Geometric algebra for vector field analysis and visualization: mathematical settings, overview and applications

Chantal Oberson Ausoni and Pascal Frey

Abstract The formal language of Clifford's algebras is attracting an increasingly large community of mathematicians, physicists and software developers seduced by the conciseness and the efficiency of this compelling system of mathematics. This contribution will suggest how these concepts can be used to serve the purpose of scientific visualization and more specifically to reveal the general structure of complex vector fields. We will emphasize the elegance and the ubiquitous nature of the geometric algebra approach, as well as point out the computational issues at stake.

1 Introduction

Nowadays, complex numerical simulations (e.g. in climate modelling, weather forecast, aeronautics, genomics, etc.) produce very large data sets, often several terabytes, that become almost impossible to process in a reasonable amount of time. Among other challenges, storage, transfer, analysis and visualization are the more crucial. This requires developing new methods and implementing new algorithms to efficiently process this large quantity of information. On the other hand, in mathematics or theoretical physics, problems are commonly posed in high-dimensional spaces and require specific methods to reduce their dimension and make the solutions understandable. In both cases, there is a critical need for an abstract, general purpose method of analysis capable of extracting the salient features of the com-

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plex data. Unfortunately, numerical algorithms are too often inadequate to perceive the mathematical properties or the general structure of the objects considered. In this chapter, we will explain how the formal language of geometric algebras may be one of these analysis tools, as it provides a unified framework bringing us closer by the topological aspects of geometrical problems, in a wide range of applications, including scientific visualization. The main strength of geometric algebra lies in the elegance and the generality (ubiquity) of its formulations, which can be injected within the classical Euclidean framework as well as in differential geometry. In this perspective, concepts and ideas introduced should not replace existing theories and tools, but complement them and shed new light on them.

Based on the work of Grassmann, Clifford's geometric algebras, born in the mid 19th-century, consider algebraic operators along with three main products to describe the spatial relations characteristic to geometric primitives in a coordinate-free approach. The many possibilities offered by Clifford algebras and geometric algebras (hereafter denoted GA), and especially their geometrically intuitive aspects, have been emphasized by numerous scientists. For instance, the physicist D. Hestenes has acknowledged their importance to relativistic physics [20]. Likewise, the mathematicians G.-C. Rota [17], I.R. Porteous [27] and J. Snygg [32], among others, have largely promoted the geometric compactness and simplicity of GA, hence contributing to broaden the field to further applications in computer graphics and scientific visualization.

The next section will briefly present the main concepts and the basic manipulation rules of Clifford and geometric algebras. Then, the specific case of vector fields defined on d-dimensional spaces or on differential manifolds will be addressed in Section 3. In the last section, we will show how geometric algebra can be efficiently used to understand the algebraic structure of vector fields and implemented.

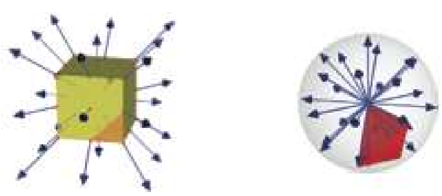

Fig. 1 Sampling a vector field over a cube (left) and summing the trivectors on the unit sphere (right), to compute an approximation of the index, see 4.2 (reprinted from [9]). Note that such a trivector is a volume in space and not a triple of vectors.

2 Clifford and geometric algebras

Leaning on the earlier concepts of Grassmann's exterior algebra and Hamilton's quaternions, Clifford intended his *geometric algebra* to describe the geometric prop-

erties of vectors, planes and eventually higher dimensional objects. Basically, Clifford algebra for $\mathbb{R}^n$ is the minimal enlargement of $\mathbb{R}^n$ to an associative algebra with unit capturing the metric, geometric and algebraic properties of Euclidean space [16]. In general, geometric algebras are distinguished from Clifford algebras by their restriction to real numbers and their emphasis on geometric interpretation and physical applications.
Note. Our intent in this section is to give an elementary and coherent account of the main concepts of Clifford and geometric algebras. The reader who is interested in the theoretical aspects of geometric algebras is referred to the textbooks [16, 20, 19], among others. Computational aspects of geometric algebra and its usability in research or engineering applications are discussed in [9, 22]. We privileged a continuous and straightforward digest, deliberately avoiding the conventional succession of definitions and theorems commonly found in most textbooks. Furthermore, most of the concepts in this section are presented in a general setting. The material in this section is intended to be fairly basic but readers unfamiliar with abstract mathematical concepts should skip the formal definition, as well as the advanced concepts in 2.1.2 and 2.2.1.

2.1 Clifford algebra

Clifford algebra can be introduced in many ways; the approach we follow here separates the algebraic structure from the geometric interpretation of the product.

2.1.1 Basic notions and definitions

Formal definition. Let V be a vector space over a field K, and let $Q : V \to K$ be a quadratic form on V. A Clifford algebra $Cl(V,Q)$ is an associative algebra over K, with identity element $\mathbb{1}$, together with a linear map $i : V \to Cl(V,Q)$ satisfying, for all $v \in V$, the *contraction rule* $i(v)^2 = Q(v)\mathbb{1}$, such that the following universal property is fulfilled [24]:

> Given any other associative algebra A over K and any linear map $j : V \to A$ such that, for all $v \in V$, $j(v)^2 = Q(v)\mathbb{1}_A$, there is a unique algebra homomorphism $f : Cl(V,Q) \to A$, for which the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\;i\;} & Cl(V,Q) \\ & {\scriptstyle j}\searrow & \downarrow{\scriptstyle f} \\ & & A \end{array}$$

Note that the existence and the uniqueness (up to unique isomorphism) of a Clifford algebra for every pair (V,Q) can be established by considering a quotient algebra of a tensor algebra.

The product defining the Clifford algebra will be called *geometric product* and denoted as: $u\,v$, for $u,v \in Cl(V,Q)$ (with a small space between the factors). One usually considers V as a linear subspace of $Cl(V,Q)$, thus dropping the inclusion in the definition of the Clifford algebra, leading $uu = u^2 = Q(u)$. Consequently, the vector space V is not closed under multiplication as, for example, uu is a scalar and not an element of V. The contraction rule also implies that every $v \in V$ has an inverse $v^{-1} = \frac{v}{Q(v)}$, unless Q is degenerate.

