^2 \operatorname{div} \widehat{\mathbf{X}} - 2 \sum_{i,j=1}^{n+1} (\partial_i \tilde{u}_* \cdot \partial_j \tilde{u}_*) \partial_i \widehat{\mathbf{X}}_j \right) dx + \int_{\Sigma} \theta \operatorname{div}_{\Sigma} \widehat{\mathbf{X}} d\mathcal{H}^{n-1} = 0.$$

Finally, by symmetry of \tilde{u}_* and $\widehat{\mathbf{X}}$, we have

$$\frac{1}{2} \int_{\widetilde{\Omega}} \left(|\nabla \tilde{u}_*|^2 \operatorname{div} \widehat{\mathbf{X}} - 2 \sum_{i,j=1}^{n+1} (\partial_i \tilde{u}_* \cdot \partial_j \tilde{u}_*) \partial_j \widehat{\mathbf{X}}_i \right) dx = \int_{\Omega} \left(|\nabla u_*|^2 \operatorname{div} \mathbf{X} - 2 \sum_{i,j=1}^{n+1} (\partial_i u_* \cdot \partial_j u_*) \partial_j \mathbf{X}_i \right) dx,$$

and

$$\int_{\Sigma} \theta \operatorname{div}_{\Sigma} \widehat{\mathbf{X}} d\mathcal{H}^{n-1} = \int_{\Sigma} \theta \operatorname{div}_{\Sigma} \mathbf{X} d\mathcal{H}^{n-1},$$

and the conclusion follows. \square

7. ASYMPTOTICS FOR THE FRACTIONAL GINZBURG-LANDAU EQUATION

The purpose of this section is to apply our previous results on the Ginzburg-Landau boundary equation to the asymptotic analysis, as $\varepsilon \downarrow 0$, of solutions of the fractional Ginzburg-Landau equation. We start with the analogue of Theorem 6.1 and Theorem 6.4 in the fractional setting when no exterior condition is imposed. We then prove Theorem 1.1. We conclude this section with the particular case of minimizers of the Ginzburg-Landau 1/2-energy under a Dirichlet exterior condition.

7.1. Asymptotics without exterior Dirichlet condition. We start in this subsection with a general case where no exterior condition is imposed. We obtain here the most important convergence results.

Theorem 7.1. *Let $\omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, and let $\{v_k\}_{k \in \mathbb{N}} \subseteq \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n)$ be such that for every $k \in \mathbb{N}$, $|v_k| \leq 1$, and v_k weakly solves*

$$(-\Delta)^{\frac{1}{2}} v_k = \frac{1}{\varepsilon_k} (1 - |v_k|^2) v_k \quad \text{in } \omega. \quad (7.1)$$

If $\sup_k \mathcal{E}_{\varepsilon_k}(v_k, \omega) < \infty$, then there exist a (not relabeled) subsequence and $v_ \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ a bounded weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω such that $v_k \rightharpoonup v_*$ weakly in $H^{1/2}(\omega) \cap L^2_{\text{loc}}(\mathbb{R}^n)$ as $k \rightarrow \infty$. In addition, there exist a nonnegative Radon measure μ_{sing} on ω , a countably \mathcal{H}^{n-1} -rectifiable relatively closed set $\Sigma \subseteq \omega$ of locally finite $(n-1)$ -dimensional Hausdorff measure in ω , and a Borel function $\theta : \Sigma \rightarrow (0, \infty)$ such that*

- (i) $|\nabla v_k^\varepsilon|^2 \mathcal{L}^{n+1} \llcorner \mathbb{R}_+^{n+1} \xrightarrow{*} |\nabla v_*|^2 \mathcal{L}^{n+1} \llcorner \mathbb{R}_+^{n+1} + \mu_{\text{sing}}$ locally weakly* as Radon measures on $\mathbb{R}_+^{n+1} \cup \omega$;
- (ii) $\frac{(1 - |v_k|^2)^2}{\varepsilon_k} \rightarrow 0$ in $L^1_{\text{loc}}(\omega)$;
- (iii) $\mu_{\text{sing}} = \theta \mathcal{H}^{n-1} \llcorner \Sigma$;
- (iv) $v_* \in C^\infty(\omega \setminus \Sigma)$ and $v_k \rightarrow v_*$ in $C^\ell_{\text{loc}}(\omega \setminus \Sigma)$ for every $\ell \in \mathbb{N}$;
- (v) if $n \geq 2$, the limiting 1/2-harmonic map v_* and the defect measure μ_{sing} satisfy

$$\left[\frac{d}{dt} \mathcal{E}(v_* \circ \phi_t, \omega) \right]_{t=0} = \frac{1}{2} \int_{\Sigma} \operatorname{div}_{\Sigma} X d\mu_{\text{sing}}$$

for all vector fields $X \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ compactly supported in ω , where $\{\phi_t\}_{t \in \mathbb{R}}$ denotes the flow on \mathbb{R}^n generated by X ;

- (vi) if $n = 1$, the set Σ is locally finite in ω and $v_* \in C^\infty(\omega)$.

Proof. Step 1. From the assumptions $|v_k| \leq 1$ and $\sup_k \mathcal{E}_{\varepsilon_k}(v_k, \omega) < \infty$, we first deduce from Lemma 2.1 that the sequence $\{v_k\}$ is bounded in $L^2(\mathbb{R}^n, \mathfrak{m})$, where the measure \mathfrak{m} is defined in (2.10). Therefore, we can find a subsequence and $v_* \in L^2(\mathbb{R}^n, \mathfrak{m})$ such that $v_k \rightharpoonup v_*$ weakly in $L^2(\mathbb{R}^n, \mathfrak{m})$. In particular, $v_k \rightharpoonup v_*$ weakly in $L^2_{\text{loc}}(\mathbb{R}^n)$. On the other hand, the uniform energy bound also shows that $|v_k|^2 \rightarrow 1$ in $L^2(\omega)$, and $\{v_k\}$ is bounded in $H^{1/2}(\omega)$. Hence $v_k \rightharpoonup v_*$ weakly in $H^{1/2}(\omega)$, and from the compact embedding $H^{1/2}(\omega) \hookrightarrow L^2(\omega)$, it implies that $v_k \rightarrow v_*$ strongly in $L^2(\omega)$. In particular, $|v_*| = 1$ a.e. in ω .

We now claim that $v_* \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$, and more precisely that

$$\mathcal{E}(v_*, \omega) \leq \liminf_{k \rightarrow \infty} \mathcal{E}(v_k, \omega) < \infty.$$

In view of the weak convergence of $\{v_k\}$ in $H^{1/2}(\omega)$, it remains to show that

$$\iint_{\omega \times \omega^c} \frac{|v_*(x) - v_*(y)|^2}{|x - y|^{n+1}} dx dy \leq \liminf_{k \rightarrow \infty} \iint_{\omega \times \omega^c} \frac{|v_k(x) - v_k(y)|^2}{|x - y|^{n+1}} dx dy. \quad (7.2)$$

We fix $R > 0$ and $0 < \delta < R/2$ such that $\omega \subseteq D_{R/2}$. Set $\omega_\delta := \{x \in \mathbb{R}^n : \text{dist}(x, \omega) < \delta\}$, and write

$$\begin{aligned} \iint_{\omega \times (\omega_\delta)^c \cap D_R} \frac{|v_k(x) - v_k(y)|^2}{|x - y|^{n+1}} dx dy &= \int_\omega |v_k(x)|^2 w_1(x) dx \\ &+ \int_{(\omega_\delta)^c \cap D_R} |v_k(x)|^2 w_2(x) dx - 2 \int_{(\omega_\delta)^c \cap D_R} v_k(x) \cdot f_k(x) dx =: I_k + II_k + III_k, \end{aligned} \quad (7.3)$$

where

$$w_1(x) := \int_{(\omega_\delta)^c \cap D_R} \frac{dy}{|x - y|^{n+1}}, \quad w_2(x) := \int_\omega \frac{dy}{|x - y|^{n+1}}, \quad f_k(x) := \int_\omega \frac{v_k(y) dy}{|x - y|^{n+1}}.$$

