Polynomial Sufficient Conditions of Well-Behavedness for Weighted Join-Free and Choice-Free Systems

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***Abstract*—Join-Free Petri nets, whose transitions have at most one input place, model systems without synchronizations while Choice-Free Petri nets, whose places have at most one output transition, model systems without conflicts. These classes respectively encompass the state machines (or S-systems) and the marked graphs (or T-systems).**

Whereas a structurally bounded and structurally live Petri net graph is said to be "well-formed", a bounded and live Petri net is said to be "well-behaved". Necessary and sufficient conditions for the well-formedness of Join-Free and Choice-Free nets have been known for some time, yet the behavioral properties of these classes are still not well understood. In particular efficient sufficient conditions for liveness have not been found until now.

In this paper, we extend results on weighted T-systems to the class of weighted Petri nets and present transformations which preserve the feasible sequences of transitions and reduce the initial marking. We introduce a notion of "balancing" that makes possible the transformation of conservative systems into so-called "1-conservative systems" while retaining the feasible transition sequences. This transformation leads to polynomial sufficient conditions of liveness for well-formed Join-Free and Choice-Free nets.

***Keywords*-Join-Free; Choice-Free; S-system; State Machine; T-system; Petri net; weighted net; liveness; boundedness; well-formedness; well-behavedness; balancing; polynomial sufficient condition; Synchronous Data Flow.**

INTRODUCTION

A. Models and Analysis

Petri nets have proved useful to model discrete event systems possibly with conflicts, synchronization and concurrency [1]. However, their expressiveness, although not Turing-complete [1], [2], comes at the cost of a high analysis complexity. The reachability and liveness problems, among others, are both EXPSPACE-hard while the k-boundedness problem is NP-hard [3].

Synchronous Data Flow (SDF) were introduced by Lee and Messerschmitt [4] while Cyclo-Static Data Flow (CSDF, extending SDF with phases) were studied by Bilsen *et al.* [5]. They are special data flow models for concurrent applications to be executed on parallel architectures and have been used in many—often multimedia—applications, such as a MP3 playback [6]. Weighted T-systems [7], [8] are Petri nets having the same modeling power as the SDF, thus the methods developed for one model can be used for the other. However, as programs become more complex, the expressiveness of models has to be extended. An objective is thus to generalize results common to SDF and T-systems in order to treat more complex applications involving choices.

Petri nets allow to model complex applications but fundamental properties are hard to analyze for this class. In order to reduce this complexity, we focus on simpler classes that are expressive enough to model many real applications and simple enough to permit efficient analysis methods.

Embedded systems must be well-designed in order to work properly. They must preserve their functionalities and use a limited amount of memory over time. The corresponding notion in Petri nets is that of well-formedness, ensuring the existence of an initial configuration for the system that lets it work as intended.

Such structural guarantees are fundamental for applications. They exist for weighted T-systems and can be found efficiently (in polynomial time) in this case [8]. Well-formedness is solved for non-weighted Free-Choice nets [9] and even larger non-weighted classes [10]. In the weighted case, the problem is solved for the class of the Equal-Conflict systems with homogeneous weights, *i.e.* identical output values for every place [11]. This class contains the weighted Choice-Free systems, which extend weighted T-systems. There exist polynomial necessary and sufficient conditions of well-formedness for weighted Choice-Free systems [12], [13] as well as weighted Join-Free systems [12]. Systems belonging to both Choice-Free and Join-Free classes are named Fork-Attribution (FA) and inherit the structural properties of both classes. Figure 1 represents the inclusion relations between the special classes of weighted Petri nets considered in this paper.

Well-formedness ensures that the structure of the system will induce a reliable behavior. The next fundamental aim is to find an initialization of the well-formed system, making it

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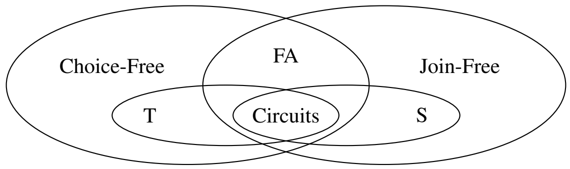


Figure 1. Some classes and subclasses of weighted systems.

usable in practice, thus well-behaved. Such a configuration has thus to be found in polynomial time for any well-formed system of the considered class. This initial state must be sufficient for well-behavedness, as sufficiency ensures that the system will work as intended. One would like it to approach the necessity threshold in order to minimize resources. This objective is often hard to reach and sufficient polynomial conditions of well-behavedness already constitute a key improvement. However, even such sufficient but not necessary conditions are not easy to discover. One has been found for T-systems [8]. In weighted Choice-Free and some larger classes, only exponential conditions are proposed [11], [12], which cannot be applied to real systems. Well-behavedness is also not mastered for weighted Join-Free systems. Existing results encompass polynomial and non-polynomial characterizations of liveness for several ordinary (non-weighted) classes [14]–[16] and non-polynomial ones for homogeneous classes [17]–[19], as well as necessary conditions for liveness and boundedness [20].

Our contribution is to provide two polynomial sufficient conditions of well-behavedness for the weighted Choice-Free and Join-Free classes. Thus, we extend the expressiveness of the models usable in real applications, whose well-behavedness may be ensured efficiently. Moreover, we show that conservativeness, induced by well-formedness in these two classes, is a major property that leads to these sufficient conditions. Such weighted and conservative systems can be transformed into 1-conservative [3] systems by modifying only weights. The key idea consists in finding a sufficient condition for weighted 1-conservative strongly connected Join-Free systems, extending the existing condition of T-systems. This novel condition is then used to construct a sufficient condition for well-formed Choice-Free systems.

B. Models and Applications

Join-Free, Choice-Free and S-systems are appealing not only from a theoretical point of view but also because they allow to model useful applications.

