Hydrodynamical form for the one-dimensional Gross-Pitaevskii equation

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Abstract

We establish a well-posedness result for the hydrodynamical form (HGP) of the one dimensional Gross-Pitaevskii equation (GP) via the classical form of this equation. The result established in this way proves that (HGP) is locally well-posed since the solution of (GP) can be vanished at some $t \neq 0$.

Key words: Non-linear Schrödinger equation, Gross-Pitaevskii equation.
Classification code: 35C07, 35C08.

1 Introduction

In this paper, we focus on the equation

$$\begin{cases} \partial_t \eta = 2\partial_x((1-\eta)v), \\ \partial_t v = \partial_x \left(\eta - v^2 - \frac{\partial_x \eta}{2(1-\eta)} + \frac{(\partial_x \eta)^2}{4(1-\eta)^2} \right), \end{cases} \tag{HGP}$$

where $(\eta, v) : (I \subset \mathbb{R}) \times \mathbb{R} \to \mathbb{R}^2$, with the condition

$$\max_{x \in \mathbb{R}} \eta(t,x) < 1, \quad \forall t \in I. \tag{1.1}$$

This equation forms some variant of the one-dimensional Gross-Pitaevskii equation

$$i\partial_t \Psi + \partial_x^2 \Psi + (1 - |\Psi|^2)\Psi = 0. \tag{GP}$$

Indeed, any non-vanishing solution Ψ to (GP), can be written, at least formally, as

$$\Psi = |\Psi| \exp(i\varphi).$$

Then the two functions $\eta = 1 - |\Psi|^2$ and $v = \partial_x \varphi$ are solutions to (HGP). Our goal is to establish non-formal links between these two formulations.

We begin by defining the spaces of resolution for (GP) and (HGP), and by establishing a link between these two spaces. Let $k \in \mathbb{N}$. We define the space E^k by

$$E^k = \left\{ u \in L^\infty(\mathbb{R}, \mathbb{C}), \quad \text{tel que} \quad 1 - |u|^2 \in L^2(\mathbb{R}) \quad \text{et} \quad u' \in H^k(\mathbb{R}) \right\}.$$

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We endow the space E^k with two distances

$$d^k(u_1,u_2) = \|u_1-u_2\|_{L^\infty(\mathbb{R})} + \||u_1|^2-|u_2|^2\|_{L^2(\mathbb{R})} + \sum_{j=1}^{k+1} \|(u_1-u_2)^{(j)}\|_{L^2(\mathbb{R})},$$

$$d^k_{loc}(u_1,u_2) = \|u_1-u_2\|_{L^\infty(-1,1)} + \||u_1|^2-|u_2|^2\|_{L^2(\mathbb{R})} + \sum_{j=1}^{k+1} \|(u_1-u_2)^{(j)}\|_{L^2(\mathbb{R})}.$$

We also define the space $X^k(\mathbb{R}) = H^{k+1}(\mathbb{R}) \times H^k(\mathbb{R})$, equipped with the norm

$$\|(\eta,v)\|^2_{X^k} = \|\eta\|^2_{H^{k+1}} + \|v\|^2_{H^k}.$$

Consider the subset

$$\mathcal{NV}^k(\mathbb{R}) = \{(\eta,v)\in X^k(\mathbb{R}), \quad \max_{x\in\mathbb{R}} \eta(x) < 1\} \subset X^k(\mathbb{R})$$

and

$$A^k := \{u \in E^k \quad \text{t.q.} \quad u(x)\neq 0 \ \forall x \in \mathbb{R}\} \subset E^k.$$

Then we establish the following result

Proposition 1.1 *The application*

$$\begin{array}{llll} \Phi: & \mathcal{NV}^k(\mathbb{R})\times \mathbb{R}/(2\pi\mathbb{Z}) & \longrightarrow & A^k \\ & ((\eta,v),\theta) & \longmapsto & u(x) := \sqrt{1-\eta(x)}\exp\left(i(\theta+\int_0^x v(s)\,ds)\right) \end{array}$$

is a bijection whose inverse is given by

$$\begin{array}{llll} \Phi^{-1}: & A^k & \longrightarrow & \mathcal{NV}^k(\mathbb{R})\times \mathbb{R}/(2\pi\mathbb{Z}) \\ & u & \longmapsto & ((\eta(x),v(x)),\theta) := ((1-|u(x)|^2, \langle \frac{i}{\bar{u}(x)}, u'(x)\rangle_{\mathbb{C}}), \arg(u(0))). \end{array}$$

