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Abstract

We investigate in this paper a scalar reaction diffusion equation with a nonlinear reaction term depending on $x - ct$. Here, c is a prescribed parameter modeling the speed of a climate change and we wonder whether a population will survive or not, that is, we want to determine the large-time behavior of the associated solution.
This problem has been solved recently when the nonlinearity is of KPP type. We consider in the present paper general reaction terms, that are only assumed to be negative at infinity. Using a new variational approach, we construct two thresholds $0 < \underline{c} \leq \overline{c} < \infty$ determining the existence and the non-existence of traveling waves. Numerics support the conjecture $\underline{c} = \overline{c}$. We then prove that any solution of the initial-value problem converges at large times, either to 0 or to a travelling wave. In the case of bistable nonlinearities, where the steady state 0 is assumed to be stable, our results lead to constrasting phenomena with respect to the KPP framework. Lastly, we illustrate our results and discuss several open questions through numerics.

2010 *Mathematics Subject Classification: 35B40, 35C07, 35J20, 35K10, 35K57, 92D52*
Keywords: Reaction diffusion equations, traveling waves, forced speed, energy functional, long time behavior.

Acknowledgments

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Program (FP/2007-2013) / ERC Grant Agreement n.321186 - ReaDi- Reaction-Diffusion Equations, Propagation and Modelling. Juliette Bouhours is funded by a PhD fellowship "Bourse hors DIM" of the "Région Ile de France".

1 Introduction and main results

1.1 Motivation: models on climate change

Reaction diffusion problem are often used to model the evolution of biological species. In 1937, Kolmogorov, Petrovsky and Piskunov in [13], Fisher in [8] used reaction diffusion to investigate the propagation of a favorable gene in a population. One of the main notions introduced in [13, 8] is the notion of traveling waves, i.e solution of the form $u(t,x) = U(x-ct)$ for $x \in \mathbb{R}$, $t > 0$ and some constant $c \in \mathbb{R}$. Since then a lot of papers have been dedicated to reaction diffusion equations and traveling waves in settings modeling all sorts of phenomena in biology.
In this paper we are interested in the following problem,

$$\begin{cases} u_t - u_{xx} = f(x-ct, u), & x \in \mathbb{R}, t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{P}$$

where u_0 is bounded, nonnegative and compactly supported.
This problem has been proposed in [2] to model the effect of climate change on biological species. In this setting u is the density of a biological population that is sensitive to climate change. We assume that the North Pole is found at $+\infty$ whereas the equator is at $-\infty$, which gives a good framework to study the effect of global warming on the distribution of the population. The dependence on z in the reaction term takes into account the notion of favorable/unfavorable area depending on the latitude for populations which are sensitive to the climate/temperature of the environment. The constant c can be seen as the speed of the climate change. In such a setting, one will be interested to know when the population can keep track with its favorable environment despite the climate change and thus persists at large times. In [2] Berestycki et al studied the existence of non trivial traveling waves solution in dimension 1 when f satisfies the KPP property: $s \in \mathbb{R}^+ \mapsto f(z,s)/s$ is decreasing for all $z \in \mathbb{R}$. They proved that in this framework, the persistence of the population depends on the sign of the principal eigenvalue of the linearized equation around the trivial steady state 0. Their results have been extended by to $\mathbb{R}^N$ in [4] and to infinite cylinders in [5] Berestycki and Rossi. In [19] Vo studies the same type of problem with more general classes of unfavorable media toward infinity.

A similar model was developed by Popatov and Lewis in [16] and by Berestycki, Desvillettes and Diekmann in[1] in order to investigate a two-species competition system facing a climate change. These papers studied the effect of the speed of the climate change on the coexistence between the competing species. In [1] the authors pointed out the formation of a spatial gap between the two species when one is forced to move forward to keep up with the climate change and the other has limited invasion speed. The persistence of a species facing a climate change was also investigated mathematically through an integrodifference model by Zhou and Kot in [21].

The particularity of all these papers is the KPP assumption for the reaction term, where the linearized equation around 0 determines the behavior of the solution of the nonlinear equation. As far as we know, such questions were only investigated numerically for other types of nonlinearities by Roques et al in [18], where the authors were mainly interested in the effects of the geometry of the domain (in dimension 2) on the persistence of the population considering KPP and bistable nonlinearities.

1.2 Framework

In this paper we are interested in this persistence question, when the evolution of the density of the population is modelled by a reaction diffusion equation, with more general hypotheses on the nonlinearity f in the favorable area. Indeed we point out that we consider general nonlinearities f, without assuming f to satisgy the KPP property.
We will assume that f is a Carathéodory function satisfying the following hypotheses,

$$f(z,0)=0, \tag{1.1}$$

$$s\mapsto f(z,s) \text{ is Lipschitz-continuous uniformly with respect to } z\in\mathbb{R}, \tag{1.2}$$

$$\exists M>0\,|\,f(z,s)\le 0, \quad \forall s\ge M \text{ and } z\in\mathbb{R}, \tag{1.3}$$

$$\exists R>0,\delta>0,\ \ f(z,s)\le -\delta s, \quad \forall |z|>R, s\in(0,M). \tag{1.4}$$

Assumption (1.1) means than when the population vanishes then no reaction takes place, i.e 0 is a steady state of the problem which corresponds to the extinction of the population. Hypothesis (1.3) models some overcrowding effect: the resources being limited , the environment becomes unfavorable when the population grows above some threshold $M>0$. The last assumption (1.4) gives information on the boundedness of the favorable environment and postulates that outside a bounded region the environment is strictly unfavorable.

1.3 Main results

Up to a change of variable ($z:=x-ct$) Problem (P) is equivalent to

$$\begin{cases} u_t-u_{zz}-cu_z=f(z,u), & z\in\mathbb{R}, t>0,\\ u(0,z)=u_0(z), & z\in\mathbb{R}, \end{cases} \tag{$\widetilde{P}$}$$

In our paper we investigate the existence of traveling waves solutions of (P), i.e nonnegative solution of the form $u(t,x)=U(x-ct)$ for all $x\in\mathbb{R}$, $t>0$ with $U\not\equiv 0$, $U(\pm\infty)=0$. This particular solutions are non trivial solutions of the following stationary problem

$$\begin{cases} -U_{zz}-cU_z=f(z,U), & z\in\mathbb{R},\\ U(z)\ge 0, & z\in\mathbb{R},\\ U(\pm\infty)=0. \end{cases} \tag{S}$$

Solutions of (S) are also the stationary solutions of Problem ($\widetilde{P}$) and notice that 0 is a solution of (S) but not a traveling wave solution. We have the following theorem,

Theorem 1.1 *Assuming that there exists $u\in H^1(\mathbb{R})$ such that*

$$E_0[u]:=\int_{\mathbb{R}}\left(\frac{u_z^2}{2}-F(z,u)\right)dz<0, \quad \textit{with } F(z,s):=\int_0^s f(z,t)dt,$$

then there exist $\overline{c}\ge\underline{c}>0$, such that

- *for all* $c \in (0, \underline{c})$, (P) *has a traveling wave solution* $U_c \in H_c^1(\mathbb{R}) = H^1(\mathbb{R}, e^{cz}dz)$ *with* $E_c[U_c] < 0$*,*

- *For all* $c > \overline{c}$*,* (P) *has no traveling wave solution, that is 0 is the only solution of* (S)*.*

The proof of Theorem 1.1 is based on a variational approach, used in [14] to prove the existence of traveling front for gradient like systems of equations. We use the same variational formula but in the case of scalar equations and when f depends on $z = x - ct$.

Then we will be interested in the convergence of the Cauchy problem toward some traveling waves solution.

Theorem 1.2 *Let* $u_0 \in H^2(\mathbb{R})$ *and* u_0 *bounded, compactly supported. Then the unique solution* u *of* (P) *satisfies* $u \in L^2([0, T[, H_c^1(\mathbb{R}))$*,* $u_t \in L^2([0, T[, L_c^2(\mathbb{R}))$*, for all* $T > 0$*, and* $t \mapsto u(t, \cdot - ct)$ *converges to a solution of* (S) *as* $t \to +\infty$*.*

Note that the limit of u in the previous theorem could be the trivial solution 0. And if $t \mapsto u(t, \cdot - ct)$ converges to 0 as $t \to +\infty$ this implies that the population goes exctinct, whereas if $t \mapsto u(t, \cdot - ct)$ converges to $U_c > 0$ non trivial solution of (S) as $t \to +\infty$ this means that there is persistence of the population and converge to a traveling wave solution.

After proving these two main theorems, we study the existence of traveling wave solutions and the behavior of the solution of the Cauchy problem $(\widetilde{P})$ depending on the linear stability of 0. Then we study the solution u of $(\widetilde{P})$ for particular f, δ and c.

- We prove that, as in the KPP framework, when 0 is linearly unstable the solution u of $(\widetilde{P})$ converges to a traveling wave solution. We also prove that in contrast with the KPP case, in bistable-like framework when 0 is linearly stable there still exists a traveling wave solution (see Proposition 4.3 and Corollary 4.4). This result emphasizes the particularity of the KPP framework where the linearity of 0 determines the existence of traveling wave solutions, which is not true in the general framework.

- In the last section we first study numerically the existence of a threshold c^* such that if $c < c^*$ the population survives, i.e the solution of the Cauchy problem $(\widetilde{P})$ converges toward traveling waves for large times, while if $c > c^*$ the population dies, i.e the solution of the Cauchy problem $(\widetilde{P})$ converges to 0 for large times, for f KPP, monostable and bistable in the favorable area. In view of the numerical results, we state the following conjecture:

  Conjecture 1 *Let* $\underline{c} \le \overline{c}$ *be defined by Theorem 1.1 then* $\underline{c} = \overline{c} = c^*$*.*

  We also plot the shape of the profile for different values of the parameter δ and f bistable.

  Then we give an example of nonlinearity f such that there exist several locally stable traveling wave solutions and illustrate this result with numerical simulations displaying the shape of the profile for different times.

Organization of the paper

Theorem 1.1 concerning the stationary framework is proved using a variational method in section 2. Sections 3 and 4 are devoted to the study of the Cauchy problem (P). We prove Theorem 1.2 in section 3 and discuss the linear stability of 0 and its consequences on the convergence of the Cauchy problem in section 4. We give some examples and discuss possible improvements of our results with numerical insight in section 5.

2 A variational approach to traveling waves

The variational structure of traveling waves solutions of homogeneous reaction-diffusion equation is known since the pioneering work of Fife and Mc Leod [7]. However, this structure has only been fully exploited quite recently in order to derive existence and stability results for traveling waves in bistable equations in parallel by Heinze [12], Lucia, Muratov and Novaga [14] and then by Risler [17] for gradient systems (see also [10, 9] for various other applications). The situation we consider in the present paper is different. First, we deal with heterogeneous reaction-diffusion equations. The homogeneity was indeed a difficulty in earlier works, since the invariance by translation caused a lack of compactness. Here, the behavior of the nonlinearity at infinity will somehow trap minimizing sequences in the favorable habitats where f is positive. Second, we consider general nonlinearities, including monostable ones. The variational approach is not a relevant tool in order to investigate such equations when the coefficients are homogeneous since traveling waves do not decrease sufficiently fast at infinity and thus have an infinite energy. Here, again, the behavior of the nonlinearity at infinity forces an admissible exponential decay and we could thus define an energy and make use of it.

We are interested in the existence of traveling wave solution of equation (P), i.e $u(t,x) = U(x-ct) = U(z)$, and U is a solution of the ordinary differential equation

$$\begin{cases} -U'' - cU' = f(z,U), & z \in \mathbb{R}, \\ U > 0 & \text{in } \mathbb{R}, \\ U(z) \to 0 & \text{as } |z| \to +\infty. \end{cases}$$

To study existence of non trivial traveling waves, we introduce the energy functional defined as follow

$$E_c[u] = \int_{\mathbb{R}} e^{cz} \left\{ \frac{u_z^2}{2} - F(z,u) \right\} dz, \qquad \forall u \in H_c^1(\mathbb{R}), \tag{2.1}$$

where $H_c^1(\mathbb{R}) = H^1(\mathbb{R}, e^{cx}dx)$ and

$$F(z,s) = \int_0^s f(z,t)dt.$$

One can notice that (1.1) and (1.2) ensure that $\int_{\mathbb{R}} F(z,u)e^{cz}dz$ is well defined for all $u \in H_c^1(\mathbb{R})$. We start by proving the first part of the Theorem and by pointing out the link between solutions of (S) and the functional E_c.