1) S-systems: They are a simple subclass of weighted Petri nets, in which programs read (resp. write) in a single memory, while memories can be shared between programs. They allow to model asynchronous parallel algorithms on several processors. Asynchronous methods diminish the synchronization points between processors, eliminating idle time at the expense of extra computations. However, such methods perform better than their synchronous counterparts for several practical problems [21]. Computational models as well as an associated convergence theory have been developed [21]. A powerful and simple model reads data from a shared memory, computes a function and overwrites data in common memory with the corresponding updated values. This computational model applies to a wide range of problems, including solving nonsingular linear systems [21], and is represented by a Petri net in figure 2.

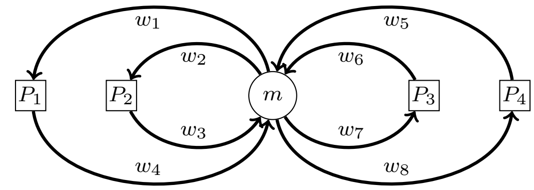


Figure 2. The S-net models one shared memory as the place m, in which four asynchronous parallel processors read and write to compute an asynchronous iteration. Weights denote amounts of read or written data.

2) Join-Free systems: They are closely related to Process Rewrite Systems (PRS) and more specifically to Basic Parallel Processes (BPP, context-free multiset rewrite systems, commutative context-free processes). Such systems allow to model multithreaded programs [22], [23]. Results on Join-Free systems with a weight restriction, named communication-free, have served to gain insight into commutative context-free grammars and BPP [24].

3) Choice-Free systems: Flow applications are usually modeled with weighted T-systems (equivalently with SDFs) [6]. The weighted Choice-Free model adds the possibility to write asynchronously in a memory, thus is strictly more powerful than the T-system model.

This paper is organized as follows. In Section I, we recall classical definitions, notations and properties. In Section II, we define the *scaling* and *balancing* of systems together with the trimming down to *useful tokens*, which are polynomial time transformations that preserve the set of feasible firing sequences. Moreover, balancing is compared to the notion of *normalization* that was developed for T-systems. In Section III (resp. IV), we use balancing to present a polynomial sufficient condition of liveness for well-formed balanced Join-Free nets (resp. well-formed Choice-Free nets). Besides, the sufficient condition of liveness for these Join-Free nets is shown to induce a necessary and sufficient condition of liveness for well-formed ordinary Join-Free nets. Finally, in Section V, we show that neither one of the sufficient conditions of liveness is necessary in the weighted case.

I. Definitions, notations and properties

We recall in this section some basic definitions and results concerning P/T nets, and systems. The first subsection is devoted to notations about weighted nets. The definitions of special classes of nets considered by our study, namely Choice-Free, Join-Free and some of their subclasses, are then recalled. The third subsection provides some definitions dealing with markings and firing sequences, and their relationships. The last one recalls some definitions and results related to liveness and boundedness.

A. Weighted and ordinary nets

A *(weighted) net* is a triple $N = (P, T, W)$ where:

- the sets P and T are finite and disjoint, T contains only transitions and P only places,
- $W : (P \times T) \cup (T \times P) \mapsto \mathbb{N}$ is a positive function.

$P \cup T$ is the set of the elements of the net.

An arc is present from a place p to a transition t (resp. a transition t to a place p) if $W(p,t) > 0$ (resp. $W(t,p) > 0$). An *ordinary* net is a weighted net whose weighting function W is valued in $\{0, 1\}$.

The *incidence matrix* of a net $N = (P, T, W)$ is a place-transition matrix C defined as

$$\forall p \in P, \forall t \in T, C[p,t] = W(t,p) - W(p,t)$$

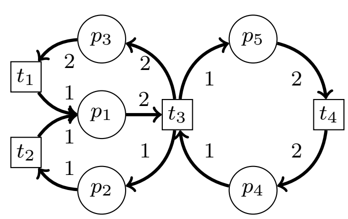
where the weight of any non-existing arc is 0.

The *pre-set* of the element x of $P \cup T$ is the set $\{w | W(w,x) > 0\}$, denoted by ${}^\bullet x$. By extension, for any subset E of P or T, ${}^\bullet E = \bigcup_{x \in E} {}^\bullet x$.

The *post-set* of the element x of $P \cup T$ is the set $\{y | W(x,y) > 0\}$, denoted by $x^\bullet$. Similarly, $E^\bullet = \bigcup_{x \in E} x^\bullet$.

A P-subnet $S = (P', T', W')$ of a net $N = (P, T, W)$ is generated by a subset of places $P' \subseteq P$ and is such that $T' = {}^\bullet P' \cup P'^\bullet$. W' is the restriction of W to P' and T'.

We denote by max_p^N the maximum output weight of p in the net N and by gcd_p^N the greatest common divisor of all input and output weights of p in the net N. The simpler notation max_p and gcd_p is used when no confusion is possible. We denote by $\mathbb{1}^n$ the vector of size n whose components are all equal to 1. Figure 3 presents a weighted net and its corresponding incidence matrix. Figure 4 pictures two subnets of the net from Figure 3.



$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix} \end{array}$$

Figure 3. A weighted net and the corresponding incidence matrix.

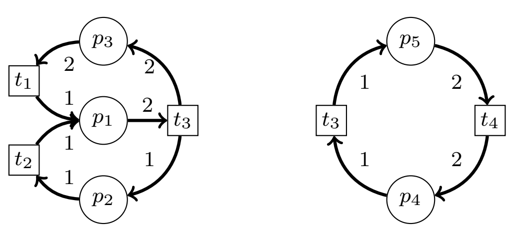


Figure 4. Two P-subnets of the net pictured by Fig. 3, defined respectively by the sets of places $\{p_1, p_2, p_3\}$ and $\{p_4, p_5\}$.