Intuitive interpretation of the geometric product. One can classically consider the product of two elements $a,b \in V$ as the sum of a symmetric and an antisymmetric part:

$$ab = \underbrace{\frac{1}{2}(ab+ba)}_{<a,b>} + \underbrace{\frac{1}{2}(ab-ba)}_{a \wedge b}.$$

- In this setting, the symmetric part $< a,b >$ corresponds to the bilinear form ϕ associated to the quadratic form Q thanks to the polarization identity: $\phi(a,b) = \frac{1}{2}(Q(a+b) - Q(a) - Q(b)) = \frac{1}{2}((a+b)(a+b) - aa - bb) =< a,b >$, this, of course, as a consequence of the contraction rule $v^2 = Q(v)$. When Q is non-degenerate, it is an *inner product*.
- The antisymetric part $a \wedge b$ has, if non-zero, to be understood as a new entity, that is neither a scalar nor a vector. For Q non-degenerate, the so defined *outer product* has a very simple interpretation: $a \wedge b$, for $a,b \in V$, geometrically represents an oriented plane segment, and can be characterized by an algebraic area (the usual area of the parallelogram with the vectors a and b as sides) and the *attitude* (angular position) of this plane[1].

The graded Clifford algebra. Consider again the Clifford algebra $Cl(V,Q)$, V and Q like above. We define G_0 as the inclusion of the scalars K in $Cl(V,Q)$. Given an orthonormal basis $\{e_1, e_2, \dots\}$ of V, let G_n be the part of $Cl(V,Q)$ generated from the products $\prod_{j=1}^{n} e_{i_j}$, for $1 \leq i_1 < \cdots < i_n$. The direct sum $\bigoplus_{n=0}^{\infty} G_n$ is then the *graded* Clifford algebra. The elements of G_n are called *n-vectors*, where n is the *grade*. Elements can be of “mixed grade”, like the product ab of two elements in V, which is a sum of a scalar (grade 0) and a bivector (grade 2). A *multivector* A can be decomposed as a sum $A = \sum_{r=0}^{\infty} A_r$, where $A_r = \langle A \rangle_r$ is of grade r.

Extension of the definition of outer product. The outer product of two multivectors A_k (grade k) and B_ℓ (grade l) is defined as the grade $|k+\ell|$- part of the product $A_k B_\ell$, writing $A_k \wedge B_\ell = \langle A_k B_\ell \rangle_{k+\ell}$. This product extends by linearity on the whole Clifford algebra. For any $n \leq dim(V)$, *n-blades* are defined recursively as outer products of n vectors $a_1 \wedge \cdots \wedge a_n = (a_1 \wedge \cdots \wedge a_{n-1}) \wedge a_n$. By convention, 0-blades are scalars. A n-blade is a n-vector, but the converse is not true. More precisely [15], a sum of two blades A and B is another blade iff they are of the same grade k **and** share a common factor of grade $k-1$ or k.

[1] The geometric interpretation of the decompostion of the geometric product in outer and inner products will be explained again for $V = \mathbb{R}^3$ at the beginning of 2.2.

2.1.2 Advanced concepts

Factorization of blades with the geometric product yields two equivalent forms for a blade: one based on the outer product, the other on the geometric product. Actually, for any arbitrary quadratic form Q, given a k-blade A_k, it is possible to find an orthogonal basis $\{v_1,\cdots,v_k\}$ of this blade[2]. It implies the double formulation $A_k = v_1 \wedge v_2 \wedge \cdots \wedge v_k = v_1 v_2 \cdots v_k$. For example, if $a,b \in V$, with $Q(a)$ non-zero, we have $a \wedge b = a \wedge \left(b - \frac{<a,b>}{Q(a)}a\right) = a\left(b - \frac{<a,b>}{Q(a)}a\right)$.

The *meet* and *join* are non-linear operations, corresponding to the blade intersection and union. Suppose we have an orthogonal factorization of two blades A and B, i.e., they are given with their orthogonal factorizations $A = A'C$ and $B = CB'$, C being the largest common factor. In this very simple case[3], $M = A \cap B = C$ and $J = A \cup B = (A'C) \wedge B'$.

Two important involutions are defined on $Cl(V,Q)$: *reversion* and *grade involution*. On a r-blade $A = (a_1 \wedge a_2 \wedge \cdots \wedge a_r)$, the reversion $A^\dagger$ consists of reversing the order of the constitutive vectors (or, because the outer product is antisymmetric on vectors, changing the sign $r(r-1)/2$ times); the grade involution $A^\#$ consists of reversing the sign of every constitutive vector:

$$A^\dagger = a_r \wedge a_{r-1} \wedge \cdots \wedge a_1 = (-1)^{r(r-1)/2} a_1 \wedge a_2 \wedge \cdots \wedge a_r \qquad A^\# = (-1)^r A\,.$$

The reversion and grade involution extend by linearity on $Cl(V,q)$: if $A = \sum_{r=0}^{\infty} A_r$,

$$A^\dagger = \sum_{r=0}^{\infty} (-1)^{r(r-1)/2} A_r \qquad A^\# = \sum_{r=0}^{\infty} (-1)^r A_r\,.$$

The *even* (resp. *odd*) multivectors are the ones with $A^\# = A$ (resp. $A^\# = -A$).

Using the reversion and the selection of the scalar part $\langle\cdot\rangle_0$, let us define a bilinear form on $Cl(V,Q)$. On blades A_k and B_ℓ, we set:

$$A_k * B_\ell = \begin{cases} \langle A_k^\dagger B_\ell \rangle_0 & \text{if } k = \ell \neq 0 \\ A_0 \cdot B_0 & \text{if } k = \ell = 0 \\ 0 & \text{else} \end{cases}\,.$$

Extending it linearly to multivectors A and B, we obtain the general formula $A * B = \langle A^\dagger B \rangle_0$. Proof of the equivalence between both formulations can be found in [20], p.13. On vectors, this bilinear form clearly corresponds to the inner product: $a * b = <a,b>$. When Q is non-degenerate, it is non-degenerate, and it is sometimes called *Clifford scalar product*.

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[2] A general demonstration (also valid for a degenerate Q) is given for example in [8], page 88. In Euclidean spaces, the well-known Gram-Schmidt orthogonalization can be used.

[3] The dualization introduced in 2.2 makes more general equations for M and J possible.

2.2 Geometric algebras

The case $V = \mathbb{R}^n$ and Q non-degenerate leads to a series of specific definitions and results. As a matter of fact, we have for example:

- Every non-zero blade A_r has an inverse $\frac{A_r^\dagger}{A * A} = \frac{A_r^\dagger}{\langle A^\dagger A \rangle_0}$.
- If, in addition, Q is positive definite, then we can define the modulus of element A as $|A| = \sqrt{A^\dagger * A} = \sqrt{\langle A^\dagger A \rangle_0}$, since for an element $a_1 \cdots a_r$, $(a_1 \cdots a_r)^\dagger (a_1 \cdots a_r) = Q(a_r) \cdots Q(a_1) \geq 0$.