Lemma 2.1 *Let $u \in H^1_c(\mathbb{R})$ nonnegative, u is a critical point of the energy functional E_c if and only if u is a solution of* (S). *Moreover $u \in W^{2,p}_{loc}(\mathbb{R})$, for all $p > 1$.*

Proof: The first part of the proof is classical. Standard arguments yield that E_c is C^1 and that its differential at u is given, for all $w \in H^1_c(\mathbb{R})$, by

$$dE_c[u](w) = \int_R e^{cz}\left\{u_z w_z - f(z,u)w\right\} dz. \tag{2.2}$$

Moreover letting $v(z) := u(z)e^{\frac{cz}{2}}$ for all $z \in \mathbb{R}$, then

$$v'' = \frac{c^2}{4}v - f(z, e^{-\frac{cz}{2}}v)e^{\frac{cz}{2}}, \quad \text{in } \mathbb{R},$$

and

$$\frac{v''(z)}{v(z)} \geq \delta + \frac{c^2}{4}, \quad \text{if } z \leq -R.$$

As $u \in H^1_c(\mathbb{R})$, $v(z) \to 0$ as $z \to -\infty$ and we can apply [4, Lemma 2.2] we have that

$$v(z)e^{-\sqrt{\frac{c^2}{4}+\delta}z} \underset{z\to-\infty}{\longrightarrow} 0,$$

which implies that

$$u(z) \leq e^{\gamma z}, \quad \forall z \leq R^-,$$

for some $R^- < -R$ and $\gamma > 0$. This implies that $u(z) \to 0$ as $z \to -\infty$ and as $u \in H^1_c(\mathbb{R})$, $u(z) \to 0$ as $z \to +\infty$. Thus $u \in H^1_c(\mathbb{R})$ is a critical point of E_c iff u is a weak solution of (S). Using classical Sobolev embeddings, and taking w smooth we prove the end of the lemma. $\square$

Let state a Poincaré type inequality that will be useful in the sequel, which is due to [14].

Lemma 2.2 (Lemma 2.1 in [14]) *For all $u \in H^1_c(\mathbb{R})$,*

$$\frac{c^2}{4}\int_{\mathbb{R}} e^{cz}u^2 dz \leq \int_{\mathbb{R}} e^{cz}u_z^2 dz \tag{2.3}$$

Now notice that we can always assume that a global minimizer of E_c is bounded and non negative.

Remark 2.3 *Considering $\widetilde{u} = \min\{u, M\}$, we have*

$$F(z,u) - F(z,\widetilde{u}) = \int_{\widetilde{u}}^{u} f(z,s)ds = \begin{cases} 0 & \text{if } u < M, \\ \displaystyle\int_M^u f(z,s)ds & \text{otherwise}, \end{cases}$$
$$\leq 0.$$

Thus

$$E_c[u] \geq \int_{\mathbb{R}} e^{cz} \left\{ \frac{u_z^2}{2} - F(z, \widetilde{u}) \right\} dz \geq \int_{\mathbb{R}} e^{cz} \left\{ \frac{\widetilde{u}_z^2}{2} - F(z, \widetilde{u}) \right\} dz = E_c[\widetilde{u}].$$

As we want to minimize the energy functional, $\widetilde{u}$ will always be a better candidate than u. Similarly, taking $\widetilde{u} = \max\{0, u\}$ instead of u gives a lower energy.

Hypothesis (1.1) ensures that $E_c(0) = 0$ and thus $\inf\limits_{u \in H_c^1(\mathbb{R})} E_c[u] \leq 0$. Moreover, the following lemma yields that $\inf\limits_{u \in H_c^1(\mathbb{R})} E_c[u] > -\infty$.

Lemma 2.4 *For all $c > 0$, there exists $C > 0$ such that for all $u \in H_c^1(\mathbb{R})$, $E_c[u] \geq -C$.*

Proof: We can assume that $0 \leq u \leq M$ using Remark 2.3. For all $u \in H_c^1(\mathbb{R})$, using assumption (1.4),

$$E_c[u] \geq \int_{B_R} e^{cz} \left\{ \frac{u_z^2}{2} - F(z, u) \right\} dz + \int_{\mathbb{R} \setminus B_R} e^{cz} \left\{ \frac{u_z^2}{2} + \frac{\delta u^2}{2} \right\} dz \geq E_c^R[u],$$

where $E_c^R[u] = \displaystyle\int_{B_R} e^{cz} \left\{ \frac{u_z^2}{2} - F(z, u) \right\} dz$. This implies that $\inf\limits_{u \in H_c^1(\mathbb{R})} E_c[u] \geq \inf\limits_{u \in H_c^1(-R,R)} E_c^R[u]$. Using the assumptions on f, there exists $C_0 > 0$ such that $-F(z, s) > -C_0$ for all $z \in B_R$ and $s \in [0, M]$. Thus there exists $C > 0$ such that $E_c^R[u] \geq -C$ for all $u \in H_c^1(-R, R)$ and then

$$\inf_{u \in H_c^1(\mathbb{R})} E_c[u] \geq -C.$$

□

Proposition 2.5 *There exists $u_\infty \in H_c^1(\mathbb{R})$ such that $E_c[u_\infty] = \min\limits_{u \in H_c^1(\mathbb{R})} E_c[u]$.*

To prove Proposition 2.5 we consider $(u_n)_n$ a minimizing sequence of E_c in $H_c^1(\mathbb{R})$, i.e such that $E_c[u_n] \to \inf\limits_{u \in H_c^1(\mathbb{R})} E_c[u] > -\infty$ as $n \to +\infty$. In view of Remark 2.3 we can assume that u_n is bounded for n large enough.

Lemma 2.6 *There exist $N \in \mathbb{N}$, $C_1 > 0$, locally bounded with respect to c, such that for all $n > N$,*

$$\|u_n\|_{H_c^1(\mathbb{R})}^2 = \int_R e^{cz} \left\{ u_z^2 + u^2 \right\} dz \leq \frac{1 + C_1}{\min\{\frac{1}{2}, \frac{\delta}{2}\}}.$$

Proof of Lemma 2.6: For all $u \in H_c^1(\mathbb{R})$, bounded,

$$\begin{aligned} E_c[u] &\geq \int_{B_R} e^{cz} \left\{ \frac{u_z^2}{2} - F(z, u) \right\} dz + \int_{\mathbb{R} \setminus B_R} e^{cz} \left\{ \frac{u_z^2}{2} + \frac{\delta u^2}{2} \right\} dz, \\ &= \int_{B_R} e^{cz} \left\{ -F(z, u) - \frac{\delta u^2}{2} \right\} dz + \int_{\mathbb{R}} e^{cz} \left\{ \frac{u_z^2}{2} + \frac{\delta u^2}{2} \right\} dz, \\ &\geq -C_1 + \min\{\frac{1}{2}, \frac{\delta}{2}\} \|u\|_{H_c^1(\mathbb{R})}^2, \end{aligned}$$

where $C_1 = -\frac{1}{c}(C_0 - \frac{\delta M^2}{2})(e^{cR} - e^{-cR})$, with C_0 as in the proof of Lemma 2.4. Moreover as $E_c[u_n] \to \inf_{u \in H_c^1(\mathbb{R})} E_c[u] \leq 0 = E_c[0]$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $E_c[u_n] \leq 1$. Then using the previous computation we obtain the Lemma. □

One can now prove Proposition 2.5.

Proof of Proposition 2.5: From Lemma 2.6, if (u_n) is a minimizing sequence of E_c in $H_c^1(\mathbb{R})$ then (u_n) is bounded in $H_c^1(\mathbb{R})$. Thus up to a subsequence (u_n) converges weakly to some $u_\infty \in H_c^1(\mathbb{R})$. One has:

$$\int_{\mathbb{R}} e^{cz}(u_\infty)_z^2 dz \leq \liminf_{n \to +\infty} \int_{\mathbb{R}} e^{cz}(u_n)_z^2 dz. \tag{2.4}$$

Moreover as $u_n \in (0, M)$, classical Sobolev injections yield that

$$u_n \to u_\infty \text{ in } C_{loc}(\mathbb{R}) \text{ as } n \to +\infty. \tag{2.5}$$

As F is bounded, the dominated convergence theorem gives, for all $T \in \mathbb{R}$,

$$\int_{-\infty}^{T} e^{cz} F(z, u_n) dz \to \int_{-\infty}^{T} e^{cz} F(z, u_\infty) dz \quad \text{as } n \to +\infty. \tag{2.6}$$

Thus, as $-\int_T^{+\infty} e^{cz} F(z, u_n) dz \geq 0$, for all $T > R$,

$$\begin{aligned}
\liminf_{n \to +\infty} E_c[u_n] &\geq \int_{\mathbb{R}} e^{cz} \frac{(u_\infty)_z^2}{2} dz + \int_{-\infty}^{T} -e^{cz} F(z, u_\infty) dz, \\
&= E_c[u_\infty] + \int_T^{+\infty} e^{cz} F(z, u_\infty) dz, \\
&\geq E_c[u_\infty] - \int_T^{+\infty} C e^{cz} u_\infty^2 dz,
\end{aligned}$$

the last inequality following from (1.2). As, for all $\varepsilon > 0$, there exists $T > R$ such that $\int_T^{+\infty} C e^{cz} u_\infty^2 dz < \varepsilon$, since $u_\infty \in H_c^1$, we have

$$E_c[u_\infty] \leq \liminf_{n \to +\infty} E_c[u_n] = \inf_{u \in H_c^1(\mathbb{R})} E_c[u],$$

and the Proposition is proved. □

We have proved that the minimum is reached in H_c^1. This implies that there exists a solution U of (S) such that $E_c[U] = \inf_{u \in H_c^1} E_c[u]$.

We will now assume that

$$\exists u \in H^1(\mathbb{R}) \mid \quad E_0[u] = \int_{\mathbb{R}} \left\{ \frac{u_z^2}{2} - F(z, u) \right\} dz < 0. \tag{H_0}$$

Proposition 2.7 *The function* $c \geq 0 \mapsto \inf_{u \in H^1_c(\mathbb{R})} E_c[u]$ *is continuous.*

Proof: Let $(c_n)_n$ a sequence in $\mathbb{R}$ such that $c_n \to c$ as $n \to +\infty$. From Proposition 2.5 we know that for all $n \in \mathbb{N}$ there exists $u_n \in H^1_{c_n}(\mathbb{R})$ such that $\inf_{u \in H^1_{c_n}(\mathbb{R})} E_{c_n}[u] = E_{c_n}[u_n]$. Let $v_n := e^{\frac{cz}{2}} u_n \in H^1(\mathbb{R})$ and notice that

$$E_{c_n}[u_n] = \int_{\mathbb{R}} \Big\{ \frac{(v_n)_z^2}{2} + \frac{c_n^2}{8} v_n^2 - e^{c_n z} F(z, e^{-\frac{c_n z}{2}} v_n) \Big\} dz =: \widetilde{E}_{c_n}[v_n].$$

Moreover the sequence $(v_n)_n$ is uniformly bounded in $H^1(\mathbb{R})$ by Lemma 2.6, as $(c_n)_n$ is uniformly bounded, thus up to a subsequence, $v_n \rightharpoonup v_\infty$ weakly in $H^1(\mathbb{R})$ as $n \to +\infty$. Moreover for all $v \in H^1(\mathbb{R})$, $\widetilde{E}_{c_n}[v] \to \widetilde{E}_c[v]$ as $n \to +\infty$. As v_n is a minimizer, for all $v \in H^1(\mathbb{R})$,

$$\widetilde{E}_{c_n}[v_n] \leq \widetilde{E}_{c_n}[v]. \tag{2.7}$$

Passing to the limit and using the same arguments as in the proof of Proposition 2.5 we obtain

$$\widetilde{E}_c[v_\infty] \leq \liminf_{n \to +\infty} \widetilde{E}_{c_n}[v_n] \leq \widetilde{E}_c[v], \quad \forall v \in H^1(\mathbb{R}). \tag{2.8}$$

This implies that $\widetilde{E}_c[v_\infty] = \inf_{v \in H^1(\mathbb{R})} \widetilde{E}_c[v]$, and letting $u_\infty = e^{-\frac{cz}{2}} v_\infty$ we get the Proposition. □

By the continuity of $c \mapsto \inf_{u \in H^1_c(\mathbb{R})} E_c[u]$ and using Proposition 2.5, the first part of Theorem 1.1 is proved.