B. Special classes of nets

$N = (P, T, W)$ is a (weighted) *Choice-Free net* if any place has at most one output transition, *i.e.* $\forall p \in P$, $|p^\bullet| \leq 1$. A *T-net* is a Choice-Free net such that any place has at most one input transition, *i.e.* $\forall p \in P$, $|{}^\bullet p| \leq 1$.

$N = (P, T, W)$ is a (weighted) *Join-Free net* if any transition has at most one input place, *i.e.* $\forall t \in T$, $|{}^\bullet t| \leq 1$. An *S-net* is a Join-Free net such that any transition has at most one output place, *i.e.* $\forall t \in T$, $|t^\bullet| \leq 1$.

A *Fork-Attribution* net (or FA net) is both a Join-Free and a Choice-Free net.

Note that a transition of a T-net may have several input places. Thus, T-nets are not included in FA nets. Similarly, a place of an S-net may have several output transitions, thus S-nets also are not included in FA nets. The nets presented in Figure 3 and Figure 4 are Choice-Free. The net on the right side of Figure 4 is a T-net.

The *dual* of a net is defined by reversing the arcs and swapping places and transitions. This transformation amounts to transposing the incidence matrix.

Choice-Free and Join-Free classes are dual. S and T classes are also dual. However transforming a net into its dual does not necessarily provide a simple way to deduce behavioral properties of one net from the other.

C. Markings and firing sequences

A *marking* M of a net N is a mapping $M : P \to \mathbb{N}$. A *system* is a couple (N, M_0) where N is a net and M_0 the initial marking of N.

A marking M of a net N *enables* a transition $t \in T$ if $\forall p \in {}^\bullet t, M(p) \geq W(p,t)$. A marking M *enables* a place $p \in P$ if M enables all its output transitions. The marking M' obtained from M by the firing of an enabled transition t is defined by $\forall p \in P, M'(p) = M(p) - W(p,t) + W(t,p)$. We note $M \xrightarrow{t} M'$.

A *firing sequence* σ of length $n \geq 1$ on the set of transitions T is a mapping $\{1, \ldots, n\} \to T$. A sequence is *infinite* if its domain is countably infinite. A firing sequence $\sigma = t_1 t_2 \cdots t_n$ is *feasible* if the successive markings obtained $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \cdots \xrightarrow{t_n} M_n$ are such that, for any $i \in \{1, \cdots, n\}$, M_{i-1} enables transition t_i. We note $M_0 \xrightarrow{\sigma} M_n$. A marking M' is said to be *reachable* from the marking M if there exists a feasible firing sequence σ such that $M \xrightarrow{\sigma} M'$. The set of reachable markings from M is denoted by $[M\rangle$.

The *Parikh vector* $\vec{\sigma} : T \to \mathbb{N}$ associated with a finite sequence of transitions σ maps every transition t of T to the number of occurrences of t in σ.

D. Liveness and boundedness

Liveness and boundedness are two basic properties ensuring that all transitions of a system $S = (N, M_0)$ can always be fired and that the overall number of tokens remains bounded.

More formally,

- A system S is *live* if for every marking M in $[M_0\rangle$ and for every transition t, there exists a marking M' in $[M\rangle$ enabling t.
- S is *bounded* if there exists an integer k such that the number of tokens in each place never exceeds k. Formally,

$$\exists k \in \mathbb{N}\ \forall M \in [M_0\rangle\ \forall p \in P,\ M(p) \leq k\,.$$

S is *k-bounded* if, for any place $p \in T$,

$$k \geq \max\{M(p) | M \in [M_0\rangle\}\,.$$

- A system S is *well-behaved* if it is live and bounded.

A marking M is live (resp. bounded) for a net N if the system (N, M) is live (resp. bounded).

Furthermore, the structure of a net N may be studied to ensure the existence of an initial marking M_0 such that (N, M_0) is live and bounded:

- A net N is *structurally live* if there exists a marking M_0 such that the system $S = (N, M_0)$ is live.
- A net N is *structurally bounded* if the system $S = (N, M_0)$ is bounded for every M_0.
- A net is *well-formed* if it is structurally live and structurally bounded.

Our study focuses on well-formed nets. For Choice-Free and Join-Free nets, these properties are related to the consistency and the conservativeness properties defined as follows using the incidence matrix C of a net:

- A net N with incidence matrix C is *consistent* if there exists a vector $X \in \mathbb{N}^{|T|}$ such that $X \geq \mathbb{1}^{|T|}$ and $CX = 0$.
- A net N with incidence matrix C is *conservative* if there exists a vector $Y \in \mathbb{N}^{|P|}$ such that $Y \geq \mathbb{1}^{|P|}$ and ${}^tYC = 0$.

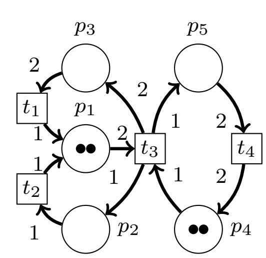
The next theorem expresses a necessary and sufficient condition of well-formedness for Choice-Free and Join-Free nets.

Theorem 1 ([12]): Suppose that N is a (weighted) Join-Free or Choice-Free net. The properties

1) N is consistent and conservative
2) N is well-formed

are equivalent. Moreover, any connected and well-formed Join-Free or Choice-Free net is strongly connected.

Figure 5 shows a well-formed Choice-Free system.



$$\begin{pmatrix} 2\ 2\ 1\ 1\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix} = {}^t\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Figure 5. The weighted Choice-Free net is both consistent (right vector ${}^t(2, 2, 2, 1)$) and conservative (left vector $(2, 2, 1, 1, 1)$), thus well-formed.

II. Transformations preserving firing sequences

This section aims at presenting several polynomial transformations that preserve the feasible firing sequences. They will be used in the sequel to prove the sufficient conditions of liveness. We first define the *scaling* of a system by a vector. The notions of *1-conservativeness* and *balancing* are then introduced. We present the *useful tokens* property allowing to reduce the initial number of tokens without modifying the feasible sequences. *Normalization* of T-systems is finally recalled and compared to balancing.