By the strong $L^2(\omega)$ -convergence of v_k toward v_* , f_k converges strongly in $L^2((\omega_\delta)^c \cap D_R)$ to the function $f_*(x) := \int_\omega \frac{v_*(y) dy}{|x - y|^{n+1}}$. From the weak $L^2_{\text{loc}}(\mathbb{R}^n)$ -convergence of v_k , we deduce that

$$\int_\omega |v_*(x)|^2 w_1(x) dx \leq \liminf_{k \rightarrow \infty} I_k, \quad \int_{(\omega_\delta)^c \cap D_R} |v_*(x)|^2 w_2(x) dx \leq \liminf_{k \rightarrow \infty} II_k,$$

and

$$\lim_{k \rightarrow \infty} III_k = -2 \int_{(\omega_\delta)^c \cap D_R} v_*(x) \cdot f_*(x) dx.$$

Therefore,

$$\iint_{\omega \times (\omega_\delta)^c \cap D_R} \frac{|v_*(x) - v_*(y)|^2}{|x - y|^{n+1}} dx dy \leq \liminf_{k \rightarrow \infty} \iint_{\omega \times \omega^c} \frac{|v_k(x) - v_k(y)|^2}{|x - y|^{n+1}} dx dy,$$

and (7.2) follows letting $R \rightarrow \infty$ and $\delta \rightarrow 0$.

We end this first step showing that $v_n^e \rightharpoonup v_*^e$ weakly in $H^1_{\text{loc}}(\mathbb{R}_+^{n+1} \cup \omega)$. Indeed, we start deducing from Lemma 2.5 that $v_n^e \rightharpoonup v_*^e$ weakly in $L^2_{\text{loc}}(\mathbb{R}_+^{n+1})$. On the other hand, the uniform energy bound together with Lemma 2.7 and standard estimates on harmonic functions, shows that $\{v_k^e\}$ is bounded in $H^1_{\text{loc}}(\mathbb{R}_+^{n+1} \cup \omega)$, whence the announced weak convergence.

Step 2. Let us now consider an increasing sequence $\{\Omega_l\}_{l \in \mathbb{N}}$ of bounded admissible open sets such that $\overline{\partial^0 \Omega_l} \subseteq \omega$ for every $l \in \mathbb{N}$, $\cup_l \Omega_l = \mathbb{R}_+^{n+1}$, and $\cup_l \partial^0 \Omega_l = \omega$. By (2.11), Step 1, and the results in Section 3, $v_k^e \in H^1(\Omega_l; \mathbb{R}^m) \cap L^\infty(\Omega_l)$ satisfies $|v_k^e| \leq 1$ and solves

$$\begin{cases} \Delta v_k^e = 0 & \text{in } \Omega_l, \\ \frac{\partial v_k^e}{\partial \nu} = \frac{1}{\varepsilon_k} (1 - |v_k^e|^2) v_k^e & \text{on } \partial^0 \Omega_l, \end{cases}$$

for every $l \in \mathbb{N}$. In addition, we have proved in Step 1 that $\sup_k E_{\varepsilon_k}(v_k^e, \Omega_l) < \infty$ for every $l \in \mathbb{N}$. Therefore, we can find a further subsequence such that the conclusions of Theorems 6.1 & 6.4 hold in every Ω_l , and v_*^e is the limiting $(\mathbb{S}^{m-1}, \partial^0 \Omega_l)$ -boundary harmonic map in each Ω_l by Step 1. This yields the announced conclusions on the defect measure μ_{sing} and on the concentration set Σ stated in (i), (ii), (iii), and (iv). The stationarity relation stated in (v) between v_* and μ_{sing} is in turn a direct consequence of Theorem 6.4 and Lemma 4.16. Then it only remains to prove that v_* is a weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω . By Proposition 4.2, it is enough to check that $\langle (-\Delta)^{\frac{1}{2}} v_*, \varphi \rangle_\omega = 0$ for every $\varphi \in H^{1/2}_0(\omega; \mathbb{R}^m) \cap L^\infty(\omega)$ compactly supported in ω satisfying $v_* \cdot \varphi = 0$ a.e. in ω . Given such a test function φ , we consider an arbitrary extension $\Phi \in H^1(\mathbb{R}_+^{n+1}; \mathbb{R}^m) \cap L^\infty(\mathbb{R}_+^{n+1})$ of φ which is compactly supported in $\mathbb{R}_+^{n+1} \cup \omega$. Then $\text{supp } \Phi \subseteq \Omega_l \cup \partial^0 \Omega_l$ for l large enough. Since v_*^e is a weak $(\mathbb{S}^{m-1}, \partial^0 \Omega_l)$ -boundary harmonic map in Ω_l , we infer from Lemma 2.9 that

$$\langle (-\Delta)^{\frac{1}{2}} v_*, \varphi \rangle_\omega = \int_{\Omega_l} \nabla v_*^e \cdot \nabla \Phi dx = 0,$$

and the proof is complete. \square

Remark 7.2. We notice that Theorem 6.1 actually implies that $v_k^e \rightarrow v_*^e$ in $C^\ell_{\text{loc}}(\mathbb{R}_+^{n+1} \cup (\omega \setminus \Sigma))$ for every $\ell \in \mathbb{N}$. In view of Lemma 2.9, it shows that $(-\Delta)^{\frac{1}{2}} v_k \rightarrow (-\Delta)^{\frac{1}{2}} v_*$ in $C^\ell_{\text{loc}}(\omega \setminus \Sigma)$ for every $\ell \in \mathbb{N}$.

7.2. Asymptotics with Dirichlet exterior condition. This section is devoted to proof of Theorem 1.1. To this aim, we consider for the rest of this subsection a smooth bounded open set $\omega \subseteq \mathbb{R}^n$, and a smooth exterior Dirichlet condition $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfying $|g| = 1$ in $\mathbb{R}^n \setminus \omega$.

Given an arbitrary sequence $\varepsilon_k \downarrow 0$, we consider $\{v_k\}_{k \in \mathbb{N}} \subseteq H_g^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ such that for each $k \in \mathbb{N}$, v_k weakly solves (7.1), and $\sup_k \mathcal{E}_{\varepsilon_k}(v_k, \omega) < \infty$. We recall that under such assumptions, we have proved in Section 3 that $v_k \in C_{\text{loc}}^{0,\alpha}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \partial\omega)$ for every $0 < \alpha < 1/2$, and that $|v_k| \leq 1$ in \mathbb{R}^n . Therefore, we can apply Theorem 7.1 to find a (not relabeled) subsequence such that $v_k \rightharpoonup v_*$ weakly in $H^{1/2}(\omega) \cap L_{\text{loc}}^2(\mathbb{R}^n)$ for some map $v_* \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ which is a bounded weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω , and such that all the conclusions of Theorem 7.1 hold. Since v_k is constantly equal to g outside ω , we also infer that $v_* \in H_g^{1/2}(\omega; \mathbb{R}^m)$.

It now remains to prove that

- (a) $v_k - v_* \rightarrow 0$ weakly in $H_{00}^{1/2}(\omega)$;
- (b) μ_{sing} is a finite measure on ω , $\mathcal{H}^{n-1}(\Sigma) < \infty$, and statements (i) and (iii) in Theorem 1.1 hold.

