Bernoulli variational problem and beyond

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Abstract

The question of 'cutting the tail' of the solution of an elliptic equation arises naturally in several contexts and leads to a singular perturbation problem under the form of a strong cut-off. We consider both the PDE with a drift and the symmetric case where a variational problem can be stated.

It is known that, in both cases, the same critical scale arises for the size of the singular perturbation. More interesting is that in both cases another critical parameter (of order one) arises that decides when the limiting behaviour is non-degenerate. We study both theoretically and numerically the values of this critical parameter and, in the symmetric case, ask if the variational solution leads to the same value as for the maximal solution of the PDE. Finally we propose a weak formulation of the limiting Bernoulli problem which incorporates both Dirichlet and Neumann boundary condition.

Key words: Free boundary problem; Bernoulli problem; Γ-convergence; Maximal solution

Mathematics Subject Classification (2010): 35J20; 35R35; 35B25;

1 Introduction

The so-called 'tail problem' arises in several aspects of physics and biology and leads to penalize small population densities either in stochastic individual based models or in population models based on PDEs which is our interest here. The 'tail problem' is usually addressed by penalizing small populations and leads to analyze singular perturbation problems where the limit has a bounded support and thus a free boundary. There are several possible 'natural rescalings' which penalize more or less strongly the solution. Here we consider the following rescaling for an elliptic problem (the simplest possible in order to address our issue)

$$-\Delta u_\varepsilon + b(x).\nabla u_\varepsilon + u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}\mathbb{1}_{\{u_\varepsilon \le \mu\,\varepsilon^\alpha\}} = f \ge 0 \qquad x \in \mathbb{R}^d, \tag{1.1}$$

with a homogenisation parameter μ. Examples where this equation arises are high activation energy in combustion [9, 18] or the Bernoulli variational problem (when $b \equiv 0$) [2, 1]. An alternative scaling

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arises in 'adaptive dynamics' [16, 14] and leads to a parabolic equation with another rescaling

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon \Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon} \mathbb{1}_{\{u_\varepsilon < e^{\varphi/\varepsilon}\}} = u_\varepsilon g(x), \tag{1.2}$$

which converges to a free boundary problem for Hamilton-Jacobi equations. Because these rescalings are so different, we can expect other distinguished limits, e. g., a small diffusion limit in the limit $\varepsilon \to 0$ in (1.1).

This type of questions leads to study weak formulations in this limit while an important literature has been devoted to strong solutions in order to study the regularity of the free boundary; for $\alpha = 1$, the limit is the well-known Bernoulli-problem [3, 5, 7, 6]. A two-phase version of the problem has been investigated in [2] with variational techniques and in [11] a forcing term has been included. A related semilinear problem is studied in [15] by a least supersolution approach. A parabolic version motivated from flame propagation was studied in [9] and with forcing term in [12, 13].

These papers establish that $\alpha = 1$ is the critical scale; then in the limit $\varepsilon \to 0$ the task is to find a set Ω such that we can solve both Dirichlet and Neumann boundary value problems simultaneously

$$\begin{cases} -\Delta u + b(x).\nabla u + u = f & x \in \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = -\frac{\mu}{\sqrt{2}} & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In a seminal paper [17], J. Serrin shows that the only possible domain Ω where one can solve the Poisson equation $\Delta u = -1$ with both zero Dirichlet condition and constant Neumann data is the ball. With a geometric motivation, in [10], the authors classify all flat surfaces with smooth boundary on which there exist positive harmonic functions having zero Dirichlet data and constant (nonzero) Neumann data.

For f with low regularity in (1.3), we will address the questions of existence of a nontrivial solution, in particular in the non-variational case it is natural to consider the *maximal solution* u^+. Are there always solutions or are size conditions needed on the parameter μ? We will also address the question of uniqueness; when $b \equiv 0$, is u^+ the limiting (in the sense of Γ-convergence) variational solution?

This paper is structured as follows: in section 2 we review the variational solution of the equation (1.1) without the drift term i.e. $b \equiv 0$ for the whole range of α. For the most interesting case, $\alpha = 1$, there is a threshold μ_- such that below this value we have a nontrivial solution and above this value only the trivial solution $u \equiv 0$. In section 3 we consider the maximal solution for (1.1) with the drift term. We show that there is a nontrivial solution for μ small and only the trivial solution for μ large. Finally in section 5 we show on different numerical examples that the maximal and variational solution are in general different.

2 Variational approach

In this section we take $b \equiv 0$ and $f \in L^1(\mathbb{R}^d)_+ \cap L^2(\mathbb{R}^d)$. On $H^1(\mathbb{R}^d)$ we consider the functional with values in $\mathbb{R} \cup \{\infty\}$

$$E_\varepsilon(u) = \frac{1}{2}\int |\nabla u|^2\, dx + \frac{1}{2}\int u^2\, dx - \int f u\, dx + \frac{1}{2\varepsilon^2}\int u^2 \mathbb{1}_{\{u \le \varepsilon^\alpha \mu\}}\, dx + \frac{\mu^2 \varepsilon^{2\alpha}}{2\varepsilon^2}\int \mathbb{1}_{\{u > \varepsilon^\alpha \mu\}}\, dx. \tag{2.1}$$

Notice that, because $\displaystyle\int fu\,dx \leq \frac{1}{2}\int [f^2+u^2]\,dx$, we have

$$E_\varepsilon(0)=0, \qquad E_\varepsilon(u) \geq -\frac{1}{2}\|f\|_2^2 \qquad \text{for all } u \in H^1(\mathbb{R}^d). \tag{2.2}$$

We recall in the Appendix A a standard argument for the existence of a minimizer.

Proposition 2.1 (Elementary properties of the minimizers) *Let u_ε be a minimizer of $E_\varepsilon(u)$. Then it follows that*
a)

$$\begin{cases} \frac{1}{\varepsilon^2}\int u_\varepsilon^2 \mathbb{I}_{\{u_\varepsilon \leq \varepsilon^\alpha \mu\}}\,dx + \varepsilon^{2\alpha-2}\mu^2 \int \mathbb{I}_{\{u_\varepsilon > \varepsilon^\alpha \mu\}}\,dx \leq \|f\|_2^2 \ , \\ \\ \|u_\varepsilon\|_{H^1(\mathbb{R}^d)} \leq 4\|f\|_2 \ , \end{cases} \tag{2.3}$$

b) u_ε solves the PDE

$$-\Delta u_\varepsilon + u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}\mathbb{I}_{\{u_\varepsilon \leq \mu\,\varepsilon^\alpha\}} = f \geq 0 \qquad x \in \mathbb{R}^d, \tag{2.4}$$

c) for $f \in L^p(\mathbb{R}^d)$ with $p > d/2$, we have with a constant independent of ε

$$\|u_\varepsilon\|_\infty \leq C. \tag{2.5}$$

Proof. a) Because of $\displaystyle\int fu\,dx \leq \frac{1}{2}\int [f^2+u^2]\,dx$ and $\displaystyle\int fu\,dx \leq \int f^2\,dx + \frac{1}{4}\int u^2\,dx$, the estimates follows from $E_\varepsilon(u_\varepsilon) \leq 0$.
b) A simple way to see this, is to define

$$G_\mu(v) := \begin{cases} \frac{1}{2}v^2 & \text{for } \ 0 \leq v \leq \mu, \\ \\ \frac{1}{2}\mu^2 & \text{for } \qquad v > \mu. \end{cases} \tag{2.6}$$

Then, we may write the last two terms in the functional (2.1) as

$$\varepsilon^{2(\alpha-1)}\int G_\mu(\frac{u}{\varepsilon^\alpha})\,dx,$$

and notice that $G'_\mu(v) = v\mathbb{I}_{\{v\leq\mu\}}$.
c) Applying elliptic regularity and embedding, gives this estimate.

We consider a family u_ε of minimizers and turn to the study of the behaviour as $\varepsilon \to 0$. For future use, we define

$$\nu_\varepsilon := \frac{u_\varepsilon}{\varepsilon^2}\mathbb{I}_{\{u_\varepsilon \leq \varepsilon^\alpha \mu\}}.$$

Integrating the PDE (2.4) we have

$$\int u_\varepsilon\,dx + \int \nu_\varepsilon\,dx = \int f\,dx \tag{2.7}$$

and thus there is a $\nu_0 \in \mathcal{M}^+(\mathbb{R}^d)$ such that $\nu_\varepsilon \to \nu_0$ in $w{-}\mathcal{M}^+(\mathbb{R}^d)$ after extraction of a subsequence.
In the following subsections we will prove

Theorem 2.2 (Characterisation of the limit) *In the limit $\varepsilon \to 0$, we have:*
- *for $\alpha < 1$,*
$$u_\varepsilon \rightharpoonup 0 \qquad \text{in } w - H^1(\mathbb{R}^d),$$
- *for $\alpha > 1$,*
$$u_\varepsilon \rightharpoonup u^0 \qquad \text{in } w - H^1(\mathbb{R}^d),$$
where u^0 is defined by
$$-\Delta u^0 + u^0 = f \quad \text{in } \mathbb{R}^d,$$
- *for $\alpha = 1$, u_ε converges weakly in H^1 towards the minimizer of*
$$E_0^\mu(u) = \frac{1}{2}\int |\nabla u|^2\, dx + \frac{1}{2}\int u^2\, dx - \int f u\, dx + \frac{\mu^2}{2} meas\{u > 0\}. \tag{2.8}$$

Remark 2.3 *At least, formally the Euler-Lagrange equations for the minimisers of E_0^μ are*
$$-\Delta u + u = f \text{ in } \{u > 0\}$$
$$u = 0, |\nabla u|^2 = \mu^2/2 \text{ on } \partial\{u > 0\}.$$
See [1] for details.

The proof is given in the next three subsections. Before starting, let us recall useful facts about Γ-convergence following [4]

Definition 2.4 (Γ-convergence) *Let (X, d) be a metric space and let $F_\varepsilon, F : X \to [-\infty, \infty]$. Then we define Γ- convergence of F_ε to F at x if*
(i) **(liminf inequality)** *for every sequence (x_ε) converging to x*
$$F(x) \le \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon), \tag{2.9}$$
(ii) **(limsup inequality)** *there exists a sequence (x_ε) converging to x such that*
$$F(x) \ge \limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon). \tag{2.10}$$

Definition 2.5 (Equi-coercivity) *We will say that a sequence $F_\varepsilon, F : X \to [-\infty, \infty]$ is equi-coercive if for all $t \in \mathbb{R}$ there exists a compact set K_t such that $\{F_\varepsilon \le t\} \subset K_t$ for all ε.*

Theorem 2.6 (Fundamental theorem of Γ-convergence) *Let (X, d) be a metric space, let (F_ε) be a equi-coercive sequence of functions on X, and $F = \Gamma - \lim_{\varepsilon \to 0} F_\varepsilon$; then*
$$\exists \min_X F = \lim_{\varepsilon \to 0} \inf_X F_\varepsilon.$$
Moreover, if (x_ε) is a pre-compact sequence such that $\lim_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) = \lim_{\varepsilon \to 0} \inf_X F_\varepsilon$, then every limit of a subsequence of (x_ε) is a minimum point for F.

According to (2.3) all minimisers of E_ε are in
$$X := \left\{ u \in H^1(\mathbb{R}^d) \middle| \|u\|_{H^1} \le 4\|f\|_2 \right\}.$$
So we work in the space X with a metric d which induces the weak topology (X is a bounded subset of a Sobolev space and thus X with the weak topology metrizable).

Note that E_ε is equi-coercive on X since X is bounded.