The application Φ is continuous if we provide A^k with the metric d^k_{loc}, but it is not continuous if we provide A^k with the metric d^k. The application Φ^{-1} is locally Lipschitz-continuous if we provide A^k with the metric d^k, but it is not so if we provide A^k with the metric d^k_{loc}.

We define the Ginzburg-Landau energy for (HGP) by

$$H(\eta,v) = \frac{1}{8}\int_{\mathbb{R}} \frac{(\partial_x\eta)^2}{1-\eta} + \frac{1}{2}\int_{\mathbb{R}} (1-\eta)v^2 + \frac{1}{4}\int_{\mathbb{R}}\eta^2,$$

and the momentum

$$P(\eta,v) = \frac{1}{2}\int_{\mathbb{R}} \eta v.$$

These two quantities are well defined on $\mathcal{NV}^0(\mathbb{R})$.

Our main result is the next theorem.

Theorem 1.2 *Let $k\in\mathbb{N}$ and $(\eta_0,v_0)\in\mathcal{NV}^k(\mathbb{R})$. There exist maximal $T_*, T^* > 0$ and a unique solution $(\eta,v)\in\mathcal{C}(-T_*,T^*,\mathcal{NV}^k)$ to equation (HGP) such that $(\eta(0,.),v(0,.)) = (\eta_0,v_0)$. Moreover, T_*, T^* are characterized by*

$$\lim_{t\to T^*}\max_{x\in\mathbb{R}} \eta(t,x) = 1, \quad \textit{when} \quad T^* < +\infty,$$

$$\lim_{t\to -T_*}\max_{x\in\mathbb{R}} \eta(t,x) = 1, \quad \textit{when} \quad -T_* > -\infty.$$

For all $t\in]T_*,T^*[$, the application $(\eta_0,v_0)\mapsto(\eta(t,.),v(t,.))$ is continuous from $\mathcal{NV}^0(\mathbb{R})$ to itself. The energy H and the momentum P are constant along the flow.

1.1 Motivation

A stability result, in the energy space, for sums of solitons of equation (GP) was established in [1] when their speeds are mutually distinct and distinct from zero. A soliton of speed c is a solution to (GP) of the form

$$\Psi(t,x) = U_c(x-ct),$$

where U_c is the solution to the ordinary differential equation

$$-icU_c' + U_c'' + U_c(1-|U_c|^2) = 0. \tag{1.2}$$

The finite energy solutions of (1.2) can be explicitly calculated. If $|c| \geq \sqrt{2}$, all of them are identically constant. If $|c| < \sqrt{2}$, there exists a family of non-constant solutions with finite energy. Such solutions are given by the expression

$$U_c(x) = \sqrt{\frac{2-c^2}{2}} \tanh\left(\frac{2-c^2}{2} x\right) + i\frac{c}{\sqrt{2}}.$$

Notice that for $c \neq 0$, U_c does not vanish on $\mathbb{R}$, which is important since the stability analysis established in [1] requires such solutions. In fact, we need in what follows a reformulation of (GP) which only makes sense for such solutions.