Proof of (a). We first notice that $v_k - v_* \in H_{00}^{1/2}(\omega; \mathbb{R}^m)$ for every $k \in \mathbb{N}$ by (2.4). Hence,

$$[v_k - v_*]_{H^{1/2}(\mathbb{R}^n)}^2 = \mathcal{E}(v_k - v_*, \omega) \leq 2(\mathcal{E}(v_k, \omega) + \mathcal{E}(v_*, \omega)),$$

so that $\{v_k - v_*\}$ is bounded in $H_{00}^{1/2}(\omega)$. Since $v_k - v_* \rightarrow 0$ weakly in $H^{1/2}(\omega)$, it remains to show that

$$\iint_{\omega \times \omega^c} \frac{(v_k(x) - v_*(x)) \cdot \varphi(x)}{|x - y|^{n+1}} dx dy \rightarrow 0$$

for every $\varphi \in H_{00}^{1/2}(\omega; \mathbb{R}^m)$. First notice that $v_k \rightarrow v_*$ strongly in $L^2(\omega)$ by the compact embedding $H^{1/2}(\omega) \hookrightarrow L^2(\omega)$. Then, by density of smooth maps compactly supported in ω , it suffices to consider the case where $\varphi \in \mathcal{D}(\omega)$. For such a test function φ , the assertion is easily proved using the dominated convergence Theorem and the fact that $v_k \rightarrow v_*$ a.e. in ω (up to a further subsequence if necessary). \square

Proof of (b). From the uniform energy bound, we may find a further subsequence such that

$$e(v_k, \omega) \cdot \mathcal{L}^n \llcorner \omega \xrightarrow{*} e(v_*, \omega) \cdot \mathcal{L}^n \llcorner \omega + \mu_{\text{def}}$$

weakly* as Radon measures on ω for some finite nonnegative measure μ_{def} . We thus have to prove that $\mu_{\text{def}} \equiv \mu_{\text{sing}}$. It will then show that μ_{sing} is finite, and that $\mathcal{H}^{n-1}(\Sigma) < \infty$ (since $\mathcal{H}^{n-1}(\Sigma)$ is controlled by the total variation of μ_{sing} , see (6.8)).

Let us now fix $\varphi \in \mathcal{D}(\omega)$ arbitrary. We notice that

$$\begin{aligned} \int_{\omega} e(v_k, \omega) \varphi dx &= \langle (-\Delta)^{\frac{1}{2}} v_k, \varphi v_k \rangle_{\omega} - \frac{\gamma_n}{2} \iint_{\omega \times \omega} \frac{(v_k(x) - v_k(y)) \cdot v_k(y) (\varphi(x) - \varphi(y))}{|x - y|^{n+1}} dx dy \\ &\quad - \gamma_n \iint_{\omega \times \omega^c} \frac{(v_k(x) - g(y)) \cdot g(y) \varphi(x)}{|x - y|^{n+1}} dx dy \\ &=: I_k - II_k - III_k. \end{aligned}$$

We consider a function $\Phi \in C^\infty(\overline{\mathbb{R}_+^{n+1}})$ compactly supported in $\mathbb{R}_+^{n+1} \cup \omega$ such that $\Phi|_{\mathbb{R}^n} = \varphi$, and set $K := \text{supp}(\Phi)$. We observe that Φv_k^e is a smooth function compactly supported by K , so that Lemma 2.9 yields

$$\langle (-\Delta)^{\frac{1}{2}} v_k, \varphi v_k \rangle_{\omega} = \int_{\mathbb{R}_+^{k+1}} \nabla v_k^e \cdot \nabla(\Phi v_k^e) dx = \int_{\mathbb{R}_+^{n+1}} |\nabla v_k^e|^2 \Phi dx + \int_K \nabla v_k^e \cdot (v_k^e \nabla \Phi) dx.$$

From (a) we deduce that $v_k^e \rightharpoonup v_*^e$ weakly in $H_{\text{loc}}^1(\overline{\mathbb{R}_+^{n+1}})$, so that $\nabla v_k^e \rightharpoonup \nabla v_*^e$ weakly in $L^2(K)$ and $v_k^e \rightarrow v_*^e$ strongly in $L^2(K)$. Together with item (i) in Theorem 7.1, it yields

$$\begin{aligned} \langle (-\Delta)^{\frac{1}{2}} v_k, \varphi v_k \rangle_{\omega} &\xrightarrow{k \rightarrow \infty} \int_{\mathbb{R}_+^{n+1}} |\nabla v_*^e|^2 \Phi dx + \int_{\omega} \varphi d\mu_{\text{sing}} + \int_K \nabla v_*^e \cdot (v_*^e \nabla \Phi) dx \\ &= \int_{\mathbb{R}_+^{n+1}} \nabla v_*^e \cdot \nabla(\Phi v_*^e) dx + \int_{\omega} \varphi d\mu_{\text{sing}}. \end{aligned}$$

By Lemma 2.9 again, we have thus proved that

$$\langle (-\Delta)^{\frac{1}{2}} v_k, \varphi v_k \rangle_{\omega} \xrightarrow{k \rightarrow \infty} \langle (-\Delta)^{\frac{1}{2}} v_*, \varphi v_* \rangle_{\omega} + \int_{\omega} \varphi d\mu_{\text{sing}}, \quad (7.4)$$

where we have used the (elementary) fact that $\varphi v_* \in H_{00}^{1/2}(\omega; \mathbb{R}^m)$.

On the other hand, we have that $v_k \rightarrow v_*$ a.e. on \mathbb{R}^n , eventually after the extraction of a further subsequence. Using $\varphi \in \mathcal{D}(\omega)$ and the weak convergence of v_k in $H^{1/2}(\omega)$, we deduce that

$$II_k \rightarrow \frac{\gamma_n}{2} \iint_{\omega \times \omega} \frac{(v_*(x) - v_*(y)) \cdot v_*(y) (\varphi(x) - \varphi(y))}{|x - y|^{n+1}} dx dy \quad (7.5)$$

and by dominated convergence,

$$III_k \rightarrow \gamma_n \iint_{\omega \times \omega^c} \frac{(v_*(x) - g(y)) \cdot g(y) \varphi(x)}{|x - y|^{n+1}} dx dy \quad (7.6)$$

as $k \rightarrow \infty$. Gathering (7.4), (7.5), and (7.6) leads to

$$\int_{\omega} e(v_k, \omega) \varphi dx \xrightarrow{k \rightarrow \infty} \int_{\omega} e(v_*, \omega) \varphi dx + \int_{\omega} \varphi d\mu_{\text{sing}},$$

and thus $\mu_{\text{def}} = \mu_{\text{sing}}$ by the arbitrariness of φ .

To derive item (iii), we just notice that

$$\begin{aligned} \int_{\omega} \frac{(1 - |v_k|^2)}{\varepsilon_k} \varphi dx &= \int_{\omega} \frac{(1 - |v_k|^2)^2}{\varepsilon_k} \varphi dx + \frac{1}{\varepsilon_k} \int_{\omega} (1 - |v_k|^2) v_k \cdot (\varphi v_k) dx \\ &= \int_{\omega} \frac{(1 - |v_k|^2)^2}{\varepsilon_k} \varphi dx + \langle (-\Delta)^{\frac{1}{2}} v_k, \varphi v_k \rangle_{\omega}, \end{aligned}$$

so that the announced convergence follows from item (ii), (7.4), and Remark 4.3. \square

Remark 7.3. In view of Remarks 4.3 & 7.2, we have

$$\frac{1 - |v_k(x)|^2}{\varepsilon_k} \rightarrow \frac{\gamma_n}{2} \int_{\mathbb{R}^n} \frac{|v_*(x) - v_*(y)|^2}{|x - y|^{n+1}} dy \text{ in } C_{\text{loc}}^{\ell}(\omega \setminus \Sigma)$$

for every $\ell \in \mathbb{N}$.

7.3. Asymptotics for Dirichlet minimizers. We finally consider solutions of the minimization problem (3.2), and we show that, in this case, no concentration occurs by minimality.

