Geometric algebra for vector fields analysis and visualization: mathematical settings, overview and applications

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Abstract The formal language of Clifford's algebras is attracting an increasingly large community of mathematicians, physicists and software developers seduced by the conciseness and the efficiency of this compelling system of mathematics. This contribution will suggest how these concepts can be used to serve the purpose of scientific visualization and more specifically to reveal the general structure of complex vector fields. We will emphasize the elegance and the ubiquitous nature of the geometric algebra approach, as well as point out the computational issues at stake.

1 Introduction

Nowadays, complex numerical simulations (e.g. in climate modelling, weather forecast, aeronautics, genomics, etc.) produce very large data sets, often several terabytes, that become almost impossible to process in a reasonable amount of time. Among other challenges, storage, transfer, analysis and visualization are the more crucial. This requires developing new methods and implementing new algorithms to efficiently process this large quantity of information. On the other hand, in mathematics or theoretical physics, problems are commonly posed in high-dimensional spaces and require specific methods to reduce their dimension and make the solutions understandable. In both cases, there is a critical need for an abstract, general purpose method of analysis capable of extracting the salient features of the complex data. Unfortunately, numerical algorithms are too often inadequate to perceive the mathematical properties or the general structure of the objects considered. In this chapter, we will explain how the formal language of geometric algebras may be

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one of these long sought analysis tools, as it provides a unified framework bringing us closer by the topological aspects of geometrical problems, in a wide range of applications, including scientific visualization.

Based on the work of Grassman, Clifford's geometric algebras, born in the mid 19th-century, consider algebraic operators along with three main products to describe the spatial relations characteristic to geometric primitives in a coordinate-free approach. The many possibilities offered by Clifford algebras and Geometric Algebras (hereafter denoted GA), and especially their geometrically intuitive aspects, have been emphasized by the physicist D. Hestenes who recognized their importance to relativistic physics [16]. Since then, geometric algebras have also found applications in computer graphics and scientific visualization, owing to their geometric compactness and simplicity.

The next section will briefly present the main concepts and the basic manipulation rules of Clifford and geometric algebras. Then, the specific case of vector fields defined on d-dimensional spaces or on differential manifolds will be addressed in Section 3. In the last section, we will show how geometric algebra can be efficiently used to understand the algebraic structure of vector fields and implemented.

2 Clifford and geometric algebras

Leaning on the earlier concepts of Grassman's exterior algebra and Hamilton's quaternions, Clifford intended his *geometric algebra* to describe the geometric properties of vectors, planes and eventually higher dimensional objects[1]. Basically, Clifford algebra for $\mathbb{R}^n$ is the minimal enlargement of $\mathbb{R}^n$ to an associative algebra with unit capturing the metric, geometric and algebraic properties of Euclidean space [13]. In general, geometric algebras are distinguished from Clifford algebras by their restriction to real numbers and their emphasis on geometric interpretation and physical applications.

Our intent in this section is to give an elementary and coherent account of the main concepts of Clifford and geometric algebras. The reader who is interested in the theoretical aspects of geometric algebras is referred to the textbooks [13, 16, 15], among others. Computational aspects of geometric algebra and its usability in research or engineering applications are discussed in [6, 18].

2.1 *Clifford algebra*

Clifford algebra can be introduced in many ways; the axiomatic approach we follow here separates the algebraic structure from the geometric interpretation.

[1] The material in this section is intended to be fairly basic but readers unfamiliar with abstract mathematical concepts may skip this formal introduction as well as the parts written in smaller characters.

2.1.1 Basic notions and definitions

Let V be a vector space over a field K, and let $Q : V \to K$ be a quadratic form on V. A Clifford algebra $Cl(V,Q)$ is an associative algebra over K, with unity 1, together with a linear map $i : V \to Cl(V,Q)$ verifying, for all $v \in V$, the *contraction rule* $i(v)^2 = Q(v)1$, such that the following universal property is satisfied [20]:

> Given any other associative algebra A over K and any linear map $j : V \to A$ such that, for all $v \in V$, $j(v)^2 = Q(v)1_A$, there is a unique algebra homomorphism $f : Cl(V,Q) \to A$, for which the following diagram commutes:
>
> $$\begin{tikzcd} V \arrow[r, "i"'] \arrow[dr, "j"'] & Cl(V,Q) \arrow[d, "f"] \\ & A \end{tikzcd}$$

Note that the existence and the uniqueness (up to unique isomorphism) of a Clifford algebra for every pair (V,Q) can be established by considering a quotient algebra of a tensor algebra. The product defining the Clifford algebra will be called *geometric product* and denoted as: $u\,v$, for $u, v \in Cl(V,Q)$ (with a small space between the factors). One usually considers V as a linear subspace of $Cl(V,Q)$, thus dropping the inclusion in the definition of the Clifford algebra, leading to write $u\,u = Q(u)$. Consequently, the vector space V is not closed under multiplication as, for example, $u\,u$ is a scalar and not an element of V. The contraction rule also implies that every $v \in V$ has an inverse $v^{-1} = \frac{v}{Q(v)}$, unless Q is degenerate.

To better understand the geometric product, one can classically define it as the sum of an *inner product* (symmetric part) and an *outer product* (antisymmetric part):

$$a\,b = \underbrace{\frac{1}{2}(a\,b + b\,a)}_{<a,b>} + \underbrace{\frac{1}{2}(a\,b - b\,a)}_{a \wedge b}.$$

In this setting, the inner product corresponds to the bilinear form ϕ associated to the quadratic form Q thanks to the polarizing identity:

$$\phi(u,v) = \frac{1}{2}(Q(u+v) - Q(u) - Q(v)) = \frac{1}{2}((u+v)\,(u+v) - u\,u - v\,v) = < u, v > .$$

By definition, the element $a \wedge b$, if non zero, is called a *2-blade*, and has to be understood as a new entity, that is neither a scalar nor a vector. Geometrically, it represents an oriented plane segment, and can be characterized by an algebraic area (the usual area of the parallelogram with sides a and b in $\mathbb{R}^d$) and the *attitude* (angular position) of this plane. Similarly, one can define n-blades, for any $n \leq dim(V)$. By convention, 0-blades are scalars.