This proposition will prove the second part of the Theorem

Proposition 2.8 *There exists* $\overline{c} > 0$ *such that for all* $c > \overline{c}$*, 0 is the only solution of equation* (S)*.*

Proof: Define

$$g(z,u) = \left(\sup_{s \geq u} \frac{f(z,s)}{s} \right) \times u, \quad \forall z \in \mathbb{R}, u \in \mathbb{R}^+. \tag{2.9}$$

Then g satisfies the following assumptions:

$$g(z,0) = 0, \quad \forall z \in \mathbb{R}, \tag{2.10}$$

$$u \mapsto g(z,u) \text{ is Lipschitz-continuous uniformly with respect to } z \in \mathbb{R}, \tag{2.11}$$

$$g(z,s) \leq 0, \quad \forall z \in \mathbb{R}, s \geq M, \tag{2.12}$$

$$u \mapsto \frac{g(z,u)}{u} \text{ is decreasing } \forall z \in \mathbb{R}, \tag{2.13}$$

$$g(z,u) \leq -\delta u, \quad \forall |z| > R. \tag{2.14}$$

Hence we know from [4, Theorem 3.2] that there exists $\overline{c} > 0$ such that if v is a solution of

$$-v_{zz} - c v_z = g(z,v) \quad \text{in } \mathbb{R}, \tag{2.15}$$

for $c > \bar{c}$, then $v \equiv 0$. Moreover for all $z \in \mathbb{R}$, $s \in \mathbb{R}$, $g(z,s) \geq f(z,s)$. Take $c > \bar{c}$ and let u a solution of (S), then u is a subsolution of the associated KPP equation, i.e

$$-u'' - cu' \leq g(z,u) \quad \text{in } \mathbb{R}.$$

Let $M > 0$ be as in condition (1.3), then $w(z) = M$ for all $z \in \mathbb{R}$ is a super solution of the associated KPP problem, i.e

$$-w'' - cw' \geq g(z,w) \quad \text{in } \mathbb{R},$$

and we can take M large enough such that $u \leq M$ in $\mathbb{R}$. Thus there exists v a solution of the KPP problem (2.15), such that $u(z) \leq v(z) \leq M$ for all $z \in \mathbb{R}$. But as $c > \bar{c}$, $v \equiv 0$, which implies that (S) has no positive solution as soon as $c > \bar{c}$ and the Proposition is proved. □

3 Convergence of the Cauchy problem

In this section we come back to the parabolic problem (P), that we remind below

$$\begin{cases} u_t - u_{xx} = f(x-ct,u) & x \in \mathbb{R}, t > 0, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

where $u_0 \in H^2(\mathbb{R})$ is non negative, bounded and compactly supported.
Letting $z := x - ct$, u satisfies the following problem

$$\begin{cases} \partial_t u - \partial_{zz} u - c\partial_z u = f(z,u) & \forall z \in \mathbb{R}, t > 0, \\ u(0,z) = u_0(z), & \text{for all } z \in \mathbb{R}. \end{cases} \qquad (\widetilde{P})$$

Defining $v(t,z) = u(t,z)e^{\frac{c}{2}z}$ for all $t > 0$, $z \in \mathbb{R}$, then v satisfies the following equation

$$v_t - v_{zz} + \frac{c^2}{4}v^2 = e^{\frac{c}{2}z} f(z, e^{-\frac{c}{2}z}v).$$

Classical arguments yield that as $v(0,\cdot) \in H^2(\mathbb{R})$, there exists a unique v, weak solution of the previous equation such that $v \in L^2((0,T), H^2(\mathbb{R}))$ and $v_t \in L^2((0,T), L^2(\mathbb{R}))$, for all $T > 0$. And thus as soon as $u_0 \in H^2(\mathbb{R})$, there exists a unique $u \in L^2([0,T[, H^2_c(\mathbb{R}))$, with $u_t \in L^2((0,T), L^2_c(\mathbb{R}))$ for all $T > 0$, solution of $(\widetilde{P})$. Moreover $u(t,x) > 0$ for all $t > 0$, $x \in \mathbb{R}$. We will now prove Theorem 1.2 on the convergence of solution of $(\widetilde{P})$ as $t \to +\infty$. In [15] Matano proves the convergence of solutions of one dimensional semilinear parabolic equations in bounded domain using a geometric argument and the maximum principle and extended this result in [6] to unbounded domain for homogeneous f. Their method relies on classification of solutions for homogeneous problems and uses a reflexion principle which cannot be applied in our case. An alternative proof of this result was first given by Zelenyak in[20] using a variational approach. In [11] Hale and Raugel proved an abstract convergence result in gradient like systems which might apply in the present framework. It roughly states that if the kernel of the linearized equation near any equilibrium has dimension 0 or 1, then the solution of the Cauchy problem converges. We prove such an intermediate step in Lemma 3.6. We chose to prove directly the convergence of the Cauchy problem in section 3.2 using arguments inspired from Zelenyak's paper [20]. But we had to deal with some additional difficulties coming from the fact that our equation is set in $\mathbb{R}$, which induced a lack of compactness and the necessity of finding some controls at infinity. All of this is detailed in section 3.2. In the next section we start by pointing out the convergence up to a subsequence of the solution u of $(\widetilde{P})$.

3.1 Convergence up to a subsequence

Proposition 3.1 *Let $u \in L^2([0,T[, H_c^1(\mathbb{R}))$ for all $T > 0$, be the solution of $(\widetilde{P})$. Then there exists a sequence $(t_n)_n$ that goes to infinity as $n \to +\infty$, such that $u(t_n, z)$ converges to a solution of* (S) *as $n \to +\infty$ locally in $z \in \mathbb{R}$.*

Proof of Proposition 3.1: Standard arguments show that $u(t,\cdot) \in H_c^1(R)$ for all $t > 0$, $t \mapsto E_c[u(t,\cdot)]$ is C^1 and

$$\begin{aligned}
\frac{d}{dt}E_c[u(t,\cdot)] &= \int_{\mathbb{R}} e^{cz}\left\{u_{zt}u_z - f(z,u)u_t\right\}dz,\\
&= \int_{\mathbb{R}} (e^{cz}u_z)u_{tz}dz - \int_{\mathbb{R}} e^{cz}f(z,u)u_t dz,\\
&= -\int_{\mathbb{R}} (cu_z + u_{zz})e^{cz}u_t dz - \int_{\mathbb{R}} e^{cz}f(z,u)u_t dz,\\
&= \int_{\mathbb{R}} (-cu_z - u_{zz} - f(z,u))e^{cz}u_t dz,\\
&= \int_{\mathbb{R}} -(u_t)^2 e^{cz}dz \le 0.
\end{aligned}$$

We know from Proposition 2.4 that $E_c[u]$ is bounded from below. It implies that $E_c[u] \to C$ as $t \to +\infty$, and there exists $(t_n)_n$, such that $t_n \to +\infty$ and $\dfrac{d}{dt}E_c[u](t_n) \to 0$ as $n \to +\infty$, i.e $\|u_t(t_n,\cdot)\|_{L_c^2(\mathbb{R})} \to 0$ as $n \to +\infty$, which implies from standard arguments, that up to extraction $u_t(t_n, z) \to 0$ as $n \to +\infty$ for almost every $z \in \mathbb{R}$. Using Schauder Theory, we have that $(u(t_n, z))_n$ converges toward u_∞ a stationary solution of $(\widetilde{P})$, i.e a solution of (S), up to extraction. □

Now we investigate the uniqueness of the limit u_∞.

3.2 Uniqueness of the limit

We want to prove that, considering compactly supported initial data u_0, the solution of our parabolic problem $(\widetilde{P})$ admits a unique limit. Define the ω-limit set:

$$\Omega(u_0) = \bigcap_{t>0} \overline{\{u(\tau,\cdot), \quad \tau \ge t\}}.$$

The closure is taken with respect to the topology of $H_c^2(\mathbb{R})$.
Let first prove the following Lemma,

Lemma 3.2 *If $w \in \Omega(u_0)$, then w is a solution of the stationary equation*

$$-w_{zz} - cw_z = f(z,w) \quad \text{in } \mathbb{R}.$$

Proof : If $w \in \Omega(u_0)$, then there exists a sequence $(t_n)_{n\geq 1}$ that converges to $+\infty$ as $n \to +\infty$ such that $u(t_n, z) \to w(z)$ in $H^1_c(\mathbb{R})$ as $n \to +\infty$. Let $u^n(t,z) = u(t+t_n, z)$ for all $t > 0$ and $z \in \mathbb{R}$, then using parabolic estimates, $u^n \to \bar{w}$ as $n \to +\infty$ (up to a subsequence) with $\bar{w}$ solution of $(\widetilde{P})$ such that $\bar{w}(0,z) = w(z)$ for all $z \in \mathbb{R}$. Moreover as $E_c[u]$ is decreasing in t and bounded from below $E_c[u^n(t,\cdot)] \to C$ as $n \to +\infty$ and thus $E_c[\bar{w}] = C$ for all $t > 0$. We have

$$\frac{d}{dt}E_c[\bar{w}] = 0,$$

this implies that $\int_{\mathbb{R}} e^{cz}(\bar{w}_t)^2 dz = 0$.We thus obtain that $\bar{w} = w$ is a stationary solution of $(\widetilde{P})$, i.e a solution of (S) and we have proved the Lemma. □

We can now state the main result of this section, from which Theorem 1.2 is immediatly derived.

Theorem 3.3 *The solution u of Problem $(\widetilde{P})$ converges exponentially to u_∞ a solution of* (S) *as $t \to +\infty$, i.e $u \rightharpoonup u_\infty$ weakly in $H^2_c(\mathbb{R})$ as $t \to +\infty$ and*

$$\|u(t,\cdot) - u_\infty\|_{L^2_c(\mathbb{R})} \leq C_1 e^{-C_2 t}, \quad \text{for all } t \text{ large enough},$$

with C_1, C_2 positive constants.

We will need to prove some Lemmas before starting the proof of the Theorem.
Take $w_0 \in \Omega(u_0)$. Let

$$\begin{array}{rccl} F: & H^2_c(\mathbb{R}) & \to & L^2_c(\mathbb{R}), \\ & w & \mapsto & w'' + cw' + f(z,w). \end{array} \tag{3.1}$$

We know that w_0 is a stationary solution of (S), in other words, $F(w_0) = 0$. Define the linear operator:

$$\begin{array}{rccl} \mathcal{L} := DF(w_0): & H^2_c(\mathbb{R}) & \to & L^2_c(\mathbb{R}), \\ & h & \mapsto & h'' + ch' + f'_u\big(z, w_0(z)\big)h. \end{array}$$

Lemma 3.4 *Assume that $w \in H^2_c(\mathbb{R})$ is a positive, bounded solution of $-w'' - cw' \leq -\delta w$ on $\mathbb{R}\backslash(-R,R)$, then $w(z) \leq Me^{\lambda_+(z+R)}$ for all $z \leq -R$ and $w(z) \leq Me^{\lambda_-(z-R)}$ for all $z > R$, where $\lambda_- < 0 < \lambda_+$ are the solutions of $\lambda^2 + \lambda c = \delta$. and $M = ||w||_{L^\infty(\mathbb{R})}$.*

Proof. Define

$$\phi_-(z) := Me^{\lambda_+(z+R)}, \quad \forall\, z < -R,$$
$$h(z) := w(z) - \phi(z), \quad \forall\, z < -R.$$

Then h is solution of

$$\begin{cases} -h'' - ch' + \delta h \leq 0 & \text{for all } z < -R, \\ h(-\infty) = 0, \quad h(-R) \leq 0. \end{cases}$$

Let assume that h achieves a maximum at $z_0 \in (-\infty, -R)$. This would imply that $h(z_0) \leq 0$ and thus $h \leq 0$ in $(-\infty, -R]$. Otherwise h is monotone on $(-\infty, -R)$, which also implies that $h \leq 0$ in $(-\infty, -R]$ and the first inequality is proved. The inequality on $[R, \infty)$ is proved similarly. $\square$

Lemma 3.5 *There exists $z_- \in \mathbb{R}$ such that, if $w_1, w_2 \in H^2_c(\mathbb{R})$ are two positive, bounded, solutions of $w'' + cw' + f(z, w) = 0$ over $\mathbb{R}$ with $w_1(z) = w_2(z)$ for some $z \leq z_-$, then $w_1 \equiv w_2$.*

Proof. Let $u := (w_1 - w_2)^2$. This function satisfies

$$\begin{aligned} u'' + cu' &= 2(w_1' - w_2')^2 + 2\big(-f(z, w_1) + f(z, w_2)\big)(w_1 - w_2) \\ &\geq -2f_u'(z, 0)u - 2\big| - f(z, w_1) + f(z, w_2) - f_u'(z, 0)(w_2 - w_1)\big||w_1 - w_2|. \end{aligned}$$

On the other hand, Lemma 3.4 and the $\mathcal{C}^1$ smoothness of $f(z, s)$ with respect to s yields that there exists z_- such that

$$\forall z \leq z_-, \quad |f(z, w_2) - f(z, w_1) - f_u'(z, 0)(w_2 - w_1)| \leq \frac{\delta}{2}|w_2 - w_1|$$

where δ is the constant defined by (1.4). We thus get

$$\forall z \leq z_-, \quad u'' + cu' \geq -2f_u'(z, 0)u - \delta u \geq \delta u$$

decreasing z_- once more if necessary.