- In $\mathbb{R}^3$, the existence of an inverse vector has a very clear interpretation. For a given vector $v \in \mathbb{R}^3$ and a given scalar a, the equation $< v, w >= a$ defines the affine plane $w_0 + v^\perp$. Likewise, given v and a bivector A, the equation $v \wedge w = A$ defines the affine line $w_0 + \lambda v$. In both cases, there is no unique solution. However, in the setting of geometric algebra, the equation $v w = A$ leads to the unique solution $w = v^{-1} A$ (corresponding to the intersection of a plane $< v, w >= A_0$ and of a line $v \wedge w = A_2$).

Such a Clifford algebra, in the case $V = \mathbb{R}^n$ and Q non-degenerate, is called *geometric algebra*. Let (p,q) be the signature of the quadratic form Q, i.e., Q diagonalizes in $Q(v) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2$ (Sylvester's law of inertia). We write $\mathbb{R}^{p,q}$ for V and $Cl_{p,q}$ for the associated geometric (Clifford) algebra.

Taking a basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$, using the element $\mathbb{1}$ to span the scalars and all products $\prod_{j=1}^{r} e_{i_j}$ for $1 \leq i_1 < \cdots < i_r \leq n$ $(r \in \mathbb{N}_n)$ to span the multivectors, the set $\{\mathbb{1}, e_1, e_2, \ldots, e_n, e_1 e_2, e_1 e_3, \ldots, e_1 e_2 \ldots e_n\}$ will form a basis for $Cl_{p,q}$, with $2^n = \sum_{r=0}^{n} \binom{n}{r}$ elements. The element $\mathrm{I}_n = e_1 e_2 \ldots e_n$ is called *pseudoscalar* and is defined to a scalar multiple, since all n-blades are proportional.

2.2.1 Duality and reciprocal frames

The dual A^* of a multivector A is defined as $A^* = A\,\mathrm{I}_n^{-1}$. The duality operation transforms a r-vector A_r into an $(n-r)$-vector $A_r\,\mathrm{I}_n^{-1}$; in particular, it maps scalars into pseudoscalars. The duality relation states $(A \wedge B)^* = A \rfloor B^*$, where $\rfloor$ denotes the left contraction [4]. The inclusion of an element x in a given subspace $\mathscr{A}$ specified by a blade A can be defined in two ways:

- the direct way: $x \in \mathscr{A} \iff x \wedge A = 0$
- the dual way: $x \in \mathscr{A} \iff x \rfloor A^* = 0$.

Given a basis $\{b_1, ..., b_n\}$ of $\mathbb{R}^{p,q}$, $n = p + q$, we can define a *reciprocal frame* $\{b^1, ..., b^n\}$, through the formula $b^i = (-1)^{i-1} (b_1 \cdots \wedge b_{i-1} \wedge \check{b}_i \wedge b_{i+1} \cdots \wedge b_n)\,\mathrm{I}_n^{-1}$, where $\mathrm{I}_n = b_1 \wedge ... \wedge b_n$ and the ˇ -sign mentions the element removed from the list.

[4] For two blades A and B of grades a and b, the left contraction $A \rfloor B$ is $\langle A B \rangle_{b-a}$ when $a \leq b$, it is zero otherwise. When blade A is contained in blade B, it equals the geometric product $A B$ [9].

The two basis are mutually orthogonal: $< b_i, b^j >= \delta^i_j$. Since the reciprocal of an orthonormal basis is itself, this definition is needed only in non-Euclidean cases. It is also useful in differential geometry.

A vector of $\mathbb{R}^{p,q}$ can be written $a = \sum_{i=1}^n a_i e^i$ or $a = \sum_{i=1}^n a^i e_i$ with $a_i =< a, e_i >$ and $a^i =< a, e^i >$. If we have a multivector basis $\{e_\alpha | \alpha \in \{1, \cdots, 2^n\}$, we can also define a reciprocal frame $\{e^\alpha | \alpha \in \{1, \cdots, 2^n\}\}$.

2.2.2 Versors, rotors, spinors and rotations

One of the main features of GA is its ability to deal with the rotations. Indeed, a unique object R can be used to compute the rotation of any subspace X, writing a conjugation with the geometric product:

$$\mathscr{R}(X) = R X R^{-1}.$$

The equation $x = axa^{-1}$ gives the reflection of an arbitrary vector x along the a-line (a invertible). Its opposite $x = -axa^{-1}$ gives the reflection in the dual hyperplane $A = a^*$. Two consecutive reflections form a simple rotation, which can be written as follows: $x" = -bx'b^{-1} = baxa^{-1}b^{-1} = (ba)x(ba)^{-1}$. It is a rotation of twice the angle between a and b in the plane containing a and b. The element ab is called a *2-versor*. In general, a *k-versor* is a multivector that can be written as the geometric product of k invertible vectors $v = v_1 v_2 ... v_k$. By the Cartan-Dieudonné Theorem [6, 7], every isometry of $\mathbb{R}^{p,q}$ can be reduced to at most $n = p + q$ reflections in hyperplanes. It means that we can write every orthogonal transformation f with a k-versor U ($k \leq n$) and the conjugation: $f(x) = (-1)^k U x U^{-1}$.

In all spaces of signatures $(n, 0)$, $(0, n)$, $(n-1, 1)$ or $(1, n-1)$, including the Euclidean spaces, every rotation can be written in exponential form[5]:

$$\mathscr{R}(x) = SxS^\dagger \text{ with } S = e^{\frac{1}{2}(i_1\theta_1 + \cdots + i_m\theta_m)}, i_1, \cdots, i_m \text{ orthogonal 2-blades}.$$

Note that a rotation of a non-Euclidean space is defined to be an orthogonal transformation of determinant one continuously connected to identity. The element S given by the exponential form of preceding equation is a *rotor*, i.e., an even versor S satisfying $SS^\dagger = 1$.

A linear map f: $V \to V$ can be extended in a function $\underline{f} : Cl(V, Q) \to Cl(V, Q)$ while preserving the outer product:

$$\underline{f}(a_1 \wedge a_2 \wedge \cdots \wedge a_r) = f(a_1) \wedge f(a_2) \wedge \cdots \wedge f(a_r).$$

It is then called an *outermorphism*. In particular, the reflection of a blade A_k in a dual hyperplane a^* is $(-1)^k a A_k a^{-1}$ and the rotation of a blade by a rotor is $RA_kR^\dagger$ according to the previous equations for vectors.