Given an orthonormal basis $\{e_1, e_2, \dots\}$ of the space V, we define G_0 as the inclusion of the scalars K in $Cl(V,Q)$ and G_n to be the part of $Cl(V,Q)$ generated from the products $\prod_{j=1}^{n} e_{i_j}$, for $1 \leq i_1 < \dots < i_n$. The direct sum $\bigoplus_{n=0}^{\infty} G_n$ is then the *graded* Clifford algebra. The elements of G_n are called *n-vectors*, where n represents the *grade*. Elements can be of "mixed grade", like the product ab, which is a sum of a scalar (grade 0) and a bivector (grade 2). A generic *multivector* A can be decomposed as a sum $A = \sum_{n=0}^{\infty} A_r$, where $A_r = \langle A \rangle_r$ is of grade r. A n-blade is a n-vector, but the converse is not true. We take for granted here that a sum of two blades A and B is another blade iff they are of the same grade k and share a common factor of grade $k-1$ or k.

Factorization of blades with the geometric product yields two equivalent forms for a blade: one based on the outer product, the other on the geometric product. Actually, for any arbitrary metric, given a k-blade A_k, it is possible to find an orthogonal basis $\{v_1, \cdots, v_k\}$ of this blade, implying $A_k = v_1 \wedge v_2 \wedge \cdots \wedge v_k = v_1 v_2 \cdots v_k$.
The *meet* and *join* are non-linear operations, corresponding to the blade intersection and union. Suppose we have an orthogonal factorization of two blades A and B, i.e. they are given with their orthogonal factorizations $A = A'C$ and $B = CB'$, C being the largest common factor. In this very simple case, $M = A \cap B = C$ and $J = A \cup B = (A'C) \wedge B'$.

An extension of the inner product, sometimes called *Clifford scalar product*, can be defined between two blades A_k of grade k and B_l of grade l as follows:

$$< A_k, B_l >= \begin{cases} \langle A_k B_l \rangle_{|k-l|} & \text{if } k,l > 0 \\ 0 & \text{else} \end{cases} .$$

Similarly, it is possible to define the outer product of two blades A_k and B_l as: $A_r \wedge B_s = \langle A_k B_l \rangle_{k+l}$.

2.1.2 Advanced concepts

Two important involutions are defined on $Cl(V,Q)$: *reversion* and *grade involution*.

On a r-blade $A = (a_1 \wedge a_2 \wedge \cdots \wedge a_r)$, the reversion $A^\dagger$ consists of reversing the order of the constitutive vectors (or, because the outer product is antisymmetric on vectors, changing the sign $r(r-1)/2$ times); the grade involution $A^\#$ consists of reversing the sign of every constitutive vector:

$$A^\dagger = a_r \wedge a_{r-1} \wedge \cdots \wedge a_1 = (-1)^{r(r-1)/2} a_1 \wedge a_2 \wedge \cdots \wedge a_r \qquad A^\# = (-1)^r A \,.$$

On a generic element $A = \sum_{r=0}^{\infty} A_r$ of $Cl(V,q)$, the reversion and grade involution are extended linearly from the previous definitions

$$A^\dagger = \sum_{r=0}^{\infty} (-1)^{r(r-1)/2} A_r \qquad A^\# = \sum_{r=0}^{\infty} (-1)^r A_r \,.$$

The *even* (resp. *odd*) multivectors are the ones with $A^\# = A$ (resp. $A^\# = -A$).

Using the reversion and the selection of the scalar part $\langle \cdot \rangle_0$, let us define a bilinear form on $Cl(V,Q)$. On blades A_k and B_l, we set:

$$A_k * B_l = \begin{cases} \langle A_k^\dagger B_l \rangle_0 & \text{if } k = l \neq 0 \\ A_0 \cdot B_0 & \text{if } k = l = 0 \\ 0 & \text{else} \end{cases} .$$

Extending it linearly to multivectors A and B, we obtain the generic formula $A * B = \langle A^\dagger B\rangle_0$. Proof of the equivalence between both formulations can be found in [16], p.13. On vectors, this bilinear form clearly corresponds to the inner product: $a * b = < a, b >$.

2.2 Geometric algebras

The case $V = \mathbb{R}^n$ and Q non degenerate lead to a series of specific definitions and results. As a matter of fact, we have for example:

If $K = \mathbb{R}$ and Q is non-degenerate, every non-zero blade A_r has an inverse $\frac{A_r^\dagger}{A*A} = \frac{A_r^\dagger}{\langle A^\dagger A\rangle_0}$.
If $K = \mathbb{R}$ and Q is positive definite on V, then we can define the modulus of element A as $|A| = \sqrt{A^\dagger * A} = \sqrt{\langle A^\dagger A\rangle_0}$, since for an element $a_1 \cdots a_r$, $(a_1 \cdots a_r)^\dagger (a_1 \cdots a_r) = Q(a_r) \cdots Q(a_1) \geq 0$.
In $\mathbb{R}^3$, the existence of an inverse vector has a very clear interpretation. For a given vector $v \in \mathbb{R}^3$ and a given scalar a, the equation $< v, w >= a$ defines the affine plane $w_0 + v^\perp$. Likewise, given v and a bivector A, the equation $v \wedge w = A$ defines the affine line $w_0 + \lambda v$. In both cases, there is no unique solution. However, in the setting of geometric algebra, the equation $vw = A$ leads to the unique solution $w = v^{-1}A$ (corresponding to the intersection of a plane $< v, w >= A_0$ and of a line $v \wedge w = A_2$).