It now follows from this inequation that u cannot reach any local maximum over $(-\infty, z_-)$. As $u(-\infty) = 0$ and $u \geq 0$, it implies that u is nondecreasing. Lastly, if $w_1(z) = w_2(z)$ for some $z \leq z_-$, then $u(z) = 0$ and thus $u \equiv 0$, meaning that $w_1 \equiv w_2$. $\square$

Lemma 3.6

$$\mathrm{dimKer}\mathcal{L} \in \{0, 1\}.$$

Proof. The Cauchy theorem yields that

$$\mathrm{Ker}\mathcal{L} = \{h \in H^2_c(\mathbb{R}), \quad h'' + ch' + f_u'\big(z, w_0(z)\big)h = 0\}$$

has at most dimension 2. If it has dimension 2, then it would mean that for all $z_0 \in \mathbb{R}$ and for all couple (h_0, h_1), the solution of

$$h'' + ch' + f_u'\big(z, w_0(z)\big)h = 0, \quad h(z_0) = h_0, \quad h'(z_0) = h_1$$

belongs to $H^2_c(\mathbb{R})$. In particular $h(-\infty) = 0$.

But now the same arguments as in the proof of Lemma 3.5 yields that h^2 is nondecreasing over $(-\infty, z_-)$ and thus one reaches a contradiction by taking $z_0 < z_-$ and (h_0, h_1) such that $h_0 h_1 < 0$. $\square$

Lemma 3.7 *Assume that there exists $v \in H^2_c(\mathbb{R})$ such that $\mathcal{L}_w v = 0$ in $\mathbb{R}$. Then there exists a constant $C = C(w)$ such that for all $g \in L^2_c(\mathbb{R})$, if $u \in H^2_c(\mathbb{R})$ satisfies $\mathcal{L}_w u = g$ in $\mathbb{R}$ and $\int_{\mathbb{R}} e^{cz} u(z) v(z) dz = 0$, then*

$$\|u\|_{H^2_c(\mathbb{R})} \leq C \|g\|_{L^2_c(\mathbb{R})}.$$

Moreover, if W is a family of solutions $w \in H^1_c(\mathbb{R})$ of (S) *such that* $\text{Ker}\mathcal{L}_w \neq \{0\}$ *for all $w \in W$ and $\sup_{w \in W} \|w\|_{H^1_c(\mathbb{R})} < \infty$, then the constant C can be chosen to be the same for all $w \in W$.*

Proof. Clearly the operator

$$\begin{array}{rccc} T: & (\text{Ker}\mathcal{L})^{\perp} & \to & \text{Im}\mathcal{L} \\ & h & \mapsto & \mathcal{L}h \end{array}$$

is invertible and continuous. Hence the bounded inverse theorem yields that its inverse is continuous. Taking C its continuity constant, this means that for all $g \in L^2_c(\mathbb{R})$ such that there exists $u \in (\text{Ker}\mathcal{L})^{\perp}$ satisfying $\mathcal{L}u = g$, one has $\|u\|_{H^2_c(\mathbb{R})} \leq C \|g\|_{L^2_c(\mathbb{R})}$ and the result follows.

Next, let first prove that there exists $C > 0$ such that if W is a family of solutions $w \in H^1_c(\mathbb{R})$ of (S) such that $\text{Ker}\mathcal{L}_w \neq \{0\}$ for all $w \in W$ and $\sup_{w \in W} \|w\|_{H^1_c(\mathbb{R})} < \infty$, then

$$\|u'\|_{L^2_c(\mathbb{R})} \leq C \|g\|_{L^2_c(\mathbb{R})}.$$

Assume that this is not true, there would exist a sequence $(w_n)_n$ of solutions of (S), bounded in $H^1_c(\mathbb{R})$, such that $\text{Ker}\mathcal{L}_{w_n} \neq \{0\}$ for all n and the associated constants $C_n = C(w_n)$ converge to $+\infty$ as $n \to +\infty$. In other words, there exist $v_n \in \text{Ker}\mathcal{L}_{w_n}$ for all n and two sequences $(u_n)_n$ in $H^2_c(\mathbb{R})$ and $(g_n)_n$ in $L^2_c(\mathbb{R})$ such that $\mathcal{L}_{w_n} u_n = g_n$ in $\mathbb{R}$, $\int_{\mathbb{R}} e^{cz} u_n(z) v_n(z) dz = 0$, $\|u_n'\|_{L^2_c(\mathbb{R})} = 1$ for all n and $\lim_{n \to +\infty} \|g_n\|_{L^2_c(\mathbb{R})} = 0$. Up to multiplication, we can assume that $\|v_n'\|_{L^2_c(\mathbb{R})} = 1$.
As $(w_n)_n$ is bounded in $H^1_c(\mathbb{R})$, we can assume, up to extraction, that it converges locally uniformly to some function $w_\infty \in H^1_c(\mathbb{R})$. Similarly, the Poincaré inequality stated in Lemma 2.2 yields that $(u_n)_n$ and $(v_n)_n$ are indeed bounded in $H^1_c(\mathbb{R})$ and we can thus define their weak limits u_∞ and v_∞ in $H^1_c(\mathbb{R})$. As $u_n'' = -cu_n' - f_u'\big(z, w_n(z)\big) u_n + g_n$, multiplying by $u_n e^{cz}$ and integrating over $\mathbb{R}$, as $\|u_n'\|_{L^2(\mathbb{R})} = 1$, we get

$$1 - \int_{\mathbb{R}} e^{cz} f_u'(z, w_n) u_n^2 dz = \int_{\mathbb{R}} e^{cz} u_n g_n dz.$$

As u_n converge weakly in L^2_c and $g_n \to 0$ in L^2_c as $n \to +\infty$, the right-hand side converges to 0 as $n \to +\infty$. Assuming $u_n \rightharpoonup 0$ in L^2_c yields a contradiction. Indeed, using Lemma 3.4 for all n, for all $\varepsilon > 0$ there exists $r > 0$ such that $w_n(z) < \varepsilon$ for all $|z| > r$. As $f(z, \cdot)$ is C^1, for ε small enough, $f_u'(z, w_n) < 0$ for all $|z| > r$. And we obtain

$$1 - \int_{-r}^{r} e^{cz} f_u'(z, w_n) u_n^2 dz \leq \int_{\mathbb{R}} e^{cz} u_n g_n dz,$$

which yields a contradiction when we let $n \to +\infty$, as $u_n \to 0$, strongly in $L^2_c([-r,r])$. This implies that $u_\infty \not\equiv 0$.
Using the same arguments with v_n, as $\|v_n'\|_{L^2(\mathbb{R})} = 1$ for all n, we have that

$$1 - \int_{\mathbb{R}} e^{cz} f_u'(z, w_n) v_n^2 dz = 0.$$

On the other hand, it follows from Lemma 3.4 that one can apply the dominated convergence theorem using the bounds $v_n(z) \le M$ for all $z < R$ and $v_n(z) \le Me^{\lambda_-(z-R)}$ for all $z > R$, since $c < -2\lambda_-$. We thus obtain

$$1 - \int_{\mathbb{R}} e^{cz} f_u'(z, w_\infty) v_\infty^2 dz = 0.$$

Moreover, classical elliptic regularity estimates yield that v_∞ satisfies $\mathcal{L}_{w_\infty} v_\infty = 0$ in $\mathbb{R}$. Integrating by parts, we get

$$\int_{\mathbb{R}} e^{cz} \Big\{ (v_\infty')^2 - f_u'(z, w_\infty) v_\infty^2 \Big\} dz = 0.$$

We thus conclude that $\|v_\infty'\|_{L^2_c(\mathbb{R})} = 1$. As $L^2_c(\mathbb{R})$ is an Hilbert space, this indeed implies that $(v_n')_n$ converges strongly to v_∞' in $L^2_c(\mathbb{R})$ as $n \to +\infty$. Using the Poincaré type inequality given in Lemma 2.2 we have that $v_n \to v_\infty$ in L^2_c as $n \to +\infty$. This implies that $\int_{\mathbb{R}} e^{cz} u_\infty(z) v_\infty(z) dz = 0$, $\mathcal{L}_{w_\infty} u_\infty = 0$ and $\mathcal{L}_{w_\infty} v_\infty = 0$ over $\mathbb{R}$. Hence, $\dim\mathrm{Ker}\mathcal{L}_{w_\infty} = 2$, which contradicts Lemma 3.6.

Thus there exists $C > 0$ such that if W is a family of solutions $w \in H^1_c(\mathbb{R})$ of (S) such that $\mathrm{Ker}\mathcal{L}_w \neq \{0\}$ for all $w \in W$ and $\sup_{w \in W} \|w\|_{H^1_c(\mathbb{R})} < \infty$, then

$$\|u'\|_{L^2_c(\mathbb{R})} \le C \|g\|_{L^2_c(\mathbb{R})}.$$

Now to prove the last assertion of the Lemma we argue by contradiction and assume that it is not true. Then there would exists a sequence $(w_n)_n$ of solutions of (S), bounded in $H^1_c(\mathbb{R})$, such that $\mathrm{Ker}\mathcal{L}_{w_n} \neq \{0\}$ for all n and the associated constants $C_n = C(w_n)$ converge to $+\infty$ as $n \to +\infty$. In other words, there exist $v_n \in \mathrm{Ker}\mathcal{L}_{w_n}$ for all n and two sequences $(u_n)_n$ in $H^2_c(\mathbb{R})$ and $(g_n)_n$ in $L^2_c(\mathbb{R})$ such that $\mathcal{L}_{w_n} u_n = g_n$ in $\mathbb{R}$, $\int_{\mathbb{R}} e^{cz} u_n(z) v_n(z) dz = 0$, $\|u_n\|_{H^2_c(\mathbb{R})} = 1$ for all n and $\lim_{n \to +\infty} \|g_n\|_{L^2_c(\mathbb{R})} = 0$.
But using the previous inequality we know that $\|u_n'\|_{L^2_c} \le C \|g_n\|_{L^2_c}$, which implies $u_n \to 0$ in $H^1_c(\mathbb{R})$ as $n \to +\infty$ by Lemma 2.2 and $u_n'' = -cu_n' - f_u'(x, w_n) u_n - g_n$, which is impossible because $\|u_n\|_{H^2_c(\mathbb{R})} = 1$. This concludes the proof. □

Lemma 3.8 *Assume that for some $T > 0$, there exist two constants $K, C > 0$ such that for all $t \in [0, T]$,*

$$\int_t^\infty \int_{\mathbb{R}} e^{cz} u_t^2(s, z) ds dz \le K e^{-Ct}.$$

Then for all $0 \le t \le \tau \le T$, one has:

$$||u(t, \cdot) - u(\tau, \cdot)||_{L^2_c(\mathbb{R})} \le \frac{\sqrt{K}}{1 - e^{-C/2}} e^{-Ct/2}.$$