A. Scaling of systems

We define the *scaling*, multiplying weights and initial markings by strictly positive rational numbers.

Definition 1: The multiplication of all input and output weights of a marked place p together with its marking by a strictly positive rational y is the *scaling of the place p* if the resulting input and output weights and marking are integers. If each place p of a system is scaled by the component $Y[p]$ of a vector Y, the *system* is said to be *scaled by Y*.
Figure 6 shows the scaling of a marked place by 2.

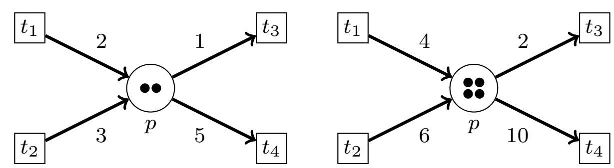


Figure 6. The marked place on the left is scaled by 2, yielding the place on the right.

Theorem 2: Let $S = ((P, T, W), M_0)$ be a system and Y a vector of $|P|$ strictly positive rational components. Scaling S by Y preserves the feasible sequences of firings.

Proof: Let $S' = ((P, T, W'), M_0')$ be the system obtained by scaling $S = ((P, T, W), M_0)$ with Y. Let $\sigma = \sigma_1 t$ be a finite sequence of firings. We prove that σ is equivalently feasible in S and S', by induction on the size of σ.

If σ_1 is empty, then suppose M_0 enables t, meaning that $M_0(p) \geq W(p, t)$ for all input places p of t.
This is equivalent to

$$\forall p \in {}^\bullet t, M_0'(p) = Y[p] \cdot M_0(p) \geq Y[p] \cdot W(p, t) = W'(p, t)$$

and M_0' enables t in S'. Now suppose that σ_1 is not empty and is feasible in both S and S'. The firing sequence σ is feasible in S if and only if

$$\forall p \in {}^\bullet t,\ M_0(p) + \sum_{t_i \in {}^\bullet p} W(t_i, p) \cdot \vec{\sigma}_1(t_i) - \sum_{t_i \in p^\bullet} W(p, t_i) \cdot \vec{\sigma}_1(t_i) \geq W(p, t)$$

which is equivalent to

$$\forall p \in {}^\bullet t,\ Y[p] \cdot M_0(p) + \sum_{t_i \in {}^\bullet p} Y[p] \cdot W(t_i, p) \cdot \vec{\sigma}_1(t_i) - \sum_{t_i \in p^\bullet} Y[p] \cdot W(p, t_i) \cdot \vec{\sigma}_1(t_i) \geq Y[p] \cdot W(p, t)$$

and

$$\forall p \in {}^\bullet t,\ M'_0(p) + \sum_{t_i \in {}^\bullet p} W'(t_i, p) \cdot \vec{\sigma}_1(t_i) - \sum_{t_i \in p^\bullet} W'(p, t_i) \cdot \vec{\sigma}_1(t_i) \geq W'(p, t)\,.$$

Thus, t is equivalently enabled in S'. ■

B. Balancing and 1-conservativeness in weighted Petri Nets

The notion of 1-conservativeness appeared in [3] as a restriction of the conservativeness. In the following, balancing is defined as a scaling that yields 1-conservative systems and applies to conservative systems.

Definition 2: A transition t is 1-conservative if

$$\sum_{p \in {}^\bullet t} W(p, t) = \sum_{p \in t^\bullet} W(t, p)\,.$$

If all the transitions of a net are 1-conservative, the net is said to be 1-conservative.

Lemma 1: 1-conservativeness implies conservativeness.

Proof: Any 1-conservative net $N = (P, T, W)$ satisfies

$$\forall t \in T, \sum_{p \in {}^\bullet t} W(p, t) = \sum_{p \in t^\bullet} W(t, p)$$

which is equivalent to

$$\forall t \in T, \sum_{p \in t^\bullet} W(t, p) + \sum_{p \in {}^\bullet t} -W(p, t) = 0\,.$$

We deduce that ${}^t\mathbb{1}^{|P|}$ is a conservativeness vector for N. ■

Now we define *balancing*, which transforms a system into a 1-conservative system having the same set of feasible firing sequences.

Definition 3: Let S be a system. *Balancing* S consists in scaling S by a vector Y of strictly positive rational numbers such that the resulting system is 1-conservative.

This transformation can help gain insight into conservative systems as shown by the next lemma and theorem.

Lemma 2: A system is conservative if and only if it can be balanced.

Proof: Consider a conservative system with incidence matrix C, then by definition there exists a vector $Y \geq \mathbb{1}^{|P|}$ of natural numbers such that ${}^tY \cdot C = 0$. Multiplying every component $C[p, t]$ by $Y[p]$ yields an incidence matrix C' satisfying for every transition t, ${}^t\mathbb{1}^{|P|} \cdot C'[t] = 0$ and the new system is 1-conservative. Moreover, C' and the new initial marking contain only integers. Now, if the system can be balanced, there exists a vector Y with only strictly positive rational numbers that annuls every column of C. Multiplying the components of Y by the least common multiple of their denominators gives a conservative vector, proving the lemma. ■

Theorem 3: Balancing preserves the feasible sequences.

Proof: Balancing is a scaling that fulfills one more condition over the incidence matrix. Thus, Theorem 2 applies, proving the claim. ■

Corollary 1: A conservative system is live if and only if one of its balancings is live.

