Robustness for a Liouville type theorem in exterior domains

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Abstract

We are interested in the robustness of a Liouville type theorem for a reaction diffusion equation in exterior domains. Indeed H. Berestycki, F. Hamel and H. Matano (2009) proved such a result as soon as the domain satisfies some geometric properties. We investigate here whether their result holds for perturbations of the domain. We prove that as soon as our perturbation is close to the initial domain in the $C^{2,\alpha}$ topology the result remains true while it does not if the perturbation is not smooth enough.

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1 Introduction and main results

1.1 Problem and motivations

This paper investigates the exterior domain problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N \backslash K, \\ \partial_\nu u = 0 & \text{on } \partial K, \\ 0 < u \leq 1 & \text{in } \mathbb{R}^N \backslash K, \\ u(x) \to 1 \quad \text{as } |x| \to +\infty & \text{uniformly in } x \in \mathbb{R}^N \backslash K, \end{cases} \tag{1.1}$$

where K is a compact set of $\mathbb{R}^N$, f is a bistable non-linearity.
This problem is motivated by the construction of generalized transition fronts for the associated parabolic problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{for all } t \in \mathbb{R}, \quad x \in \mathbb{R}^N \backslash K, \\ \partial_\nu u = 0 & \text{for all } t \in \mathbb{R}, \quad x \in \partial K, \end{cases} \tag{1.2}$$

such that

$$\sup_{x\in\mathbb{R}^N\setminus K}|u(t,x)-\phi(x_1-ct)|\to 0 \text{ as } t\to-\infty,$$

where ϕ is a planar traveling wave connecting 1 to 0, i.e

$$\begin{cases}-\phi''-c\phi'=f(\phi) & \text{in } \mathbb{R},\\ \phi(-\infty)=1, \quad \phi(+\infty)=0.\end{cases} \tag{1.3}$$

It is proved in [**?**] that the unique solution of (1.2) converges toward a solution of (1.1) as $t\to+\infty$. Thus problem (1.1) determines whether there is a complete invasion or not, that is whether $u(t,x)\to 1$ as $t\to+\infty$ for all $x\in\mathbb{R}^N\setminus K$. More precisely, complete invasion is shown to hold if and only if (1.1) has no solution different from 1. In [**?**], Berestycki, Hamel and Matano have shown that if K is star shaped or directionally convex the unique solution of (1.1) is 1 (see at the end of this section for precise definitions of star shaped or directionally convex domain). The present paper examines under which conditions this Liouville type theorem is robust under perturbations of the domain. This is shown here to strongly depend on the smoothness of the perturbations that are considered. Indeed our main result is to show that it is true for $C^{2,\alpha}$ perturbations but not for C^0 ones. This is stated precisely in the next section. We leave as an open problem to determine what is the optimal space of regularity of the perturbation for which the result remains true.

In this paper f is assumed to be a $C^{1,1}([0,1])$ function such that

$$f(0)=f(1)=0, \quad f'(0)<0, \quad f'(1)<0, \tag{1.4a}$$

and there exists $\theta\in(0,1)$ such that,

$$f(s)<0 \quad \forall s\in(0,\theta), \quad f(s)>0 \quad \forall s\in(\theta,1). \tag{1.4b}$$

Moreover we suppose that f satisfies the following positive mass property,

$$\int_0^1 f(\tau)d\tau>0. \tag{1.5}$$

Before stating the main results, let explain what we mean by star-shaped or directionally convex obstacles.

Definition 1.1 K is called star-shaped, if either $K=\emptyset$, or there is $x\in\overset{\circ}{K}$ such that, for all $y\in\partial K$ and $t\in[0,1)$, the point $x+t(y-x)$ lies in $\overset{\circ}{K}$ and $\nu_K(y)\cdot(y-x)\geq 0$, where $\nu_K(y)$ denotes the outward unit normal to K at y.

Definition 1.2 K is called directionally convex with respect to a hyperplane P if there exists a hyperplane $P=\{x\in\mathbb{R}^N, x\cdot e=a\}$ where e is a unit vector and a is some real number, such that

- for every line Σ parallel to e the set $K\cap\Sigma$ is either a single line or empty,
- $K\cap P=\pi(K)$ where $\pi(K)$ is the orthogonal projection of K onto P.