[5] Quite naturally, the exponential of a blade A is defined with the usual power series $\sum_{k=0}^\infty \frac{A^k}{k!}$. The additivity $\exp(A+B) = exp(A)\,exp(B)$ is not true in general. The circular and hyperbolic functions of blades are also defined with power series.

2.2.3 Geometric calculus

Differentiation. We consider a finite-dimensional vector space V with quadratic form Q and a multivector-valued function $F : U \subset V \longrightarrow Cl(V,Q)$. It comes of no surprise that the directional derivative of F in direction r is simply:

$$F_r(x) = \lim_{s\to 0} \frac{F(x+s\,r)-F(x)}{s}\,.$$

This expression will be most of the time written $(r*\nabla)F$ instead of F_r, expressing the idea of a scalar product between r and the operator ∇, seen as a vector, as will be clearer below. The linearity in r is straightforward; the sum, the geometric product and the grade are preserved. If we want to differentiate a multivector-valued function $F : U \subset V \longrightarrow Cl(V,Q)$ directly relative to the variable, we consider a base $\{e_1,\cdots,e_m\}$ of V and the coordinate functions of the vector x in this basis $x = \sum_{i=1}^n x^i e_i$. The directional derivatives along the basis directions, $(e_i*\nabla) = \frac{\partial}{\partial_{x_i}}$, combine into a total change operator [6] as:

$$\nabla = \sum_{i=1}^m e^i\,(e_i*\nabla) \quad \text{meaning} \quad \nabla F(x) = \sum_{i=1}^m e^i\,\frac{\partial F(x)}{\partial_{x_i}}\,.$$

Note that we also have to define the differentiation from the right, because of the non-commutativity: for a function F, $F(x)\nabla = \sum_{i=1}^m (e_i*\nabla)F(x)\,e^i = \sum_{i=1}^m \frac{\partial F(x)}{\partial_{x_i}}\,e^i$. Thanks to the geometric product, we can write ∇ as: $\nabla F = \nabla\wedge F + <\nabla, F>$. In the case of a vector-valued function F, we have the usual definitions of the divergence and curl operators:

$$\mathrm{curl}(F) := \nabla\wedge F = \frac{1}{2}(\nabla F - F\nabla) \quad \text{and} \quad \mathrm{div}(F) := <\nabla, F> = \frac{1}{2}(\nabla F + F\nabla)\,.$$

To write the product rule, accents are necessary to specify on what factor the differentiation acts: $\nabla(F\,G) = \grave{\nabla}\grave{F}\,G + \grave{\nabla}F\,\grave{G}$. The definition of a differentiation with respect to a multivector, for a function $F : U \subset Cl(V,Q) \longrightarrow Cl(V,Q)$, is quite straightforward, given a reciprocal frame for the whole space $Cl(V,Q)$.

Integration. Consider again a multivector-valued function F; the line integral is

$$\int_C F(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^n \overline{F}^j \Delta x^j\,, \quad \text{with} \quad \overline{F}^j = \frac{1}{2}\left(F(x_i) + F(x_{i-1})\right)$$

where the chords $\Delta x^i = x_i - x_{i-1}$ correspond to a subdivision of the curve C. The measure dx is said to be a *directed measure*, since it is vector-valued. The product between $F(x)$ and dx is the geometric product. If F is vector-valued,

$$\int_C F(x)\,dx = \int_C <F(x), dx> + \int_C F(x)\wedge dx\,.$$

[6] This explains the notation $F_r = (r*\nabla)F$ for the directional differentiation.

Similarly, if $D \subset \mathbb{R}^2$ is a triangulated planar domain, $\overline{F}^k$ is the average of F over the k-th simplex,

$$\int_D F(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \overline{F}^k \Delta x^k\,.$$

The surface measure of the k-th simplex given by vertices x_0, x_1, x_2 is

$$\Delta x^k = \frac{1}{2}\left(x_0 \wedge x_1 + x_2 \wedge x_0 + x_1 \wedge x_2\right).$$

This integral definition can be generalized to higher dimensions [5].

The fundamental theorem states:

$$\oint_{\partial V} F\,dS = \int_V \grave{F}\,\grave{\nabla}\,dX\,, \quad \text{for a function } F \text{ defined over a volume } V\,.$$

2.2.4 Clifford Convolution and Clifford Fourier Transform

For F and H two multivector-valued functions $F,H : \mathbb{R}^m \to Cl_{p,q}$, the left- and the right-Clifford Convolution of the functions write respectively:

$$(H *_\ell F)(x) = \int_{\mathbb{R}^m} H(x')F(x-x')|dx'|\,, \quad (H *_r F)(x) = \int_{\mathbb{R}^m} H(x-x')F(x')|dx'|\,.$$

The quantity $|dx|$ is used to make the integral grade-preserving since dx is a vector within Clifford algebra. Modifying $x-x'$ into $x+x'$, we get the left- and right- *Clifford correlations* [10]. The Clifford convolutions generalize the known convolution of scalar-valued functions.

A vector field F can be smoothed through convolution with a scalar field, for example a Gaussian kernel. In the case of two vector fields, the formula for the geometric product leads to the integration of a scalar function $< H(x-x'), F(x') >$ and a bivector function $H(x-x') \wedge F(x')$ [29].

In the case of a multivector-valued function $F : \mathbb{R}^3 \to Cl_{3,0}$, it is possible to define the *Clifford Fourier Transform* (CFT) of F and its inverse as follows:

$$\mathscr{F}\{F\}(u) = \int_{\mathbb{R}^3} F(x)\, e^{-2\pi \mathrm{I}_3 <x,u>}|dx|\,, \quad \mathscr{F}^{-1}\{F\}(x) = \int_{\mathbb{R}^3} F(u) e^{2\pi \mathrm{I}_3 <x,u>}|du|\,.$$

The function $e^{-2\pi \mathrm{I}_3 <x,u>} = \cos(2\pi < x,u >) + \mathrm{I}_3 \sin(2\pi < x,u >)$ is often called *Clifford Fourier kernel*.

The convolution theorem is also valid for the Clifford Fourier Transform and Clifford convolutions as defined here. For example, using the left convolution,

$$\mathscr{F}\{H *_\ell F\}(u) := \mathscr{F}\{H\}(u)\,\mathscr{F}\{F\}(u)\,.$$

As mentioned before, the reader willing to get a deeper understanding of the mathematical basics about Clifford algebras and geometric algebras is referred to [16, 20, 19]. In the next section, we will focus on the analysis of vector fields in the context of GA.