2.1 The case $\alpha < 1$

In order to identify the weak limit, whose existence follows from (2.3), we decompose

$$u_\varepsilon = u_\varepsilon \mathbb{1}_{\{u_\varepsilon \le \varepsilon^\alpha \mu\}} + u_\varepsilon \mathbb{1}_{\{u_\varepsilon > \varepsilon^\alpha \mu\}}.$$

Multiplying u_ε with a test function $\psi(x)$, integrating and using (2.7) again, we have

$$\int \psi u_\varepsilon \mathbb{1}_{\{u \le \varepsilon^\alpha \mu\}}\, dx = \varepsilon^2 \int \nu_\varepsilon \psi\, dx \to 0$$

and, using (2.3) again and (2.5),

$$\int |\psi u_\varepsilon \mathbb{1}_{\{u > \varepsilon^\alpha \mu\}}|\, dx \le \|u_\varepsilon\|_\infty \|\psi\|_\infty \int \mathbb{1}_{\{u > \varepsilon^\alpha \mu\}}\, dx = O(\varepsilon^{2-2\alpha}).$$

2.2 The case $\alpha > 1$

Here we show that E_ε Γ-converges toward

$$E_{\min}(u) = \frac{1}{2}\int |\nabla u|^2\, dx + \frac{1}{2}\int u^2\, dx - \int f u\, dx \tag{2.11}$$

and

$$u_\varepsilon \rightharpoonup u^0 \qquad \text{in } w-H^1(\mathbb{R}^d). \tag{2.12}$$

(i) **(liminf inequality)** We take u and a sequence u_ε in X such that $u_\varepsilon \to u$ in $w-H^1(\mathbb{R}^d)$. First, by convexity it follows that

$$E_\varepsilon(u_\varepsilon) \ge E_{\min}(u_\varepsilon) \ge E_{\min}(u).$$

Therefore, we have

$$E_{\min}(u) \le \liminf_{\varepsilon} E_{\min}(u_\varepsilon) \le \liminf_{\varepsilon} E_\varepsilon(u_\varepsilon).$$

(ii) **(limsup inequality)** For each $u \in X$, we consider a smooth cut-off function

$$\phi(r) := \begin{cases} 1 & \text{for } r < \frac{1}{2}, \\ 0 & \text{for } r > 1, \end{cases} \tag{2.13}$$

and, with $R_\varepsilon := \varepsilon^{\frac{1-\alpha}{d}} \to \infty$, we define a cut-off version of u as

$$U_\varepsilon(x) := u(x)\phi(|x|/R_\varepsilon).$$

Clearly $U_\varepsilon \to u$ in $H^1(\mathbb{R}^d)$. Moreover, we compute

$$E_\varepsilon(U_\varepsilon) = \frac{1}{2}\int |\nabla U_\varepsilon|^2\, dx + \frac{1}{2}\int U_\varepsilon^2\, dx - \int f U_\varepsilon\, dx + \underbrace{\frac{1}{2\varepsilon^2}\int U_\varepsilon^2 \mathbb{1}_{\{U_\varepsilon \le \varepsilon^\alpha \mu\}}\, dx + \frac{\varepsilon^{2\alpha}\mu^2}{2\varepsilon^2}\int \mathbb{1}_{\{U_\varepsilon > \varepsilon^\alpha \mu\}}\, dx}_{:= I_1}.$$

Since $U_\varepsilon(x) = 0$ for $|x| \ge R_\varepsilon$, we obtain

$$I_1 \le C\varepsilon^{2\alpha-2} R_\varepsilon^d = C\varepsilon^{\alpha-1}$$

and thus,

$$E_{\min}(u) = \limsup_{\varepsilon \to 0} E_\varepsilon(U_\varepsilon).$$

Therefore minimisers of E_ε converge to the minimiser of $E_{\min}$ in $H^1(\mathbb{R}^d)$ which is given by u^0.

2.3 The case $\alpha = 1$. The Γ-limit.

To show Γ-convergence of E_ε toward E_0^μ we define

$$F_\varepsilon(u) := \frac{1}{2\varepsilon^2} \int u^2 \mathbb{1}_{\{u \le \varepsilon\mu\}}\, dx, \qquad H_\varepsilon(u) := \frac{\mu^2}{2} \int \mathbb{1}_{\{u > \varepsilon\mu\}}\, dx.$$

(i) **(liminf inequality)** We take $u_\varepsilon \rightharpoonup u$ in w-$H^1(\mathbb{R}^d)$ and define

$$L := \liminf_{\varepsilon \to 0} \ \mathrm{meas}\{u_\varepsilon > \mu\varepsilon\}.$$

Then there is a sequence such that

$$L = \lim_{k \to \infty} \int \mathbb{1}_{\{u_{\varepsilon_k} > \mu\varepsilon_k\}}$$

and a further subsequence (denoted the same way) such that $u_{\varepsilon_k} \to u$ pointwise a.e. on $\mathbb{R}^d$. The Fatou lemma implies

$$L \ge \int \liminf_{k\to\infty} \mathbb{1}_{\{u_{\varepsilon_k} > \mu\varepsilon_k\}}\, dx \ge \int \mathbb{1}_{\{u>0\}}\, dx.$$

In other words, $H_0(u) \le \liminf H_\varepsilon(u_\varepsilon)$. Together with $F_\varepsilon(u) \ge 0$ and the lower semi-continuity of $\int |\nabla u|^2 + u^2\, dx$, this gives

$$E_0^\mu(u) \le \liminf_\varepsilon E_\varepsilon(u_\varepsilon). \tag{2.14}$$

(ii) **(limsup inequality)** For $u \in H^1(\mathbb{R}^d)$ we define $U_\varepsilon := u$. We want to show

$$E_0^\mu(u) \ge \limsup_{\varepsilon\to 0} E_\varepsilon(U_\varepsilon). \tag{2.15}$$

If $\mathrm{meas}\{u > 0\} = \infty$, then (2.15) holds. Otherwise $\mathrm{meas}\{u > 0\} < \infty$: Then, we have $\{u > \mu\varepsilon\} \subset \{u > 0\}$ and thus

$$H_\varepsilon(u) \le \frac{\mu^2}{2} \mathrm{meas}\{u > 0\} = H_0(u).$$

Moreover, the family of functions

$$v_\varepsilon := \frac{1}{2\varepsilon^2} u^2 \mathbb{1}_{\{u \le \varepsilon\mu\}}$$

converges to 0 pointwise. Since also $0 \le v_\varepsilon \le \frac{1}{2}\mathbb{1}_{\{u>0\}}\mu^2$, by the Lebesgue theorem, $F_\varepsilon(U_\varepsilon) \to 0$ for $\varepsilon \to 0$. Again (2.15) holds.

2.4 The case $\alpha = 1$. The minimizer.

We study the dependence of the minimizer on μ. There is a threshold μ_{var}. If the parameter μ is below μ_{var}, minimizers are non-trivial whereas above μ_{var} the minimizer is identically 0. This is formalised in the next theorem.

Theorem 2.7 (Extinction/non-extinction depending on μ) *For $f \in L^1_+(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we define*

$$\mu_{\mathrm{var}}^2 := \sup_{\mathrm{meas}(\Omega)<\infty} \left\{ \frac{\int_\Omega f u\, dx}{\mathrm{meas}(\Omega)} : -\Delta u + u = f \ \text{ on } \Omega, u \in H_0^1(\Omega) \right\}. \tag{2.16}$$

We obtain

- $\min\limits_{H^1(\mathbb{R}^d)} E_0^\mu < 0$ *for* $\mu < \mu_{\rm var}$,
- $\min\limits_{H^1(\mathbb{R}^d)} E_0^\mu = 0$ *for* $\mu \geq \mu_{\rm var}$,
- $u_{\rm var} \equiv 0$ *is the unique minimiser for* $\mu > \mu_{\rm var}$,
- $\mu_{\rm var} > 0$ *for* $f \not\equiv 0$,
- $\mu_{\rm var} \leq \|f\|_\infty$ *for* $f \in L^\infty(\mathbb{R}^d)$,
- $\mu_{\rm var} \leq C(d)\|f\|_d$ *for* $f \in L^d(\mathbb{R}^d)$ *and* $d > 2$.

To define (2.16), we understand here that the value of the functional is $-\infty$ when $H_0^1(\Omega)$ is empty.
Proof.

For $\mu < \mu_{\rm var}$**.** There is a sequence (Ω_k) s.t. $\mu_k^2 := \frac{\int_{\Omega_k} f u_k\, dx}{|\Omega_k|} \uparrow \mu_{\rm var}^2$. So there is a k such that $\mu_k > \mu$ and

$$E_0^\mu(u_k) = \frac{1}{2}\left(\mu^2|\Omega_k| - \int_{\Omega_k} f u_k\, dx\right) < \frac{|\Omega_k|}{2}\left(\mu_k^2 - \frac{\int_{\Omega_k} f u_k\, dx}{|\Omega_k|}\right) = 0.$$

For $\mu > \mu_{\rm var}$**.** Assume that we have a minimiser $u_{\rm var} \not\equiv 0$ with support $\Omega_{\rm var}$. Because $u_{\rm var}$ also minimizes the energy functional with $\Omega_{\rm var}$ fixed, we can use that it solves the elliptic PDE in $\Omega_{\rm var}$. Then since the energy of minimiser is non-positive, we have

$$E_0^\mu(u_{\rm var}) = \frac{1}{2}\left(\mu^2|\Omega_{\rm var}| - \int_{\Omega_{\rm var}} f u_{\rm var}\, dx\right) = \frac{|\Omega_{\rm var}|}{2}\left(\mu^2 - \frac{\int_{\Omega_{\rm var}} f u_{\rm var}\, dx}{|\Omega_{\rm var}|}\right) \leq 0.$$

This is a contradiction with the definition of $\mu_{\rm var}$. Therefore, the unique minimiser is $u_{\rm var}^\mu \equiv 0$.

Proof that $\min_{H^1(\mathbb{R}^d)} E_0^{\mu_{\rm var}} = 0$**.** Moreover, assume there is a minimiser $u_{\rm var}^{\mu_{\rm var}}$ such that $E^{\mu_{\rm var}} < 0$ then also $E^\mu(u_{\rm var}^{\mu_{\rm var}}) < 0$ for μ close to $\mu_{\rm var}$ and $\mu > \mu_{\rm var}$. So this leads to a contradiction with the case $\mu > \mu_{\rm var}$ above. Therefore $\min\limits_{H^1(\mathbb{R}^d)} E^{\mu_{\rm var}} = 0$.

For $f \not\equiv 0$ We fix a bounded open set Ω such that $f \not\equiv 0$ on Ω. Then, for a solution $u \in H_0^1(\Omega)$ to the equation in (2.16) we have

$$\int_\Omega [|\nabla u|^2 + u^2]\, dx = \int_\Omega f u\, dx > 0.$$

This gives $\mu_{\rm var} > 0$.

For $f \in L^\infty(\mathbb{R}^d)$**.** With this equality and Hölder we obtain

$$\left(\int uf\, dx\right)^2 \leq \int f^2\, dx \int u^2\, dx \leq \int f^2\, dx \int uf\, dx. \tag{2.17}$$

So it follows that

$$\int uf\, dx/\mathrm{meas}(\Omega) \leq \int f^2\, dx/\mathrm{meas}(\Omega) \leq (\|f\|_\infty)^2 \tag{2.18}$$

and therefore $\mu_{\rm var} \leq \|f\|_\infty$.