1.2 Main results

Our main result is the following Theorem

Theorem 1.3 *Let $(K_\varepsilon)_{0<\varepsilon\leq 1}$ be a family of $C^{2,\alpha}$ compact manifolds of $\mathbb{R}^N$. Assume that $K_\varepsilon \to K$ for the $C^{2,\alpha}$ topology as $\varepsilon \to 0$, and K is either star-shaped or directionally convex with respect to some hyperplane P. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the unique solution of (1.1) is $u_\varepsilon \equiv 1$*

This theorem means that for obstacles that are compact sets in $\mathbb{R}^N$ and close enough (in the $C^{2,\alpha}$ sense) to some star-shaped or directionally convex domains, the unique solution of (1.1) is the constant 1. And thus a sufficient condition for the Liouville theorem to be robust under perturbation is the $C^{2,\alpha}$ convergence. On the other hand one can prove that the C^0 convergence of the perturbation is not enough for the result to stay true. This is stated in the Theorem below.

Theorem 1.4 *There exists $(K_\varepsilon)_\varepsilon$ a family of compact manifolds of $\mathbb{R}^N$ such that $K_\varepsilon \to B_{R_0}$ for the C^0 topology as $\varepsilon \to 0$, and for all $\varepsilon > 0$ the unique solution u_ε of* (1.1) *is such that $0 < u_\varepsilon < 1$ in $\mathbb{R}^N \backslash K_\varepsilon$.*

Remark 1.5 *When we write $K_\varepsilon \to K$ for the X topology we mean that for each x_0 in ∂K, and for some $r > 0$ such that $\partial K_\varepsilon \cap B_r(x_0) \neq \emptyset$ there exists a couple of parametrization of K_ε and K, ψ_ε and ψ, $X(B_r(x_0))$ functions such that $\|\psi_\varepsilon - \psi\|_{X(B_r(x_0))} \to 0$ as $\varepsilon \to 0$.*
*For more details about the $C^{2,\alpha}$ topology one can look at [**?**], chapter 6.*

Before proving the previous statements, let give some examples of domains $(K_\varepsilon)_\varepsilon$ and K to illustrate our results.

1.3 Examples of domains

We assume that $N = 2$ and we construct two families of obstacles; one which converges to a star shaped domain and the other which converges to a directionally convex domain. The black plain line represents the limit K and the thin parts represent the small perturbations (of order ε).

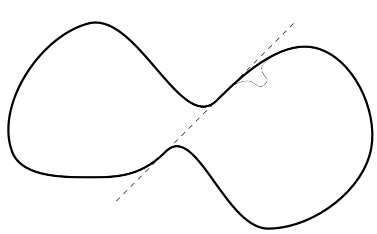

Figure 1: Obstacle that converges toward a star-shaped domain

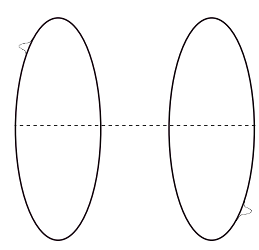

Figure 2: Obstacle that converges toward a directionally convex domain

The long-dashed lines are used during the construction of K. For the star shaped domain, it is on this line that we could find the center(s) of the domain (i.e the point x in Definition

1.1). For the directionally convex domain it represents the hyperplane P. We can clearly see that for all $\varepsilon > 0$, K_ε is not star-shaped for figure 1 and not directionally convex for figure 2. One need to be careful on the shape of the perturbations. Indeed in figure 3 below K_ε converges to an ellipse as $\varepsilon \to 0$ but the convergence of K_ε is not $C^{2,\alpha}$ (see section 3 for more details) but only C^0 which is not enough for the Liouville theorem to remain valid.

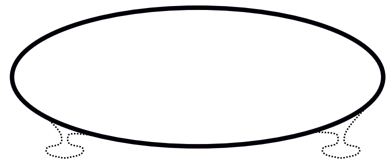

Figure 3: Obstacles converging only in the C^0 topology

We will prove Theorem 1.3 in section 2 below and Theorem 1.4 in section 3.

2 Robustness of the result for $C^{2,\alpha}$ perturbations

In this section we prove the robustness of the Liouville result when the perturbation is close to a star shaped or directionally convex domain in the $C^{2,\alpha}$ topology. To prove Theorem 1.3, we will need the following Proposition:

Proposition 2.1 *For all $0 < \delta < 1$, if u_ε is a solution of* (1.1)*, then there exists $R = R_\delta > R_0$, such that $u_\varepsilon(x) \geq 1 - \delta$ for all $|x| \geq R$ and for all $0 < \varepsilon < 1$.*

This proposition means that u_ε converges toward 1 as $|x| \to +\infty$ uniformly in ε.
Let first admit this result and prove Theorem 1.3.