3 Vector fields in geometric algebra

Our main focus in this paper is the analysis of vector fields, more precisely of steady, linear and non-linear vector fields in Euclidean space and on manifolds. One is classically interested in streamlines, critical points with their classification, separatrices, leading to the topological graph of a vector field. We will show how the analysis of vector fields can benefit from the richer context of geometric algebra.

3.1 Vector fields on domains of Euclidean space

Classically, vector fields are mappings of the form $v : U \subset \mathbb{R}^n \to \mathbb{R}^n$, where U is an open set. Using the notions of GA defined in Section 2.2, if $Cl^1_{n,0}$ is the set of 1-vectors of $Cl_{n,0}$, a map $v : \mathbb{R}^n \longrightarrow Cl^1_{n,0}$ is also a vector field. This definition can be easily extended to bivector, trivector, or spinor fields, for example.

The identification of vector fields (satisfying the Lipschitz regularity condition) with ordinary differential equations ${}^{dx}/_{dt} = v(x)$ can also be transposed from the classical to the GA setting. The Lipschitz continuity condition can be written in this frame, i.e., there exists a scalar constant $K \geq 0$ such that

$$\|v(x_1) - v(x_2))\|_n \leq K\|x_1 - x_2\|_n \quad \text{for all} \quad x_1, x_2 \in U\,.$$

Furthermore, the defined derivation and integration make it possible to state the existence of an unique solution (*streamline* or *integral curve*) through a given point, exactly like in the classical frame. In 2D and 3D, drawing the streamlines is a very classical way to represent a vector field. In order to avoid occlusions and empty areas, departure points (*seeds*) for these curves are to be placed efficiently.

Let us consider a small example. To a given classical vector field, we can associate curvature and torsion scalar fields: the curvature (resp. torsion) in a point is the curvature (resp. torsion) of the unique streamline in this point [35]. The curvature field associated to a vector field can be used for the seeding, or can be displayed as a further scalar value in the form of isosurfaces or by color coding. In the GA settings, instead of scalar fields, a curvature bivector field and a torsion trivector field can be defined. Visualizing the curvature bivector along a streamline, we get what is called the *Frenet ribbon* [4], see figure 2 for such a representation of the vector field.

3.2 Vector fields on differential manifolds

Now we turn to vector fields on differential manifolds, having in mind to embed the differential geometry formalism into geometric calculus. For a more detailed presentation of this combined approach, see [20, 32].

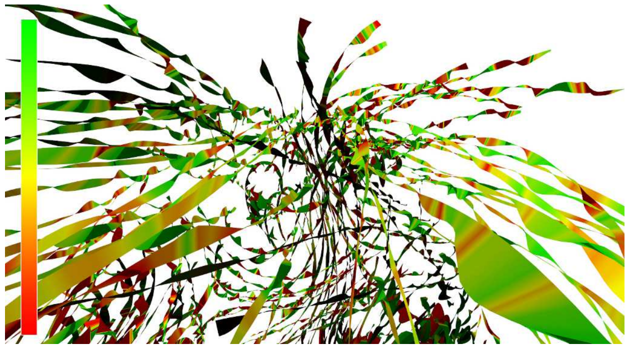

Fig. 2 Frenet ribbons constructed from a discrete vector field. The colour encodes the torsion (reprinted from [4]).

In differential geometry, if M is a smooth manifold, a tangent vector in $p \in M$ is a derivation, i.e., a linear operator D on C_p^∞ (the algebra of germs of smooth functions at p) satisfying the Leibniz rule $D(f \cdot g) = f \cdot D(g) + g \cdot D(f)$. The tangent space of M in p is T_pM, the set of such derivations. A vector field is a function assigning to every $p \in M$ an element of T_pM.

In $\mathbb{R}^3$, in a more intuitive way, we can imagine giving in each point p of a surface S a vector tangent $v(p)$ to the surface. The link between this v and the associated derivation D_v is the derivative $D_v(f)(p) = Df(p)(v(p))$. The operator point of view makes it easier to manipulate vector fields and compose them with other operators. Furthermore a discretization can be made without working with coordinates [2].

To translate this definition into GA, we give the tangent spaces a Clifford algebra structure. Taking a chart (U, ϕ) around $p \in M$, the derivations e_i^p defined by

$$e_i^p(f) = \frac{\partial}{\partial x_i}(f \circ \phi^{-1})|_{x=\phi(p)}$$

form a basis for T_pM. Forming the blades of these basis vectors, we can build a geometric algebra structure on T_pM.

With a little more abstraction, a vector field can classically be seen as a *section* of the *tangent bundle*, a particular *vector bundle*: Taking TM to be the disjoint union of tangent spaces on M, $TM = \sqcup_{x \in M} T_xM$, and $\pi : TM \to M$ defined by $\pi(v) = x$ for $x \in T_xM$, we can see M as the base space, TM as the total space and π as the projection, these three elements defining a fibre bundle called the tangent bundle. The section is a continuous map s with $\pi \circ s = id_M$, meaning $s(x) \in T_xM$, hence what we understand as a vector field. The adding of a geometric algebra structure can be done in the general case of a vector bundle on a manifold with some metrics, using a construction very similar as the one made in 2.1.1: quotienting a tensor algebra with a two-sided ideal.

Scalar fields, vector fields, bivector fields, spinor fields on surfaces, for example, are natural extensions of this definition of vector fields (or can be seen as sections of the Clifford tangent bundle, see above), and, as long as M is simply connected, it is also the case for rotation fields $r : M \to SO(n)$, since they can be lifted to spinor fields.

Since every differentiable manifold is locally diffeomorphic to an Euclidean space (via the charts), the existence and uniqueness of streamlines is also granted on manifolds, within or outside GA context.

3.3 Critical points, orbits and topological graph

The topological graph is an important tool of analysis: it goes one step further than the streamline representation and decomposes the vector field domain into regions of similar behavior. The critical points and closed orbits (with their type, like defined below) and the separatrices (streamlines or surfaces between areas of different behavior) form the topological graph of the vector field, that eventually describes the underlying structure of the field in a more efficient way as a collection of streamlines. Such a graph does not take into account the norm of the vector field [12].