Proof: This Lemma is similar to Lemma 4 in Zelenyak paper [20, Lemma 4]. As our solutions are defined on the full line $\mathbb{R}$ instead of a segment, we obtain a control in L^2 instead of L^1. Assume first that $|t-\tau| \leq 1$. Then

$$\begin{aligned} ||u(t,\cdot)-u(\tau,\cdot)||^2_{L^2_c(\mathbb{R})} &= \int_{\mathbb{R}} e^{cz}\Big|\int_t^\tau u_t(s,z)ds\Big|^2 dz \\ &\leq \int_{\mathbb{R}}\int_t^\tau (\tau-t)e^{cz}u_t^2(s,z)dsdz \\ &\leq \int_{\mathbb{R}}\int_t^\infty e^{cz}u_t^2(s,z)dsdz \\ &\leq Ke^{-Ct}. \end{aligned}$$

Next, if $|t-\tau|>1$, let $N=[\tau-t]$ the integer part of $\tau-t$. We compute:

$$\begin{aligned} ||u(t,\cdot)-u(\tau,\cdot)||_{L^2_c(\mathbb{R})} &\leq \sum_{n=0}^{N-1} ||u(t+n,\cdot)-u(t+n+1,\cdot)||_{L^2_c(\mathbb{R})} + ||u(t+N,\cdot)-u(\tau,\cdot)||_{L^2_c(\mathbb{R})} \\ &\leq \sum_{n=0}^{N-1}\sqrt{K}e^{-C(t+n)/2} + \sqrt{K}e^{-C(t+N)/2} \\ &\leq \frac{\sqrt{K}}{1-e^{-C/2}}e^{-Ct/2}, \end{aligned}$$

which ends the proof. □

Proof of Theorem 3.3: Let assume that $\Omega(u_0)$ is not an isolated point. Using Lemma 3.5 we can choose R large enough such that $\Omega(u_0)$ is parametrized by the value of the function in $-R$, i.e $\Omega(u_0)=\{w(\alpha,\cdot),\quad w(\alpha,-R)=\alpha$ and w is a stationary solution$\}$. As u is bounded, the quantities $0\leq\alpha_1=\liminf_{t\to+\infty}u(t,-R)<\alpha_2=\limsup_{t\to+\infty}u(t,-R)$ are well-defined and classical connectedness and compactness arguments yield that $\Omega(u_0)$ is the curve $\{w(\alpha,\cdot),\alpha\in[\alpha_1,\alpha_2]\}$.

For each $w\in\Omega(u_0)$, $v=\dfrac{\partial w}{\partial\alpha}$ exists in $H^1_c(\mathbb{R})$ and is solution of

$$v(\alpha,-R)=1 \quad\text{and}\quad \mathcal{L}_w v = v''+cv'+f'_u(x,w)v=0 \quad\text{ over } \mathbb{R}.$$

We have $v(\alpha,\cdot)\not\equiv 0$ in $\mathbb{R}$. Now let define for fixed $t>0$,

$$\alpha(t)=\arg\inf\big\{||u(t,\cdot)-w(\alpha,\cdot)||_{L^2_c(\mathbb{R})},\alpha\in[\alpha_1,\alpha_2]\big\} \underset{t\to+\infty}{\longrightarrow} 0.$$

For each $t>0$, if the inf is attained at an interior point $\alpha(t)\in(\alpha_1,\alpha_2)$, then $\dfrac{\partial}{\partial\alpha}||u(t,\cdot)-w(\alpha,\cdot)||^2_{L^2_c(\mathbb{R})}\Big|_{\alpha=\alpha(t)}=0$, and thus

$$\int_{\mathbb{R}} e^{cz}\big(u(t,z)-w(\alpha,z)\big)\frac{\partial w}{\partial\alpha}\Big|_{\alpha=\alpha(t)}dz=0.$$

We thus have for all $t>0$ such that $\alpha(t)\in(\alpha_1,\alpha_2)$:

$$\mathcal{L}_{w(\alpha(t),\cdot)}v=0,\qquad \int_{\mathbb{R}} e^{cz}(u-w)v|_{\alpha=\alpha(t)}dz=0 \quad\text{ and }\quad \mathcal{L}_{w(\alpha(t),\cdot)}(u-w)=g,$$

with

$$g(t,x):=u_t(t,x)+b(t,x)(u(t,x)-w\big(\alpha(t),x)\big),$$

$$b(t,x):=f'_u\big(x,w(\alpha(t),x)\big)-\frac{f\big(x,u(t,x)\big)-f\big(x,w(\alpha(t),x)\big)}{u(t,x)-w(\alpha(t),x)}.$$

Lemma 3.7 thus applies and gives

$$||u(t,\cdot)-w(\alpha(t),\cdot)||_{H^2_c(\mathbb{R})}\le C||u_t(t,\cdot)||_{L^2_c(\mathbb{R})}+C||b(t,\cdot)||_{L^2_c(\mathbb{R})}||u(t,\cdot)-w(\alpha(t),\cdot)||_{L^2_c(\mathbb{R})},$$

for all $t>0$ such that $\alpha(t)\in(\alpha_1,\alpha_2)$. But as $f=f(x,u)$ is of class $\mathcal{C}^1$ with respect to u uniformly in x and as $\lim_{t\to+\infty}||u(t,\cdot)-w(\alpha(t),\cdot)||_{L^2_c(\mathbb{R})}=0$, one has $||b(t,\cdot)||_{L^2_c(\mathbb{R})}\to 0$ as $t\to+\infty$ and it thus follows that, even if it means increasing C, for all admissible $t>0$, one has

$$||u(t,\cdot)-w(\alpha(t),\cdot)||_{H^2_c(\mathbb{R})}\le C||u_t(t,\cdot)||_{L^2_c(\mathbb{R})}.$$

and C is bounded independently of $\alpha(t)\in(\alpha_1,\alpha_2)$.

Now ending the proof as in Zelenyak [20], we have that for all $t>0$ and any $w\in\Omega(u_0)$, the solution u of our parabolic problem satisfies

$$\begin{aligned}E_c[u(t,\cdot)]-E_c[w]&=\frac{1}{2}\int_{\mathbb{R}}e^{cz}\big(u_z^2(t,z)-w_z^2(z)\big)dz-\int_{\mathbb{R}}e^{cz}\big(F(z,u(t,z))-F(z,w(z))\big)dz\\&=\frac{1}{2}\int_{\mathbb{R}}e^{cz}(u_z-w_z)^2dz+\int_{\mathbb{R}}e^{cz}(u_z-w_z)w_zdz\\&\quad-\int_{\mathbb{R}}e^{cz}f\big(z,w(z)\big)(u(t,z)-w(z))dz+\int_{\mathbb{R}}e^{cz}C(t,z)(u(t,z)-w(z))^2dz,\end{aligned}$$

where $C=C(t,z)$ is a bounded and measurable function since $f=f(z,u)$ is of class $\mathcal{C}^1$ with respect to u, uniformly in z. As w is a stationary solution of (S), integrating by parts, we get

$$\begin{aligned}E_c[u(t,\cdot)]-E_c[w]&=\tfrac{1}{2}\int_{\mathbb{R}}e^{cz}(u_z-w_z)^2dz+\int_{\mathbb{R}}e^{cz}C(t,x)(u(t,z)-w(z))^2dz\\&\le \sup\{\tfrac{1}{2},\|C\|_{L^\infty(\mathbb{R})}\}||u(t,\cdot)-w||^2_{H^1_c(\mathbb{R})}.\end{aligned}$$

Next, for all $t>0$ such that $\alpha(t)\in(\alpha_1,\alpha_2)$, gathering the previous inequalities, one gets

$$\frac{d}{dt}(E_c(u(t,\cdot)-E_c^\infty)=-||u_t(t,\cdot)||^2_{L^2_c(\mathbb{R})}\le -C||u(t,\cdot)-w(\alpha(t),\cdot)||^2_{H^2_c(\mathbb{R})}\le -C(E_c[u(t,\cdot)]-E_c^\infty),\tag{3.2}$$

where $E_c^\infty:=\lim_{t\to+\infty}E_c[u(t,\cdot)]$ is equal to $E_c[w]$ for all $w\in\Omega[u_0]$ since the energy converges as $t\to+\infty$.

Now let $\alpha_0 \in (\alpha_1, \alpha_2)$ and take a sequence $(t_n)_n$ such that $\lim_{n\to+\infty} t_n = +\infty$ and $\lim_{n\to+\infty} u(t_n, x) = w(\alpha_0, x)$. There exists $\eta > 0$ such that

$$||w(\alpha_0, \cdot) - w(\alpha_1, \cdot)||_{L^2_c(\mathbb{R})} > \eta \quad \text{ and } \quad ||w(\alpha_0, \cdot) - w(\alpha_2, \cdot)||_{L^2_c(\mathbb{R})} > \eta.$$

Choose N large enough such that $||u(t_N, \cdot) - w(\alpha_0, \cdot)||_{L^2_c(\mathbb{R})} \le \frac{\eta}{8}$ and for all $t \ge t_N$

$$\sqrt{E_c^\infty - E_c[u(t, \cdot)]} \le (1 - e^{-C/2})\frac{\eta}{8}.$$

We set

$$\bar{t} = \inf\left\{t \ge t_N, \quad ||u(t, \cdot) - w(\alpha_0, \cdot)||_{L^2_c(\mathbb{R})} \ge \min\{||u(t, \cdot) - w(\alpha_1, \cdot)||_{L^2_c(\mathbb{R})}, ||u(t, \cdot) - w(\alpha_2, \cdot)||_{L^2_c(\mathbb{R})}\}\right\}.$$

Clearly $\alpha(t) \neq \alpha_1$ and $\alpha(t) \neq \alpha_2$, that is, $\alpha(t)$ is an interior point, for all $t \in [t_N, \bar{t})$. Hence, inequality (3.2) holds for all $t \in [t_N, \bar{t})$, i.e

$$E_c^\infty - E_c[u(t, \cdot)] \le (E_c^\infty - E_c[u(t_N, \cdot)])\, e^{-C(t-t_N)}.$$

By Lemma 3.8, one has for all $t_N \le t \le \tau \le \bar{t}$:

$$||u(t, z) - u(\tau, z)||_{L^2_c(\mathbb{R})} \le \frac{\sqrt{E_c^\infty - E_c[u(t_N, \cdot)]}}{1 - e^{-C/2}} e^{-C(t-t_N)/2} \le \frac{\eta}{8} e^{-C(t-t_N)/2}. \tag{3.3}$$

If $\bar{t}$ is finite then from the previous inequality we obtain that

$$||u(\bar{t}, \cdot) - w(\alpha_0, \cdot)||_{L^2_c(\mathbb{R})} \le ||u(\bar{t}, \cdot) - u(t_N, \cdot)||_{L^2_c(\mathbb{R})} + ||u(t_N, \cdot) - w(\alpha_0, \cdot)||_{L^2_c(\mathbb{R})} \le \frac{\eta}{4} \tag{3.4}$$

and, for $k = 1$ and $k = 2$:

$$||u(\bar{t}, \cdot) - w(\alpha_k, \cdot)||_{L^2_c(\mathbb{R})} \ge ||w(\alpha_k, \cdot) - w(\alpha_0, \cdot)||_{L^2_c(\mathbb{R})} - ||u(\bar{t}, \cdot) - w(\alpha_0, \cdot)||_{L^2_c(\mathbb{R})} \ge \eta - \frac{\eta}{4} = \frac{3}{4}\eta. \tag{3.5}$$

Comparing (3.4) and (3.5) we conclude that $\inf ||u(\bar{t}, \cdot) - w(\alpha, \cdot)||_{L^2_c(\mathbb{R})}$ cannot be attained for $\alpha = \alpha_k$,$(k = 1, 2)$, and thus $\bar{t} = \infty$. We thus conclude that (3.3) holds for all $\tau \ge t \ge t_N$ which proves that u converges strongly in L^2_c. □

4 On the stability of the trivial steady state 0

In this section we discuss the different behaviors of the solution of $(\widetilde{P})$ depending on the stability of 0 and the initial condition u_0. We first define what we mean by stability of the trivial steady state 0.