Theorem 7.4. *Let $\omega \subseteq \mathbb{R}^n$ be a smooth bounded open set, and let $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a smooth map satisfying $|g| = 1$ in $\mathbb{R}^n \setminus \omega$, and such that*

$$H_g^{1/2}(\omega; \mathbb{S}^{m-1}) := \left\{ v \in H_g^{1/2}(\omega; \mathbb{R}^m) : |v| = 1 \text{ a.e. in } \omega \right\} \neq \emptyset. \quad (7.7)$$

Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, and let $\{v_k\}_{k \in \mathbb{N}} \subseteq H_g^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega)$ be such that for each $k \in \mathbb{N}$,

$$v_k \in \operatorname{argmin} \left\{ \mathcal{E}_{\varepsilon_k}(v, \omega) : v \in H_g^{1/2}(\omega; \mathbb{R}^m) \cap L^4(\omega) \right\}. \quad (7.8)$$

Then there exist a (not relabeled) subsequence and $v_* \in H_g^{1/2}(\omega; \mathbb{R}^m)$ a minimizing 1/2-harmonic map into \mathbb{S}^{m-1} in ω such that $v_k - v_* \rightarrow 0$ strongly in $H_{00}^{1/2}(\omega)$. In addition,

- (i) $\mathcal{E}_{\varepsilon_k}(v_k, \omega) \rightarrow \mathcal{E}(v_*, \omega)$;
- (ii) $\frac{1 - |v_k(x)|^2}{\varepsilon_k} \rightarrow \frac{\gamma_n}{2} \int_{\mathbb{R}^n} \frac{|v_*(x) - v_*(y)|^2}{|x - y|^{n+1}} dy$ in $\mathcal{D}'(\omega)$;
- (iii) $v_n \rightarrow v_*$ in $C_{\text{loc}}^{\ell}(\omega \setminus \operatorname{sing}(v_*))$ for every $\ell \in \mathbb{N}$;
- (iv) if $n = 1$, then $\omega \cap \operatorname{sing}(v_*) = \emptyset$, while $\dim_{\mathcal{H}}(\operatorname{sing}(v_*) \cap \omega) \leq n - 2$ for $n \geq 3$, and $\operatorname{sing}(v_*) \cap \omega$ is discrete for $n = 2$.

Proof. By assumption (7.7) there exists $\tilde{g} \in H_g^{1/2}(\omega; \mathbb{R}^m)$ such that $|\tilde{g}| = 1$ a.e. in ω . Then we infer from (7.8) that for all $k \in \mathbb{N}$,

$$\mathcal{E}_{\varepsilon_k}(v_k, \omega) \leq \mathcal{E}_{\varepsilon_k}(\tilde{g}, \omega) = \mathcal{E}(\tilde{g}, \omega).$$

Therefore $\sup_k \mathcal{E}_{\varepsilon_k}(v_k, \omega) < \infty$, and we can extract a (not relabeled) subsequence such that the conclusions of Theorem 1.1 do hold for a map $v_* \in \widehat{H}_g^{1/2}(\omega; \mathbb{R}^m)$ which is a bounded weak 1/2-harmonic map into \mathbb{S}^{m-1} in ω , and a finite measure μ_{sing} on ω . By the compact embedding $H^{1/2}(\omega) \hookrightarrow L^2(\omega)$, we may assume that $v_k \rightarrow v_*$ strongly in $L^2(\omega)$. Since $v_k = v_* = g$ outside ω , we deduce that $v_k \rightarrow v_*$ a.e.

in \mathbb{R}^n , eventually up to a further subsequence. By (7.7) and the lower semicontinuity of the 1/2-Dirichlet energy with respect to a.e. pointwise convergence, we have

$$\mathcal{E}(v, \omega) \geq \liminf_{k \rightarrow \infty} \mathcal{E}_{\varepsilon_k}(v_k, \omega) \geq \liminf_{k \rightarrow \infty} \mathcal{E}(v_k, \omega) \geq \mathcal{E}(v_*, \omega)$$

for all $v \in H_g^{1/2}(\omega; \mathbb{S}^{m-1})$. Hence v_* is a minimizing 1/2-harmonic map into \mathbb{S}^{m-1} in ω , and item (iv) follows from Theorem 1.2. On the other hand, the minimality of v_k yields

$$\mathcal{E}_{\varepsilon_k}(v_k, \omega) \leq \mathcal{E}_{\varepsilon_k}(v_*, \omega) = \mathcal{E}(v_*, \omega).$$

Combining the two previous inequalities leads to item (i). In turn, it implies that $\mu_{\text{sing}} = 0$, and then yields item (ii). The conclusion in item (iii) is a straightforward consequence of Theorem 6.1 - item (iii) - together with the fact that $\mu_{\text{sing}} = 0$. It now only remains to prove the strong convergence of $v_k - v_*$ to 0 in $H_{00}^{1/2}(\omega)$. To this purpose, we have to prove that

$$\lim_{k \rightarrow \infty} \langle (-\Delta)^{\frac{1}{2}} v_k, v_* \rangle_{\omega} = \langle (-\Delta)^{\frac{1}{2}} v_*, v_* \rangle_{\omega} = 2\mathcal{E}(v_*, \omega). \quad (7.9)$$

Indeed, if (7.9) holds, then item (i) yields

$$\|v_k - v_*\|_{H^{1/2}(\mathbb{R}^n)}^2 = \mathcal{E}(v_k - v_*, \omega) = \mathcal{E}(v_k, \omega) + \mathcal{E}(v_*, \omega) - \langle (-\Delta)^{\frac{1}{2}} v_k, v_* \rangle_{\omega} \xrightarrow{k \rightarrow \infty} 0.$$

To show (7.9) we first write

$$\begin{aligned} \langle (-\Delta)^{\frac{1}{2}} v_k, v_* \rangle_{\omega} &= \frac{\gamma_n}{2} \iint_{\omega \times \omega} \frac{(v_k(x) - v_k(y)) \cdot (v_*(x) - v_*(y))}{|x - y|^{n+1}} dx dy \\ &\quad + \gamma_n \iint_{\omega \times \omega^c} \frac{(v_k(x) - v_*(y)) \cdot (v_*(x) - v_*(y))}{|x - y|^{n+1}} dx dy =: I_k + II_k. \end{aligned}$$

Since $v_k \rightharpoonup v_*$ weakly in $H^{1/2}(\omega)$, we have

$$I_k \xrightarrow{k \rightarrow \infty} \frac{\gamma_n}{2} \iint_{\omega \times \omega} \frac{|v_*(x) - v_*(y)|^2}{|x - y|^{n+1}} dx dy.$$

Next we set $V_k(x, y) := (v_k(x) - v_*(y)) \cdot (v_*(x) - v_*(y))$. Then $\{V_k\}$ is bounded in $L^2(\omega \times \omega^c, \mathbf{\Lambda})$ for the weighted measure $\mathbf{\Lambda} := |x - y|^{-n-1} \mathcal{L}_x^n \otimes \mathcal{L}_y^n$. Extracting a further subsequence if necessary, V_k is thus weakly converging in $L^2(\omega \times \omega^c, \mathbf{\Lambda})$ to some function V . Using the strong convergence in $L^2(\omega)$ of v_k , we can argue as in the proof of Lemma 8.2 in Appendix A to show that $V(x, y) = (v_*(x) - v_*(y)) \cdot (v_*(x) - v_*(y))$. As a consequence,

$$\lim_{k \rightarrow \infty} \gamma_n \iint_{\omega \times \omega^c} \frac{(v_k(x) - v_*(y)) \cdot (v_*(x) - v_*(y))}{|x - y|^{n+1}} dx dy = \gamma_n \iint_{\omega \times \omega^c} \frac{|v_*(x) - v_*(y)|^2}{|x - y|^{n+1}} dx dy,$$

and (7.9) is proved. \square

8. APPENDIX A

The purpose of this first appendix is to give detailed proofs of the different auxiliary results stated in Section 2.