Proof: Any conservative system can be balanced by Lemma 2. The resulting system has the same feasible firing sequences by Theorem 3. Thus, if for every reachable marking M and every transition t of one of the systems, there exists a sequence feasible at M that contains t, then the whole sequence is feasible in the other system, in which t can consequently be fired. We deduce that the liveness of one system is equivalent to the liveness of the other one. ■

Finding an adequate (conservative) scaling vector for a well-formed Choice-Free or Join-Free net consists in finding a solution to ${}^tX \cdot C = 0, X \geq \mathbb{1}^{|P|}$, or ${}^tC \cdot X = 0, X \geq \mathbb{1}^{|P|}$ where the entries of C are integers and those of X are naturals [12]. A rational solution X can be found with a linear program in weakly polynomial time [25]. Multiplying the components of X by the product of their denominators leads to a scaling vector solution with a polynomial increase of the number of bits.

C. Useful Tokens in Weighted Petri Nets

Definition 4: A weighted Petri net is said to satisfy the *useful tokens condition* if every place p is initially marked with a multiple of gcd_p.

The following theorem shows that any initial marking can be modified to satisfy this condition in such a way that the set of feasible sequences is not modified.

Theorem 4: The marking $M_0(p)$ of every place p of a system $S = (N, M_0)$ can be replaced by

$$\left\lfloor \frac{M_0(p)}{gcd_p} \right\rfloor \cdot gcd_p$$

without modifying the feasible firing sequences of S.

Proof: Let N be a net. Consider two markings M_0 and M'_0 such that

$$\forall p \in P,\ M'_0(p) = \left\lfloor \frac{M_0(p)}{gcd_p} \right\rfloor \cdot gcd_p\,.$$

Let r_p be the remainder of the division of $M_0(p)$ by gcd_p. We get $M_0(p) = M'_0(p) + r_p$. Consider a feasible firing sequence $\sigma = \sigma_1 t$ such that $M_0 \xrightarrow{\sigma_1} M$ and $M'_0 \xrightarrow{\sigma_1} M'$. Denote by v_p the integer

$$\sum_{t_i \in {}^\bullet p} W(t_i, p) \cdot \vec{\sigma}_1(t_i) - \sum_{t_i \in p^\bullet} W(p, t_i) \cdot \vec{\sigma}_1(t_i)\,.$$

Since all input and output weights of p are multiples of gcd_p, v_p is a multiple of gcd_p. Now suppose that σ is a feasible firing sequence for M_0. This implies that t is enabled by M, hence the inequality :

$$\forall p \in {}^\bullet t, M_0(p) + v_p \geq W(p,t)$$

is equivalent to

$$\forall p \in {}^\bullet t, M_0'(p) + r_p + v_p \geq W(p,t).$$

Now, for every place p, $M_0'(p)$, v_p and $W(p,t)$ are multiples of gcd_p and r_p is strictly smaller than gcd_p. Equivalently,

$$\forall p \in {}^\bullet t, M_0'(p) + v_p \geq W(p,t)$$

and t can be fired at M', which completes the proof. ■

D. Normalization and Balancing

Normalization was introduced in the context of weighted T-systems to obtain a sufficient condition of liveness [8]. This transformation is explicited in a different way here starting from the consistency of a well-formed T-system.

Definition 5: A transition t is *normalized* if all the input and output weights of t are equal. A system is normalized if all its transitions are normalized.

Normalizing consists in transforming a system into a normalized one by means of an appropriate system scaling.

The next theorem shows that normalization can be performed on any consistent weighted T-system.

Theorem 5 ([8]): Every consistent weighted T-system can be normalized.

Proof: Suppose that $S = (N, M_0)$ is a weighted consistent T-system with incidence matrix C. There exists by definition a vector $X \geq \mathbb{1}^{|T|}$ of integers such that $C \cdot X = 0$.

Since the T-system is consistent, its places either have one input and one output transition or are isolated. Only non isolated places need be considered for normalization.

Since any place p has exactly one input transition t and one output transition t', we observe that

$$W(t,p) \cdot X[t] = W(p,t') \cdot X[t'] .$$

Denote by K the least common multiple of the values $X[t]$, $t \in T$. For any place p with input transition t, we set

$$\alpha_p = \frac{K}{X[t] \cdot W(t,p)} .$$

Now we prove that the weighted T-system $S' = (N', M_0')$ obtained by scaling S with $(\alpha_1, \cdots, \alpha_{|P|})$ is normalized. Indeed, for any place $p \in {}^\bullet t'$ such that ${}^\bullet p = \{t\}$,

$$W'(p,t') = W(p,t') \cdot \alpha_p = \frac{W(t,p) \cdot X[t] \cdot K}{X[t'] \cdot X[t] \cdot W(t,p)} = \frac{K}{X[t']} .$$

Renaming t' to t, we get

$$W'(p,t) = \frac{K}{X[t]} .$$

Similarly, for any place $p \in t^\bullet$,

$$W'(t,p) = W(t,p) \cdot \alpha_p = \frac{K}{X[t]}$$

and S' is normalized. ■

Figure 7 illustrates the differences between balanced and normalized weighted nets. On the left, the Choice-Free net is balanced but neither normalized nor normalizable. The net in the middle is a normalized and balanced circuit. The third one is a normalized but not balanced T-system.

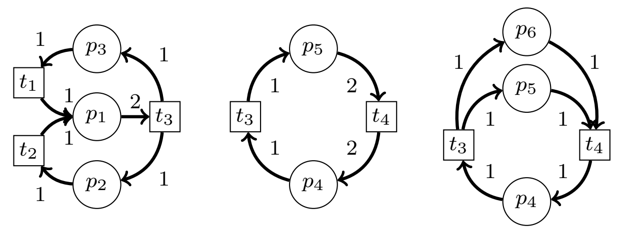


Figure 7. Possibilities to normalize or balance a net depend on its structure.

III. Well-behavedness of Join-Free systems

We present in this section a polynomial sufficient condition for the liveness of a well-formed Join-Free system. This condition will be used in Section IV to deduce a sufficient condition for the liveness of Choice-Free systems.