For $f \in L^d(\mathbb{R}^d)$ and $d > 2$. We define $p := 2d/(d+2)$ and $p^* := 2d/(d-2)$, and estimate

$$\int_\Omega fu \le \|f\|_{L^p(\Omega)} \|u\|_{p^*} \le C(d)\|f\|_p \|\nabla u\|_2 \le C(d)\|f\|_{L^p(\Omega)} \left(\int_\Omega fu\right)^{1/2}.$$

Using Hölder with $q = (d+2)/d$ and $q^* = (d+2)/2$, we have

$$\frac{\int_\Omega fu}{|\Omega|} \le \frac{C(d)^2}{|\Omega|} \left(\int_\Omega |f|^p\right)^{2/p} \le \frac{C(d)^2}{|\Omega|} \|f\|^2_{pq^*} \left(\int_\Omega dx\right)^{2/(pq)}.$$

Since $2 = pq$ and $pq^* = d$, we obtain

$$\mu_{\text{var}} \le C(d)\|f\|_d.$$

Remark 2.8 *The estimate*

$$\mu_{\text{var}} \le C(d)\|f\|_d,$$

brings us closer to the L^1-norm in $d = 1$ which appears in the examples of section 4.

3 The PDE approach

We now study the PDE problem given by equation (1.1) with $\alpha = 1$. Our goal is to show that u_ε converges to some $u_0 \in H^1(\mathbb{R}^d)$, with support $\Omega \ne \mathbb{R}^d$ and that it satisfies a weak formulation of the Neumann boundary condition in (1.3).

We use several types of assumptions that we present together even though they are used separately in this section

$$\beta := \|b\|_\infty, \qquad \text{div} b \le 1 - \gamma, \text{ with } \gamma > 0, \qquad f \in L^2 \cap L^1, \qquad f \ge 0, \ f \ne 0. \tag{3.1}$$

It is also convenient to assume better lower and upper controls as

$$\exists R > 0, \ \underline{f}_R > 0, \text{ such that } \quad f(x) \ge \underline{f}_R \quad \text{for} \quad |x| \le R, \tag{3.2}$$

$$f(x) \le F(|x|), \quad F' \le 0, \quad F(|\cdot|) \in L^1(\mathbb{R}). \tag{3.3}$$

All integrals in this section are over $\mathbb{R}^d$ unless stated otherwise.

A first observation is that solutions to equation (1.1) are far from unique, a consequence of a non-monotonic nonlinearity,

Theorem 3.1 (Maximal, minimal solutions) *Assume* (3.1)*. There is a maximal solution u_ε^+ and we have*

$$\|u_\varepsilon^+\|_1 \le \frac{1}{\gamma}\|f\|_1, \qquad \|\nabla u_\varepsilon^+\|_2 \le C(\|f\|_2, \|f\|_1, \gamma). \tag{3.4}$$

With the additional assumptions (3.2), (3.3)*, for μ small enough u_ε^+ does not vanish as $\varepsilon \to 0$ and for μ large enough $u_\varepsilon^+ \le \mu\varepsilon$. Moreover, assuming* (3.1) *and $f \in L^\infty(\mathbb{R}^d)$, there is for ε small enough a minimal solution $u_\varepsilon^- = O(\varepsilon^2)$.*

In other words, and by monotonicity in μ, there is a critical value μ_+ below which the limit $\varepsilon \to 0$ gives rise to a non-vanishing solution; we will discard later that it can be u^0, the solution to the linear problem in the full space as defined in (3.6).

In particular cases, it is easy to build other solutions than the minimal and maximal. For a multi-bump data f it is possible to switch, on each bump, from one strategy to the other in the construction below. Various examples are built in Sections 4 and 5 that show explicitly the processes at work. In particular the question of the 'optimal norm' on f to measure μ_+ appears to be rather complex.

3.1 Minimal and maximal solutions

This section is devoted to the proof of the first statements in Theorem 3.1.

Proof. The minimal solution is simply defined by

$$-\Delta u_\varepsilon^- + b(x).\nabla u_\varepsilon^- + u_\varepsilon^- + \frac{u_\varepsilon^-}{\varepsilon^2} = f \qquad x \in \mathbb{R}^d. \tag{3.5}$$

By the maximum principle and for ε small enough, we have

$$u_\varepsilon^- \leq \varepsilon^2 \|f\|_\infty < \varepsilon\mu.$$

To build the maximal solution, consider the construction by induction

$$-\Delta u^0 + b(x).\nabla u^0 + u^0 = f \qquad x \in \mathbb{R}^d, \tag{3.6}$$

$$-\Delta u_\varepsilon^{k+1} + b(x).\nabla u_\varepsilon^{k+1} + u_\varepsilon^{k+1} + \frac{u_\varepsilon^{k+1}}{\varepsilon^2}\mathbb{1}_{\{u_\varepsilon^k \leq \mu\,\varepsilon\}} = f \qquad x \in \mathbb{R}^d. \tag{3.7}$$

Since we have

Lemma 3.2 (Comparison principle) *Consider the solutions v^1, v^2 of the equations*

$$-\Delta v^i + b(x).\nabla v^i + c^i(x) v^i = f^i \qquad x \in \mathbb{R}^d \quad i = 1,\ 2.$$

If $c^1 \leq c^2$ and $f^1 \geq f^2$, then $v_1 \geq v_2$.

Obviously $u_\varepsilon^1 \leq u^0$ therefore $\mathbb{1}_{\{u_\varepsilon^1 \leq \mu\,\varepsilon\}} \geq \mathbb{1}_{\{u^0 \leq \mu\,\varepsilon\}}$. Then, applying again this comparison principle, we have for all k, $u_\varepsilon^{k+1} \leq u_\varepsilon^k$ and

$$u_\varepsilon^k \searrow u_\varepsilon^+, \qquad \text{for } k \to \infty.$$

Again, by comparison principle, any solution is less that u^0, thus less than u_ε^1...etc therefore u_ε^+ is the maximal solution of (1.1).

The uniform bounds are also standard. The L^1 bound is obtained by mere integration of (1.1) because

$$\int u_\varepsilon^+ [1 - \text{div} b]\, dx \leq \int f\, dx.$$

The H^1 bound follows then by integration against u_ε^+, writing $2\displaystyle\int u_\varepsilon^+ b(x).\nabla u_\varepsilon^+\, dx = -\int \text{div} b\ (u_\varepsilon^+)^2\, dx$ and using

$$\int |\nabla u_\varepsilon^+|^2\, dx + \frac{1}{2}\int [1 - \text{div} b](u_\varepsilon^+)^2\, dx + \frac{1}{2}\int (u_\varepsilon^+)^2\, dx \leq \int f u_\varepsilon^+\, dx.$$

□

As a consequence, we may extract a subsequence such that

$$u_\varepsilon^+ \underset{\varepsilon \to 0}{\longrightarrow} u^+ \qquad w\text{–}H^1(\mathbb{R}^d).$$

Notice there is no monotonicity in ε. The next question we address is to understand when u^+ does not vanish.

3.2 Non-extinction for μ small enough

Next, we give a uniform lower bound on u_ε^+ ensuring that it does not vanish with ε. This is the case for μ small or, equivalently, f large. To present a precise version of the statement in Theorem 3.1 we need additional ingredients. The first one is a size condition on f on some ball of radius R that we center at 0 to simplify notations.

Then we need a radial auxiliary function $\underline{u}_R(r)$, $r = |x|$, defined by

$$\begin{cases} -\Delta \underline{u}_R + \beta |\nabla \underline{u}_R| + \underline{u}_R = 1, \qquad |x| < R, \\ \underline{u}_R = 0 \qquad \text{on } \{|x| = R\}. \end{cases} \tag{3.8}$$

Proposition 3.3 (Non-extinction for μ small) *Let us assume* (3.1), (3.2) *and that μ is small enough such that*

$$\mu < \underline{f}_R \, |\frac{d\underline{u}_R}{dr}(R)|, \tag{3.9}$$

then for ε small enough, u_ε^+ is uniformly controlled from below as $u_\varepsilon^+ \geq \underline{f}_R \underline{u}_R$.

Proof. Our aim is to prove that for each of the iterates u_ε^k converging to the maximal solution, we have $u_\varepsilon^k \geq \underline{f}_R \, u_R$ under the condition (3.9). To do so, we define the (radially symmetric) solution of the equation

$$-\Delta w_\varepsilon + \beta |\nabla w_\varepsilon| + w_\varepsilon + \frac{w_\varepsilon}{\varepsilon^2} \mathbb{I}_{\{|x| \geq R\}} = \underline{f}_R \, \mathbb{I}_{\{|x|<R\}} \qquad x \in \mathbb{R}^d. \tag{3.10}$$

Properties of w_ε are established in Appendix B. In particular, equation (3.10) is linear. Furthermore, w_ε is a decreasing function of $|x|$ and thus, as $\varepsilon \to 0$, $w_\varepsilon \searrow \underline{u}_R$ for $|x| \leq R$ and 0 outside this ball.

This function w_ε is a subsolution of (1.1) under the condition $\{|x| \geq R\} \supset \{w_\varepsilon \leq \mu\varepsilon\}$, which is satisfied, because w_ε is decreasing, if

$$w_\varepsilon(R) \leq \mu\varepsilon. \tag{3.11}$$

In Appendix B, we prove that

$$\frac{d}{dr} w_\varepsilon(R) \underset{\varepsilon \to 0}{\longrightarrow} \underline{f}_R \, \frac{d}{dr} \underline{u}_R(R), \qquad \frac{w_\varepsilon(R)}{\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} -\underline{f}_R \, \frac{d}{dr} \underline{u}_R(R).$$

Therefore (3.11) holds true for μ small enough.

It remains to argue that, obviously, $w_\varepsilon \leq u^0$, therefore $\{|x| \geq R\} \supset \{w_\varepsilon \leq \mu\varepsilon\} \supset \{u^0 \leq \mu\varepsilon\}$. Thus $w_\varepsilon \leq u_\varepsilon^1$ and iterating the argument gives $w_\varepsilon \leq u_\varepsilon^k$. Since w_ε is a supersolution to (3.8) we conclude

$$\underline{u}_R \leq w_\varepsilon \leq u_\varepsilon^k,$$

and our result is proved. $\square$

3.3 Extinction for μ large enough

We continue the proof of Theorem 3.1 and prove extinction for μ large. A precise statement is

Proposition 3.4 (Extinction for μ large) *Let us assume* (3.1), (3.3) *and that μ be large enough (depending on F). Then for ε small enough, we have*

$$u_\varepsilon^+ \le \mu\varepsilon.$$

Proof. We are going to build a radially symmetric supersolution at all steps of the iterative process. We choose the function $U_\varepsilon^+(|x|)$ as the maximal solution of

$$-\Delta U_\varepsilon^+ - \beta|\nabla U_\varepsilon^+| + U_\varepsilon^+ + \frac{U_\varepsilon^+}{\varepsilon^2}\mathbb{1}_{\{U_\varepsilon^+\le\mu\varepsilon\}} = F(|x|), \qquad x\in\mathbb{R}^d, \tag{3.12}$$

in radial coordinates this is

$$\begin{cases} -\frac{d^2}{dr^2}U_\varepsilon^+ - \frac{d-1}{r}\frac{d}{dr}U_\varepsilon^+ - \beta|\frac{d}{dr}U_\varepsilon^+| + U_\varepsilon^+ + \frac{U_\varepsilon^+}{\varepsilon^2}\mathbb{1}_{\{U_\varepsilon^+\le\mu\varepsilon\}} = F(|x|), \\ U_\varepsilon^{+\prime}(0)=0, \qquad U_\varepsilon^+(\infty)=0, \end{cases} \tag{3.13}$$

and we prove that in fact $U_\varepsilon^+ < \mu\varepsilon$ in $\mathbb{R}^d$ for μ large. To do so, we argue by contradiction and suppose it is wrong. Then, we may define $R_\varepsilon>0$ as

$$U_\varepsilon^+(R_\varepsilon)=\mu\varepsilon, \qquad U_\varepsilon^+(r)<\mu\varepsilon \text{ for } r>R_\varepsilon, \qquad \frac{d}{dr}U_\varepsilon^+(R_\varepsilon)\le 0.$$