Proof of Lemma 2.1. Let $\omega \subseteq \mathbb{R}^n$ be a bounded open set, and $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$. Let us fix $x_0 \in \omega$ and $\rho > 0$ such that $D_{2\rho}(x_0) \subseteq \omega$. Then,

$$\iint_{D_{2\rho}^c(x_0) \times D_{\rho}(x_0)} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy \leq C\mathcal{E}(v, D_{2\rho}(x_0)).$$

Next we estimate

$$\begin{aligned} \iint_{D_{2\rho}^c(x_0) \times D_{\rho}(x_0)} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy &\geq \iint_{D_{2\rho}^c(x_0) \times D_{\rho}(x_0)} \frac{||v(x)| - |v(y)||^2}{(|x - x_0| + |y - x_0|)^{n+1}} dx dy \\ &\geq C_{\rho} \iint_{D_{2\rho}^c(x_0) \times D_{\rho}(x_0)} \frac{|v(x)|^2 - 2|v(y)|^2}{(|x - x_0| + 1)^{n+1}} dx dy, \end{aligned}$$

which yields

$$\int_{D_{2\rho}^c(x_0)} \frac{|v(x)|^2}{(|x - x_0| + 1)^{n+1}} dx \leq C_{\rho} \left(\mathcal{E}(v, D_{2\rho}(x_0)) + \|v\|_{L^2(D_{\rho}(x_0))}^2 \right).$$

Hence,

$$\int_{\mathbb{R}^n} \frac{|v(x)|^2}{(|x - x_0| + 1)^{n+1}} dx \leq C_\rho \left(\mathcal{E}(v, D_{2\rho}(x_0)) + \|v\|_{L^2(D_{2\rho}(x_0))}^2 \right),$$

and the proof is complete. \square

Proof of Lemma 2.7, Part 1. We prove in this first part that $v^e \in L_{\text{loc}}^2(\overline{\mathbb{R}_+^{n+1}})$. We fix $R > 0$ arbitrary, $x_0 \in \omega$ and $\rho > 0$ such that $D_{2\rho}(x_0) \subseteq \omega$. We claim that $v^e \in L^2(B_R^+(x_0))$. Using Jensen's inequality we estimate for $x = (x', x_{n+1}) \in B_R^+(x_0)$,

$$\begin{aligned} |v^e(x)|^2 &\leq \gamma_n \int_{\mathbb{R}^n} \frac{x_{n+1}|v(z)|^2}{(|x' - z|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} dz \\ &\leq \gamma_n \int_{D_{5R}^c(x_0)} \frac{x_{n+1}|v(z)|^2}{(|z - x_0|^2 - 2R|z - x_0|)^{\frac{n+1}{2}}} dz + \gamma_n \int_{D_{5R}(x_0)} \frac{x_{n+1}|v(z)|^2}{(|x' - z|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} dz \\ &\leq C_R \int_{\mathbb{R}^n} \frac{|v(z)|^2}{(|z - x_0| + 1)^{n+1}} dz + \gamma_n \int_{D_{5R}(x_0)} \frac{x_{n+1}|v(z)|^2}{(|x' - z|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} dz. \end{aligned} \quad (8.1)$$

Then we estimate for $0 < x_{n+1} < R$,

$$\begin{aligned} &\int_{D_R(x_0)} \left(\int_{D_{5R}(x_0)} \frac{x_{n+1}|v(z)|^2}{(|x' - z|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} dz \right) dx' \\ &= \int_{D_R(x_0)} \left(\int_{D_{5R}(x')} \frac{x_{n+1}|v(x' - y + x_0)|^2}{(|y - x_0|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} dy \right) dx' \\ &\leq \int_{D_{6R}(x_0)} \left(\int_{D_R(x_0)} \frac{x_{n+1}|v(y + x_0 - x')|^2}{(|y - x_0|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} dx' \right) dy \\ &\leq \left(\int_{D_{6R}(x_0)} \frac{x_{n+1}}{(|y - x_0|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} dy \right) \left(\int_{D_{7R}(x_0)} |v(z)|^2 dz \right) \\ &\leq C \int_{D_{7R}(x_0)} |v(z)|^2 dz. \end{aligned} \quad (8.2)$$

Combining (8.1) and (8.2) we deduce from Lemma 2.1 that

$$\int_{B_R^+(x_0)} |v^e(x)|^2 dx \leq C_R \int_{\mathbb{R}^n} \frac{|v(z)|^2}{(|z - x_0| + 1)^{n+1}} dz \leq C_{R,\rho} \left(\mathcal{E}(v, D_{2\rho}(x_0)) + \|v\|_{L^2(D_{2\rho}(x_0))}^2 \right), \quad (8.3)$$

which ends this first part. \square

Remark 8.1 (Proof of Lemma 2.5). Notice that the first inequality in (8.3) shows the continuity of the linear operator \mathfrak{F}_R defined in (2.13).

To complete the proof of Lemma 2.7, we shall need the following smooth approximation result.

Lemma 8.2. *Let $\omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$. There exists a sequence $\{v_k\}_{k \in \mathbb{N}}$ of smooth functions such that $v_k \rightarrow v$ strongly in $L_{\text{loc}}^2(\mathbb{R}^n)$, and such that for any relatively compact open subset $\omega' \subseteq \omega$ with Lipschitz boundary,*

- (i) $v_k \in \widehat{H}^{1/2}(\omega'; \mathbb{R}^m)$ for k large enough;
- (ii) $\mathcal{E}(v_k - v, \omega') \rightarrow 0$ as $k \rightarrow \infty$;
- (iii) $v_k^e \rightarrow v^e$ in $L_{\text{loc}}^2(\overline{\mathbb{R}_+^{n+1}})$ as $k \rightarrow \infty$.

Proof. Let us fix a sequence $\varepsilon_k \downarrow 0$, and a nonnegative function $\varrho \in C_c^\infty(\mathbb{R}^n)$ compactly supported in D_1 satisfying $\int_{D_1} \varrho dx = 1$. Setting $\varrho_k(x) := \varepsilon_k^{-n} \varrho(x/\varepsilon_k)$, we define

$$v_k(x) := \int_{D_{\varepsilon_k}} \varrho_k(z) v(x+z) dz.$$

Then $v_k \in C^\infty(\mathbb{R}^n; \mathbb{R}^m)$, $v_k \rightarrow v$ strongly in $L_{\text{loc}}^2(\mathbb{R}^n)$. Extracting a subsequence if necessary, we may assume that $v_k \rightarrow v$ a.e. in \mathbb{R}^n .

Let us now consider a relatively compact open subset $\omega' \subseteq \omega$ with Lipschitz boundary. Set

$$\delta_0 := \min \left\{ \text{dist}(\omega', \partial\omega), \sup_{x \in \omega'} \text{dist}(x, \partial\omega') \right\},$$

and define for $0 < \delta < \delta_0$,

$$\omega_\delta := \{x \in \omega : \text{dist}(x, \omega') < \delta\}, \quad \omega_{-\delta} := \{x \in \omega' : \text{dist}(x, \partial\omega') > \delta\}.$$

For $0 < \varepsilon_k < \delta < \delta_0$, we estimate through Jensen's inequality,

$$\begin{aligned} \iint_{\omega' \times \omega'} \frac{|v_k(x) - v_k(y)|^2}{|x - y|^{n+1}} dx dy &\leq \iint_{\omega' \times \omega'} \left(\int_{D_{\varepsilon_k}} \varrho_k(z) \frac{|v(x+z) - v(y+z)|^2}{|x - y|^{n+1}} dx dy \right) dz \\ &\leq \int_{D_{\varepsilon_k}} \varrho_k(z) \left(\iint_{(z+\omega') \times (z+\omega')} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy \right) dz \\ &\leq \iint_{\omega_\delta \times \omega_\delta} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy. \end{aligned} \quad (8.4)$$

Hence $\{v_k\}$ is bounded in $H^{1/2}(\omega'; \mathbb{R}^m)$, and since $v_k \rightarrow v$ in $L^2(\omega')$ we deduce that $v_k \rightharpoonup v$ weakly in $H^{1/2}(\omega'; \mathbb{R}^m)$. Now we infer from (8.4) and Fatou's lemma that

$$\begin{aligned} \iint_{\omega' \times \omega'} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy &\leq \liminf_{k \rightarrow +\infty} \iint_{\omega' \times \omega'} \frac{|v_k(x) - v_k(y)|^2}{|x - y|^{n+1}} dx dy \\ &\leq \limsup_{k \rightarrow +\infty} \iint_{\omega' \times \omega'} \frac{|v_k(x) - v_k(y)|^2}{|x - y|^{n+1}} dx dy \leq \iint_{\omega_\delta \times \omega_\delta} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy. \end{aligned}$$

Since $[v]_{H^{1/2}(\omega_\delta)}^2 \rightarrow [v]_{H^{1/2}(\omega')}^2$ as $\delta \downarrow 0$ by monotone convergence, we conclude from this last inequality and the arbitrariness of δ that $[v_k]_{H^{1/2}(\omega')}^2 \rightarrow [v]_{H^{1/2}(\omega')}^2$. Therefore $v_k \rightarrow v$ strongly in $H^{1/2}(\omega')$.