2.1 Proof of Theorem 1.3

As u_ε is uniformly bounded for all $\varepsilon > 0$, using Schauder estimates, we know that up to a subsequence $u_{\varepsilon_n} \to u_0$ in C^2_{loc} as $n \to +\infty$ and u_0 satisfies:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^N \backslash K, \\ \nu \cdot \nabla u = 0 & \text{on } \partial K. \end{cases} \tag{2.1}$$

Using Proposition 2.1 we get $\lim\limits_{|x| \to +\infty} u_0(x) = 1$. And K is either star-shaped or directionally convex. We now recall the following results from [**?**]:

Theorem 1 (Theorem 6.1 and 6.4 in [?]) *Let f be a Lipschitz-continuous function in $[0,1]$ such that $f(0) = f(1) = 0$ and f is nonincreasing in $[1-\delta, 1]$ for some $\delta > 0$. Assume that*

$$\forall\, 0 \leq s < 1, \quad \int_s^1 f(\tau) d\tau > 0. \tag{2.2}$$

Let Ω be a smooth, open, connected subset of $\mathbb{R}^N$ (with $N \geq 2$) with outward unit normal ν, and assume that $K = \mathbb{R}^N \backslash \Omega$ is compact. Let $0 \leq u \leq 1$ be a classical solution of

$$\begin{cases} \Delta u = f(u) & \textit{in } \Omega, \\ \nu \cdot \nabla u = 0 & \textit{on } \partial\Omega, \\ u(x) \to 1 & \textit{as } |x| \to +\infty. \end{cases} \tag{2.3}$$

If K is star shaped or directionally convex, then

$$u \equiv 1 \textit{ in } \overline{\Omega}. \tag{2.4}$$

It thus follows that $u_0 \equiv 1$. It also proves that the limit u_0 is unique and thus $u_\varepsilon \to u_0$ as $\varepsilon \to 0$ in C^2_{loc} (and not only along a subsequence).

Now we need to prove that there exists $\varepsilon_0 > 0$ such that $u_\varepsilon \equiv 1$ for all $0 < \varepsilon < \varepsilon_0$. Let assume that for all $\varepsilon > 0$, $u_\varepsilon \not\equiv 1$ in $\Omega_\varepsilon = \mathbb{R}^N \backslash K_\varepsilon$. Then there exists $x_0 \in \overline{\Omega}_\varepsilon$ such that $u_\varepsilon(x_0) = \min_{x \in \Omega_\varepsilon} u_\varepsilon(x) < 1$. As u_ε is a solution of (1.1), the Hopf lemma yields that,

$$\text{if } x_0 \in \partial K_\varepsilon \text{ then } \frac{\delta u_\varepsilon}{\delta \nu}(x_0) < 0,$$

which is impossible due to Neuman boundary conditions. Hence $x_0 \in \Omega_\varepsilon$.

If $u_\varepsilon(x_0) > \theta$,

$$-\Delta u_\varepsilon(x_0) = f(u_\varepsilon(x_0)) > 0,$$

which is impossible since x_0 is a minimizer. So, for all $0 < \varepsilon < 1$,

$$0 \leq \min_{x \in \Omega_\varepsilon} u_\varepsilon(x) \leq \theta,$$

which contradicts $u_\varepsilon \to u_0 \equiv 1$. Thus there exists ε_0 such that for all $\varepsilon < \varepsilon_0$, $u_\varepsilon \equiv 1$. □

2.2 Proof of Proposition 2.1

We will now prove Proposition 2.1, using the following lemma:

Lemma 2.2 *There exists $\omega = \omega(r)$ with $r \in \mathbb{R}^+$ such that*

$$\begin{cases} -\omega''(r) = f(\omega(r)), & \forall r \in \mathbb{R}^+_*, \\ \omega(0) = 0, \ \omega'(0) > 0, \\ \omega' > 0, \ 0 < \omega < 1 & \textit{in } \mathbb{R}^*_+, \\ \lim_{r \to +\infty} \omega(r) = 1. \end{cases} \tag{2.5}$$

This is a well known result. In deed, by a shooting argument, if ω is a solution of the initial value problem

$$\begin{cases} -\omega'' = f(\omega) & \text{in } (0, +\infty), \\ \omega(0) = 0, \\ \omega'(0) = \sqrt{2F(1)}, \end{cases}$$

where $F(z) = \int_0^z f(s)ds$, it is easily seen that ω is also a solution of (2.5).