We first derive some information for $r\ge R_\varepsilon$. From the maximum principle on the equation on $U_\varepsilon^{+\prime}$ (differentiating (3.13)), we conclude that $U_\varepsilon^{+\prime}(r)\le 0$ on $[R_\varepsilon,\infty)$. Then, we consider two cases:

$\frac{d}{dr}U_\varepsilon^+\mathbb{1}_{\{[R_\varepsilon,\infty)\}}$ is uniformly bounded Integrating equation (3.13) between R_ε and $R' := (d-1)/(\beta+1)$, we obtain that the integral

$$\int_{R_\varepsilon}^{R'}[\beta-\frac{d-1}{r}](\frac{d}{dr}U_\varepsilon^+)\,dr$$

is uniformly bounded. Also we have

$$\int_{R_\varepsilon}^{\infty}(\frac{d}{dr}U_\varepsilon^+)^2\,dr \le C\int_{R_\varepsilon}^{\infty}-\frac{d}{dr}U_\varepsilon^+\,dr = O(\varepsilon),$$

and therefore the functions $\frac{d}{dr}U_\varepsilon^+\mathbb{1}_{\{[R_\varepsilon,\infty)\}}$ tend to 0 a.e. as $\varepsilon\to 0$. So together the two integrals

$$\int_{R_\varepsilon}^{\infty}[\beta-\frac{d-1}{r}](\frac{d}{dr}U_\varepsilon^+)^2\,dr \text{ and } \int_{R_\varepsilon}^{\infty}F\frac{d}{dr}U_\varepsilon^+\,dr$$

tend to 0 as $\varepsilon\to 0$. Following the arguments in Appendix B, using the equality

$$\left(\frac{d}{dr}U_\varepsilon^+(r)\right)^2 + 2\int_{r'}^{\infty}[\beta-\frac{d-1}{r}](\frac{d}{dr}U_\varepsilon^+)^2\,dr = 2\int_{r'}^{\infty}F\frac{d}{dr}U_\varepsilon^+\,dr + (1+\frac{1}{\varepsilon^2})U_\varepsilon^+(r)^2 \le \mu^2(1+\varepsilon^2),$$

we also conclude

$$\lim_{\varepsilon\to 0} -\frac{d}{dr}U_\varepsilon^+(R_\varepsilon)=\mu.$$

For $r<R_\varepsilon$ and ε small enough, we now have the boundary value problem $\frac{d}{dr}U_\varepsilon^+(R_\varepsilon)\le -\mu/2$ and for μ large enough the solution is negative at $r=R_\varepsilon$, thus a contradiction. To see this we write $U_\varepsilon^+<V$ (see below) and we need to show the Lemma 3.5.

$\frac{d}{dr}U_\varepsilon^+\mathbb{I}_{\{[R_\varepsilon,\infty)\}}$ **is unbounded** In this case, there is a sequence $\hat R_\varepsilon \geq R_\varepsilon$ such that $\frac{d}{dr}U_\varepsilon^+(\hat R_\varepsilon)$ tends to $-\infty$. Integrating equation (3.13) between R_ε and $\hat R_\varepsilon$ gives

$$\frac{d}{dr}U_\varepsilon^+(R_\varepsilon) + \int_{R_\varepsilon}^{\hat R_\varepsilon} \left[\beta \frac{d}{dr}U_\varepsilon^+ + U_\varepsilon^+ + \frac{U_\varepsilon^+}{\varepsilon^2}\mathbb{I}_{\{U_\varepsilon^+<\mu\varepsilon\}} - F\right] dr = \frac{d}{dr}U_\varepsilon^+(\hat R_\varepsilon) + \int_{R_\varepsilon}^{\hat R_\varepsilon} \frac{d-1}{r}\frac{d}{dr}U_\varepsilon^+ \, dr.$$

Since the integral term on the right hand side is bounded and the integral term on the left hand side is negative, the sequence $\frac{d}{dr}U_\varepsilon^+(R_\varepsilon)$ tends to $-\infty$. Also in this case applying Lemma 3.5 leads to a contradiction.

Lemma 3.5 *The radial solution of*

$$\begin{cases} -\Delta V - \beta|\nabla V| + V = F(|x|), & |x| < R, \\ \frac{d}{dr}V(R) = -\mu/2, \end{cases}$$

satisfies $V(R) < 0$ for μ large enough depending on $\int F$ but independently of $R > 0$.

Since $\frac{d}{dr}V$ does not change sign thanks to arguments in Appendix B, it becomes a statement on a linear ODE which is left to the reader. □

3.4 Weak formulation of the Neumann boundary condition

We now consider a solution u_ε to the elliptic PDE (1.1) with assumption (3.1) and study its limit u_0 as ε vanishes. From the arguments in Theorem 3.1, and by a simple integration of the equation, we have the uniform bounds

$$\int [\,|\nabla u_\varepsilon|^2 + u_\varepsilon]\, dx \leq C_0, \qquad \int_{\{u_\varepsilon<\varepsilon\mu\}} \frac{u_\varepsilon}{\varepsilon^2}\, dx \leq C_0, \qquad u_\varepsilon \leq u^0. \tag{3.14}$$

We denote

$$\Omega_\varepsilon = \{u_\varepsilon > \varepsilon\mu\}, \qquad \Lambda_\varepsilon = \mathbb{I}_{\{u_\varepsilon\leq\varepsilon\mu\}}\frac{u_\varepsilon}{\varepsilon^2}.$$

After extractions, we have

$$\begin{cases} u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u_0 & \text{in } L^2(\mathbb{R}^d), \quad w-H^1(\mathbb{R}^d), \\ \Lambda_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \Lambda_0 & \text{weakly in bounded measures} \\ \mathbb{I}_{\{\Omega_\varepsilon\}} \underset{\varepsilon\to 0}{\rightharpoonup} Q(x) & L^\infty(\mathbb{R}^d) - w* . \end{cases} \tag{3.15}$$

Passing to the limit we find, setting $\Omega = \{u_0 > 0\}$, that

$$\mathbb{I}_{\{\Omega\}} \leq w* - \lim_{\varepsilon\to 0} \mathbb{I}_{\{\Omega_\varepsilon\}} = Q(x) \leq 1, \qquad Q \in L^1(\mathbb{R}^d), \tag{3.16}$$

and, thanks to (3.14), the nonnegative bounded measure Λ_0 is such that

$$-\Delta u_0 + u_0 + \Lambda_0 = f. \tag{3.17}$$

With the definition of the energy function G_μ in (2.6), we can complete the a priori estimates in (3.14) with the

Theorem 3.6 (Bounds for G_μ and Ω_ε) *Here we assume* (3.1) *and that* $|x|\ |b| \in L^\infty$, $|x|f \in L^2$. *Then*

$$\int_{\mathbb{R}^d} G_\mu(\frac{u_\varepsilon}{\varepsilon})dx \quad \text{and} \quad |\Omega_\varepsilon| \quad \text{are uniformly bounded.} \tag{3.18}$$

Proof. For a smooth test function $\Phi \in \mathbb{R}^d$, multiply equation (1.1) by $\Phi(x).\nabla u_\varepsilon$ and integrate by parts in the whole space. We find

$$\int_{\mathbb{R}^d} \big[-\text{div}\Phi\frac{|\nabla u_\varepsilon|^2}{2}+\nabla u_\varepsilon.D\Phi.\nabla u_\varepsilon-\text{div}\Phi\frac{u_\varepsilon^2}{2}-\text{div}\Phi\ G_\mu(\frac{u_\varepsilon}{\varepsilon})\big]\,dx=\int_{\mathbb{R}^d}[f-b.\nabla u_\varepsilon]\Phi(x)\nabla u_\varepsilon\,dx. \tag{3.19}$$

With the choice $\Phi = x$ and thanks to the estimates in (3.14), we find (3.18). □

One can strengthen the above limits in assuming further regularity on u_ε to ensure continuity in the limit. This follows for instance from the Lipschitz bounds in [11] (see also [8]). Indeed we have

Lemma 3.7 *Assume additionally that u_ε is uniformly equi-continuous, then*

$$u_0\ \Lambda_0 = 0, \tag{3.20}$$

$$\nabla u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \nabla u_0 \qquad \text{in} \ \ L^2(\mathbb{R}^d), \tag{3.21}$$

$$Q = \mathbb{1}_{\{\Omega\}}, \qquad \Omega=\{u_0>0\}, \qquad |\Omega|<\infty. \tag{3.22}$$

Proof. To simplify the notations, we take $b = 0$ in the first two steps of this proof. Thanks to the second bound in (3.14),

$$\int_{\mathbb{R}^d} u_\varepsilon\ \Lambda_\varepsilon\,dx = \int_{\mathbb{R}^d} u_\varepsilon(x)\frac{u_\varepsilon(x)}{\varepsilon^2}\mathbb{1}_{\{u_\varepsilon(x)\le\mu\varepsilon\}}\,dx \le \mu\varepsilon\int_{\mathbb{R}^d}\Lambda_\varepsilon.$$

Because we assume uniform continuity, we conclude that in the limit $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} u_0\ \Lambda_0\,dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} u_\varepsilon\ \Lambda_\varepsilon\,dx = 0$$

and the first statement is proved.

Now we multiply the equation (3.17) by u_0 and integrating by parts gives

$$\int_{\mathbb{R}^d}\big[|\nabla u_0|^2+|u_0|^2+u_0\ \Lambda_0\big]\,dx=\int_{\mathbb{R}^d} f\ u_0\,dx$$

and thus

$$\int_{\mathbb{R}^d}\big[|\nabla u_0|^2+u_0^2\big]\,dx=\int_{\mathbb{R}^d} fu_0\,dx.$$

We compare this result with the same manipulation at the ε level. Multiplying the equation on u_ε by u_ε and integrating by parts gives

$$\int_{\mathbb{R}^d}\big[|\nabla u_\varepsilon|^2+u_\varepsilon^2+\frac{u_\varepsilon^2}{\varepsilon^2}\mathbb{1}_{\{u_\varepsilon(x)\le\mu\varepsilon\}}\big]\,dx=\int_{\mathbb{R}^d} fu_\varepsilon\,dx.$$

But we have $\frac{u_\varepsilon^2}{\varepsilon^2}\mathbb{1}_{\{u_\varepsilon(x)\leq \mu\varepsilon\}} \leq \mu\varepsilon\frac{u_\varepsilon}{\varepsilon^2}\mathbb{1}_{\{u_\varepsilon(x)\leq \mu\varepsilon\}}$ and from the bound in (3.14) the integral vanishes. Therefore we conclude that (after extraction)

$$\int_{\mathbb{R}^d} f u_0\, dx = \int_{\mathbb{R}^d} \left[|\nabla u_0|^2 + u_0^2\right] dx \leq \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^d} |\nabla u_\varepsilon|^2 + \int_{\mathbb{R}^d} u_0^2\, dx = \int_{\mathbb{R}^d} f u_0\, dx.$$

This proves that the L^2 norm of the gradient converges and thus there is strong convergence and (3.21) holds.