Let $\mathcal{L}$ be the linearized operator around 0:

$$-\mathcal{L}u := -u'' - cu' - f_s(z, 0)u,$$

defined for all $u \in H^1(\mathbb{R})$. It is easy to check (using Lemma 3.4) that the operator $\mathcal{L}$ admits a principal eigenfunction in $H^1_c(\mathbb{R})$, that is there exist (λ_c, ϕ) such that

$$\begin{cases} -\mathcal{L}\phi = \lambda_c \phi & \text{in } \mathbb{R}, \\ \phi > 0 & \text{in } \mathbb{R}, \\ \phi \in H^1_c(\mathbb{R}). \end{cases} \tag{4.1}$$

This eigenvalue λ_c is also characterized as the generalized eigenvalue of $\mathcal{L}$:

$$\lambda_c(-\mathcal{L}, \mathbb{R}) := \sup\left\{\lambda \in \mathbb{R},\ \exists \phi \in W^{2,1}_{\text{loc}}(\mathbb{R}),\ \phi > 0, (\mathcal{L} + \lambda)\phi \le 0 \text{ a.e in } \mathbb{R}\right\}. \tag{4.2}$$

One can look at [4] and references therein for more details about generalized eigenvalue. We know from [4, Proposition 1 - section 2] that, if we denote by $\lambda(r)$ the principal eigenvalue of our problem on B_r with Dirichlet boundary condition, then $\lambda(r) \to \lambda_c$ as $r \to +\infty$ and there exists $\phi_c \in W^{2,p}_{\text{loc}}(\mathbb{R})$, $1 \le p < +\infty$, the principal eigenfunction solution of (4.1).

Letting $v(x) = u(x)e^{\frac{cx}{2}}$, then

$$-\mathcal{L}u = 0 \iff -\widetilde{\mathcal{L}}v = -v'' + \frac{c^2}{4}v - f_s(z, 0)v = 0,$$

where $\widetilde{\mathcal{L}}$ is self adjoint. From [4, 3]

$$\lambda_c(-\mathcal{L}, \mathbb{R}) = \lambda_c(-\widetilde{\mathcal{L}}, \mathbb{R}) = \inf_{\phi \in H^1(\mathbb{R}), \phi \not\equiv 0} \frac{\int_{\mathbb{R}} \phi'(x)^2 + (\frac{c^2}{4} - f_s(x, 0))\phi(x)^2 dx}{\int_{\mathbb{R}} \phi(x)^2 dx}. \tag{4.3}$$

If we define λ_0 as the generalized eigenvalue corresponding to $c = 0$, i.e when the medium does not move with time, then we have that

$$\lambda_c = \lambda_0 + \frac{c^2}{4}.$$

We will say that 0 is linearly stable (respectively unstable) if $\lambda_c \ge 0$ (respectively $\lambda_c < 0$). Let notice that if 0 is stable in the steady frame, i.e $\lambda_0 > 0$, then 0 is necessarily stable in the moving frame.

4.1 Convergence to a non trivial traveling wave solution when 0 is linearly unstable

In this section we want to prove that when 0 is linearly unstable, i.e $\lambda_0 < 0$ and $c < 2\sqrt{-\lambda_0}$, for $u_0 \not\equiv 0$ non negative initial condition, the solution u of $(\widetilde{P})$ converges to a non trivial traveling wave solution as time goes to infinity.

Proposition 4.1 *Let assume that* $\lambda_0 < 0$ *and that* f *satisfies* (1.1)-(1.4)*, then for all* $c < 2\sqrt{-\lambda_0}$,

$$\inf_{u \in H^1_c(\mathbb{R})} E_c[u] < 0,$$

i.e there exists a non trivial traveling wave solution of (S)

Proof of Proposition 4.1: Take λ such that $\lambda_0 < \lambda < -c^2/4$. It follows from (4.3) that there exists $\phi_0 \in H^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \big(\phi_0'(x)^2 - f_s(x,0)\phi_0^2(x)\big)dx \leq \lambda \int_{\mathbb{R}} \phi_0^2(x)dx.$$

Let

$$\phi_n(z) = \frac{\phi_0(z)e^{-\frac{c}{2}z}}{n} \ \ \forall z \in \mathbb{R}.$$

Then we have the following computation:

$$\begin{aligned}
E_c[\phi_n] &= \int_{\mathbb{R}} \left\{ \frac{|\phi_0'(z)e^{-\frac{c}{2}z})|^2}{2n^2} - F\left(z, \frac{\phi_0(z)e^{-\frac{c}{2}z}}{n}\right)\right\} e^{cz}dz, \\
&= \int_{\mathbb{R}} \frac{(\phi_0'(z))^2}{2n^2} + \frac{c^2}{4}\frac{(\phi_0(z))^2}{2n^2} \\
&\quad - \left(F(z,0) + F_s(z,0)\frac{\phi_0(z)e^{-\frac{c}{2}z}}{n} + F_{ss}(z,0)\frac{(\phi_0(z)e^{-\frac{c}{2}z})^2}{2n^2} + o\Big(\frac{(\phi_0(z)e^{-\frac{c}{2}z})^2}{n^2}\Big)\right) e^{cz}dz, \\
&= \int_{\mathbb{R}} \frac{(\phi_0'(z))^2}{2n^2} + \frac{c^2}{4}\frac{(\phi_0(z))^2}{2n^2} \\
&\quad - \left(f(z,0)\frac{\phi_0(z)e^{-\frac{c}{2}z}}{n} + f_s(z,0)\frac{(\phi_0(z)e^{-\frac{c}{2}z})^2}{2n^2} + o\Big(\frac{(\phi_0(z)e^{-\frac{c}{2}z})^2}{n^2}\Big)\right) e^{cz}dz, \\
&\leq \int_{\mathbb{R}} (\lambda + \frac{c^2}{4})\frac{(\phi_0(z))^2}{2n^2}dz + o(\frac{1}{n^2}).
\end{aligned}$$

This implies that

$$\min_{u\in H_c^1(\mathbb{R})} E_c[u] \leq (\lambda + \frac{c^2}{4})\int_{\mathbb{R}} \frac{(\phi_0(z))^2}{2n^2}dz + o(\frac{1}{n^2}) < 0,$$

for n large enough. The Proposition is proved. □

And we have the following Proposition to characterize the behavior of u as time goes to infinity.

Proposition 4.2 *If $\lambda_0 < 0$, for all $c < 2\sqrt{-\lambda_0}$, the solution u of $(\widetilde{P})$ converges to a non trivial solution of* (S) *as $t \to +\infty$.*

Proof of Proposition 4.2: We will use the same argument as in [4, section 2.4]. We know that $\lambda(R) \to \lambda_c$ as $R \to +\infty$, and $\lambda_c < 0$ thus for R large enough $\lambda(R) < 0$ and let $\phi_R > 0$ be the principal eigenfunction. Define

$$\underline{U} = \begin{cases} \kappa\phi_R & \text{in } B_R, \\ 0 & \text{otherwise,} \end{cases} \tag{4.4}$$

Then for κ small $\underline{U}$ is a subsolution of $(\widetilde{P})$ and $\underline{U} \leq u(\tau,\cdot)$ in $\mathbb{R}$ for some $\tau > 0$ small, $\overline{U} \equiv M \geq u_0$ in $\mathbb{R}$ and is a super solution. Then the solution u of $(\widetilde{P})$ is greater than $\underline{U}$ for all $t > 0$ and $x \in \mathbb{R}$. Moreover using Theorem 1.2 we know that u converges to $u_\infty \geq \underline{U}$ as $t \to +\infty$. And thus u converges to a non trivial traveling wave solution as $t \to +\infty$. □

4.2 Existence of traveling wave when 0 is linearly stable

In this section we use the same notations than in the previous one and assume now that

$$\lambda_c > 0.$$

We first state a result on the existence of a traveling wave with positive energy. This implies that we do not necessary have uniqueness of the profile U and that there exist profiles with positive energy.

Proposition 4.3 *Assume that $\lambda_c > 0$, if $\min_{u\in H^1_c(\mathbb{R})} E_c[u] < 0$ then there exists at least two non trivial traveling wave solution of* (S) *and one of them has a positive energy.*

An easy application of this proposition is the following Corollary.

Corollary 4.4 *Let*

$$f(z,u) = \begin{cases} f_0(u) & \textit{if } |z| < R, \\ -\delta u & \textit{otherwise,} \end{cases}$$

where $R, \delta > 0$, f_0 is a bistable function., i.e

$$\textit{There exists } \theta \in (0,1) \textit{ such that } f_0(0) = f_0(\theta) = f_0(1) = 0, \textit{ and } f_0'(0) < 0, \quad f_0'(1) < 0,$$
$$f_0(s) < 0 \textit{ for all } s \in (0,\theta),\ f_0(s) > 0 \textit{ for all } s \in (\theta, 1),$$

with positive mass:

$$\int_0^1 f_0(\tau)d\tau > 0. \tag{4.5}$$

Then for R sufficiently large, there exists $\widetilde{u} \in H^1_c(\mathbb{R})$ solution of (S) *such that $E_c[\widetilde{u}] > 0$.*

Let highlight this result which is totally different from what is known when f satisfies the KPP property. Indeed in the present framework 0 is linearly stable, nevertheless we still have the existence of traveling wave solutions. This implies that outside the KPP framework the linearity of 0 does not determine the existence of traveling wave solutions and the persistence of the population.

Proof of Corollary 4.4. As $f_s(z,0) = f_0'(0) < 0$ if $|z| < R$, $-\delta < 0$ otherwise, one has $\lambda_0 > 0$ and thus $\lambda_c = \lambda_0 + c^2/4 > 0$.

Moreover, as f_0 has a positive mass, taking

$$u_{min}(z) = \begin{cases} 1 & \text{for all } |z| < R, \\ 0 & \text{for all } |z| > R+1, \end{cases} \tag{4.6}$$

such that $u_{min} \in H^1_c(\mathbb{R})$, one can check that for R large enough $E_c[u_{min}] < 0$ and $\|u_{min}\|_{H^1_c} > r$. Proposition 4.3 applies and gives the conclusion. □

To prove Proposition 4.3 we start with the following Lemma.

Lemma 4.5 *For all $r > 0$ small enough, there exists $\gamma > 0$ such that $E_c[u] > \gamma$ for all $u \in H_c^1(\mathbb{R})$ such that $\|u\|_{H_c^1(\mathbb{R})} = r$.*

Proof of Lemma 4.5: To prove this Lemma, we just need to prove that 0 achieves a strict local minimum, i.e $dE_c[0] \equiv 0$ and $d^2E_c[0] > 0$ in the sense that for all $w \in H_c^1(\mathbb{R})$, $w \not\equiv 0$, $d^2E_c[0](w,w) > 0$, with

$$d^2E_c[0](w,w) = \int_{\mathbb{R}} e^{cz} \left\{ w_z^2 - f_s(z,0)w^2 \right\} dz.$$

Using the equalities in (4.3) with $\phi(z) = e^{cz/2}w(z)$, we get,

$$d^2E_c[0](w,w) \geq \lambda_c \|w\|_{H_c^1(\mathbb{R})},$$

for all $w \in H_c^1(\mathbb{R})$, which proves the Lemma, as λ_c is assumed to be positive. □

Now to prove Proposition 4.3, we want to use the Mountain Pass Theorem, so we need to prove that our energy functional satisfies the Palais-Smale Condition.