We first note that well-formed Join-Free systems are necessarily balanceable. The sufficient condition of liveness is then expressed on a balanced version of the system. Normalization is compared to balancing for the subclass of S-systems. Finally we examine the special case of ordinary Join-Free systems.

A. Well-formedness of Weighted Join-Free Systems

As recalled earlier, our study is restricted to well-formed Join-Free systems. By Theorem 1, such systems are consistent and conservative. According to Lemma 2 these systems are also balanceable, which can be done in polynomial time. In the following, we focus on strongly connected and balanced Join-Free systems.

B. A sufficient polynomial condition for the liveness of balanced and weighted Join-Free systems

The following technical lemma expresses a simple sufficient condition for the existence of enabled places.

Lemma 3: Let $S = ((P, T, W), M_0)$ be a balanced strongly connected Join-Free system fulfilling the useful tokens condition and the inequality

$$\sum_{p \in P} M_0(p) > \sum_{p \in P} (max_p - gcd_p) . \quad (1)$$

Then for every marking M in $[M_0\rangle$, there exists a place $p \in P$ which is enabled by M.

Proof: As M_0 fulfills the useful tokens condition, it follows that for every place p, $M_0(p)$ is a multiple of gcd_p. Moreover, all input and output weights of every place p are multiples of gcd_p. Thus, for every reachable marking M and every place p, $M(p)$ is a multiple of gcd_p.

Now suppose, by contradiction, that M is a fixed reachable marking that does not enable any place. Then

$$\forall p \in P, M(p) \leq max_p - gcd_p,$$

thus,

$$\sum_{p\in P} M(p) \leq \sum_{p\in P} (max_p - gcd_p)\,. \qquad (2)$$

Since S is balanced, every transition firing maintains the number of tokens in the system, implying that

$$\sum_{p\in P} M(p) = \sum_{p\in P} M_0(p)\,.$$

Inequality (2) is then equivalent to

$$\sum_{p\in P} M_0(p) \leq \sum_{p\in P} (max_p - gcd_p),$$

contradicting inequality (1). ■

We are now able to prove that the liveness condition given in the next theorem is sufficient.

Theorem 6: Let $S = (N, M_0)$ be a balanced strongly connected Join-Free system satisfying the useful tokens condition. S is live if

$$\sum_{p\in P} M_0(p) > \sum_{p\in P} max_p - gcd_p\,.$$

Proof: Let S be a Join-Free system meeting the conditions of the theorem. We show that S is live, *i.e.* for every reachable marking M and every transition t, there exists a finite and feasible firing sequence starting at M and leading to a marking M' enabling t. For that purpose, we prove that Algorithm 1 computes such a sequence and terminates. Tokens are arbitrarily numbered to ensure its convergence.

If $M(p) \geq W(p,t)$ then t is enabled. The algorithm terminates and σ is the requested firing sequence.

In the other case, p is not enabled. $L \neq \emptyset$ by Lemma 3 and thus, p' exists. Also note that $p \neq p'$ since $p \notin L$.

At every step of the loop, a firing occurs so as to reduce the minimal distance between the mobile token with smallest number and the place p. Notice that a firing can move several tokens at once on different paths, taking some of them away from p. Such a firing is always possible when p is not enabled, as there exists at least one enabled place at any reachable marking by Lemma 3. In so doing, a new marking M' is reached, inducing the new shortest distances of $d^{M'}$. We prove that $d^M >_{lex} d^{M'}$, following the lexical order on $\mathbb{N}^\delta$. The firing of t' ensures that $d_i^{M'} < d_i^M$. At this step, as the numbers of the other displaced tokens are greater than i, the inequality $d^M >_{lex} d^{M'}$ is true, even if for any $j > i$, d_j^M may increase. Now, as the lexical order is well-founded over $\mathbb{N}^\delta$, the algorithm terminates, σ is finite and t is enabled. ■

Data:
- The current reached marking M, which contains δ tokens numbered $1, \ldots, \delta$;
- The unique input place p of t;
- The δ-tuple $d^M = (d_1^M, \ldots, d_\delta^M)$, which associates the shortest distance d_i^M from token i to the place p according to the marking M.

Result: A finite firing sequence σ such that $M \xrightarrow{\sigma} M'$ and M' enables t.

```
σ := ε, the empty sequence;
while M(p) < W(p,t) do
    Let L be the set of the places enabled by M and
    J = {i_1 ... i_k} the set of the numbers of the tokens
    in the places of L at M;
    Let p' be the place of L containing token i where i
    is the smallest value in J;
    Let μ = p', t', p'', ..., p be one shortest path from
    p' to p;
    Fire t', send token i to p'' and sufficiently many
    other tokens of p' to the output places of t',
    according to the output weights of t';
    Upgrade M and d^M;
    σ := σ t'
end
```

Algorithm 1: The algorithm computes a feasible firing sequence starting at M so as to enable any transition t.

Figure 8 shows the application of this theorem to a balanced Join-Free system. The pictured system satisfies $\sum_p (max_p - gcd_p) = 1+1+0 = 2$. The inequality becomes $\sum_p M_0(p) > 2$ and is fulfilled by the marking on the figure.

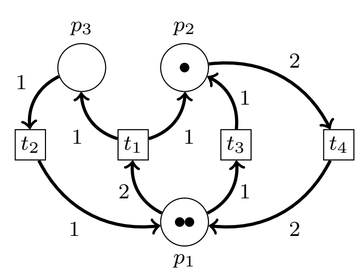


Figure 8. The initial marking of this balanced Join-Free system fulfills the conditions of Theorem 6 and is thus well-behaved.

C. Balancing, Normalization and S-systems

S-systems constitute a subclass of Join-Free systems. As shown below, the normalization that was developed for T-systems coincides with balancing for S-systems.