With this strong convergence, we may pass to the limit in (3.19) and find

$$\int_{\mathbb{R}^d} \left[-\mathrm{div}\Phi\frac{|\nabla u_0|^2}{2} + \nabla u_0.D\Phi.\nabla u_0 - \mathrm{div}\Phi\frac{u_0^2}{2} - \mathrm{div}\Phi\ \langle G_\mu\rangle\right] dx = \int_{\mathbb{R}^d} [f - b.\nabla u_0]\Phi(x)\, dx.\nabla u_0\, dx. \tag{3.23}$$

with $\langle G_\mu\rangle$ defined as the $L^\infty - w*$ limit of $G_\mu(\frac{u_\varepsilon}{\varepsilon})$. Because we may write

$$2G_\mu(\frac{u_\varepsilon}{\varepsilon}) = \frac{u_\varepsilon^2}{\varepsilon^2}\mathbb{1}_{\{u_\varepsilon(x)\leq \mu\varepsilon\}} + \mu^2\mathbb{1}_{\{\Omega_\varepsilon\}},$$

and arguing as before for the first term, we find

$$\langle G_\mu\rangle = \frac{\mu^2}{2}Q(x). \tag{3.24}$$

We compare again the result with what is obtained when we convolve the equation (3.17) with a smoothing kernel ω_δ and integrate against $\Phi.\nabla u_0 * \omega_\delta$. We find

$$\int_{\mathbb{R}^d} \left[-\mathrm{div}\Phi\frac{|\nabla u_0|^2}{2} + \nabla u_0.D\Phi.\nabla u_0 - \mathrm{div}\Phi u_0^2\right] dx + \lim_{\delta\to 0}\int_{\mathbb{R}^d} \Phi.\nabla u_0 * \omega_\delta\ \Lambda_0 * \omega_\delta\, dx = \int_{\mathbb{R}^d} [f - b.\nabla u_0]\Phi(x).\nabla u_0\, dx.$$

Consequently, for all test functions Φ,

$$-\frac{\mu^2}{2}\int_{\mathbb{R}^d} \mathrm{div}\Phi\ Q(x)\, dx = \lim_{\delta\to 0}\int_{\mathbb{R}^d} \Phi.\nabla u_0 * \omega_\delta\ \Lambda_0 * \omega_\delta\, dx.$$

When Φ has support outside $\overline{\Omega}$, the right hand side vanishes as $\delta \to 0$, which proves also that Q is supported in Ω and together with (3.16), this proves the last statement. $\square$

From this proof, and additionally to

$$\langle G_\mu\rangle = \frac{\mu^2}{2}\mathbb{1}_{\{\Omega\}}, \tag{3.25}$$

we infer

Proposition 3.8 (Weak formulation) *As $\varepsilon \to 0$, a uniformly equicontinuous limit of weak solutions to* (1.1) *satisfies the two equations*

$$-\Delta u_0 + u_0 = f \qquad in\ \ H_0^1(\Omega), \tag{3.26}$$

$$\int_\Omega \left[-\mathrm{div}\Phi\frac{|\nabla u_0|^2}{2} + \nabla u_0.D\Phi.\nabla u_0 - \mathrm{div}\Phi\frac{u_0^2}{2} - \frac{\mu^2}{2}\mathrm{div}\Phi\right] dx = \int_\Omega [f - b.\nabla u_0]\Phi(x).\nabla u_0\, dx \tag{3.27}$$

for all C^1 test functions Φ with compact support.

The equations (3.26)–(3.27) are indeed a *weak formulation* of the Neumann boundary condition

$$\frac{\partial u_0}{\partial \nu} = -\frac{\mu}{\sqrt{2}} \qquad on \quad \partial\Omega. \tag{3.28}$$

This is because if Ω is C^1 and $u_0 \in H^2$, we may integrate (3.26) against $\Phi.\nabla u_0$ over Ω and find

$$\int_\Omega \big[-\operatorname{div}\Phi \frac{|\nabla u_0|^2}{2} + \nabla u_0.D\Phi.\nabla u_0 - \operatorname{div}\Phi \frac{u_0^2}{2}\big]\, dx - \frac{1}{2}\int_{\partial\Omega} \left(\frac{\partial u_0}{\partial \nu}\right)^2 \Phi.\nu\, d\sigma = \int_\Omega [f - b.\nabla u_0]\Phi(x).\nabla u_0\, dx. \tag{3.29}$$

In comparison to (3.27), using that $-\frac{\mu^2}{2}\int_\Omega \operatorname{div}\Phi\, dx = -\frac{\mu^2}{2}\int_{\partial\Omega} \Phi.\nu\, d\sigma$ and again because we can use arbitrary test functions Φ, we find

$$\frac{\mu^2}{2} = \left(\frac{\partial u_0}{\partial \nu}\right)^2.$$

This is equivalent to writing (3.28).

Let us conclude this section by an observation. With enough regularity, we may also integrate over Ω_ε and find

$$\int_{\Omega_\varepsilon} \big[-\operatorname{div}\Phi \frac{|\nabla u_\varepsilon|^2}{2} + \nabla u_\varepsilon.D\Phi.\nabla u_\varepsilon - \operatorname{div}\Phi \frac{u_\varepsilon^2}{2}\big]\, dx - \int_{\partial\Omega_\varepsilon} \frac{\partial u_\varepsilon}{\partial \nu}\Phi(x).\nabla u_\varepsilon\, d\sigma + \frac{1}{2}\int_{\partial\Omega_\varepsilon} \Phi.\nu\big[u_\varepsilon^2 + |\nabla u_\varepsilon|^2\big]\, d\sigma$$

$$= \int_{\Omega_\varepsilon} [f - b.\nabla u_\varepsilon]\Phi(x).\nabla u_\varepsilon\, dx.$$

We examine the boundary integrals and because $u_\varepsilon = \mu\varepsilon$ on $\partial\Omega_\varepsilon$, ∇u_ε is reduced to its normal component. We find

$$-\int_{\partial\Omega_\varepsilon} \frac{\partial u_\varepsilon}{\partial \nu}\Phi(x).\nabla u_\varepsilon\, dx + \frac{1}{2}\int_{\partial\Omega_\varepsilon} \Phi.\nu\big[u_\varepsilon^2 + |\nabla u_\varepsilon|^2\big]\, dx = \frac{1}{2}\int_{\partial\Omega_\varepsilon} \left[-\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)^2 + \mu^2\varepsilon^2\right] \Phi.\nu\, d\sigma.$$

As $\varepsilon \to 0$ we find

$$\int_\Omega \big[-\operatorname{div}\Phi \frac{|\nabla u_0|^2}{2} + \nabla u_0.D^2\Phi.\nabla u_0 - \operatorname{div}\Phi \frac{u_0^2}{2}\big]\, dx - \frac{1}{2}\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial \nu}\right)^2 \Phi.\nu\, d\sigma = \int_\Omega [f - b.\nabla u_0]\Phi(x).\nabla u_0\, dx, \tag{3.30}$$

a formula which in comparison to (3.29) carries information on the limit of $\frac{\partial u_\varepsilon}{\partial \nu}$ on $\partial\Omega_\varepsilon$ as $\varepsilon \to 0$.

4 Maximal solution with positive energy

In order to illustrate our theoretical results and gain some intuition, we consider several examples in one dimension. We begin with a simplified version, dropping the absorbtion term, which allows for very elementary calculations. Then we include the absorbtion term and show that the maximal solution still has positive energy and thus does not coincide with the variational solution.

4.1 Explicit construction for a simplified model

In order to simplify the formula, we consider the slightly changed equation with a truncated right hand side

$$-u'' + \frac{u}{\varepsilon^2} \mathbb{1}_{\{u \le \mu\varepsilon\}} = f(x)\, \mathbb{1}_{\{u > \mu\varepsilon\}}. \tag{4.1}$$

Notice that all the methods and results elaborated before hold true for this truncate right hand side.

Let us assume $f \ge 0$ and $f(-x) = f(x)$. If we assume additionally that the the region $\{u \ge \mu\varepsilon\}$ is an interval, then for symmetry reason it must be centered around 0 i.e. $(-R, R)$.

In this framework, we would like to understand the range (μ_-, μ_+) of values for which the non-trivial limit exists, if variational and maximal solutions agree and if there is one-to-one mapping $R \mapsto \mu$ in the 'fixed point' algorithm

$$-u'' + \frac{u}{\varepsilon^2} \mathbb{1}_{\{|x| \ge R\}} = f(|x|)\, \mathbb{1}_{\{|x| < R\}}, \qquad x \in \mathbb{R}, \tag{4.2}$$

$$\mu := \frac{u(R)}{\varepsilon}. \tag{4.3}$$

The limit problem becomes

$$-u'' = f(|x|), \qquad |x| \le R, \qquad u(R) = 0, \qquad u'(0) = 0, \qquad -u'(R) = \mu. \tag{4.4}$$

The shortcoming of dropping the absorption term is that we cannot define the maximal solution of these problems as we did by iterating from the positive solution of $-\Delta u^0 = f$; it does not exist. However we can find solutions of the nonlinear problems at hand. Namely, we have

Lemma 4.1 *For $f \in L^1_+$ and $\mu \in (0, \|f\|_1/2]$, define R by $\mu = \int_0^R f\, dx$ (independently of ε). Then there is a solution u_ε to* (4.2), (4.3) *and it satisfies $-u'_\varepsilon(R) = \mu$.*

As ε vanishes, it converges to a nontrivial solution of (4.4).

The case of a Dirac mass at 0 is also included with the notational convention $\int_0^r \delta(x) = 1/2$.

Proof. For $x > R$, the solution u is given by $u_\varepsilon(x) = be^{-x/\varepsilon}$.

Now we match the derivative of the solution at R:

$$u'_\varepsilon(R^-) = u'_\varepsilon(R^+) = -\frac{b}{\varepsilon} e^{-R/\varepsilon}. \tag{4.5}$$

Also, since $u'_\varepsilon(0) = 0$ for an even function, we have

$$-u'_\varepsilon(R^-) = -\int_0^R u''_\varepsilon\, dx = \int_0^R f\, dx.$$

Combining this with (4.5), we obtain

$$b = \varepsilon e^{R/\varepsilon} \int_0^R f\, dx.$$

It remains to identify the values at $x = R$. For $0 \le r \le R$, u is defined up to an additive constant that we can adapt for continuity. Therefore, we only check on the right

$$\mu = \frac{u_\varepsilon(R)}{\varepsilon} = \frac{b}{\varepsilon} e^{-R/\varepsilon} = \int_0^R f\, dx.$$

Since R is fixed, the limit $\varepsilon \to 0$ follows immediately. Note that u_ε and its limit only differ by a small additive constant in the interval $(0, R)$. $\square$

4.2 Computing the energy of these solutions

We can calculate that these solutions have positive energy. It is given by

$$E_\varepsilon(u) \quad = \quad \frac{1}{2}\int |u'|^2\,dx \;-\; \int f\,\mathbb{1}_{\{|x|<R\}} u\,dx \;+\; \frac{1}{2\varepsilon^2}\int u^2\,\mathbb{1}_{\{u\le\varepsilon\mu\}}\,dx \;+\; \frac{1}{2}\mu^2\int \mathbb{1}_{\{u>\varepsilon\mu\}}\,dx. \tag{4.6}$$

We start with the part for $x > R$

$$E_{\varepsilon,>R}(u) = \frac{b^2}{\varepsilon^2}\int_R^\infty e^{\frac{-2x}{\varepsilon}}\,dx + \frac{b^2}{\varepsilon^2}\int_R^\infty e^{\frac{-2x}{\varepsilon}}\,dx = \frac{b^2}{\varepsilon}e^{\frac{-2R}{\varepsilon}},$$

which gives, the expression for b,

$$E_{\varepsilon,>R}(u) = \frac{\varepsilon}{4}\left(\int_{-R}^{R} f\,dx\right)^2 = \varepsilon\mu^2.$$

For $x < R$, the energy is reduced to

$$E_{\varepsilon,<R}(u) = \frac{1}{2}\int |u'|^2\,dx - \int f\,\mathbb{1}_{\{|x|<R\}} u\,dx + \mu^2 R. \tag{4.7}$$

To continue, we compute

$$\int_0^R f\,\mathbb{1}_{\{|x|<R\}} u\,dx = -\int_0^R u'' u\,dx = -(u'u)(R) + \int_0^R (u')^2\,dx = \varepsilon\mu^2 + \int_0^R (u')^2\,dx$$

and it follows that

$$E_{\varepsilon,<R}(u) = -\int_0^R (u')^2\,dx - 2\varepsilon\mu^2 + \mu^2 R.$$

Altogether we arrive at

$$E_\varepsilon(u) = -\int_0^R (u')^2\,dx - \varepsilon\mu^2 + \mu^2 R.$$

By concavity of u, $|u'(r)| < \mu$ on $[0,r)$ and thus $E_\varepsilon(u)$ is strictly positive for ε small enough.