Proof of Proposition 2.1. Now we introduce a function f_δ with the same hypothesis as f but such that $f_\delta \leq f$, $f_\delta = f$ in $[0, 1-\delta]$ and $f_\delta(1 - \frac{\delta}{2}) = 0$. Notice that $\int_0^{1-\frac{\delta}{2}} f_\delta(z)dz > 0$ for δ small. Using the same arguments than in Lemma 2.2 there exists $\omega = \omega_\delta$ such that

$$\begin{cases} -\omega_\delta''(x) = f_\delta(\omega_\delta(x)) & \text{in } (0, +\infty), \\ \omega_\delta(0) = 0, \ \ \omega_\delta(+\infty) = 1 - \frac{\delta}{2}, & \\ 0 < \omega_\delta < 1 - \frac{\delta}{2} & \text{in } (0, +\infty), \\ \omega_\delta' > 0 & \text{in } (0, +\infty). \end{cases} \tag{2.6}$$

As K_ε is a compact set of $\mathbb{R}^N$ there exists R_0 such that $K_\varepsilon \subset B_{R_0}$ for all $\varepsilon > 0$.
Next, for any $R > R_0$ let consider $z(x) = \omega_\delta(|x| - R)$, for every $|x| \geq R$. One gets:

$$-\Delta z < f(z) \text{ in } \mathbb{R}^N \backslash B_R. \tag{2.7}$$

We want to prove that

$$\omega_\delta(|x| - R_0) < u_\varepsilon(x), \qquad \forall x \in \mathbb{R}^N, |x| \geq R_0.$$

We know from (1.1) that $u_\varepsilon(x) \to 1$ as $|x| \to +\infty$. Hence there exists $A = A(\varepsilon) > 0$ such that $u_\varepsilon(x) \geq 1 - \dfrac{\delta}{3} > \omega_\delta(|x| - A)$, for all $|x| \geq A$. Consider

$$\overline{R} = \inf \left\{ R \geq R_0; u_\varepsilon(x) > \omega_\delta(|x| - R), \text{ for all } |x| \geq R \right\}. \tag{2.8}$$

As $\overline{R} \geq R_0$ and $K_\varepsilon \subset B_{R_0}$, u_ε is always defined in $\left\{|x| > \overline{R}\right\}$. One will prove that $\overline{R} = R_0$. As ω_δ is increasing, we know that

$$\forall R \geq A \ \ u_\varepsilon(x) \geq \omega(|x| - R), \ \ \forall |x| \geq R.$$

Hence $\overline{R} \leq A$.
Assume that $\overline{R} > R_0$. Then there are two cases to study:

- either $\inf \left\{ u_\varepsilon(x) - \omega_\delta(|x| - \overline{R}), \ \ \forall |x| > \overline{R} \right\} > 0$, (1)
- or $\inf \left\{ u_\varepsilon(x) - \omega_\delta(|x| - \overline{R}), \ \ \forall |x| > \overline{R} \right\} = 0$. (2)

In the first case (1), one gets $u_\varepsilon(x) > \omega_\delta(|x| - \overline{R})$ for all $|x| > \overline{R}$. As ∇u_ε and ω_δ' are bounded, there exists $R^* < \overline{R}$ such that $u_\varepsilon(x) \geq \omega_\delta(|x| - R^*)$ for all $|x| > R^*$, and $u_\varepsilon(x_0) = \omega_\delta(|x_0| - R^*)$ for some $|x_0| > R^*$. This contradicts the optimality of $\overline{R}$.

In the second case (2), there necessarily exists x_0 with $|x_0| > \overline{R}$ such that $u_\varepsilon(x_0) = \omega_\delta(|x_0| - \overline{R})$. Let $v(x) = u_\varepsilon(x) - \omega_\delta(|x| - \overline{R})$, for all $|x| > \overline{R}$. As u_ε is a solution of (1.1) and using (2.7), v satisfies:

$$\begin{cases} -\Delta v > f(v) & \text{in } \left\{|x| > \overline{R}\right\}, \\ v > 0 & \text{on } \left\{|x| = \overline{R}\right\}. \end{cases} \tag{2.9}$$

From the maximum principle $v(x) \geq 0$, for all $|x| \geq \overline{R}$. But there exists x_0 such that $|x_0| > \overline{R}$ and $v(x_0) = 0$ which implies that $v(\cdot) \equiv 0$. This is impossible because $v(\cdot) > 0$, for all $|x| = \overline{R}$.