Similarly to (8.4), we estimate

$$\iint_{\omega' \times (\omega')^c} \frac{|v_k(x) - v_k(y)|^2}{|x - y|^{n+1}} dx dy \leq \iint_{\omega_\delta \times \omega_{-\delta}^c} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy.$$

Arguing as above, we apply Fatou's lemma as $k \rightarrow \infty$, and then the monotone convergence theorem as $\delta \downarrow 0$ to deduce that

$$\iint_{\omega' \times (\omega')^c} \frac{|v_k(x) - v_k(y)|^2}{|x - y|^{n+1}} dx dy \xrightarrow{k \rightarrow \infty} \iint_{\omega' \times (\omega')^c} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy. \quad (8.5)$$

We have thus proved that

$$\mathcal{E}(v_k, \omega') \xrightarrow{k \rightarrow \infty} \mathcal{E}(v, \omega').$$

In particular, $v_k \in \widehat{H}^{1/2}(\omega'; \mathbb{R}^m)$ for k large enough.

Setting $V_k(x, y) := (v_k(x) - v_k(y))$, (8.5) implies that the sequence $\{V_k\}$ is bounded in $L^2(\omega' \times (\omega')^c, \mathbf{\Lambda})$ for the weighted measure $\mathbf{\Lambda} := |x - y|^{-n-1} \mathcal{L}^n \otimes \mathcal{L}^n$. Hence we can extract a subsequence such that $V_k \rightharpoonup w$ weakly $L^2(\omega' \times (\omega')^c, \mathbf{\Lambda})$. On the other hand, from the convergence of v_k to v in $L^2_{\text{loc}}(\mathbb{R}^n)$ we deduce that for all $F \in L^2(\omega' \times (\omega')^c, \mathbf{\Lambda})$ compactly supported in $\omega' \times (\overline{\omega'})^c$,

$$\begin{aligned} \iint_{\omega' \times (\omega')^c} \frac{V(x, y) \cdot F(x, y)}{|x - y|^{n+1}} dx dy &= \lim_{k \rightarrow \infty} \iint_{\omega' \times (\omega')^c} \frac{V_k(x, y) \cdot F(x, y)}{|x - y|^{n+1}} dx dy \\ &= \iint_{\omega' \times (\omega')^c} \frac{(v(x) - v(y)) \cdot F(x, y)}{|x - y|^{n+1}} dx dy. \end{aligned}$$

Functions with compact support in $\omega' \times (\overline{\omega'})^c$ being dense in $L^2(\omega' \times (\omega')^c, \mathbf{\Lambda})$, we conclude that

$$\iint_{\omega' \times (\omega')^c} \frac{V(x, y) \cdot F(x, y)}{|x - y|^{n+1}} dx dy = \iint_{\omega' \times (\omega')^c} \frac{(v(x) - v(y)) \cdot F(x, y)}{|x - y|^{n+1}} dx dy$$

for all $F \in L^2(\omega' \times (\omega')^c, \mathbf{\Lambda})$. The Riesz Representation Theorem then yields $V(x, y) = (v(x) - v(y))$. As a consequence,

$$\iint_{\omega' \times (\omega')^c} \frac{(v_k(x) - v_k(y)) \cdot (v(x) - v(y))}{|x - y|^{n+1}} dx dy \xrightarrow{k \rightarrow \infty} \iint_{\omega' \times (\omega')^c} \frac{|v(x) - v(y)|^2}{|x - y|^{n+1}} dx dy. \quad (8.6)$$

Combing (8.6) with the convergence of v_n in $H^{1/2}(\omega'; \mathbb{R}^m)$, we infer that

$$\langle (-\Delta)^{\frac{1}{2}} v_k, v \rangle_{\omega'} \xrightarrow{k \rightarrow \infty} \langle (-\Delta)^{\frac{1}{2}} v, v \rangle_{\omega'} = 2\mathcal{E}(v, \omega'),$$

and we finally conclude

$$\mathcal{E}(v_k - v, \omega') = \mathcal{E}(v, \omega') - \langle (-\Delta)^{\frac{1}{2}} v_k, v \rangle_{\omega'} + \mathcal{E}(v_k, \omega') \xrightarrow{k \rightarrow \infty} 0,$$

which proves (ii).

It now remains to show (iii). We consider an arbitrary radius $R > 0$, $x_0 \in \omega'$ and $\rho > 0$ such that $D_{2\rho}(x_0) \subseteq \omega'$. Noticing that $(v_k - v)^e = v_k^e - v^e$, we deduce from (8.3) that

$$\begin{aligned} \int_{B_R^+(x_0)} |v_k^e - v^e|^2 dx &\leq C_{R,\rho} \left(\mathcal{E}(v_k - v, D_{2\rho}(x_0)) + \|v_k - v\|_{L^2(D_{2\rho}(x_0))}^2 \right) \\ &\leq C_{R,\rho} \left(\mathcal{E}(v_k - v, \omega') + \|v_k - v\|_{L^2(D_{2\rho}(x_0))}^2 \right) \rightarrow 0, \end{aligned}$$

and the proof is complete. \square

Proof of Lemma 2.7, Part 2. Step 1. Let us first consider the case where $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m) \cap C^\infty(\mathbb{R}^n)$. Then v^e is smooth in $\overline{\mathbb{R}}_+^{n+1}$. Let $x_0 \in \omega$ and $\rho > 0$ such that $D_{2\rho}(x_0) \subseteq \omega$. We consider a cut-off function $\chi \in C^\infty(\mathbb{R}^{n+1}; [0, 1])$ satisfying $\chi = 1$ in $B_\rho(x_0)$, and $\chi = 0$ in $\mathbb{R}^{n+1} \setminus B_{2\rho}(x_0)$. We set $\Phi := \chi^2 v^e$ and $\varphi := \Phi|_{\mathbb{R}^n} = \chi^2 v$. From the harmonicity of v^e in \mathbb{R}_+^{n+1} we infer that

$$\int_{\mathbb{R}_+^{n+1}} \nabla v^e \cdot \nabla \Phi dx = \int_{\mathbb{R}^n} \frac{\partial v^e}{\partial \nu} \cdot \varphi dx = \lim_{\delta \downarrow 0} \int_{\mathbb{R}^n} \frac{(v(x) - v^e(x, \delta)) \cdot \varphi(x)}{\delta} dx.$$

On the other hand, applying formula (2.9) to represent v^e yields

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{(v(x) - v^e(x, \delta)) \cdot \varphi(x)}{\delta} dx &= \gamma_n \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v(x) - v(y)) \cdot \varphi(x)}{(|x - y|^2 + \delta^2)^{\frac{n+1}{2}}} dx dy \\ &= \frac{\gamma_n}{2} \iint_{D_{2\rho}(x_0) \times D_{2\rho}(x_0)} \frac{(v(x) - v(y)) \cdot (\varphi(x) - \varphi(y))}{(|x - y|^2 + \delta^2)^{\frac{n+1}{2}}} dx dy \\ &\quad + \gamma_n \iint_{D_{2\rho}(x_0) \times D_{2\rho}^c(x_0)} \frac{(v(x) - v(y)) \cdot (\varphi(x) - \varphi(y))}{(|x - y|^2 + \delta^2)^{\frac{n+1}{2}}} dx dy. \end{aligned}$$