In the variational approach, one can compute that $\mu_{\text{var}} = \int_0^\infty f$. The positive energy calculation rises an apparent contradiction with Theorem 2.7 because minimizers are solutions to the elliptic equation. This is explained because $\min E_0^\mu = -\infty$ (due to the lack of absorbtion term) for $\mu < \mu_{\text{var}}$ and a minimizing sequence can be defined in the following way: there is N such that $\mu' := \int_0^N f > \mu$ and a u_N with

$$-u_N'' = f(|x|), \qquad |x| \le R, \qquad u(N) = 0, \qquad u'(0) = 0, \qquad -u'(N) = \mu'.$$

To obtain a solution u_n on intervals $(-n,n)$, we take the solution u_N, lift it up, extend it down to 0 linearly and we have again a solution as illustrated in Figure 1. On (N,n) the derivative u_n' is equal to μ', these solutions have negative energy for n large enough and their energy tends to $-\infty$ for $n\to\infty$.

4.3 An example with positive energy (with absorption term)

We now include the absorption term in the equation and we build another example where a solution of the PDE has a positive energy and thus is not the variational solution.

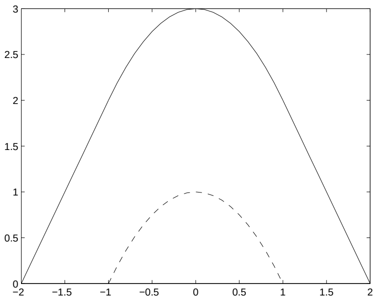


Figure 1: The figure shows a solution u_N (- - - -) as well as a lifted up and linearly extended solution (——).

Explicit construction. We consider the equation for $x \in \mathbb{R}$

$$-u_\varepsilon'' + u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2} \mathbb{I}_{\{u_\varepsilon \leq \mu\varepsilon\}} = a \mathbb{I}_{\{u_\varepsilon > \mu\varepsilon\}} := f_\varepsilon^\mu(x) \tag{4.8}$$

with a given constant $a > \mu$. For symmetry reasons we only build the decreasing solution for $x \geq 0$

$$u_\varepsilon(x) = \begin{cases} a_2 e^{-x/\delta} & x \geq R_\varepsilon, \\ a - a_1(e^x + e^{-x}) & x \leq R_\varepsilon, \end{cases} \tag{4.9}$$

and the conditions $u_\varepsilon(R_\varepsilon) = \mu\varepsilon$ and $u_\varepsilon'(R_\varepsilon)$ continuous, give the coefficients

$$\delta := \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}, \qquad e^{2R_\varepsilon} := \frac{a - \mu\varepsilon + \mu\sqrt{1+\varepsilon^2}}{a - \mu\varepsilon - \mu\sqrt{1+\varepsilon^2}},$$
$$a_1 := \frac{a - \mu\varepsilon}{g_\varepsilon + g_\varepsilon^{-1}}, \qquad g_\varepsilon = e^{R_\varepsilon}, \qquad a_2 := \mu\varepsilon e^{\frac{R_\varepsilon}{\delta}}.$$

In order to enforce positivity in the last line, we choose $a > \mu$ and ε small enough.

We define the energy functional corresponding to the nonlinear right hand side

$$F_\varepsilon(u) = \frac{1}{2} \int \left[|\nabla u|^2 + u^2 \right] dx - \int a(u - \mu\varepsilon)_+ \, dx + \frac{1}{2\varepsilon^2} \int u^2 \mathbb{I}_{\{u \leq \varepsilon\mu\}} \, dx + \frac{\mu^2}{2} \int \mathbb{I}_{\{u > \varepsilon\mu\}} \, dx. \tag{4.10}$$

To compute $F_\varepsilon(u_\varepsilon)$, we first consider $x \leq R_\varepsilon$ and use the equation to find

$$\int_0^{R_\varepsilon} \left[|u_\varepsilon'|^2 + u_\varepsilon^2 \right] dx - u_\varepsilon(R_\varepsilon) u_\varepsilon'(R_\varepsilon) = \int_0^{R_\varepsilon} \left[-u_\varepsilon u_\varepsilon'' + u_\varepsilon^2 \right] dx = \int_0^{R_\varepsilon} a u_\varepsilon \, dx.$$

This gives us the first contribution to the energy

$$F_\varepsilon(u_\varepsilon\big|_{|x|<R_\varepsilon}) = -\frac{1}{2} \int_0^{R_\varepsilon} a u_\varepsilon + \frac{1}{2}\mu\varepsilon + \frac{\mu^2}{2} R_\varepsilon + \frac{1}{2} u_\varepsilon(R_\varepsilon) u_\varepsilon'(R_\varepsilon).$$

We argue in the same way for $x > R_\varepsilon$. We compute

$$\int_{R_\varepsilon}^\infty \left[|u_\varepsilon'|^2 + u_\varepsilon^2 + \frac{u_\varepsilon^2}{\varepsilon^2} \right] dx + u_\varepsilon(R_\varepsilon) u_\varepsilon'(R_\varepsilon) = \int_{R_\varepsilon}^\infty \left[-u_\varepsilon u_\varepsilon'' + u_\varepsilon^2 + \frac{u_\varepsilon^2}{\varepsilon^2} \right] dx = 0$$

and find the second contribution

$$F_\varepsilon(u_\varepsilon\big|_{|x|>R_\varepsilon}) = -\frac{1}{2}u_\varepsilon(R_\varepsilon)u_\varepsilon'(R_\varepsilon).$$

Together we obtain

$$F_\varepsilon(u_\varepsilon) = -\frac{1}{2}\int_0^{R_\varepsilon} au_\varepsilon\,dx + \frac{1}{2}\mu\varepsilon + \frac{\mu^2}{2}R_\varepsilon = -\frac{a^2}{2}R_\varepsilon + \frac{aa_1}{2}(e^{R_\varepsilon} - e^{-R_\varepsilon}) + \frac{\mu^2}{2}R_\varepsilon + \frac{1}{2}\mu\varepsilon. \tag{4.11}$$

With straightforward calculations, we may compute the limit u_0 of u_ε as $\varepsilon \to 0$; the coefficients of interest become

$$e^{2R_0} = \frac{a+\mu}{a-\mu} =: g^2, \qquad a_1 := \frac{a}{g+g^{-1}},$$

$$u_0(x) = \begin{cases} 0 & \text{for } |x| \geq R_0, \\ a\left[1 - \frac{e^x+e^{-x}}{g+g^{-1}}\right] & \text{for } |x| \leq R_0. \end{cases} \tag{4.12}$$

Therefore the limiting energy is

$$\begin{aligned} 2F_0(u_0) &= (\mu^2 - a^2)R_0 + a^2\tfrac{g^2-1}{g^2+1} = (\mu^2-a^2)R_0 + a\mu \\ &= (\mu^2-a^2)\ln\left(\tfrac{a+\mu}{a-\mu}\right) + a\mu. \end{aligned} \tag{4.13}$$

For $\mu < a$ but close enough to a, it follows that $F_0(u_0) > \frac{a\mu}{4} > 0$.

A maximal solution for fixed ε. In order to exhibit a maximal solution with positive energy, we rather consider the equation

$$-U_\varepsilon'' + U_\varepsilon + \frac{U_\varepsilon}{\varepsilon^2}\mathbb{1}_{\{U_\varepsilon \leq \mu\varepsilon\}} = a\mathbb{1}_{\{|x|<R_\varepsilon\}} = f_\varepsilon^\mu, \tag{4.14}$$

and its maximal solution $U_\varepsilon^+ \geq u_\varepsilon$ (because u_ε is a solution). Since $(U_\varepsilon^+)'$ cannot have extrema in $[R_\varepsilon, \infty)$, there is a unique $R_\varepsilon' \geq R_\varepsilon$ such that $U_\varepsilon^+(R_\varepsilon') = \mu\varepsilon$. By the maximum principle $U_\varepsilon^+ \geq \mu\varepsilon$ on $(R_\varepsilon, R_\varepsilon')$. Therefore this solution has the form

$$U_\varepsilon^+(x) = \begin{cases} a - a_3(e^x + e^{-x}) & x \leq R_\varepsilon, \\ a_4e^x + a_5e^{-x} & R_\varepsilon \leq x \leq R_\varepsilon', \\ a_6e^{-x/\delta} & x \geq R_\varepsilon'. \end{cases} \tag{4.15}$$

At R_ε and R_ε', we have the following conditions

$$a - a_3(e^{R_\varepsilon} + e^{-R_\varepsilon}) = a_4e^{R_\varepsilon} + a_5e^{-R_\varepsilon}, \tag{4.16}$$

$$-a_3(e^{R_\varepsilon} - e^{-R_\varepsilon}) = a_4e^{R_\varepsilon} - a_5e^{-R_\varepsilon}, \tag{4.17}$$

$$a_4e^{R_\varepsilon'} + a_5e^{-R_\varepsilon'} = \mu\varepsilon, \tag{4.18}$$

$$a_4e^{R_\varepsilon'} - a_5e^{-R_\varepsilon'} = -\mu\varepsilon/\delta. \tag{4.19}$$

From (4.18) and (4.19), we obtain

$$2a_4 e^{R'_\varepsilon} = \mu\varepsilon(1 - 1/\delta), \qquad 2a_5 e^{-R'_\varepsilon} = \mu\varepsilon(1 + 1/\delta).$$

Inserting these expressions for a_4 and a_5 in (4.16) and (4.17), eliminating a_3 and defining $r := R'_\varepsilon - R_\varepsilon$ we obtain

$$2a + (a - \mu\varepsilon)\delta\left[(1 - 1/\delta)e^{-r} - (1 + 1/\delta)e^{r}\right] = \mu\varepsilon\left[(1 - 1/\delta)e^{-r} + (1 + 1/\delta)e^{r}\right].$$

This can be rewritten as $p(e^r) = 0$ with

$$p(y) := (1 + 1/\delta)[(a - \mu\varepsilon)\delta + \mu\varepsilon]y^2 - 2ay + (1 - 1/\delta)[\mu\varepsilon - (a - \mu\varepsilon)\delta].$$

We have $p(1) = 0$, the coefficient of y^2 is positive and $p'(1) = 2(1+1/\delta)[(a-\mu\varepsilon)\delta+\mu\varepsilon]-2a > 0$. So we know that the second root of p is less than 1 and this corresponds to $R'_\varepsilon < R_\varepsilon$ which is a contradiction. Therefore we have $R'_\varepsilon = R_\varepsilon$ and therefore $U^+_\varepsilon = u_\varepsilon$.