Lemma 4.6 *If $(u_n)_n$ is a sequence in $H_c^1(\mathbb{R})$ such that $E_c[u_n] \leq C$ for all $n \in \mathbb{N}$ and $dE_c[u_n] \to 0$ as $n \to +\infty$ strongly in $(H_c^1)^*$, then there exists a subsequence, that we still call $(u_n)_n$, which converges strongly in $H_c^1(\mathbb{R})$ toward a solution u of $dE_c[u] = 0$.*

Proof of Lemma 4.6: As $E_c[u_n] \leq C$ for all $n \in \mathbb{N}$ and using Lemma 2.4, we have

$$\|u_n\|_{H_c^1}^2 \leq \frac{C + C_1}{\min\{1, \delta\}},$$

which implies that, up to a subsequence, (u_n) converges weakly to $u \in H_c^1(\mathbb{R})$. Moreover for all $w \in H_c^1(\mathbb{R})$, $dE_c[u_n](w) \to 0$ as $n \to +\infty$, so

$$\begin{aligned} 0 &= \lim_{n \to +\infty} dE_c[u_n](w) \\ &= \lim_{n \to +\infty} \int_{\mathbb{R}} e^{cz} \left\{ (u_n)_z w_z - f(z,u_n)w \right\} dz \\ &= \int_{\mathbb{R}} e^{cz} \left\{ u_z w_z - f(z,u)w \right\} dz. \end{aligned}$$

Hence $dE_c[u] \equiv 0$.
Now let prove that (u_n) converges strongly to u in $H_c^1(\mathbb{R})$ as $n \to +\infty$. We just need to prove that $\|u_n\|_{H_c^1(\mathbb{R})} \to \|u\|_{H_c^1(\mathbb{R})}$ as $n \to +\infty$, since $H_c^1(\mathbb{R})$ is a Hilbert space. Taking $w = u_n$ we get

$$\int_{\mathbb{R}} e^{cz} \left\{ (u_n)_z^2 - f(z,u_n)u_n \right\} dz = \langle dE_c[u_n], u_n \rangle_{(H_c^1)^*, H_c^1} \tag{4.7}$$

And $\langle dE_c[u_n], u_n \rangle_{(H_c^1)^*, H_c^1} \leq \|dE_c[u_n]\|_{(H_c^1)^*} \|u_n\|_{H_c^1} = o(1)$, since (u_n) is bounded in $H_c^1(\mathbb{R})$. Hence, $\langle dE_c[u_n], u_n \rangle \to 0$ as $n \to +\infty$.
As $\langle dE_c[u], u \rangle = 0$, we have that

$$\int_{\mathbb{R}} e^{cz} u_z^2 ds = \int_{\mathbb{R}} e^{cz} f(z,u) u dz.$$

Using the same arguments as in Proposition 2.5 we have that for all $\varepsilon > 0$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}} e^{cz} f(z,u_n)u_n dz \leq \int_{\mathbb{R}} e^{cz} f(z,u)udz + \varepsilon.$$

This inequality and (4.7) implies that

$$\|u\|_{H^1_c(\mathbb{R})} \leq \liminf_{n\to+\infty} \|u_n\|_{H^1_c(\mathbb{R})} \leq \limsup_{n\to+\infty} \|u_n\|_{H^1_c(\mathbb{R})} \leq \|u\|_{H^1_c(\mathbb{R})} + \varepsilon,$$

for all $\varepsilon > 0$. One has proved the Lemma. □

Proof of Proposition 4.3: As assumed in the Proposition $\min_{u\in H^1_c(\mathbb{R})} E_c[u] < 0$. This minimum is reached for some $u_{\min} \in H^1_c$ such that $u_{\min} \not\equiv 0$ and $\|u_{min}\|_{H^1_c} > r$ for r defined in Lemma 4.5 small enough. Then using the Mountain Pass Theorem, there exists $\widetilde{u} \in H^1_c$ such that $dE_c[\widetilde{u}] \equiv 0$ and $E_c[\widetilde{u}] \geq \gamma$. We have proved Proposition 4.3. □

We want to prove that we can always find an initial condition $u_0 \not\equiv 0$ small enough such that u solution of $(\widetilde{P})$ converges to 0.

Proposition 4.7 *Let λ_c be the principal eigenvalue of $-\mathcal{L}$. If $\lambda_c > 0$ then there exists $u_0 \not\equiv 0$ such that the solution u of $(\widetilde{P})$ converges to 0 as $t \to +\infty$.*

Proof: We noticed in the previous section that $\lambda_c = \lambda_0 + \frac{c^2}{4}$ and if $\lambda_0 > 0$, then $\lambda_c > 0$. We know that there exists a positive function $\phi \in W^{2,p}_{\mathrm{loc}}(\mathbb{R})$, for any $1 \leq p < +\infty$, such that

$$-\phi'' - c\phi' - f_s(z,0)\phi = \lambda_c\phi \text{ in } \mathbb{R}.$$

Let $w(t,z) := \kappa\phi(z)e^{-\delta t}$ for all $t \geq 0$, $z \in \mathbb{R}$, $\kappa > 0$, $\delta > 0$ some constants that we specify later. Then w satisfies the following equation

$$w_t - w_{zz} - cw_z = (f_s(z,0) + \lambda_c - \delta)w.$$

As $\lambda_c > 0$, choosing $\delta = \dfrac{\lambda_c}{2}$, there exists $\kappa > 0$ small enough such that

$$w_t - w_{zz} - cw_z \geq f(z,w).$$

Thus if $u_0 \leq \kappa\phi$ in $\mathbb{R}$, using the weak parabolic maximum principle we have that for all $t \geq 0$, $z \in \mathbb{R}$,

$$u(t,z) \geq \kappa\phi(z)e^{-\delta t},$$

for some constants $\kappa > 0$, $\delta > 0$ small enough. This proves Proposition 4.7. □

5 Examples and discussion

5.1 Numerical simulations

In this section we illustrate the behavior of the solution of the parabolic problem considering different type of reaction terms f, different values of δ and c. We solve numerically the following problem

$$\begin{cases} \partial_t u - \partial_{zz} u - c\partial_z u = f(z,u), & \text{for } t \in [0,T], z \in [0,L], \\ u(0,x) = 1_{z\in[\frac{L}{2}-l,\frac{L}{2}+l]}, & \text{for } z \in [0,L], \\ u(t,0) = 0, & \text{for } t \in [0,T], \\ u(t,L) = 0, & \text{for } t \in [0,T], \end{cases} \tag{5.1}$$

where

$$f(z,u) = \begin{cases} f_0(u) & \text{if } \frac{L}{2} - 2l < z < \frac{L}{2} + 2l, \\ -\delta u & \text{otherwise}, \end{cases} \tag{5.2}$$

with $L > 0$, $T > 0$ and $0 < l < \dfrac{L}{10}$ some constants.

We approximate our problem $(\widetilde{P})$ by a Dirichlet boundary value problem. Indeed we know that the solution u of $(\widetilde{P})$ converges at least exponentially to 0 as $z \to \pm\infty$ and using the comparison principle we have that for L large enough, $\varepsilon > 0$, $u^0(t,z) < u(t,z) < u^\varepsilon(t,z)$ for all $t > 0$, $x \in [0,L]$, where u^0 (respectively u^ε) is the solution of $(\widetilde{P})$ for $x \in [0,L]$ with $u^0(t,0) = u^0(t,L) = 0$ (respectively $u^\varepsilon(t,0) = u^\varepsilon(t,L) = \varepsilon$) for all $t > 0$. We observed that for ε small and L large u^0 and u^ε have the same shape and behavior, which implies that Problem (5.1) is a good approximation of $(\widetilde{P})$.

5.1.1 Existence of a critical speed

We consider three types of reaction function f_0: the KPP case, the monostable case and the bistable case (see figure 1). We restrict our analysis to $[0,T] \times [0,L]$ and take T and L large enough to act as if it was $+\infty$.
In [2] and [4] the authors studied the asymptotic behavior of the parabolic solution and more precisely the existence of non trivial traveling wave solution in the KPP case, i.e $\frac{f_0(u)}{u}$ is maximal when $u = 0$. The authors proved that there exist traveling wave solutions if and only if $\lambda_0 < 0$ and $c < 2\sqrt{-\lambda_0}$, where λ_0 is the generalized eigenvalue when $c = 0$. In other words there exists a critical speed $c^* = 2\sqrt{-\lambda_0}$ such that $\underline{c} = \overline{c} = c^*$ in Theorem 1.1. In our paper we consider more general nonlinearities f and do not assume that f satisfies the KPP property. We proved in Theorem 1.1 that there exists $\underline{c} \leq \overline{c}$ such that there exist traveling wave solutions for all $c < \underline{c}$ and the only solution of (S) is 0 for all $c > \overline{c}$. We wonder if in this general framework, there still exists a critical speed c^* such that $c^* = \underline{c} = \overline{c}$. We investigate this conjecture numerically in the monostable and bistable case.
The existence of a critical speed has already been introduced in [18], where the authors highlight some monotonicity of the global population with respect to the speed c.

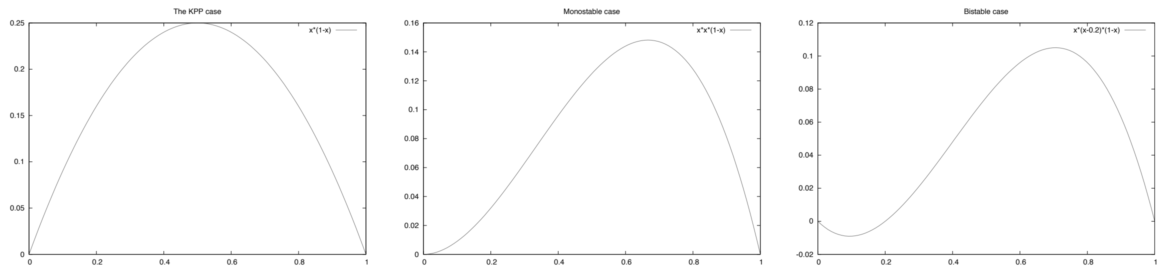

Figure 1: Different type of reaction terms, from left to right:
KPP nonlinearity: $f_0(u) = u(1-u)$, Monostable nonlinearity: $f_0(u) = u^2(1-u)$ and Bistable nonlinearity: $f_0(u) = u(1-u)(u-0.2)$.

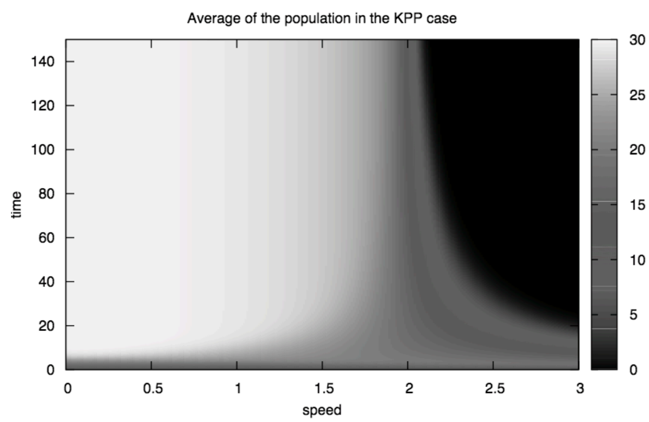

Figure 2: Average of the population $P(t) = \int_0^L u(t,x)dx$ for $c \in [0,3]$ in the KPP case (L=120)

Figure 2 displays the behavior proved analytically in [2, 4]: there exists a critical speed c^* (around 2) such that for $c < c^*$ the population survives whereas for $c > c^*$ the population dies.
In Figure 3 and 4 one can observe the same phenomenon but for lower critical speeds, which proves the existence of a such a c^* in both cases (monostable and bistable).

Let also notice that at it has been proved in Proposition 4.3 and illustrate in Corollary 4.4, we still have persistence of the population even when $\lambda_c > 0$ in the bistable cases (Figure 4 for $c \in [0, 0.4]$).

5.1.2 Shape of the solution in the moving frame

We now investigate the shape of the front when δ varies and f is bistable, i.e $f_0(u) = u(1-u)(u-0.2)$.
When δ is small (figure 6), a tail grows at the bottom of the front whereas the transition at the front edge of the front stays sharp when the speed c is small enough for the population

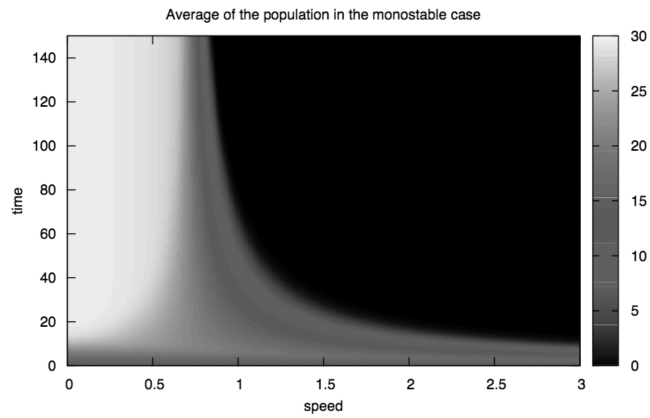

Figure 3: Average of the population $P(t) = \int_0^L u(t,x)dx$ for $c \in [0,3]$ in the monostable case (L=120)

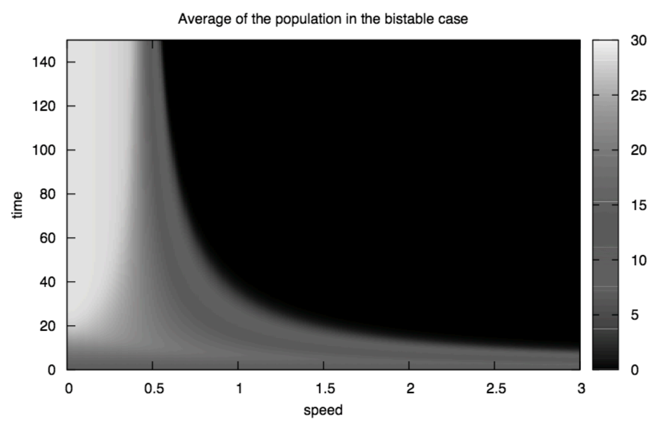

Figure 4: Average of the population $P(t) = \int_0^L u(t,x)dx$ for $c \in [0,3]$ in the bistable case (L=120)

to survive (see Figure 7, where the speed is too large and the population goes extinct). This tail is created by the movement of the favorable environment, indeed the death rate δ is too small to kill the population which reproduced quickly in the favorable zone. On the other hand when c=0, both edges of the front become less and less sharp (figure 5).