2.2.1 Particular settings

Such a Clifford algebra, in the case $V = \mathbb{R}^n$ and Q non degenerate, is called *geometric algebra*. Let (p, q) be the signature of the quadratic form Q, i.e. Q diagonalizes in $Q(v) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2$ (Sylvester's law of inertia). We write $\mathbb{R}^{p,q}$ for V and $Cl_{p,q}$ for the associated geometric (Clifford) algebra.

Taking a basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$, using the element 1 to span the scalars and all products $\prod_{j=1}^{r} e_{i_j}$ for $1 \leq i_1 < \cdots < i_r \leq n$ ($r \in \mathbb{N}_n$) to span the multivectors, the set $\{1, e_1, e_2, \ldots, e_n, e_1 e_2, e_1 e_3, \ldots, e_1 e_2 \ldots e_n\}$ will form a basis for $Cl_{p,q}$, with $2^n = \sum_{r=0}^{n} \binom{n}{r}$ elements. The element $\mathrm{I}_n = e_1 e_2 \ldots e_n$ is called *pseudoscalar* and is defined to a scalar multiple, since all n-blades are proportional.

The dual A^* of a multivector A is defined as $A^* = A\,\mathrm{I}_n^{-1}$. The duality operation transforms a r-vector A_r into an $(n-r)$-vector $A_r \mathrm{I}_n^{-1}$; in particular, it maps scalars into pseudoscalars. The duality relation states $(A \wedge B)^* = A \rfloor B^*$, where $\rfloor$ denotes the left contraction [2]. The inclusion of an element x in a given subspace $\mathscr{A}$ specified by a blade A can be defined in two ways:

- the direct way: $x \in \mathscr{A} \iff x \wedge A = 0$
- the dual way: $x \in \mathscr{A} \iff x \rfloor A^* = 0$.

[2] For two blades A and B of grades a and b, the left contraction $A \rfloor B$ is $\langle AB\rangle_{b-a}$ when $a \leq b$, it is zero otherwise. When blade A is contained in blade B, it equals the geometric product AB [6].

Given a basis $\{b_1,...,b_n\}$ of $\mathbb{R}^{p,q}$, $n = p+q$, we can define a *reciprocal frame* $\{b^1,...,b^n\}$, through the formula $b^i = (-1)^{i-1}(b_1 \cdots \wedge b_{i-1} \wedge \check{b_i} \wedge b_{i+1} \cdots \wedge b_n)\,\mathrm{I}_n^{-1}$, where $\mathrm{I}_n = b^1 \wedge ... \wedge b^n$ and the $\check{}$ -sign mentions the element removed from the list. The two basis are mutually orthogonal: $< b_i, b^j >= \delta^i_j$. Since the reciprocal of an orthonormal basis is itself, this definition is needed only in non-euclidean cases. It is also useful in differential geometry.

A vector of $\mathbb{R}^{p,q}$ can be written $a = \sum_{i=1}^n a_i e^i$ or $a = \sum_{i=1}^n a^i e_i$ with $a_i =< a, e_i >$ et $a^i =< a, e^i >$. If we have a multivector basis $\{e_\alpha | \alpha \in \{1,\cdots,2^n\}$, we can also define a reciprocal frame $\{e^\alpha | \alpha \in \{1,\cdots,2^n\}\}$.

2.2.2 Versors, rotors, spinors and rotations

One of the main features of GA is its ability to deal with the rotations. Indeed, a unique object R can be used to compute the rotation of any subspace X, writing a conjugation with the geometric product:

$$\mathscr{R}(X) = R X R^{-1}\,.$$

The equation $x = a x a^{-1}$ gives the reflection of an arbitrary vector x along the a-line (a invertible). Its opposite $x = -a x a^{-1}$ gives the reflection in the dual hyperplane $A = a^*$. Two consecutive reflections form a simple rotation, which can be written as follows: $x" = -b x' b^{-1} = b a x a^{-1} b^{-1} = (b a) x (b a)^{-1}$. It is a rotation of twice the angle between a and b in the plane containing a and b. The element $a b$ is called a *2-versor*. In general, a *k-versor* is a multivector that can be written as the geometric product of k invertible vectors $v = v_1 v_2 ... v_k$. By Cartan-Dieudonné theorem, every isometry of $\mathbb{R}^{p,q}$ can be reduced to at most $n = p+q$ reflections in hyperplanes. It means that we can write every orthogonal transformation f with a k-versor U ($k \leq n$) and the conjugation: $f(x) = (-1)^k U x U^{-1}$.

In all spaces of signatures $(n,0)$, $(0,n)$, $(n-1,1)$ or $(1,n-1)$, including the Euclidean spaces, every rotation can be written in exponential form[3]:

$$\mathscr{R}(x) = S x S^\dagger \text{ with } S = e^{\frac{1}{2}(i_1\,\theta_1 + \cdots + i_m\,\theta_m)}, i_1, \cdots, i_m \text{ orthogonal 2-blades}\,.$$

Note that a rotation of a non-Euclidean space is defined to be an orthogonal transformation of determinant one continuously connected to identity. The element S given by the exponential form of preceding equation is a *rotor*, i.e. an even versor S satisfying $S S^\dagger = 1$.

A linear map f: $V \to V$ can be extended in a function $\underline{f} : Cl(V,Q) \to Cl(V,Q)$ while preserving the outer product:

$$\underline{f}(a_1 \wedge a_2 \wedge \cdots \wedge a_r) = f(a_1) \wedge f(a_2) \wedge \cdots \wedge f(a_r)\,.$$

It is then called an *outermorphism*. In particular, the reflection of a blade A_k in a dual hyperplane a^* is $(-1)^k a A_k a^{-1}$ and the the rotation of a blade by a rotor is $R A_k R^\dagger$ according to the previous equations for vectors.