Maximal solution for limit $\varepsilon \to 0$. For $\mu < a$ let us denote the corresponding solution of (4.8) by u^μ_ε, the corresponding radius by R^μ_ε. Now we consider the equation

$$-\Delta v_\varepsilon + v_\varepsilon + \frac{v_\varepsilon}{\varepsilon^2}\mathbb{1}_{\{v_\varepsilon \le \mu\varepsilon\}} = a\mathbb{1}_{\{|x|<R_0\}} = f_0, \tag{4.20}$$

and construct its maximal solution v^+_ε as in section 3.1. Let us define its limit as v^+.

Let us take $\mu' < \mu$. We have $R^{\mu'}_0 < R^\mu_0$ and so for ε small enough, $f_0 \ge f^{\mu'}_\varepsilon$. By maximum principle, the construction algorithm in section 3.1 gives $v^+_\varepsilon \ge u^{\mu'}_\varepsilon$. So we obtain in the limit $v^+ \ge u^{\mu'}_0$ and $v^+ \ge u^\mu_0$ as $\mu' \to \mu$ since u^μ_0 is continuous in μ.

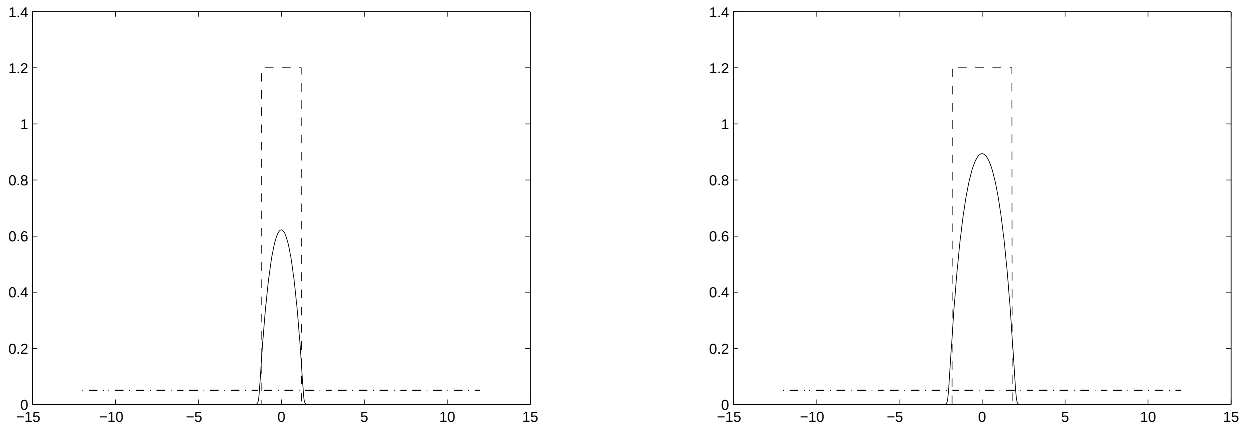
We have $e^{2R_\varepsilon} = 1 + \frac{2\mu\sqrt{1+\varepsilon^2}}{a-\mu\varepsilon-\mu\sqrt{1+\varepsilon^2}}$, so $R_\varepsilon > R_0$ and therefore $f_0 \le f_\varepsilon$. By maximum principle, the construction algorithm in section 3.1 gives $v^k_\varepsilon \le u^k_\varepsilon$ and therefore $v^+_\varepsilon \le u_\varepsilon$. So we obtain in the limit $v^+ \le u_0$.

5 Numerical computations

Our goal here is to give numerical evidence that the variational and maximal solutions are not always identical. For this reason we take $b = 0$.

We begin with 1D calculations. We illustrate the explicit solution obtained in section 4.3. We use a finite difference scheme in Matlab to implement the algorithm converging to the maximal solution on a grid with 1600 points on the domain $[-10R_0, 10R_0]$. We use $\mu = 1$, $a = 1.2$ and $\varepsilon = .05$. The algorithm runs for 80 iterations. The numerical value of R_0 is approximately 1.1989. Figure 2 (a) shows the maximal solution u^+_ε for $f = \mathbb{1}_{\{|x|\le R_0\}}$ and we see a good agreement with the condition $u(\pm R_0) = \mu\varepsilon$. The energy of the solution in the limit $\varepsilon \to 0$ is given in (4.13) as $0.0725 > 0$. In Figure 2 (b) the maximal solution u^+_ε for $f = \mathbb{1}_{\{|x|\le 1.5R_0\}}$ is plotted.

In higher dimension, except for the radial case, analytical solutions to our problem are not available. Therefore numerical results help towards an intuition whether the maximal and the variational



(a) u_ε^+ for $f = \mathbb{1}_{\{|x| \le R_0\}}$ (b) u_ε^+ for $f = \mathbb{1}_{\{|x| \le 1.5 R_0\}}$

Figure 2: Maximal solution for two different right hand sides f. The maximal solution is plotted as ______, f as $- - --$ and the line $y = \mu\varepsilon$ as $- \cdot - \cdot -$

solution are identical or not. We illustrate this issue thanks to computations done using the software FreeFem++ [19] based on $P1$ finite elements.

We consider the domain $\Omega = [0,1] \times [0,2]$ and a square grid with 400 points on the boundary $[0,1] \times \{0\}$, $[0,1] \times \{2\}$, 800 points on $\{0\} \times [0,2]$, $\{1\} \times [0,2]$. The parameters are chosen as $\varepsilon = .01$, $\mu = 1$. The right-hand side f is always taken as a indicator function of a set $[.35, .65] \times S$ where S is either an interval or the union of two intervals and f is symmetric with respect to $x = .5$.

The error is estimated in the following way: Let u be the numerical solution and $(v_i)_{i=1,\dots,M}$ the hat function basis of the finite element space P1. We define respectively the bilinear and linear forms

$$a(v,w) := \int_\Omega \left[\nabla v . \nabla w + vw + \frac{v}{\varepsilon^2} \mathbb{1}_{\{u \le \mu\, \varepsilon\}} w\right] dx, \tag{5.1}$$

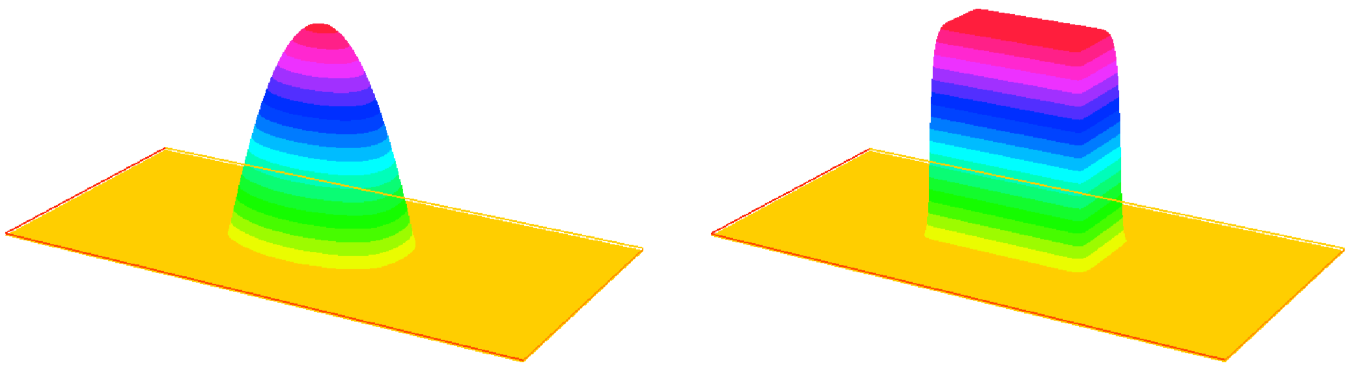
$$l(w) := \int_\Omega f w \, dx. \tag{5.2}$$

The error is calculated as the l^2-norm of the vector $\big(a(u,v_i) - l(v_i)\big)_{i=1,\dots,M}$. In all calculations shown the error is always less than 10^{-8}.

For the maximal solution we implement the iterative scheme described in section 3 with zero Dirichlet boundary conditions. For the variational solution we take advantage of the nonlinear conjugate gradient method.

Firstly, we choose $f = 10\, \mathbb{1}_{\{[.35,.65]\times[.7,1.3]\}}$ i.e. one single symmetric bump, we obtain u_ε^+ as shown in Figure 3 with energy 0.0268279 and u_{var} with energy -0.000860634. So the numerics indicates that the maximal and variational solutions are different.

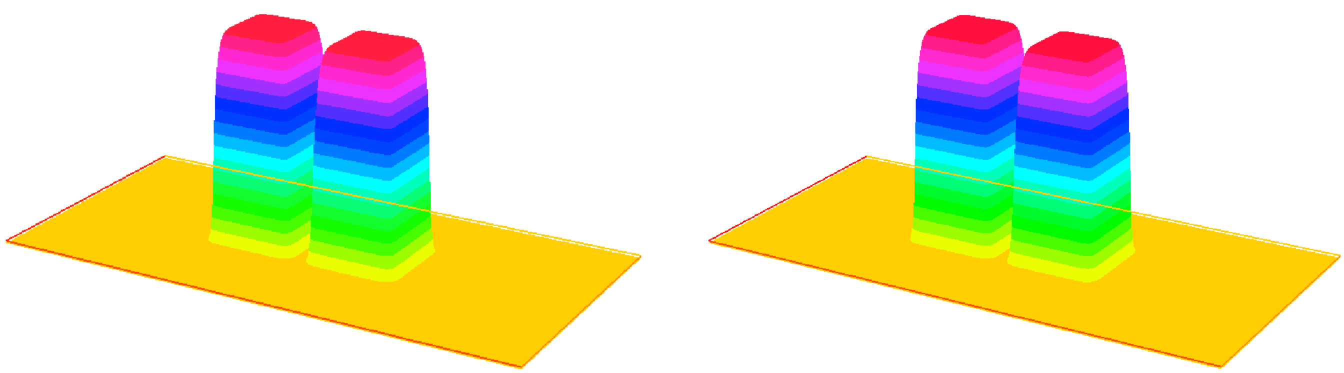
Secondly, we choose $f = 10\, \mathbb{1}_{\{[.35,.65]\times[.65,.95]\cup[.35,.65]\times[1.05,1.35]\}}$ i.e. two symmetric bumps. We obtain u_ε^+ as shown in Figure 4 with energy -0.000843616 and u_{var} with energy -0.000843724. Numerics



(a) u_ε^+ (maximum is .18). (b) u_{var} (maximum is 10^{-3}).

Figure 3: Maximal and variational solution for $f = 10\,\mathbb{1}_{\{[.35,.65]\times[.7,1.3]\}}$ i.e. one symmetric bump.

seems to indicate they are the same.



(a) u_ε^+ (maximum is 10^{-3}). (b) u_{var} (maximum is 10^{-3}).