Then we see that when $c > 0$ (small enough for the population to survive), both edges of the front become sharper and sharper as δ increases (Figures 6, 8 and 9).

5.2 Non uniqueness of stable traveling waves

We can also build f such that (S) has more than one stable solution with negative energies in the sense that the solutions are local minimizers of the energy functional.

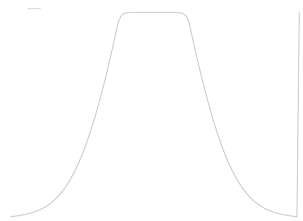

Figure 5: Solution of (5.1) for $\delta = 0.001$ and $c = 0$ for t=150. The horizontal line on the right of each figure gives the scaling corresponding to 1 on the y-axis.

Figure 6: Solution of (5.1) for $\delta = 0.001$ and $c = 0.4$ for t=150. The horizontal line on the right of each figure gives the scaling corresponding to 1 on the y-axis

Figure 7: Solution of (5.1) for $\delta = 0.001$ and $c = 0.8$ for t=150. The horizontal line on the right of each figure gives the scaling corresponding to 1 on the y-axis

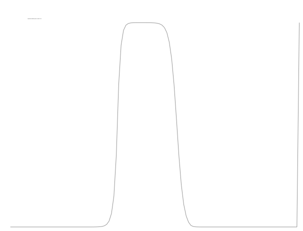

Figure 8: Solution of (5.1) for $\delta = 1$ and $c = 0.4$ for t=150. The horizontal line on the right of each figure gives the scaling corresponding to 1 on the y-axis

Proposition 5.1 *There exists $f(z,u)$ satisfying assumptions* (1.1)-(1.4), *such that there exist u^* and v^* solutions of* (S) *local minimizers of the energy functional with $E_c[v^*] <$*

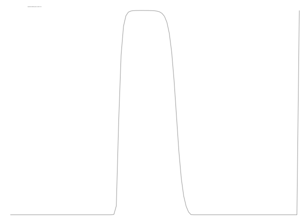

Figure 9: Solution of (5.1) for $\delta = 10$ and $c = 0.4$ for t=150. The horizontal line on the right of each figure gives the scaling corresponding to 1 on the y-axis

$E_c[u^*] < 0$.

Let f be as follow

$$f(z,u) = \begin{cases} f_0(u) & \text{if } |z| < R, \\ -\delta u & \text{otherwise,} \end{cases} \tag{5.3}$$

where f_0 is a multi-stable function, i.e there exist $0 < \theta_0 < 1 < \theta_1 < C$ such that

$$\begin{gathered} f(0) = f(\theta_0) = f(1) = f(\theta_1) = f(C) = 0, \\ f(s) < 0, \quad \text{for } s \in (0,\theta_0) \cup (1,\theta_1), \\ f(s) > 0 \quad \text{for } s \in (\theta_0, 1) \cup (\theta_1, C), \end{gathered}$$

$\int_0^1 f_0(s)ds > 0$ and $\int_0^C f_0(s)ds > \int_0^1 f_0(s)ds$ (one can look at Figure 10 for an example of f_0), and $\delta > 0$.

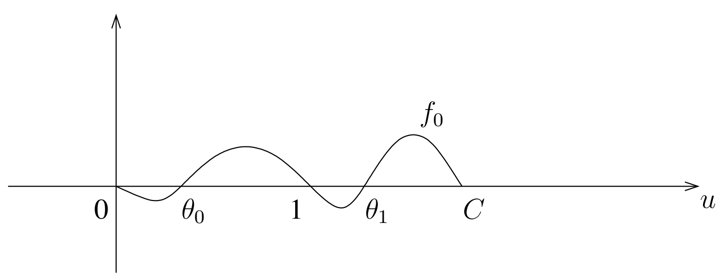

Figure 10: f_0 a multistable function such that $\int_0^1 f_0(s)ds > 0$, there exist $0 < \theta_0 < 1 < \theta_1 < C$ such that $f(0) = f(\theta_0) = f(1) = f(\theta_1) = f(C) = 0$, $f(s) < 0$, for $s \in (0,\theta_0) \cup (1,\theta_1)$ and $f(s) > 0$ for $s \in (\theta_0, 1) \cup (\theta_1, C)$ with $\int_0^C f_0(s)ds > \int_0^1 f_0(s)ds$.

We start with the proof of the following Lemma.

Lemma 5.2 *There exists* $u^* \in H_c^1(\mathbb{R})$ *a local minimizer of* $E_c[u]$ *such that* $0 < u^* < 1$ *in* $\mathbb{R}$, $E_c[u^*] < 0$ *and* u^* *is a solution of* (S).

Proof of Lemma 5.2: Let define f^* such that

$$f^*(z,u) = \begin{cases} 0 & \text{if } z \in (-R,R) \text{ and } u \notin [0,1], \\ f(z,u) & \text{otherwise,} \end{cases} \tag{5.4}$$

Using Proposition 2.5 we know that there exists u^*, traveling waves solution of (S) with f^* for some $c > 0$ such that $\min_{u \in H_c^1} E_c^*[u] = E_c^*[u^*]$, where E_c^* is the energy functional associated with f^*.

We know that $u^* \leq 1$ in $\mathbb{R}$ by Remark 2.3. Thus u^* satisfies the following equation

$$-(u^*)'' - c(u^*)' = f(z,u^*),$$

and

$$E_c[u^*] = \min_{u \in H_c^1} E_c^*[u].$$

Taking

$$u_{min}(z) = \begin{cases} 1 & \text{for all } |z| < R, \\ 0 & \text{for all } |z| > R+1, \end{cases} \tag{5.5}$$

such that $u_{min} \in H_c^1(\mathbb{R})$, one can check that for R large enough $E_c^*[u_{min}] < 0$, which implies that $E_c[u^*] = \min_{u \in H_c^1} E_c^*[u] < 0$. We have proved that there exists a solution $u^* \in H_c^1(\mathbb{R})$ of (S), such that $0 < u^*$ in $\mathbb{R}$ and $E_c[u^*] < 0$. Now let prove that u^* is a local minimizer. Using classical Sobolev injections, there exists $\rho > 0$ small enough, such that

$$\|u - u^*\|_{H_c^1(\mathbb{R})} < \rho \quad \Longrightarrow \quad \|u - u^*\|_{L^\infty(-R,R)} \leq \theta_1 - 1.$$

Now let prove that as soon as $\|u - u^*\|_{H_c^1(\mathbb{R})} < \rho$, then $E_c[u] \geq E_c[u^*]$.

$$\begin{aligned} E_c[u] &= \int_{\mathbb{R}} e^{cz} \left\{ \frac{(u')^2}{2} - F(z,u) \right\} dz, \\ &= E_c^*[u] + \int_{-R}^{R} e^{cz} \left\{ F^*(z,u) - F(z,u) \right\} dz. \end{aligned}$$

As $\|u - u^*\|_{L^\infty(-R,R)} \leq \theta_1 - 1$, $f^*(z,u) \geq f(z,u)$ for all $z \in (-R,R)$, thus

$$\int_{-R}^{R} e^{cz} \left\{ F^*(z,u) - F(z,u) \right\} dz \geq 0.$$

We have proved the Lemma. □

Proof of Propostion 5.1: Now let prove that there exists $v^* \in H_c^1(\mathbb{R})$ solution of (S) such that $E_c[v^*] < E_c[u^*] < 0$. Let u_3 be as follow,

$$u_3(z) = \begin{cases} C & \text{if } |z| < R, \\ 0 & \text{if } |z| > R+\varepsilon, \end{cases} \tag{5.6}$$

such that $u_3 \in H^1_c(\mathbb{R})$. Then

$$E_c[u_3] = -\left(\int_0^C f_0(s)ds\right)\frac{e^{cR}-e^{-cR}}{c} + \int_{R<|z|<R+\varepsilon}\left\{\frac{(u_3'(z))^2}{2} + \frac{\delta u_3(z)^2}{2}\right\} e^{cz}dz.$$

Thus choosing C close enough to 1 and $f_0 >> 0$ in $(\theta_1+\eta, C-\eta)$ for some $\eta > 0$, small, we have

$$E_c[u_3] < E_c[u^*].$$

Using Proposition 2.5, we know that there exists $v^* \in H^1_c(\mathbb{R})$ such that

$$E_c[v^*] = \min_{u \in H^1_c(\mathbb{R})} E_c[u] \leq E_c[u_3].$$

One has proved Proposition 5.1. □

We now illustrate the previous results, choosing a specific reaction term

$$f_0(u) = u(1-u)(u-0.2)(1.1-u)(1.5-u)$$

and an appropriate initial condition we get different convergences as one can see in Figures 11, 12 and 13. We computed the same problem (5.1) that in section 5.1.2, with $\delta = 1$ and $f_0(u) = u(1-u)(u-0.2)(1.1-u)(1.5-u)$. In the first two figures (11 and 12), one can see that depending on the initial condition, we get two different fronts but with a similar shape with sharp edge on both sides. On the other hand when $c > 0$ the front edge takes the shape of a stairs, indeed in the favorable environment the population moves rapidly to 1 but need more time to grow from 1 to 1.5.

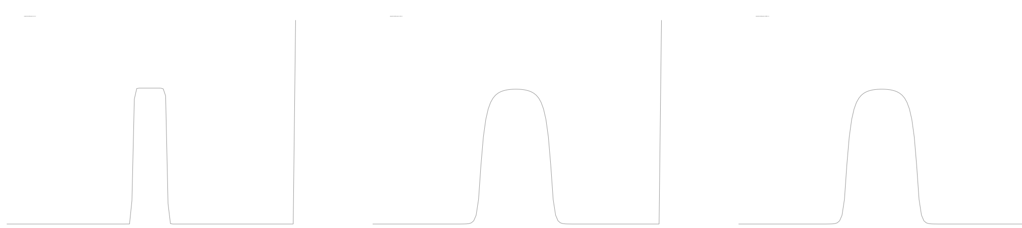

Figure 11: Solution of (5.1) for $u_0(x) = 1_{52.5<x<67.5}$ and $c = 0$ for $t = 0$, 150 and 300. The horizontal line on the right of each figure gives the scaling corresponding to 1.5 (the maximum) on the y-axis

Figure 12: Solution of (5.1) for $u_0(x) = 1.5 \times 1_{52.5<x<67.5}$ and $c = 0$ for $t = 0$, 150 and 300. The horizontal line on the right of each figure gives the scaling corresponding to 1.5 (the maximum) on the y-axis

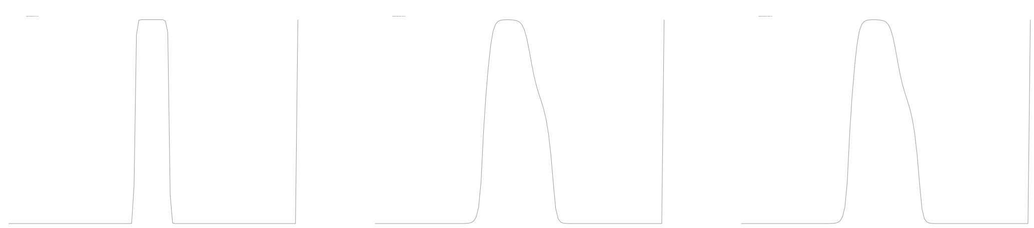

Figure 13: Solution of (5.1) for $u_0(x) = 1.5 \times 1_{52.5<x<67.5}$ and $c = 0.2$ for $t = 0$, 150 and 300. The horizontal line on the right of each figure gives the scaling corresponding to 1.5 (the maximum) on the y-axis

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