The classification of critical points finds its origin in the theory of dynamical systems. For regular critical points, i.e., for critical points with an invertible Jacobian matrix, a linear approximation of the field can be considered. Studying eigenvalues and eigenvectors of the Jacobian matrix makes the classification possible, provided none of the eigenvalues is pure imaginary. The so-called hyperbolic critical points, satisfying this condition, are isolated and are structurally stable: a small local perturbation does not modify the topology. This justifies the use of the linear approximation to describe the field's behavior around this point. In two dimensions for example, the hyperbolic critical points are sources, sinks, saddles and spirals. Unstable critical points are centers. A similar classification can be done for orbits, according to the derivative of the Poincaré map [1]. For non-linear critical points, said to be of *higher order*, the non-invertibility of the first derivative leads one to consider a higher order Taylor expansion. For the isolated ones, the index[7] can help discriminate critical points of different types. Sometimes this proves insufficient, since two critical points with same index can be of different types. The GA formalism provides an elegant alternative for the computation of the index: for example,

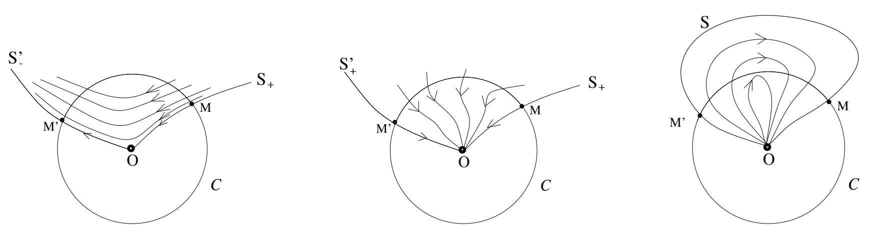

Fig. 3 Classification of sectors: hyperbolic, parabolic and elliptic sectors (reprinted from [30]).

[7] In 2D, the index corresponds the number of turns the field makes around a critical point.

in 3D,

$$ind(c) = \frac{1}{8\pi I_3} \int_{B(c)} \frac{v \wedge dv}{|v|^3},$$

for v the vector field, c the critical point, $B(c)$ an arbitrary small ball around c [25]. A corresponding discrete computation will be introduced in 4.2. Unlike the index, the ordered list of all different behavior sectors (i.e., *elliptic*, *hyperbolic* and *parabolic sectors*) makes an unambiguous classification possible [30, 18, 14, 34] (see Fig 3).

Next, we turn to a more practical view of geometric algebras, as this chapter is also intended for engineers and practitioners. In particular, we will briefly explain how GA can be implemented and the potential advantages of using Clifford algebra when, for example, dealing with rotations in spaces of high dimensions.

4 Geometric algebra for computer graphics and visualization of vector fields

Nowadays, geometric algebra is mostly recognized as a promising mathematical concept and is beginning to find broader application. Emerging computer architectures (multicore, many-core, parallel) lead us to believe that the language of GA may find a new playground and evolve towards what Hildenbrand calls Geometric Algebra Computing (GAC) [21]. However, GA is not yet a widespread method in engineering applications, mainly because of two reasons, academic and practical [26]. On the one hand, GA combines many mathematical concepts that were developed separately over the years and are taught as such in curriculum. On the other hand, most engineering applications in three-dimensional space can be dealt using standard vector and matrix algebra tools. The goal of this last section is to introduce how GA can be used advantageously in computer graphics applications and vector field analysis and visualization.

4.1 Geometric algebra for computer graphics

Computer graphics is surely the most obvious field of application of GA. In geometrical applications, operations and transformations are applied on primitives that are combined to represent an object (model). Linear geometric transformations are usually represented using matrices, vectors and scalars. But while 3×3 matrices encode the 3D-rotations about an axis through the origin, quaternions are better suited instead, because they are easier to interpret.

The quaternion representation of a rotation is a nearly minimal parametrization that requires only four scalars. Given a quaternion, one can easily read off the axis and angle of the rotation, it is not the case with the Euler angles representation. The composition of rotations in quaternion form is faster and more stable numerically (the renormalization is more efficient than with matrices). Furthermore, the interpolation in the set of quaternions $\mathbb{H}$ (for example to get an animated view of a rotated object) consists in defining a path on S^3 which is mapped to $SO(3)$. The Euler angles parametrization, from the 3-torus to $SO(3)$ is not a global diffeomorphism: the uniqueness breaks at some points (problem known as the *gimbal lock*). This is why, in graphic libraries such as OpenGL, rotations are given in terms of a rotation axis and a rotation angle and converted internally into rotation matrices.

Note that $\mathbb{H}$ forms a subalgebra of a geometric algebra [27]: it is trivially isomorphic to the even algebra $Cl^+_{3,0}$ (the set of even multivectors of $Cl_{3,0}$). We can identify the unit and the basis elements $\mathbf{i},\mathbf{j},\mathbf{k}$ of $\mathbb{H}$ with the unit and the products $e_1\,e_2$, $e_1\,e_3$ and $e_2\,e_3$ of $Cl_{3,0}$ [19]. Thus, the aforementioned quaternion representation of a rotation (of angle θ around unit axis u):

$$q = e^{\frac{1}{2}\theta(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\tfrac{1}{2}\theta + (u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})\sin\tfrac{1}{2}\theta \quad \text{with} \quad \mathscr{R}(x) = qxq^{-1}\,,$$

can be seen as a rotor of $Cl_{3,0}$. The products and the conjugation on $\mathbb{H}$ and on $Cl^+_{3,0}$ are defined likewise. Similarly, identifying the pseudoscalar I of $Cl_{2,0}$ with the imaginary unit i of $\mathbb{C}$, we have an algebra isomorphism between $Cl^+_{2,0}$ and $\mathbb{C}$.

Clearly, geometric algebra exhibits structural simplicity in the formulations, but its naive implementation may be far less efficient than classical analytical geometry implementations, especially for high dimensions. Fortunately, GA expressions can benefit from compilation (e.g. operator overloading) and parallelization techniques (including GPUs) [15, 21].

In practice, GA has been implemented in two ways. The *additive approach* encodes each multivector of $Cl_{n,0}$ with its 2^n coordinates. It leads typically to a $O(2^{2n})$ time complexity for linear operations and for products (inner, outer and geometric), and to a storage complexity in $O(2^n)$. The *multiplicative approach*, restricted to blades, stores the coordinates of the unit orthogonal factors in a matrix and the magnitude using a scalar. Although the storage complexity is smaller than in the additive approach, there is still no available strategy for an efficient implementation of addition. Nevertheless, factorization and simplification operations allow a trivial implementation of "meet" and "join" operations.

As suggested by its name, the conformal model $Cl_{4,1}$ of $\mathbb{R}^3$ can be used to represent various angle-preserving geometries. In this model, all conformal transformations can be represented by versors, especially the ones preserving the Euclidean distances.