[3] Quite naturally, the exponential of a blade A is defined with the usual power series $\sum_{k=0}^{\infty} \frac{A^k}{k!}$. The additivity $\exp(A+B) = exp(A)\,exp(B)$ is not true in general. The circular and hyperbolic functions of blades are also defined with power series.

2.2.3 Geometric calculus

Differentiation. We consider a finite dimensional vector space V with quadratic form Q and a multivector-valued function $F : U \subset V \longrightarrow Cl(V,Q)$. It comes of no surprise that the directional derivative of F in direction r is simply:

$$F_r(x) = \lim_{s\to 0} \frac{F(x+sr)-F(x)}{s}\,.$$

This expression will be most of the time written $(r*\nabla)F$ instead of F_r, expressing the idea of a scalar product between r and the operator ∇, seen as a vector, as will be clearer below. The linearity in r is straightforward; the sum, the geometric product and the grade are preserved. If we want to differentiate a multivector-valued function $F : U \subset V \longrightarrow Cl(V,Q)$ directly relative to the variable, we consider a base $\{e_1,\cdots,e_m\}$ of V and the coordinates functions of the vector x in this basis $x = \sum_{i=1}^n x^i e_i$. The directional derivatives along the basis directions, $(e_i * \nabla) = \frac{\partial}{\partial_{x_i}}$, combine into a total change operator [4] as:

$$\nabla = \sum_{i=1}^m e^i\,(e_i*\nabla) \quad \text{meaning} \quad \nabla F(x) = \sum_{i=1}^m e^i\,\frac{\partial F(x)}{\partial_{x_i}}\,.$$

Note that we also have to define the differentiation from the right, because of the non-commutativity: for a function F, $F(x)\nabla = \sum_{i=1}^m (e_i*\nabla)F(x)\,e^i = \sum_{i=1}^m \frac{\partial F(x)}{\partial_{x_i}}\,e^i$. Thanks to the geometric product, we can write ∇ as: $\nabla F = \nabla\wedge F + <\nabla, F>$. In the case of a vector-valued function F, we have the usual definitions of the divergence and curl operators:

$$\mathrm{curl}(F) := \nabla\wedge F = \frac{1}{2}(\nabla F - F\nabla) \quad \text{and} \quad \mathrm{div}(F) := <\nabla, F> = \frac{1}{2}(\nabla F + F\nabla)\,.$$

To write the product rule, accents are necessary to specify on what factor the differentiation acts: $\nabla(F\,G) = \grave{\nabla}\grave{F}\,G + \grave{\nabla}F\,\grave{G}$. The definition of a differentiation with respect to a multivector, for a function $F : U \subset Cl(V,Q) \longrightarrow Cl(V,Q)$, is quite straightforward, given a reciprocal frame for the whole space $Cl(V,Q)$.

Integration. Consider again a multivector-valued function F; the line integral is

$$\int_C F(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^n \overline{F}^j \Delta x^j\,, \quad \text{with} \quad \overline{F}^j = \frac{1}{2}\left(F(x_i)+F(x_{i-1})\right)$$

where the chords $\Delta x^i = x_i - x_{i-1}$ correspond to a subdivision of the curve C. The measure dx is said to be a *directed measure*, since it is vector-valued. The product between $F(x)$ and dx is the geometric product. If F is vector-valued,

$$\int_C F(x)\,dx = \int_C <F(x), dx> + \int_C F(x)\wedge dx\,.$$

[4] This explains the notation $F_r = (r*\nabla)F$ for the directional differentiation.

Similarly, if $D \subset \mathbb{R}^2$ is a triangulated planar domain, $\overline{F}^k$ is the average of F over the k-th simplex,

$$\int_D F(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \overline{F}^k \Delta x^k \,.$$

The surface measure of the k-th simplex given by vertices x_0, x_1, x_2 is

$$\Delta x^k = \tfrac{1}{2}\,(x_0 \wedge x_1 + x_2 \wedge x_0 + x_1 \wedge x_2)\,.$$

This integral definition can be generalized to higher dimensions [5].

The fundamental theorem states:

$$\oint_{\partial V} F\,dS = \int_V \grave{F}\,\grave{\nabla}\,dX\,, \quad \text{for a function } F \text{ defined over a volume } V\,.$$

2.2.4 Clifford Convolution and Clifford Fourier Transform

For F and H two multivector-valued functions $F, H : \mathbb{R}^m \to Cl_{p,q}$, the left- and the right-Clifford Convolution of the functions write respectively:

$$(H *_l F)(x) = \textstyle\int_{\mathbb{R}^m} H(x')F(x-x')|dx'|\,, \quad (H *_r F)(x) = \textstyle\int_{\mathbb{R}^m} H(x-x')F(x')|dx'|\,.$$

The quantity $|dx|$ is used to make the integral grade preserving since dx is a vector within Clifford algebra. Modifying $x - x'$ into $x + x'$, we get the left- and right- *Clifford correlations* [7]. The Clifford convolutions generalize the known convolution of scalar-valued functions.

A vector field F can be smoothed through convolution with a scalar field, for example a Gaussian kernel. In the case of two vector fields, the formula for the geometric product leads to the integration of a scalar function $< H(x-x'), F(x') >$ and a bivector function $H(x-x') \wedge F(x')$ [25].