Figure 4: Maximal and variational solution for $f = 10\,\mathbb{1}_{\{[.35,.65]\times[.65,.95]\cup[.35,.65]\times[1.05,1.35]\}}$ i.e. with two symmetric bumps

Thirdly, for an asymmetric set-up with $f = \mathbb{1}_{[.35,.65]\times[.7,1.]\cup[.35,.65]\times[1.05,1.35]}$, numerically, the variational and maximal solutions are different as shown in Figure 5.

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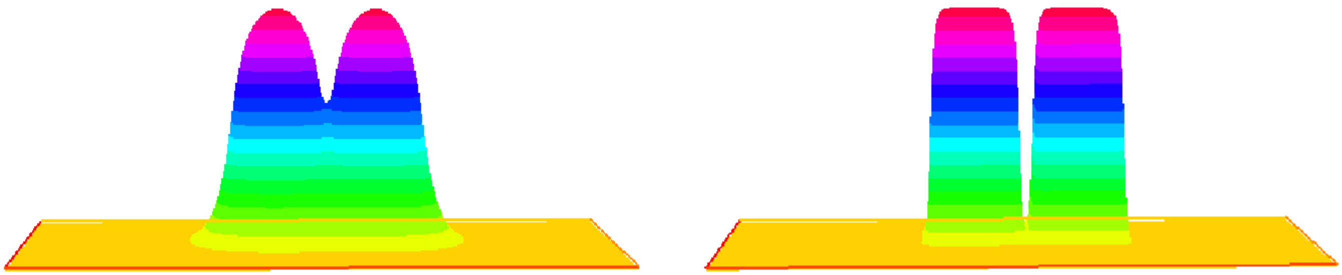
A Existence of minimisers of E_ε

For the sake of completeness we recall the argument for existence of a minimizer in the functionals of section 2 with $\varepsilon > 0$ fixed. Also, to increase readability, we only keep the index ε when convenient. Let u_k be a sequence, such that

$$E_\varepsilon(u_k) \to \inf_{u\in H^1(\mathbb{R}^k)} E_\varepsilon(u) \quad \text{as} \;\; k \to \infty.$$



(a) u_ε^+ (maximum is 0.023).

(b) $u_{\rm var}$ (maximum is 10^{-3}).

Figure 5: Maximal and variational solution for $f = 10\, \mathbb{I}_{\{[.35,.65]\times[.7,1.]\cup[.35,.65]\times[1.05,1.35]\}}$ i.e. two asymmetric bumps

With estimates (2.2), there is a $u \in H^1(\mathbb{R}^d)$ such that a subsequence u_k converges to u strongly in $L^2_{\rm loc}(\mathbb{R}^d)$ and weakly in $H^1(\mathbb{R}^d)$. We have

$$\liminf_{k\to\infty} \int \big[\frac{1}{2}|\nabla u_k|^2 + \frac{1}{2}u_k^2 - f u_k\big]\, dx \geq \int \big[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - f u\big]\, dx.$$

We write the remaining two terms in E_ε as

$$\varepsilon^{2(\alpha-1)} \int G_\mu(\frac{u}{\varepsilon^\alpha})\, dx.$$

Let us assume that

$$\liminf_{k\to\infty} \varepsilon^{2(\alpha-1)} \int G_\mu(\frac{u_k}{\varepsilon^\alpha})\, dx < \varepsilon^{2(\alpha-1)} \int G_\mu(\frac{u}{\varepsilon^\alpha})\, dx.$$

This means that there is a R and another subsequence such that

$$\lim_{k\to\infty} \varepsilon^{2(\alpha-1)} \int_{B_R(0)} G_\mu(\frac{u_k}{\varepsilon^\alpha})\, dx < \varepsilon^{2(\alpha-1)} \int_{B_R(0)} G_\mu(\frac{u}{\varepsilon^\alpha})\, dx.$$

But this contradicts the convergence in $L^2_{\rm loc}(\mathbb{R}^d)$.

B Study of w_ε

In Section 3.2 we have introduced solution of the linear equation (3.10) which reads

$$-\Delta w_\varepsilon + \beta|\nabla w_\varepsilon| + w_\varepsilon + \frac{w_\varepsilon}{\varepsilon^2}\mathbb{I}_{\{|x|\geq R\}} = \underline{f}_R\, \mathbb{I}_{\{|x|<R\}} \qquad x \in \mathbb{R}^d. \tag{B.1}$$

This section is devoted to prove some properties and asymptotic behaviour for this problem. A first observation is the obvious bounds

$$0 \leq w_\varepsilon \leq \underline{f}_R, \qquad \int w_\varepsilon\, dx \leq \underline{f}_R\, |B_R|, \qquad \int |\nabla w_\varepsilon|^2\, dx \leq C(R)\underline{f}_R^{\;2},$$

and w_ε is decreasing as $\varepsilon \searrow 0$.

Then we take advantage of the writing in radial coordinates

$$\begin{cases} -w_\varepsilon'' - \frac{d-1}{r} w_\varepsilon' + \beta|w_\varepsilon'| + w_\varepsilon + \frac{w_\varepsilon}{\varepsilon^2}\mathbb{I}_{\{|x|\geq R\}} = \underline{f}_R\, \mathbb{I}_{\{|x|<R\}}, \\ w_\varepsilon'(0) = 0, \qquad w_\varepsilon(\infty) = 0. \end{cases}$$

We notice that w_ε is decreasing because (i) $w_\varepsilon''(0) < 0$, (ii) a local minima cannot occur for $|x| \le R$ by the minimum principle and thus $w_\varepsilon' \le 0$ on $[0, R]$, (iii) a local maxima cannot occur for $|x| > R$ still by he maximum principle.

Then, we prove that w_ε' is uniformly bounded. Indeed, at a minimum point x_m one as $w_\varepsilon''(x_m) = 0$ and the equation contradicts $x_m > R$. If $x_m < R$ we conclude a bound from the equation again. For the point $x_m = R$ we conclude from the formula

$$-w_\varepsilon'(R)^2 - 2\int_0^R (\beta + \frac{d-1}{r})(w_\varepsilon')^2\, dr + w_\varepsilon^2(R) - w_\varepsilon^2(0) = 2\underline{f}_R[w_\varepsilon(R) - w_\varepsilon(0)]$$

obtained by multiplying the equation by w_ε' and integrating from 0 to R. Notice that this implies that w_ε is uniformly smooth (C^2 at least) inside the ball) and thus has a uniform limit together with its derivative.

Integrating this time between R and ∞ we find also

$$w_\varepsilon'(R)^2 - 2\int_R^\infty (\beta + \frac{d-1}{r})(w_\varepsilon')^2\, dr = \frac{w_\varepsilon^2(R)}{\varepsilon^2} + w_\varepsilon^2(R).$$

From this we conclude that $w_\varepsilon^2(R) = O(\varepsilon^2)$. Therefore we may write

$$\int_R^\infty (\beta + \frac{d-1}{r})(w_\varepsilon')^2\, dr \le C\|w_\varepsilon'\|_\infty \int_R^\infty (-w_\varepsilon')\, dr = C\|w_\varepsilon'\|_\infty\, O(\varepsilon)$$

and thus $\int_R^\infty (\beta + \frac{d-1}{r})(w_\varepsilon')^2\, dr \to 0$.

Therefore we conclude that $w_\varepsilon \searrow u_R$ (see section 3.2) and finally

$$w_\varepsilon'(R) \to u_R'(R), \qquad w_\varepsilon'(R)^2 - \frac{w_\varepsilon^2(R)}{\varepsilon^2} \to 0.$$

These are the staements we need in section 3.2.

C Variational and maximal solution

Consider two $H_0^1(\Omega)$-solutions u and u^+ (the maximal solution) of the boundary value problem for the semilinear PDE

$$-\Delta u + g(u) = f \text{ in } \Omega.$$

Both solutions satisfy

$$\int_\Omega [|\nabla u|^2 + ug(u) - fu]\, dx = 0.$$

The energy is defined by

$$E(v) = \int_\Omega [\frac{1}{2}|\nabla v|^2 + G(v) - fv]\, dx$$

For solutions the energy is also written

$$E(u) = \frac{1}{2}\int_\Omega [2G(u) - ug(u) - fu]\, dx.$$

Therefore

$$E(u^+) - E(u) = \frac{1}{2}\int_\Omega H(u^+) - H(u) - f(u^+ - u)\, dx, \tag{C.1}$$

$$H(u) := 2G(u) - ug(u).$$

A way to enforce $E(u^+) - E(u) \leq 0$, is to ask that H is decreasing. Note that

$$H'(u) = g(u) - ug'(u).$$

In our case $g(u) = \frac{u}{\varepsilon^2}\mathbb{1}_{\{u<\mu\varepsilon\}}$ and

$$H'(u) = \frac{\mu}{\varepsilon}\delta(u - \mu\varepsilon).$$

Thus, there is no direct way to guarantee that the variational and maximal solutions are equal.

References

[1] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math., 325 (1981), pp. 105–144.

[2] H. W. Alt, L. A. Caffarelli, and A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc., 282 (1984), pp. 431–461.

[3] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, *Uniform estimates for regularization of free boundary problems*, in Analysis and partial differential equations, vol. 122 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1990, pp. 567–619.

[4] A. Braides, *Γ-convergence for beginners*, vol. 22 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2002.

[5] L. A. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are $C^{1,\alpha}$*, Rev. Mat. Iberoamericana, 3 (1987), pp. 139–162.

[6] ——, *A Harnack inequality approach to the regularity of free boundaries. III. Existence theory, compactness, and dependence on X*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 15 (1988), pp. 583–602 (1989).

[7] ——, *A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz*, Comm. Pure Appl. Math., 42 (1989), pp. 55–78.

[8] L. A. Caffarelli, D. Jerison, and C. E. Kenig, *Some new monotonicity theorems with applications to free boundary problems*, Ann. of Math. (2), 155 (2002), pp. 369–404.

[9] L. A. Caffarelli and J. L. Vázquez, *A free-boundary problem for the heat equation arising in flame propagation*, Trans. Amer. Math. Soc., 347 (1995), pp. 411–441.

[10] L. Hauswirth, F. Hélein, and F. Pacard, *On an overdetermined elliptic problem*, Pacific J. Math., 250 (2011), pp. 319–334.

[11] C. Lederman and N. Wolanski, *A two phase elliptic singular perturbation problem with a forcing term*, J. Math. Pures Appl. (9), 86 (2006), pp. 552–589.

[12] ——, *A local monotonicity formula for an inhomogeneous singular perturbation problem and applications*, Ann. Mat. Pura Appl. (4), 187 (2008), pp. 197–220.

[13] ——, *A local monotonicity formula for an inhomogeneous singular perturbation problem and applications. II*, Ann. Mat. Pura Appl. (4), 189 (2010), pp. 25–46.

[14] S. Mirrahimi, G. Barles, B. Perthame, and P. E. Souganidis, *Singular hamilton-jacobi equation for the tail problem*, Accepted in SIAM Journal on Mathematical Analysis.

[15] D. R. Moreira, *Least supersolution approach to regularizing free boundary problems*, Arch. Ration. Mech. Anal., 191 (2009), pp. 97–141.

[16] B. Perthame and M. Gauduchon, *Survival thresholds and mortality rates in adaptive dynamics: conciliating deterministic and stochastic simulations*, Math. Med. Biol., 27 (2010), pp. 195–210.

[17] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal., 43 (1971), pp. 304–318.

[18] E. V. Teixeira, *A variational treatment for general elliptic equations of the flame propagation type: regularity of the free boundary*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008), pp. 633–658.

[19] www.freefem.org.