To define the conformal model $Cl_{4,1}$ of $\mathbb{R}^3$, two vectors e_+ and e_- are adjoined to the basis vectors e_1,e_2,e_3 of $\mathbb{R}^3$ embedded in $\mathbb{R}^5$. They are chosen to form an orthogonal basis, with ${e_+}^2 = Q(e_+) = 1$ and ${e_-}^2 = Q(e_-) = -1$. If we define respectively $n_0 = {}^1/_{\sqrt{2}}(e_- + e_+)$ and $n_\infty = {}^1/_{\sqrt{2}}(e_- - e_+)$, the new basis $\{e_1,e_2,e_3,n_0,n_\infty\}$ is not orthogonal ($< n_0, n_\infty >= -1$), but makes intuitive definitions for the model possible. The representation p of a point $p_b \in \mathbb{R}^3$ in the conformal model is defined by the following mapping:

$$p = F(p_b) = p_b + n_0 + \tfrac{1}{2}\,p_b^2 n_\infty\,.$$

The element n_0 has the same translation role as the origin vector e_0 in the homogeneous model. The vector n_∞ represents the point at infinity and the axis of symmetry of the *horosphere*, the set of elements defined by this equation. The Euclidean distance between two points $p_b, q_b \in \mathbb{R}^3$ is directly proportional to the squared root of the inner product $< F(p_b), F(q_b) >$ of their representations in the model $Cl_{4,1}$. The horosphere is formed of *null vectors*, i.e., vectors of zero norm, as consequence of the fact that $p^2 =< p, p >$ is proportional to $(p_b - p_b)^2 = 0$.

The spheres, planes, circles and lines of $\mathbb{R}^3$ can be expressed in the conformal model space $Cl_{4,1}$ with two different conditions, using the inner or the outer product. For the example of the sphere:

- A sphere $S(a_b, r)$ centered in a_b, with radius r corresponds to: $\tilde{S}(a_b, r) = F(a_b) - {}^1\!/_2 r^2 n_\infty \in Cl_{4,1}$ with $p_b \in S(a, r) \Longleftrightarrow < F(p_b), \tilde{S}(a_b, r) >= 0$.
- The sphere containing the four points $a_b, b_b, c_b, d_b \in \mathbb{R}^3$ corresponds to the element: $S = F(a_b) \wedge F(b_b) \wedge F(c_b) \wedge F(d_b) \in Cl_{4,1}$ with $p_b \in S \Longleftrightarrow F(p_b) \wedge S = 0$.

Since any vector $x \in Cl_{4,1}$ can be written $x = F(a_b) \pm {}^1\!/_2 r^2 n_\infty$, for an $a_b \in \mathbb{R}^3$ and a $r \in \mathbb{R}$, the building blocks of $Cl_{1,4}$ are spheres, points (spheres with radius zero) and imaginary spheres (spheres with imaginary radius). The reflection in an hyperplane corresponds to a conjugation by a vector in $Cl_{4,1}$. To the other transformations, translations, rotations and scalings, correspond rotors in exponential form (e.g. $T = e^{-{}^1\!/_2 t_b n_\infty}$ for the translation of vector t_b). All orthonormal transformations can be expressed by rotors, since translations enjoy this property.

4.2 Geometric algebra for the visualization of vector fields

For the sake of clarity, we restrict ourselves here to 2D and 3D vector fields or vector fields defined on surfaces embedded in $\mathbb{R}^3$. The objective is to show that GA allows one to perform the local analysis of the fields using differential geometry in a rather classical way, but offers more flexibility and efficiency when identifying the global structures.

With vector data defined at the vertices of a simplicial triangulation T_h or of a regular sampling (Cartesian grid), discrete equivalents of geometric and topological entities (e.g. curve, ball) are needed, as well as interpolations, giving vector values at arbitrary locations. This can be achieved in several ways but requires special attention to avoid ambiguous or non-conformal situations [23].

To compute the topological index in 2D, we recast the formulation given in Section 3.3 in a discrete setting [18]. Let $B(c)$ denote a closed polygonal curve around the critical point. For every couple of neighbor vertices (p_1, p_2), form the bivector ${}^1\!/_2(\tilde{v}(p_1) \wedge \tilde{v}(p_2))$ with the values of the normalized vector field $\tilde{v} = {}^v\!/_{\|v\|}$. The sum of all bivectors, divided by the volume of the unit disk π, will give an approximation of the winding number of v on the curve, which is in turn an approximation of the index of v in c.

It can be shown that two closed polygonal curves discretizing the same underlying continuous curve lead to the same winding number, as long as they are ε-dense (i.e., any point of the continuous curve between two neighbors will be within ε-distance of both neighbors). In a continuous setting, the index of a critical point is well defined as the winding number of every circle containing this only critical point, since a nonvanishing vector field v in the interior of a closed path γ implies a zero winding number of v on γ.

A similar computation can be done for 3D vector fields, on a triangulated surface around the critical point (see figure 1). For a triangle of neighbor vertices (p_1, p_2, p_3) on this surface, form the trivector $^1/_6(\tilde{v}(p_1) \wedge \tilde{v}(p_2) \wedge \tilde{v}(p_3))$ with the values of the normalized vector field. The normalization factor is $\frac{4}{3}\pi$ for the volume of the unit ball [9, 25]. For a vector field on a surface, the computation is less straightforward than in 2D, since vectors should be projected on a plane, before the sum is computed.

A common aforementioned technique in visualization is to integrate the vector field along a curve, the *integral line* (or streamline in a fluid). Given a Lipschitz continuous vector field v defined on an open subset $U \subset \mathbb{R}^m$, one defines curves $\gamma(t)$ on U such that for each t in an interval I, $\gamma'(t) = v(\gamma(t))$. Picard's theorem states that there exists a unique C^1-curve γ_x for each point x in U, so that $\gamma_x(0) = x$, and $\gamma_x'(t) = v(\gamma_x(t))$ for $t \in (-\varepsilon, +\varepsilon)$. These curves partition the set U into equivalent classes.

Numerically, the discretization of streamlines relies on an integration method; Euler or Runge-Kutta methods are the most common schemes to advance a point along the integral curve given its previous location and a time step δt. Any such method requires to interpolate the field vector at a new location x. The interpolation, defined on classical vector fields using barycentric coordinates, can be written exactly the same way for GA vector fields $v : \mathbb{R}^m \to Cl_{m,0}$ $(m = 2,3)$. For example, if x is contained in a simplex then the linear interpolate reads: $v(x) = \sum_{i=1}^{l} \lambda_i v_i$, where v_i (resp. λ_i) denotes the values of v at the simplex vertices (resp. corresponding barycentric coordinates). Note that the interpolation of a vector field v defined on a triangulated surface S is not straightforward, since the interpolated vectors need to be defined in the tangent planes.