In the case of a multivector-valued function $F : \mathbb{R}^3 \to Cl_{3,0}$, it is possible to define the *Clifford Fourier Transform* (CFT) of F and its inverse as follows:

$$\mathscr{F}\{F\}(u) = \textstyle\int_{\mathbb{R}^3} F(x)\,e^{-2\pi \mathbf{I}_3 <x,u>}|dx|\,, \quad \mathscr{F}^{-1}\{F\}(x) = \textstyle\int_{\mathbb{R}^3} F(u)e^{2\pi \mathbf{I}_3 <x,u>}|du|\,.$$

The function $e^{-2\pi \mathbf{I}_3 <x,u>} = \cos(2\pi < x, u >) + \mathbf{I}_3 \sin(2\pi < x, u >)$ is often called *Clifford Fourier kernel*.

The convolution theorem is also valid for the Clifford Fourier Transform and Clifford convolutions as defined here. For example, using the left convolution,

$$\mathscr{F}\{H *_l F\}(u) := \mathscr{F}\{H\}(u)\,\mathscr{F}\{F\}(u)\,.$$

As mentioned before, the reader willing to get a deeper understanding of the mathematical basics about Clifford algebras and geometric algebras is referred to [13, 16, 15]. In the next section, we will focus on the analysis of vector fields in the context of GA.

3 Vector fields in geometric algebra

Our main focus in this paper is the analysis of vector fields, more precisely of steady, linear and non-linear vector fields in Euclidean space and on manifolds. One is classically interested in streamlines, critical points with their classification, separatrices, leading to the topological graph of a vector field. We will show how the analysis of vector fields can benefit from the richer context of geometric algebra.

3.1 Vector fields on domains of Euclidean space

Classically, vector fields are applications $v : U \subset \mathbb{R}^n \to \mathbb{R}^n$. Using the notions of GA defined in Section 2.2, if $Cl^1_{n,0}$ is the set of 1-vectors of $Cl_{n,0}$, a map $v : \mathbb{R}^n \longrightarrow Cl^1_{n,0}$ is also a vector field. This definition can be easily extended to bivector, trivector, or spinor fields, for example.

The identification of vector fields (satisfying the Lipschitz regularity condition) with ordinary differential equations ${}^{dx}/_{dt} = v(x)$ can also be transposed from the classical to the GA setting. The Lipschitz continuity condition can be written in this frame,i.e. there exists a scalar constant $K \geq 0$ such that

$$\|v(x_1) - v(x_2))\|_n \leq K\|x_1 - x_2\|_n \quad \text{for all} \quad x_1, x_2 \in U \,.$$

Furthermore, the defined derivation and integration make it possible to state the existence of an unique solution (*streamline* or *integral curve*) through a given point, exactly like in the classical frame. In 2D and 3D, drawing the streamlines is a very classical way to represent a vector field. In order to avoid occlusions and empty areas, departure points (*seeds*) for these curves are to be placed efficiently.

Let us consider a small example. To a given classical vector field, we can associate curvature and torsion scalar fields: the curvature (resp. torsion) in a point is the curvature (resp. torsion) of the unique streamline in this point [31]. The curvature field associated to a vector field can be used for the seeding, or can be displayed as a further scalar value in the form of isosurfaces or by color coding. In the GA settings, instead of scalar fields, a curvature bivector and a torsion trivector fields can be defined. Visualizing the curvature bivector along a streamline, we get what is called the *Frenet ribbon* [4].

3.2 Vector fields on differential manifolds

Now we turn to vector fields on differential manifolds, having in mind to embed the differential geometry formalism into geometric calculus. For a more detailed presentation of this combined approach, see [16, 28].

In differential geometry, if M is a smooth manifold, a tangent vector is most of the time seen as a derivation, i.e. a linear operator $D : C^\infty(p) \to C^\infty(p)$[5] satisfying the Leibniz rule $D(f \cdot g) = f \cdot D(g) + g \cdot D(f)$. The tangent space of M in p is T_pM, the set of such derivations. A vector field is a function assigning to every $p \in M$ an element of T_pM.

In $\mathbb{R}^3$, in a more intuitive way, we can imagine giving in each point p of a surface S a vector tangent $v(p)$ to the surface. The link between this v and the associated derivation D_v is the derivative $D_v(f)(p) = Df(p)(v(p))$. The operator point of view makes it easier to manipulate vector fields and compose them with other operators. Furthermore a discretization can be made without working with coordinates [2].

To translate this definition into GA, we give the tangent spaces a Clifford algebra structure. Taking a chart (U, ϕ) around $p \in M$, the derivations e_i^p defined by

$$e_i^p(f) = \frac{\partial}{\partial x_i}(f \circ \phi^{-1})|_{x=\phi(p)}$$

form a basis for T_pM. Forming the blades of these basis vectors, we can be build a geometric algebra structure on T_pM.

With a little more abstraction, a vector field can classically be seen as a *section* of the *tangent bundle*, a particular *vector bundle*: Taking TM to be the disjoint union of tangent spaces on M, $TM = \sqcup_{x \in M} T_xM$, and $\pi : TM \to M$ defined by $\pi(v) = x$ for $x \in T_xM$, we can see M as the base space, TM as the total space and π as the projection, these three elements defining a fibre bundle called the tangent bundle. The section is a continuous map s with $\pi \circ s = id_M$, meaning $s(x) \in T_xM$, hence what we understand as a vector field. The adding of a geometric algebra structure can be done in the general case of a vector bundle on a manifold with some metrics, using a construction very similar as the one made in 2.1.1: quotienting a tensor algebra with a two-sided ideal.

Scalar fields, vector fields, bivector fields, spinor fields on surfaces, for example, are natural extensions of this definition of vector fields (or can be seen as sections of the Clifford tangent bundle, see above), and, as long as M is simply connected, it is also the case for rotation fields $r : M \to SO(n)$, since they can be lifted to spinor fields.

Since every differentiable manifold is locally diffeomorphic to an Euclidean space (via the charts), the existence and uniqueness of streamlines is also granted on manifolds, within or outside GA context.