Theorem 7: A strongly connected S-system is balanced if and only if it is normalized.

Proof: As every transition of the system has one input and one ouput place, it is balanced if and only if its unique input weight equals its unique output weight, characterizing a normalized system. ∎

D. The special case of ordinary Join-Free Systems

We show that the sufficient condition of liveness for well-formed Join-Free nets becomes a necessary and sufficient condition of liveness for ordinary well-formed Join-Free nets, which are S-nets.

Theorem 8: An ordinary strongly connected well-formed Join-Free net is an S-net.

Proof: By contradiction, suppose that there exists an ordinary well-formed strongly connected Join-Free net S and a transition t with $|t^\bullet| > 1$. S is strongly connected, thus there exists a circuit $c = tp_1t_1p_2t_2 \dots p_kt_k$ passing through t with $t_k = t$. As S is conservative by Theorem 1, there exists $Y \geq \mathbb{1}^{|P|}$ such that ${}^tY \cdot C = 0$. Since $|t^\bullet| > 1$, there is at least one other place $p' \in t^\bullet$ with $p' \neq p_1$. Thus

$$Y[p_k] = \sum_{p_i \in t^\bullet} Y[p_i] > Y[p_1] \,.$$

Generalizing over $i \in \{1, \dots, k\}$ with $Y[p_{k+1}] = Y[p_1]$, $Y[p_i] \geq Y[p_{i+1}]$, which contradicts $Y[p_k] > Y[p_1]$. ∎

The inequality of Theorem 6 induces a necessary and sufficient condition of liveness for ordinary well-formed strongly connected S-systems. The following theorem proves a similar condition of liveness for ordinary strongly connected Join-Free systems that may not be well-formed.

Theorem 9: An ordinary and strongly connected Join-Free system (N, M_0) having at least one place and one transition is live if and only if $\sum_p M_0(p) \geq 1$.

Proof: As the system is strongly connected and Join-Free, every transition has exactly one input place. As weights are all 1, one token is necessary and sufficient to fire any transition. Moreover, every transition has at least one output place, thus each firing preserves or increases the number of tokens in the system. Thus, at any reachable marking M, a token enables a transition and can reach any other place following a finite sequence. ∎

IV. Well-behavedness of Choice-Free systems

In this section, we recall known structural properties of well-formedness and a property relating the liveness of a Choice-Free system to the liveness of particular subnets. Exploiting these properties and the results obtained in previous sections, we deduce a polynomial sufficient condition of liveness for well-formed Choice-Free systems. We finally deduce from these new results a known sufficient condition of liveness that was developed for T-systems [8].

A. Well-formedness, liveness and FA P-subnets

We recall properties of Choice-Free and Join-Free systems and present results leading to the sufficient condition.

Theorem 10 ([12]): Any conservative and strongly connected Join-Free net is consistent. Any consistent and strongly connected Choice-Free net is conservative.

A *source* place is defined as a place with at least one output transition and without input transition. The liveness of a Choice-Free system can be stated by observing the liveness of some of its Fork-Attribution (FA) subsystems.

Theorem 11 ([12]): Let (N, M_0) be a Choice-Free system without source places. (N, M_0) is live iff for every strongly connected FA P-subnet of N, noted N', the system $(N', M_0[P'])$ is live.

Figure 4 represents all strongly connected FA P-subnets of the Choice-Free net shown in Figure 3.

Well-formed Choice-Free nets are consistent. The following property shows that consistency propagates to P-subnets.

Lemma 4: All P-subnets of a consistent net are consistent.

Proof: Let N be a consistent net and C its incidence matrix. By definition of consistency, there exists a vector $X \geq \mathbb{1}^{|T|}$ such that $C \cdot X = 0$. Thus, X annuls each row of C. By definition, a P-subnet N' of a net N is composed of a subset of places P' and the set of all their input and output transitions. Thus, the incidence matrix C' of N' is the subset of rows from C that correspond to the places of P'. Since the vector X annuls all rows of C, it annuls all rows of C' and N' is consistent. ∎

Thus, the well-formedness of a Choice-Free net induces strong structural properties for its P-subnets. Morever, FA P-subnets conform to balancing.

Lemma 5: All strongly connected FA P-subnets of a well-formed Choice-Free net are conservative.

Proof: Let N be a well-formed Choice-Free net. By Theorem 1, N is consistent. All FA P-subnets of a consistent Choice-Free net are consistent by Lemma 4. According to Theorem 10, if an FA P-subnet is consistent and strongly connected, then it is conservative. ∎

The previous section presented a live initial marking for balanced strongly connected Join-Free nets. This condition applies to balanced strongly connected FA nets.

Lemma 6: Let $S = ((P, T, W), M_0)$ be a strongly connected and balanced FA system satisfying the useful tokens condition. S is live if

$$\sum_{p \in P} M_0(p) > \sum_{p \in P} (max_p - gcd_p) \,.$$

Proof: FA nets form a subclass of Join-Free nets, thus Theorem 6 applies, which proves the claim. ∎

B. A polynomial sufficient condition for the liveness of well-formed and weighted Choice-Free systems

The next lemma gives a live initial marking for strongly connected and conservative FA nets, which are well-formed.

Lemma 7: Let $S = ((P, T, W), M_0)$ be a strongly connected and conservative FA system. S is live if

$$\forall p \in P,\ M_0(p) = max_p \,.$$

Proof: Consider the incidence matrix C of N and the system $S = (N, M_0)$. By Lemma 2, there exists a balancing vector $Y \geq \mathbb{1}^{|P|}$ for N such that Y contains only naturals. Scaling S by Y yields the balanced system $S^b = (N^b, M_0^b)$. Consequently,

$$\forall p \in P,\ max_p^{N^b} = Y[p] \cdot max_p^N,\ max_p^{N^b} \in \mathbb{N} \,.$$

We deduce that

$$\sum_{p \in P} M_0^b(p) = \sum_{p \in P} max_p^{N^b} > \sum_{p \in P} (max_p^{N^b} - gcd_p^{N^b}) \,.$$

Moreover, for every place p, max_p is a multiple of gcd_p and $M_0^b(p) = Y[p] \cdot max_p$. Thus M_0^b fulfills the useful tokens condition and by Lemma 6 the balanced FA system S^b is live. According to Corollary 1, balancing a system preserves the property of liveness, thus S is live. ■

As a consequence, a live initial marking is determined in polynomial time for well-formed Choice-Free nets.