Using the fact that $v \in \widehat{H}^{1/2}(D_{2\rho}(x_0); \mathbb{R}^m)$ and $\varphi \in H_{00}^{1/2}(D_{2\rho}(x_0); \mathbb{R}^m)$, we derive by dominated convergence that

$$\lim_{\delta \downarrow 0} \int_{\mathbb{R}^n} \frac{(v(x) - v^e(x, \delta)) \cdot \varphi(x)}{\delta} dx = \langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle_{D_{2\rho}(x_0)}.$$

As in (2.7), it then follows from Cauchy-Schwarz Inequality that

$$\int_{\mathbb{R}_+^{n+1}} \nabla v^e \cdot \nabla \Phi dx \leq \sqrt{\mathcal{E}(v, D_{2\rho}(x_0))} \sqrt{\mathcal{E}(\varphi, D_{2\rho}(x_0))} \leq \frac{1}{2} (\mathcal{E}(v, D_{2\rho}(x_0)) + \mathcal{E}(\varphi, D_{2\rho}(x_0))). \quad (8.7)$$

Now we estimate

$$\begin{aligned} \int_{\mathbb{R}_+^{n+1}} \nabla v^e \cdot \nabla \Phi dx &\geq \int_{B_{2\rho}^+(x_0)} \chi^2 |\nabla v^e|^2 dx - 2 \int_{B_{2\rho}^+(x_0)} \chi |v^e| |\nabla v^e| |\nabla \chi| dx \\ &\geq \frac{1}{2} \int_{B_{2\rho}^+(x_0)} \chi^2 |\nabla v^e|^2 dx - 2 \int_{B_{2\rho}^+(x_0)} |v^e|^2 |\nabla \chi|^2 dx \\ &\geq \frac{1}{2} \int_{B_{2\rho}^+(x_0)} |\nabla v^e|^2 dx - C_\rho \|v^e\|_{L^2(B_{2\rho}^+(x_0))}^2, \end{aligned} \quad (8.8)$$

for a constant $C_\rho > 0$ independent of v . Then (8.7) and (8.8) yield

$$\int_{B_{2\rho}^+(x_0)} |\nabla v^e|^2 dx \leq \mathcal{E}(v, D_{2\rho}(x_0)) + \mathcal{E}(\varphi, D_{2\rho}(x_0)) + C_\rho \|v^e\|_{L^2(B_{2\rho}^+(x_0))}^2, \quad (8.9)$$

and it remains to estimate the second term in the right handside of this inequality. A straightforward computation yields

$$\begin{aligned} \mathcal{E}(\varphi, D_{2\rho}(x_0)) &\leq 2\mathcal{E}(v, D_{2\rho}(x_0)) + 4 \int_{D_{4\rho}(x_0)} \left(\int_{D_{2\rho}(x_0)} \frac{|\chi^2(x) - \chi^2(y)|^2}{|x - y|^{n+1}} dx \right) |v(y)|^2 dy \\ &\quad + 4 \int_{D_{4\rho}^c(x_0)} \left(\int_{D_{2\rho}(x_0)} \frac{1}{|x - y|^{n+1}} dx \right) |v(y)|^2 dy. \end{aligned}$$

Noticing that for $y \in D_{4\rho}(x_0)$,

$$\int_{D_{2\rho}(x_0)} \frac{|\chi^2(x) - \chi^2(y)|^2}{|x - y|^{n+1}} dx \leq C_\rho \int_{D_{2\rho}(x_0)} \frac{1}{|x - y|^{n-1}} dx \leq C_\rho,$$

and that for $y \in D_{4\rho}^c(x_0)$,

$$\int_{D_{2\rho}(x_0)} \frac{1}{|x-y|^{n+1}} dx \leq \frac{C_\rho}{(|y-x_0|+1)^{n+1}},$$

we deduce that

$$\mathcal{E}(\varphi, D_{2\rho}(x_0)) \leq 2\mathcal{E}(v, D_{2\rho}(x_0)) + C_\rho \int_{\mathbb{R}^n} \frac{|v(z)|^2}{(|z-x_0|+1)^{n+1}} dz. \quad (8.10)$$

Finally, gathering (8.9) with (8.3) and (8.10), we infer that

$$\int_{B_\rho^+(x_0)} |\nabla v^e|^2 dx \leq 3\mathcal{E}(v, D_{2\rho}(x_0)) + C_\rho \int_{\mathbb{R}^n} \frac{|v(z)|^2}{(|z-x_0|+1)^{n+1}} dz.$$

In view of Lemma 2.1, we have thus proved that

$$\int_{B_\rho^+(x_0)} |\nabla v^e|^2 dx \leq C_\rho (\mathcal{E}(v, D_{2\rho}(x_0)) + \|v\|_{L^2(D_{2\rho}(x_0))}^2).$$

Step 2. Let us now consider an arbitrary $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$, and a sequence $\{v_k\} \subseteq C^\infty(\mathbb{R}^n; \mathbb{R}^m)$ given by Lemma 8.2. Let $x_0 \in \omega$ and $\rho > 0$ such that $D_{3\rho}(x_0) \subseteq \omega$. Then, for k large enough, $v_k \in \widehat{H}^{1/2}(D_{2\rho}(x_0); \mathbb{R}^m)$. Applying Step 1 to v_k we infer that for k large enough,

$$\int_{B_\rho^+(x_0)} |\nabla v_k^e|^2 dx \leq C_\rho (\mathcal{E}(v_k, D_{2\rho}(x_0)) + \|v_k\|_{L^2(D_{2\rho}(x_0))}^2).$$

By Lemma 8.2, the right hand side of this inequality is uniformly bounded with respect to k . Since $v_k^e \rightarrow v^e$ in $L^2(B_\rho^+(x_0))$ (still by Lemma 8.2), we deduce that $v_k^e \rightharpoonup v^e$ weakly in $H^1(B_\rho^+(x_0))$. By lower semicontinuity and (ii) in Lemma 8.2, we obtain

$$\int_{B_\rho^+(x_0)} |\nabla v^e|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{B_\rho^+(x_0)} |\nabla v_k^e|^2 dx \leq C_\rho (\mathcal{E}(v, D_{2\rho}(x_0)) + \|v\|_{L^2(D_{2\rho}(x_0))}^2).$$

Moreover, in view of the arbitrariness of x_0 , we conclude that

$$v^e \in H^1(B_\rho^+(x); \mathbb{R}^m) \quad \text{for all } x \in \omega \text{ and } \rho > 0 \text{ such that } D_{3\rho}(x) \subseteq \omega. \quad (8.11)$$

Finally, the conclusion $v^e \in H_{\text{loc}}^1(\mathbb{R}_+^{n+1} \cup \omega; \mathbb{R}^m)$ follows from (8.11) together with a standard covering argument. \square

Proof of Lemma 2.9. By the density of compactly supported smooth functions in (2.2), we may assume without loss of generality that $\varphi \in \mathcal{D}(\omega; \mathbb{R}^m)$. Let $\Omega \subseteq \mathbb{R}_+^{n+1}$ be an admissible bounded open set such that $\text{supp } \varphi \subseteq \partial^0 \Omega$, and $\overline{\partial^0 \Omega} \subseteq \omega$. We then consider a smooth extension Φ of φ to \mathbb{R}_+^{n+1} which is compactly supported in $\Omega \cup \partial^0 \Omega$. Let $\{v_k\} \subseteq C^\infty(\mathbb{R}^n; \mathbb{R}^m)$ be a sequence given by Lemma 8.2. Then, $v_k \in \widehat{H}^{1/2}(\partial^0 \Omega; \mathbb{R}^m)$ for k large enough. Arguing as in the proof of Lemma 2.7 (Part 2, Step 1), we show that

$$\left\langle \frac{\partial v_k^e}{\partial \nu}, \varphi \right\rangle := \int_{\mathbb{R}_+^{n+1}} \nabla v_k^e \cdot \nabla \Phi dx = \int_{\mathbb{R}^n} \frac{\partial v_k^e}{\partial \nu} \cdot \varphi dx = \langle (-\Delta)^{\frac{1}{2}} v_k, \varphi \rangle_{\partial^0 \Omega}.$$