3.3 Critical points, orbits and topological graph

The topological graph is an important tool of analysis: it goes one step further as the streamline representation and decomposes the vector field domain in areas of similar behavior. The critical points and closed orbits (with their type, like defined below) and the separatrices (streamlines or surfaces between areas of different behavior)

[5] $C^\infty(p)$ describes the smooth functions defined on some open neighborhood of p.

form the topological graph of the vector field, that eventually describes the underlying structure of the field in a more efficient way as a collection of streamlines. Such a graph does not take into account the norm of the vector field [9].

The classification of critical points finds its origin in the theory of dynamical systems. For regular critical points, i.e. for critical points with an invertible Jacobian matrix, a linear approximation of the field can be considered. Studying eigenvalues and eigenvectors of the Jacobian matrix makes the classification possible, provided none of the eigenvalues is pure imaginary. The so-called hyperbolic critical points, satisfying this condition, are isolated and are structurally stable: a small local perturbation does not modify the topology. This legitimates the linear approximation to describe the field's behavior around this point. In two dimensions for example, the hyperbolic critical points are sources, sinks, saddles and spirals. Unstable critical points are centers. A similar classification can be done for orbits, according to the derivative of the Poincaré map [1].

For non-linear critical points, said of *higher order*, the non-invertibility of the first derivate leads to consider a higher order Taylor expansion. For the isolated ones, the index[6] might help to discriminate critical points of different types. Sometimes this proves insufficient, since two critical points with same index can be of different types. The GA formalism provides an elegant alternative for the computation of the index: for example, in 3D,

$$ind(c) = \frac{1}{8\pi \mathrm{I}_3} \int_{B(c)} \frac{v \wedge dv}{|v|^3},$$

for v the vector field, c the critical point, $B(c)$ an arbitrary small ball around c [21].

Unlike the index, the ordered list of all different behavior sectors (i.e., *elliptic*, *hyperbolic* and *parabolic sectors*) makes a univoque classification possible [26, 14, 11, 30].

Next, we turn to a more practical view of geometric algebras, as this chapter is also intended for engineers and practitioners. In particular, we will briefly explain how GA can be implemented and the potential advantages of using Clifford algebra when, for example, dealing with rotations in spaces of high dimensions.

4 Geometric algebra for computer graphics and visualization of vector fields

Nowadays, geometric algebra is mostly recognized as a promising mathematical concept and is only beginning to find broader legitimacy in applications. Emerging computer architectures (multicore, many-core, parallel) lead us to believe that the language of GA may find a new playground and evolve towards what Hildenbrand calls Geometric Algebra Computing (GAC) [17]. However, GA is not yet

[6] In 2D, the index corresponds the number of turns the field makes around a critical point.

a widespread method in engineering applications, mainly because of two reasons, academic and practical [22]. On the one hand, GA combines many mathematical concepts that were developed separately over the years and are taught as such in curriculum. On the other hand, most engineering applications in three-dimensional space can be dealt using standard vector and matrix algebra tools. The goal of this last section is to introduce how GA can be used advantageously in computer graphics applications and vector field analysis and visualization.

4.1 Geometric algebra for computer graphics

It is surely the most obvious field of application of GA. In geometrical applications, operations and transformations are applied on primitives that are combined to represent an object (model). Linear geometric transformations are usually represented using matrices, vectors and scalars. But while 3×3 matrices encode the 3D-rotations about an axis through the origin, quaternions (which form a subalgebra of GA [23]) are better suited instead, because they are easier to interpret.

The quaternion representation of a rotation is a nearly minimal parametrization that requires only four scalars. Given a quaternion, one can easily read off the axis and angle of the rotation, it is not the case with the Euler angles representation. The composition of rotations in quaternion form is faster and more stable numerically (the renormalization is more efficient than with matrices). Furthermore, the interpolation in $\mathbb{H}$ (for example to get an animated view of a rotated object) consists in defining a path on S^3 which is mapped to $SO(3)$. The Euler angles parametrization, from the 3-torus to $SO(3)$ is not a global diffeomorphism: the uniqueness breaks at some points (problem known as the *gimbal lock*). This is why, in graphic libraries such as OpenGL, rotations are given in terms of a rotation axis and a rotation angle and converted internally into rotation matrices.

Note that $\mathbb{H}$ is trivially isomorphic to the even algebra $Cl^+_{3,0}$ (the set of even multivectors of $Cl_{3,0}$): we can identify the unit and the basis elements $\mathbf{i},\mathbf{j},\mathbf{k}$ of $\mathbb{H}$ with the unit and the products $e_1\,e_2$, $e_1\,e_3$ and $e_2\,e_3$ of $Cl_{3,0}$ [15]. Thus, the aforementioned quaternion representation of a rotation (of angle θ around unit axis u):

$$q = e^{\frac{1}{2}\theta(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\tfrac{1}{2}\theta + (u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})\sin\tfrac{1}{2}\theta \quad \text{with} \quad \mathscr{R}(x) = qxq^{-1}\,,$$

can be seen as a rotor of $Cl_{3,0}$. The products and the conjugation on $\mathbb{H}$ and on $Cl^+_{3,0}$ are defined likewise. Similarly, identifying the pseudoscalar I of $Cl_{2,0}$ with the imaginary unit i of $\mathbb{C}$, we have an algebra isomorphism between $Cl^+_{2,0}$ and $\mathbb{C}$.

Clearly, geometric algebra exhibits structural simplicity in the formulations, but its naive implementation may be far less efficient than classical analytical geometry implementations, especially for high dimensions. Fortunately, GA expressions can benefit from compilation (e.g. operator overloading) and parallelization techniques (including GPUs) [12, 17].