Then $\overline{R} = R_0$ which does not depend on ε and

$$\forall |x| \geq R_0 \quad u_\varepsilon(x) \geq \omega(|x| - R_0).$$

As $\omega_\delta(x) \to 1-\frac{\delta}{2}$ as $|x| \to +\infty$, there exists $\hat{R}$, independent of ε, such that for all $|x| > \hat{R}+R_0$, $u_\varepsilon(x) > \omega_\delta(|x| - R_0) \geq 1 - \delta$. One has proved Proposition 2.1. □

3 Counter example in the case of C^0 perturbations

Until now we have assumed that $K_\varepsilon \to K$ in $C^{2,\alpha}$, in order to use the Schauder estimates and ensure the convergence of u_ε as $\varepsilon \to 0$. One can wonder if we can weaken this hypothesis, i.e would the C^0 or C^1 convergence be enough?
We prove that C^0 perturbations are not smooth enough for the Liouville result to remain true.

3.1 Construction of a particular family of C^0 perturbations

In this subsection we construct a family of obstacles that are neither star-shaped nor directionally convex but converges uniformly to B_{R_0} which is convex. We want to prove that for all $\varepsilon \in]0,1]$ there exists a solution of (1.1) which is not identically equal to 1. To do so we will use the counterexample of section 6.3 in [?].

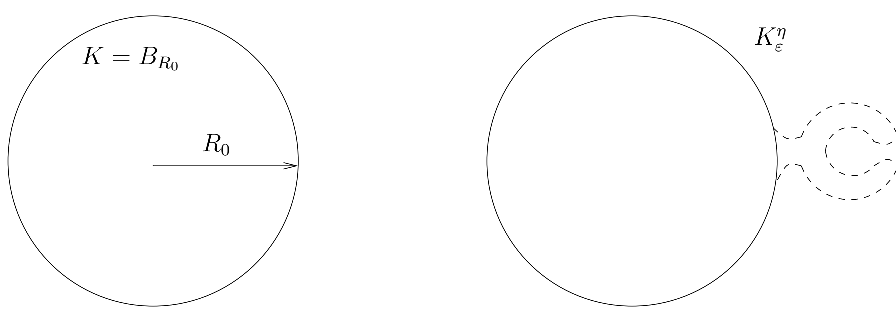

Figure 4: Liouville counterexample

Zooming on the dashed part:

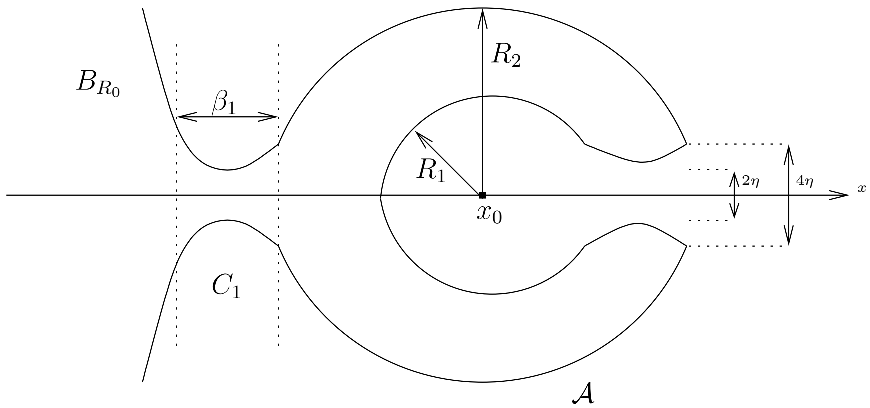

Figure 5: Zoom on the perturbation

We consider an obstacle $K_1 = K_1^\eta$ (see figure 4 and 5), such that:

$$\begin{cases} \left(\mathcal{A} \cap \{x; x_1 \leq x_1^0\}\right) \cup B_{R_0} \cup C_1 \subset K_1^\eta, \\ \mathcal{A} \cap \left\{x; x_1 > x_1^0, |x'| > 2\eta\right\} \subset K_1^\eta, \\ K_1^\eta \subset \left(\mathcal{A} \cap \left\{x; x_1 > x_1^0, |x'| > \eta\right\}\right) \cup B_{R_0} \cup \left(\mathcal{A} \cap \left\{x; x_1 \leq x_1^0\right\}\right) \cup C_1. \end{cases} \tag{3.1}$$

where $x' = (x_2, ..., x_N)$ and $\mathcal{A} = \{x : R_1 \leq |x - x^0| \leq R_2\}$, R_0, $R_1 < R_2$, are three positive constants, $x^0 = (x_1^0, 0, 0, ..., 0)$ is the center of the annular region $\mathcal{A}$ with $x_1^0 = R_0 + R_2 + \beta_1$, C_1 is some corridor that links smoothly $\mathcal{A}$ and B_{R_0} which length is β_1 and $\eta > 0$, small enough.
The family (K_ε) is constructed by downsizing K_1 such that for all $0 < \varepsilon < 1$, $\mathcal{A}_\varepsilon$ stays an annular region. We have the following lemma.

Lemma 3.1 *$K_\varepsilon \to K$ for the C^0 topology as $\varepsilon \to 0$ but not for the C^1 topology.*

This Lemma is easily proved using smooth parametrization of B_{R_0} and K_1 and noticing that for all $\varepsilon > 0$ there exists a point on the boundary of the perturbation that has a outward unit normal ortogonal to $e_1 = (1, 0, ..., 0)$.

3.2 Existence of a non constant solution u_ε of (1.1)

We want to prove that for all $0 < \varepsilon < 1$ their exists a solution $0 < u_\varepsilon < 1$ of

$$\begin{cases} -\Delta u_\varepsilon = f(u_\varepsilon) & \text{in } \mathbb{R}^N \backslash K_\varepsilon^\eta = \Omega_\varepsilon^\eta, \\ \nu \cdot \nabla u_\varepsilon = 0 & \text{on } \partial K_\varepsilon^\eta = \partial \Omega_\varepsilon^\eta, \\ u_\varepsilon(x) \to 1 \text{ as } |x| \to +\infty. \end{cases} \tag{3.2}$$

We will follow the same steps as in [?], section 6. First, let notice that it is enough to find $\omega \not\equiv 1$ solution of

$$\begin{cases} -\Delta\omega = f(\omega) & \text{in } B_R \backslash K_\varepsilon^\eta, \\ \nu \cdot \nabla\omega = 0 & \text{on } \partial K_\varepsilon^\eta, \\ \omega = 1 & \text{on } \partial B_R, \end{cases} \tag{3.3}$$

for some $R > 0$ large enough such that $K_\varepsilon^\eta \subset B_R$.
Indeed then ω extended by 1 outside B_R is a supersolution of (3.2) and one can define:

$$\psi(x) = \begin{cases} 0 & \text{if } \{|x| < R\} \backslash K_\varepsilon^\eta, \\ U(|x| - R) & \text{if } |x| \geq R, \end{cases} \tag{3.4}$$

where $U : \mathbb{R}^+ \to (0,1)$ satisfies $U'' + f(U) = 0$ in $\mathbb{R}_+^*$, $U(0) = 0$, $U'(\xi) > 0 \ \forall\ \xi \geq 0$, $U(+\infty) = 1$. It exists as soon as (1.5) is satisfied (see Lemma 2.2). As $U(|\cdot| - R)$ is a subsolution, ψ is a subsolution of (3.2).
Hence there exists a solution $\psi < u_\varepsilon < \omega$ of (3.2). If we prove that $\omega \not\equiv 1$ then $0 < u_\varepsilon < 1$ (with the maximum principle).

Now let consider our problem (3.3) and replace ω by $v = 1 - \omega$. The problem becomes

$$\begin{cases} -\Delta v = -f(1 - v) = g(v) & \text{in } B_R \backslash K_\varepsilon^\eta, \\ \nu \cdot \nabla v = 0 & \text{on } \partial K_\varepsilon^\eta, \\ v = 0 & \text{on } \partial B_R. \end{cases} \tag{3.5}$$