Not every characteristic of the field lies in the topological graph: features such as vortices, shear zones, shock waves, attachment lines or surfaces are not captured in this description and are very important elements to specify the structure of a vector field. The computation methods reviewed in [28] to extract features in vector fields are presented in the classical frame but can be extended naturally to the GA frame. Several scalar fields deliver information on the presence of vortices: the vorticity magnitude, the helicity magnitude, the pressure for example. For instance, the vorticity is exactly half of the curl defined in GA.

In some specific situations, the vector field may exhibit local patterns with repetitions over the domain. Their localization would help to apprehend the overall structure of the field. For example, in 2D, we could look for the repetition of singularities like monkey saddles, zones with axis drain, or S-shaped zones. The following approach is inspired by image processing.

Correlation. Given a 2D (resp. 3D) pattern, i.e., a vector field defined on a small square (resp. cubic) domain, we can compute the Clifford correlation (introduced in 2.2.4) between this pattern and a vector field. At each point of the domain, this function gives the similarity of the vector field (in the neighborhood of this point) with the given pattern [11]. The correlation implies a convolution (quadratic complexity), which can be replaced, via Clifford Fourier Transform, by a multiplication (linear complexity) in the frequency domain. Furthermore, since the 3D CFT can be written as a sum of four complex Fourier transforms through the identification of the pseudoscalar I_3 with the imaginary unit i, Fast Fourier Transforms can be used. However, the main drawback of this method is related to the necessity to check the presence of a given pattern in all positions, for many scales and in many orientations, or the search of the pattern will not be complete.

Invariants. Suppose that we have again a particular feature (*patch*) we want to identify in a given vector field. Let us attribute values to the different patches through a mapping. Such a mapping, if it exhibits rotation, translation and scale invariance is called shortly *RTS-invariant*. If it is, for example, not rotation invariant, then its value has to be computed for all rotated variants of the patch of interest.

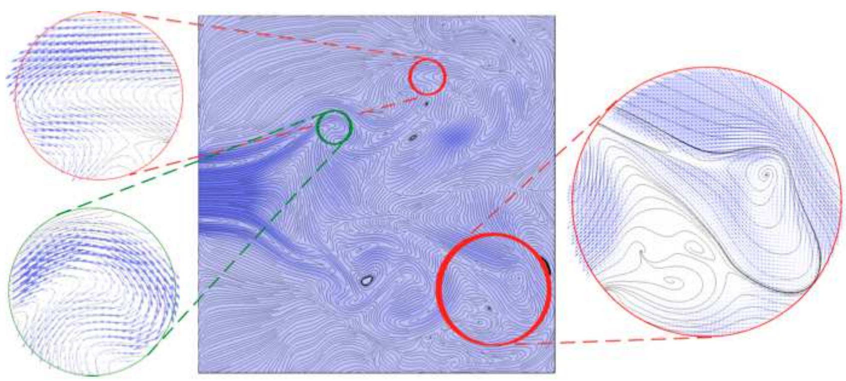

Fig. 4 Search for an S-shaped pattern in a 2D swirling jet flow dataset. The original pattern is shown in a green circle, whereas the found occurences are shown in red circles, overlapping for different scales. The method used is the comparison of computed values for a family of moment invariants on the dataset, comparing with the tabulated values for the pattern (reprinted from [31]).

A family of RTS-invariants and non-redundant *moments* of order $\leq d$ [31] can be built for 2D scalar and vector fields, using the complex numbers to get a nice formulation of the rotation invariance in the equations. On figure 4, showing a 2D swirling jet flow dataset, the occurences of a given S-shaped pattern can be seen, as obtained by this method. For 3D scalar functions, one of the ways of defining such moments is to use the spherical harmonic functions as building bricks. To extend to 3D vector fields, complex numbers are no help anymore, and quaternions generate a dimension 4 algebra. If the nice formulation of rotations in Clifford algebra and the existence of a product of vectors seems to pave the way for this generalization, the defining of building bricks (perhaps with the spherical vectorial harmonics) for the moments is the first difficulty, followed by the formulation of a rotation invariance condition. To our knowledge, the extension has not been written yet.

Several alternatives to moments as RTS-invariants are defined in literature. For example, the *harmonic power spectrum* and *harmonic bispectrum* defined in [13] for 3D vector fields rely on spherical vectorial harmonics. The theory is explained in the classical frame, using representation theory, but possibly further invariants could be defined and a substantial gain of clarity could be achieved if using GA.

Heat equation. On a Riemannian manifold M, consider the Clifford bundle obtained from the tangent bundle. The Riemannian metric $g_{ij}(p) =< e_i^p, e_j^p >$, since positive definite, leads to Euclidean tangent spaces. Let us define now a connection on the manifold ∇^E compatible with the metric (for example the Levi-Cevita connection) and extend it as ∇^C to the Clifford space such that it preserves the graduation, we define a generalized Laplacian as follows:

$$\Delta^C = \sum_{ij} g_{ij} (\nabla^C_{e_i} \nabla^C_{e_j} - \sum_k \Gamma^k_{ij} \nabla^C_{e_k}) \,.$$

Considering the heat equation $\frac{\partial s_t}{\partial t} + \Delta^C s_t = 0$, with initial condition $s_0 = s$, associated with these operators, the solution is a regularization of the section s. It can be approximated through the convolution with the heat kernel. Varying the operators (Clifford-Hodge, Clifford-Beltrami), different flows are obtained, leading to different regularizations. This approach was introduced in [3], and was applied to reducing noise in color images, see figure 5, but not yet, to the best of our knowledge, as a global approach tool for vector fields.

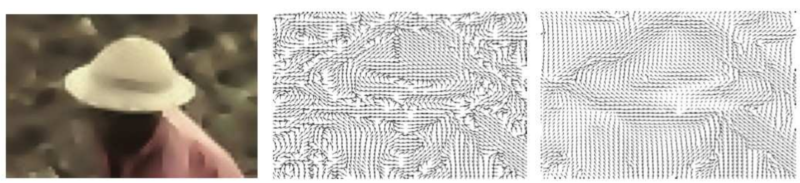

Fig. 5 A color image (left) with the corresponding unit vector field of edge orientations (middle) and a Clifford-Beltrami regularization of this vector field (reprinted from [3]).

In addition to regularization, heat kernel signatures, like they are defined for scalar fields [33], could be used to define signatures of vector field patches.

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