In practice, GA has been implemented in two ways. The *additive approach* encodes each multivector of $Cl_{n,0}$ with its 2^n coordinates. It leads typically to a $O(2^{2n})$ time complexity for linear operations and for products (inner, outer and geometric), and to a storage complexity in $O(2^n)$. The *multiplicative approach*, restricted to blades, stores the coordinates of the unit orthogonal factors in a matrix and the magnitude using a scalar. Although the storage complexity is smaller than in the additive approach, there is still no available strategy for an efficient implementation of addition. Nevertheless, factorization and simplification operations allow a trivial implementation of "meet" and "join" operations.

As suggested by its name, the conformal model $Cl_{4,1}$ of $\mathbb{R}^3$ can be used to represent various angle-preserving geometries. In this model, all conformal transformations can be represented by versors, especially the ones preserving the Euclidean distances.

To define the conformal model $Cl_{4,1}$ of $\mathbb{R}^3$, two vectors e_+ and e_- are adjoined to the basis vectors e_1, e_2, e_3 of $\mathbb{R}^3$ embedded in $\mathbb{R}^5$. They are chosen to form an orthogonal basis, with ${e_+}^2 = Q(e_+) = 1$ and ${e_-}^2 = Q(e_-) = -1$. If we define respectively $n_0 = {}^1/_{\sqrt{2}}(e_- + e_+)$ and $n_\infty = {}^1/_{\sqrt{2}}(e_- - e_+)$, the new basis $\{e_1, e_2, e_3, n_0, n_\infty\}$ is not orthogonal ($< n_0, n_\infty >= -1$), but makes intuitive definitions for the model possible. The representation p of a point $p_b \in \mathbb{R}^3$ in the conformal model is defined by the following mapping:
$p = F(p_b) = p_b + n_0 + \frac{1}{2} p_b^2 n_\infty$.
The element n_0 has the same translation role as the origin vector e_0 in the homogeneous model. The vector n_∞ represents the point at infinity and the axis of symmetry of the *horosphere*, the set of elements defined by this equation. The Euclidean distance between two points $p_b, q_b \in \mathbb{R}^3$ is directly proportional to the squared root of the inner product $< F(p_b), F(q_b) >$ of their representations in the model $Cl_{4,1}$. The horosphere is formed of *null vectors*, i.e. vectors of zero norm, as consequence of the fact that $p^2 =< p, p >$ is proportional to $(p_b - p_b)^2 = 0$.

The spheres, planes, circles and lines of $\mathbb{R}^3$ can be expressed in the conformal model space $Cl_{4,1}$ with two different conditions: an inner product and an outer product.

For example, a sphere $S(a_b, r)$ centered in a_b, with radius r corresponds to:
$\tilde{S}(a_b, r) = F(a_b) - {}^1/_2 r^2 n_\infty \in Cl_{4,1}$ with $p_b \in S(a, r) \Longleftrightarrow < F(p_b), \tilde{S}(a_b, r) >= 0$.
And the sphere containing the four points $a_b, b_b, c_b, d_b \in \mathbb{R}^3$ corresponds to the element:
$S = F(a_b) \wedge F(b_b) \wedge F(c_b) \wedge F(d_b) \in Cl_{4,1}$ with $p_b \in S \Longleftrightarrow F(p_b) \wedge S = 0$.

Since any vector $x \in Cl_{4,1}$ can be written $x = F(a_b) \pm {}^1/_2 r^2 n_\infty$, for an $a_b \in \mathbb{R}^3$ and a $r \in \mathbb{R}$, the buildings blocks of $Cl_{1,4}$ are spheres, points (spheres with radius zero) and imaginary spheres (spheres with imaginary radius). The reflection in an hyperplane corresponds to a conjugation by a vector in $Cl_{4,1}$. To the other transformations, translations, rotations and scalings, correspond rotors in exponential form (e.g. $T = e^{-{}^1/_2 t_b n_\infty}$ for the translation of vector t_b). All orthonormal transformations can be expressed by rotors, since translations enjoy this property.

4.2 Geometric algebra for the visualization of vector fields

For the sake of clarity, we restrict ourselves here to 2D and 3D vector fields or vector fields defined on surfaces embedded in $\mathbb{R}^3$. The objective is to show that GA allows to perform the local analysis of the fields using differential geometry in a rather classical way, but offers more flexibility and efficiency when identifying the global structures.

With vector data defined at the vertices of a simplicial triangulation T_h or of a regular sampling (Cartesian grid), discrete equivalents of geometric and topological entities (e.g. curve, ball) are needed, as well as interpolations, giving vector values at arbitrary locations. This can be achieved in several ways but requires special attention to avoid ambiguous or non conformal situations [19].

To compute the topological index in 2D, we recast the formulation given in Section 3.3 in a discrete setting [14]. Let $B(c)$ denote a closed polygonal curve around the critical point. For every triple of neighbor vertices (p_1, p_2, p_3), form the trivector ${}^1/_6(\tilde{v}(p_1) \wedge \tilde{v}(p_2) \wedge \tilde{v}(p_3))$ with the values of the normalized vector field $\tilde{v} = {}^v/_{\|v\|}$. The sum of all trivectors, divided by the volume of the unit disk π, will give an approximation of the winding number of v on the curve, which is in turn an approximation of the index of v in c.

It can be shown that two closed polygonal curves discretizing the same underlying continuous curve lead to the same winding number, as long as they are ε-dense (i.e. any point of the continuous curve between two neighbors will be within ε-distance of both neighbors). In a continuous setting, the index of a critical point is well defined as the winding number of every circle containing this only critical point, since a non vanishing vector field v in the interior of a closed path γ implies a zero winding number of v on γ.

A similar computation can be done for 3D vector fields, on a triangulated surface around the critical point, and with the normalization factor ${}^4/_3\pi$ for the volume of the unit ball. For a vector field on a surface, the computation is less straightforward than in 2D, since vectors should be projected on a plane, before the sum is computed.