By the proof of Lemma 2.7 (Part 2, Step 2), $v_k^e \rightharpoonup v^e$ weakly in $H^1(\Omega)$. Therefore,

$$\left\langle \frac{\partial v_k^e}{\partial \nu}, \varphi \right\rangle = \int_{\Omega} \nabla v_k^e \cdot \nabla \Phi dx \xrightarrow{k \rightarrow \infty} \int_{\Omega} \nabla v^e \cdot \nabla \Phi dx = \left\langle \frac{\partial v^e}{\partial \nu}, \varphi \right\rangle.$$

On the other hand, by (2.7) and Lemma 8.2 we have

$$\left| \langle (-\Delta)^{\frac{1}{2}} (v_k - v), \varphi \rangle_{\partial^0 \Omega} \right| \leq \|\varphi\|_{H_0^{1/2}(\partial^0 \Omega)} \sqrt{\mathcal{E}(v_k - v, \partial^0 \Omega)} \xrightarrow{k \rightarrow \infty} 0.$$

Consequently,

$$\left\langle \frac{\partial v^e}{\partial \nu}, \varphi \right\rangle = \lim_{k \rightarrow \infty} \left\langle \frac{\partial v_k^e}{\partial \nu}, \varphi \right\rangle = \lim_{k \rightarrow \infty} \langle (-\Delta)^{\frac{1}{2}} v_k, \varphi \rangle_{\partial^0 \Omega} = \langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle_{\partial^0 \Omega} = \langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle_\omega,$$

and the lemma is proved. \square

9. APPENDIX B

This purpose of this last appendix is to provide the elliptic regularity results we have been using during the proof of Lemma 5.6.

Lemma 9.1. *For $\varepsilon > 0$, let $u_\varepsilon \in H^1(B_1^+)$ be the unique (variational) solution of*

$$\begin{cases} -\Delta u_\varepsilon = 1 & \text{in } B_1^+, \\ u_\varepsilon = 1 & \text{on } \partial^+ B_1, \\ \varepsilon \frac{\partial u_\varepsilon}{\partial \nu} + u_\varepsilon = 0 & \text{on } D_1. \end{cases}$$

Then $u_\varepsilon \in C^{0,\alpha}(B_1^+) \cap C^\infty(\overline{B_1^+} \setminus \partial D_1)$ for some $0 < \alpha < 1$, $u_\varepsilon > 0$ in $\overline{B_1^+}$, and

$$u_\varepsilon \leq C\varepsilon \quad \text{on } D_{1/2},$$

for some constant $C > 0$ which only depends on the dimension.

Proof. The Hölder continuity of u_ε (up to the boundary) follows from the general results of STAMPACCHIA [73, 74] (see also [21]), while higher order regularity away from ∂D_1 follows from standard elliptic regularity theory. The fact that $u_\varepsilon > 0$ in $\overline{B_1^+}$ is an easy consequence of the maximum principle and the Hopf boundary Lemma. Indeed, assume that x_0 be a minimum point of u_ε such that $u_\varepsilon(x_0) \leq 0$. By the maximum principle $x_0 \in \partial B_1^+$, and thus $x_0 \in D_1$ since $u_\varepsilon = 1$ on $\partial^+ B_1$ and u_ε is continuous on $\overline{B_1^+}$. Since u_ε is smooth up to D_1 (away from ∂D_1), we can apply the Hopf boundary Lemma to infer that $\partial_\nu u_\varepsilon(x_0) < 0$. Then $\varepsilon \partial_\nu u_\varepsilon(x_0) + u_\varepsilon(x_0) < 0$ which contradicts the equation on D_1 .

Let us now consider the unique solution $w \in H^1(B_1^+)$ of

$$\begin{cases} -\Delta w = 1 & \text{in } B_1^+, \\ w = 0 & \text{on } \partial B_1^+. \end{cases}$$

It is well known that $w \in C^{0,1}(B_1^+) \cap C^\infty(\overline{B_1^+} \setminus \partial D_1)$ by convexity of B_1^+ (see e.g. [34]). We set $v_\varepsilon := u_\varepsilon - w$, so that $v_\varepsilon \in C^{0,\alpha}(B_1^+) \cap C^\infty(\overline{B_1^+} \setminus \partial D_1)$ solves

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{in } B_1^+, \\ v_\varepsilon = 1 & \text{on } \partial^+ B_1, \\ \varepsilon \frac{\partial v_\varepsilon}{\partial \nu} + v_\varepsilon = -\varepsilon \frac{\partial w}{\partial \nu} & \text{on } D_1. \end{cases}$$

Setting

$$\kappa := \left\| \frac{\partial w}{\partial \nu} \right\|_{L^\infty(D_1)},$$

we observe that κ only depends on the dimension. Next we define $\bar{u}_\varepsilon \in H^1(B_1^+)$ as the unique (variational) solution of

$$\begin{cases} \Delta \bar{u}_\varepsilon = 0 & \text{in } B_1^+, \\ \bar{u}_\varepsilon = 1 & \text{on } \partial^+ B_1, \\ \varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial \nu} + \bar{u}_\varepsilon = 0 & \text{on } D_1. \end{cases}$$

As previously, $\bar{u}_\varepsilon \in C^{0,\beta}(B_1^+) \cap C^\infty(\overline{B_1^+} \setminus \partial D_1)$ for some $0 < \beta < 1$, and by the maximum principle and the Hopf boundary Lemma, we have

$$0 \leq \bar{u}_\varepsilon \leq 1 \quad \text{in } \overline{B_1^+}. \quad (9.1)$$

Observe that

$$v_\varepsilon \leq \bar{u}_\varepsilon + \kappa\varepsilon \quad \text{in } \overline{B_1^+}. \quad (9.2)$$

Indeed, consider the function $h := \bar{u}_\varepsilon + \kappa\varepsilon - v_\varepsilon$. Then h satisfies

$$\begin{cases} \Delta h = 0 & \text{in } B_1^+, \\ h \geq 0 & \text{on } \partial^+ B_1, \\ \varepsilon \frac{\partial h}{\partial \nu} + h \geq 0 & \text{on } D_1, \end{cases}$$

so that $h \geq 0$ in \bar{B}_1^+ , still by the maximum principle and the Hopf boundary Lemma.

Finally, in view of (9.1) and according to [36, Theorem 6.26],

$$\|\nabla \bar{u}_\varepsilon\|_{L^\infty(B_{1/2}^+)} \leq C,$$

for a constant $C > 0$ which only depends on the dimension. In particular $|\partial_\nu \bar{u}_\varepsilon| \leq C$ on $D_{1/2}$, and thus

$$\bar{u}_\varepsilon \leq C\varepsilon \quad \text{on } D_{1/2}. \quad (9.3)$$

Since $v_\varepsilon = u_\varepsilon$ on D_1 , the conclusion now follows from (9.2) and (9.3). \square

We conclude this appendix with a standard elliptic boundary estimate for a linear Neumann problem. As showed in [17, proof of Lemma 2.2], such (classical) estimate can be easily obtained from usual boundary regularity by considering the auxiliary function $v(x', x_{n+1}) := \int_0^{x_{n+1}} u(x', t) dt$ which satisfies a Poisson equation in B_1^+ with an homogeneous Dirichlet boundary condition on D_1 .

Lemma 9.2. *Given $0 < \alpha < 1$, let $f \in C^{0,\alpha}(B_1^+)$, $g \in C^{0,\alpha}(D_1)$, and $u \in C^2(B_1^+) \cap C^{1,\alpha}(B_1^+)$ solving*

$$\begin{cases} -\Delta u = f & \text{in } B_1^+, \\ \frac{\partial u}{\partial \nu} = g & \text{on } D_1. \end{cases}$$

Then,

$$\|u\|_{C^{1,\alpha}(B_1^+)} \leq C_{r,\alpha} (\|f\|_{C^{0,\alpha}(B_1^+)} + \|g\|_{C^{0,\alpha}(D_1)} + \|u\|_{L^\infty(B_1^+)})$$

for every $0 < r < 1$ and a constant $C_{r,\alpha} > 0$ which only depends on n, α, r .

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