Using exactly the same arguments as in [?] one proves that, if we consider:

$$v_0(x) = \begin{cases} 1 & \text{if } x \in B_{R_2}(x^0) \backslash K_\varepsilon^\eta \cap \Big\{x; x_1 - x_1^0 \leq \frac{2R_1 + R_2}{3}\Big\}, \\ \frac{3}{R_2 - R_1}\big(\frac{R_1 + 2R_2}{3} - (x_1 - x_1^0)\big) & \begin{array}{l}\text{if } x \in B_{R_2}(x^0) \backslash K_\varepsilon^\eta \\ \cap \Big\{x; \frac{2R_1 + R_2}{3} \leq x_1 - x_1^0 \leq \frac{R_1 + 2R_2}{3}\Big\},\end{array} \\ 0 & \begin{array}{l}\text{if } x \in \Big[B_R \backslash \big(B_{R_2}(x^0) \cup C_\varepsilon \cup B_{R_0}(0)\big)\Big] \\ \cup \Big[B_{R_2}(x^0) \backslash K_\varepsilon^\eta \cap \{x, x_1 - x_1^0 \geq \frac{R_1 + 2R_2}{3}\}\Big],\end{array} \end{cases} \tag{3.6}$$

then for $\eta > 0$ small enough, there exists $v \in H^1(B_R \backslash K_\varepsilon^\eta) \cap \{v = 0 \text{ on } \partial B_R\} = \overline{H}_0^1$, $\delta > 0$ such that $\|v - v_0\|_{H^1} < \delta$ and v is a local minimizer of the associated energy functional in $\overline{H}_0^1$. For more clarity we will give the main step of the proof but for details and proofs see [?], section 6.3.

We introduce the energy functional in a domain D:

$$J_D(\omega) = \int_D \Big\{\frac{1}{2}|\nabla\omega|^2 - G(\omega)\Big\} dx, \tag{3.7}$$

defined for functions of $H^1(D)$, where

$$G(t) = \int_0^t g(s) ds, \tag{3.8}$$

g defined in (3.5). Using Proposition 6.6 in [?] one gets the following Corollary

Corollary 3.2 *In $B_{R_1}(x^0)$, $v_0 \equiv 1$ is a strict local minimum of $J_{B_{R_1}(x^0)}$ in the space $H^1(B_{R_1}(x^0))$. More precisely, their exist $\alpha > 0$ and $\delta > 0$ for which*

$$J_{B_{R_1}(x^0)}(v) \geq J_{B_{R_1}(x^0)}(v_0) + \alpha \|v - v_0\|^2_{H^1(B_{R_1}(x^0))}, \tag{3.9}$$

for all $v \in H^1(B_{R_1}(x^0))$ such that $\|v - v_0\|^2_{H^1(B_{R_1}(x^0))} \leq \delta$.

And then using Proposition 6.8 of [?] and Corollary 3.2 one gets the following Corollary

Corollary 3.3 *There exist $\gamma > 0$ and $\eta_0 > 0$ (which depend on ε) such that for all $0 < \eta < \eta_0$ and $v \in H^1_0$ such that $\|v - v_0\|^2_{H^1(B_{R_1}(x^0))} = \delta$, then*

$$J_{B_R \backslash K^\eta_\varepsilon}(v_0) < J_{B_R \backslash K^\eta_\varepsilon}(v) - \gamma.$$

The proof of this corollary relies on the existence of a channel of width of order $\eta > 0$ opening on the interior of the annular region $\mathcal{A}$ (third assumption in (3.6)). This condition cannot be satisfied if the convergence of the obstacle is C^1 (see Lemma 3.1).
The functional $J_{B_R \backslash K^\eta_\varepsilon}$ admits a local minimum in the ball of radius δ around v_0 in $H^1(B_R \backslash K^\eta_\varepsilon) \cap \{v = 0 \text{ on } \partial B_R\}$. This yields a (stable) solution v of (3.5) for small enough $\eta > 0$. Furthermore, provided that δ is chosen small enough, this solution does not coincide either with 1 or with 0 in $B_R \backslash K^\eta_\varepsilon$.
We have proved that for all $\varepsilon \in]0, 1]$, $0 < u_\varepsilon < 1$.

One has proved that C^0 convergence of the domain is not sufficient and thus Theorem 1.4.

One can conclude that if the perturbation is smooth in the $C^{2,\alpha}$ topology, we still have a Liouville type result for reaction diffusion equation in exterior domain. Whereas one can construct counterexample of this Liouville result for C^0 perturbations. One question that is still open is thus the optimal space of regularity of the perturbation for the results to remain true under perturbation. For instance is the C^1 convergence of the perturbation enough to get the result?

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