Theorem 12: Let $S = ((P, T, W), M_0)$ be a well-formed Choice-Free system. S is well-behaved if

$$\forall p \in P,\ M_0(p) = max_p \,.$$

Proof: Let $N^s = (P^s, T^s, W^s)$ be any of the strongly connected FA P-subnets of N and $S^s = (N^s, M_0^s)$ where M_0^s is the restriction of M_0 to P^s. N^s is conservative (Lemma 5) and S^s is live (Lemma 7). Thus, the marking M_0 makes any strongly connected FA P-subnet live. N^s is well-formed thus without source place and by Theorem 11 (N, M_0) is live. ■

Figure 9 pictures a well-behaved Choice-Free system. Indeed, this system is well-formed (see Fig. 5) and the marking of each place p equals max_p. As the system is Choice-Free, each place has only one output transition. Consequently, it is sufficient to initially mark each place with its output weight.

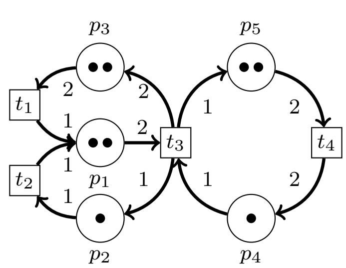


Figure 9. The Choice-Free system is well-formed, thus the sufficient initial marking of Theorem 12 makes the system well-behaved. In order to construct this live initialization, each place p is marked with max_p tokens.

C. Balancing, Normalization and T-systems

For T-systems, a sufficient condition of liveness was expressed in [8]. We prove here that this result can be viewed as a direct consequence of Lemma 6.

Lemma 8: All strongly connected FA P-subnets of a strongly connected T-net are circuits.

Proof: Let us consider the strongly connected FA P-subnet N of a strongly connected T-net and suppose by contradiction that a transition t of N has at least two output places p_1 and p_2. Let p be the unique input place of t. Places p_1 and p_2 belong to two different paths leading to p as the net is strongly connected. These paths are not allowed to merge as two inputs of a place since the net is a T-net. They cannot be two inputs of a transition either, since the net is FA thus Join-Free. ■

We recall a known sufficient condition of liveness for T-systems that we prove now, differently from [8].

Theorem 13 ([8]): Let $S = ((P, T, W), M_0)$ a strongly connected and normalized T-system that fulfills the useful tokens condition. S is live if every circuit C of S satisfies

$$\sum_{p \in C \cap P} M_0(p) > \sum_{p \in C \cap P} (max_p - gcd_p) \,.$$

Proof: By definition of the normalization, if N is a normalized T-net then its circuits are balanced. Moreover, balanced circuits are strongly connected and balanced FA nets, thus Lemma 6 applies. T-nets are Choice-Free nets and S is strongly connected thus without source place. By Lemma 8 and Theorem 11, the claim is proved. ■

V. Sufficient conditions are not necessary

Both previous sufficient conditions of liveness for Join-Free and Choice-Free systems are not necessary. This is shown through a counter example that comes from the T-system [8] pictured in Figure 10. It consists of a live marked circuit that does not fulfill the sufficient conditions. Indeed,

$$\sum_p (max_p - gcd_p) = (14 - 2) + (21 - 7) + (6 - 3) = 29 \,.$$

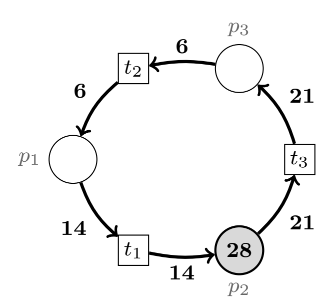


Figure 10. This circuit is a live Join-Free and Choice-Free system but does not fulfill their sufficient conditions.

The reachability graph of Figure 11 shows that every transition can be fired from any reachable marking after a finite firing sequence, thus the circuit of Figure 10 is live.

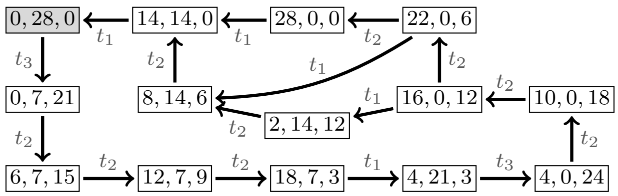


Figure 11. The reachability graph of the circuit shows all its feasible sequences. From any reachable marking, there exists a finite feasible sequence that contains every transition, implying liveness.

VI. Conclusion and perspectives

We presented a polynomial sufficient condition of liveness for well-formed Choice-Free systems. Moreover, any well-formed strongly connected Join-Free net can be balanced and we gave a polynomial sufficient condition of liveness for such systems. We unified a set of theoretical results over subclasses of Petri nets, S and T-systems in particular. Three transformations that simplify weights or markings have been investigated. One of them is normalization, which was introduced in the context of T-systems and presented in this paper as a particular case of our theory. The second transformation is balancing, applying to conservative nets. Finally, the reduction to useful tokens, introduced for T-systems by [8], has been extended to all Petri nets. The developed theory provides efficient and simple algorithms ensuring the well-behavedness of systems. Such methods can be used in real cases, to design embedded systems. Future work would extend this theory to larger classes of nets and consider timed versions of the subclasses studied.

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