A common aforementioned technique in visualization is to integrate the vector field along a curve, the *integral line* (or streamline in a fluid). Given a Lipschitz continuous vector field v defined on an open subset $U \subset \mathbb{R}^m$, one defines curves $\gamma(t)$ on U such that for each t in an interval I, $\gamma'(t) = v(\gamma(t))$. Picard's theorem states that there exists a unique C^1-curve γ_x for each point x in U, so that $\gamma_x(0) = x$, and $\gamma_x'(t) = v(\gamma_x(t))$ for $t \in (-\varepsilon, +\varepsilon)$. These curves partition the set U into equivalent classes.

Numerically, the discretization of streamlines relies on an integration method; Euler or Runge-Kutta methods are the most common schemes to advance a point along the integral curve given its previous location and a time step δt. Any such method requires to interpolate the field vector at a new location x. The interpolation, defined on classical vector fields using barycentric coordinates, can be written exactly the same way for GA vector fields $v : \mathbb{R}^m \to Cl_{m,0}$ $(m = 2, 3)$. For example, if x is contained in a simplex then the linear interpolate reads: $v(x) = \sum_{i=1}^{l} \lambda_i v_i$, where v_i (resp. λ_i) denotes the values of v at the simplex vertices (resp. corresponding barycentric coordinates) . Note that the interpolation of a vector field v defined on a triangulated surface S is not straightforward, since the interpolated vectors need to be defined in the tangent planes.

Not every characteristic of the field lies in the topological graph: features such as vortices, shear zones, shock waves, attachment lines or surfaces are not captured in this description and are very important elements to specify the structure of a vector field. The computation methods reviewed in [24] to extract features in vector fields are presented in the classical frame but can be extended naturally to the GA frame. Several scalar fields deliver information on the presence of vortices: the vorticity magnitude, the helicity magnitude, the pressure for example. For instance, the vorticity is exactly half of the curl defined in GA.

In some specific situations, the vector field may exhibit local patterns with repetitions over the domain. Their localization would help to apprehend the overall structure of the field. For example, in 2D, we could look for the repetition of singularities like monkey saddles, zones with axis drain, or S-shaped zones. The following approach is inspired by image processing.

Correlation. Given a 2D (resp. 3D) pattern, i.e. a vector field defined on a small square (resp. cubic) domain, we can compute the Clifford correlation (introduced in 2.2.4) between this pattern and a vector field. At each point of the domain, this function gives the similarity of the vector field (in the neighborhood of this point) with the given pattern [8]. The correlation implies a convolution (quadratic complexity), which can be replaced, via Clifford Fourier Transform, by a multiplication (linear complexity) in the frequency domain. Furthermore, since the 3D CFT can be written as a sum of four complex Fourier transforms through the identification of the pseudoscalar I_3 with the imaginary unit i, Fast Fourier Transforms can be used. However, the main drawback of this method is related to the necessity to check the presence of a given pattern in all positions, for many scales and in many orientations, or the search of the pattern will not be complete.

Invariants. Suppose that we have again a particular feature (*patch*) we want to identify in a given vector field. Let us attribute values to the different patches through a mapping. Such a mapping, if it exhibits rotation, translation and scale invariance is called shortly *RTS-invariant*. Is it, for example, not rotation invariant, then its value has to be computed for all rotated variants of the patch of interest.

> A family of RTS-invariants and non redundant *moments* of order $\leq d$ [27] can be built for 2D scalar and vector fields, using the complex numbers to get a nice formulation of the rotation invariance in the equations. For 3D scalar functions, one of the ways of defining such moments is to use the spherical harmonic functions as building bricks. To extend to 3D vector fields, complex numbers are no help anymore, and quaternions generate a dimension 4 algebra. If the nice formulation of rotations in Clifford algebra and the existence of a product of vectors seems to pave the way for this generalization, the defining of building bricks (perhaps with the spherical vectorial harmonics) for the moments is the first difficulty, followed by the formulation of a rotation invariance condition. To our knowledge, the extension has not been written yet.

Several alternatives to moments as RTS-invariants are defined in literature. For example, the *harmonic power spectrum* and *harmonic bispectrum* defined in [10] for 3D vector fields rely on spherical vectorial harmonics. The theory is explained in the classical frame, using representation theory, but possibly further invariants could be defined and a substantial gain of clarity could be achieved if using GA.

Heat equation. On a Riemannian manifold M, consider the Clifford bundle obtained from the tangent bundle. The Riemannian metric $g_{ij}(p) =< e_i^p, e_j^p >$, since positive definite, leads to Euclidean tangent spaces. Let define now a connection on the manifold ∇^E compatible with the metric (for example the Levi-Cevita connection) and extend it as ∇^C to the Clifford space such that it preserves the graduation, we define a generalized Laplacian as follows:

$$\Delta^C = \sum_{ij} g_{ij}(\nabla^C_{e_i}\nabla^C_{e_j} - \sum_k \Gamma^k_{ij}\nabla^C_{e_k}) .$$

Considering the heat equation $\frac{\partial s_t}{\partial t} + \Delta^C s_t = 0$, with initial condition $s_0 = s$, associated with these operators, the solution is a regularization of the section s. It can be approximated through the convolution with the heat kernel. Varying the operators (Clifford-Hodge, Clifford-Beltrami), different flows are obtained, leading to different regularizations. This approach was introduced in [3], and was applied to reduce noise in color images (translating the RGB intensities in a surface and a vector field on this surface), but not yet, to the best of our knowledge, as a global approach tool for vector fields.
In addition to regularization, heat kernel signatures, like they are defined for scalar fields [29], could be used to define signatures of vector field patches.

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