1.2 Strategy of proof

Let $k \in \mathbb{N}$. We define the space E^k by

$$E^k = \left\{ u \in L^\infty(\mathbb{R},\mathbb{C}), \quad \text{tel que} \quad 1-|u|^2 \in L^2(\mathbb{R}) \quad \text{et} \quad u' \in H^k(\mathbb{R}) \right\}.$$

We provide two distances on the space E^k, namely:

$$d^k(u_1,u_2) = \|u_1-u_2\|_{L^\infty(\mathbb{R})} + \||u_1|^2 - |u_2|^2\|_{L^2(\mathbb{R})} + \sum_{j=1}^{k+1} \|(u_1-u_2)^{(j)}\|_{L^2(\mathbb{R})},$$

$$d^k_{loc}(u_1,u_2) = \|u_1-u_2\|_{L^\infty(-1,1)} + \||u_1|^2 - |u_2|^2\|_{L^2(\mathbb{R})} + \sum_{j=1}^{k+1} \|(u_1-u_2)^{(j)}\|_{L^2(\mathbb{R})}.$$

The space E^k equipped with the distance d^k is a complete space. The strategy of proof consists of proving certain equivalence between (GP) and (HGP) which takes into account the spaces of resolution $\mathcal{C}(I,E^k)$ and $\mathcal{C}(I,\mathcal{NV}^k(\mathbb{R}))$, respectively. For that purpose, it will be required to show that the Cauchy problem for (GP) is well posed in E^k. Given any initial data $(\eta_0,v_0) \in \mathcal{NV}^k(\mathbb{R})$ of (HGP), we construct $\Psi_0 \in E^k$ from (η_0,v_0). Considering the solution Ψ of (GP) whose initial data is Ψ_0, we construct a solution $(\eta,v) \in \mathcal{C}(I,\mathcal{NV}^k(\mathbb{R}))$ for (HGP) with $(\eta(0,.),v(0,.)) = (\eta_0,v_0)$, and we define the time interval $I=[T_1,T_2]$ in such a way that $\inf_{(t,x)\in I\times\mathbb{R}} |\Psi(t,x)| > 0$. This gives the existence in the space $\mathcal{C}(I,\mathcal{NV}^k(\mathbb{R}))$. To prove the uniqueness, we show that every solution $(\eta,v) \in \mathcal{C}(I,\mathcal{NV}^k(\mathbb{R}))$ of (HGP) allows us to construct a solution $\Psi \in \mathcal{C}(I,E^k)$ of (GP); hence the uniqueness of such solution yields that of (η,v). The repetition of this procedure prove the existence of T_*, T^* mentioned in Theorem 1.2. The continuity property will be proved according to the following diagram

$$\begin{tikzcd}
(\eta_0,v_0) \arrow[r, "1"] \arrow[d] & \Psi_0 \arrow[d, "2"] \\
(\eta(t,.),v(t,.)) & \Psi(t,.) \arrow[l, "3"]
\end{tikzcd}$$

where arrow (1) represents the continuity of the application $(\eta_0, v_0) \mapsto \Psi_0 = \sqrt{1-\eta_0}\exp(i\int_0^x v_0)$ from $(\mathcal{NV}^k, \|.\|_{X^k})$ to (E^k, d^k_{loc}), arrow (2) represents the continuity of $\Psi(t,.)$ with respect to the initial data Ψ_0 in (E^k, d^k_{loc}) for all $t \in]-T_*, T^*[$ and arrow (3) represents the continuity of the application $\Psi \mapsto (\eta, v)$.

Concerning the equation (GP), we will show the following

Theorem 1.3 *Let $k \in \mathbb{N}$. For all $R > 0$, there exists $T = T(k,R) > 0$ such that if $\Psi_0 \in E^k$ satisfies*

$$\sum_{j=1}^{k+1} \|\Psi_0^{(j)}\|_{L^2} + \|1 - |\Psi_0|^2\|_{L^2} \leq R,$$

then there exists a unique solution $\Psi \in \mathcal{C}([-T,T],(E^k, d^k))$ to the Cauchy problem

$$\begin{cases} i\partial_t \Psi + \partial_{xx}\Psi + (|\Psi|^2 - 1)\Psi = 0 & \text{on } (-T,T)\times\mathbb{R}, \\ \Psi(0,\cdot) = \Psi_0. \end{cases}$$

Moreover, the energy

$$\Sigma(\Psi(t)) := \frac{1}{2}\int_{\mathbb{R}} |\Psi'|^2(t,x)\,dx + \frac{1}{4}\int_{\mathbb{R}} (1 - |\Psi(t,x)|^2)^2\,dx$$

is constant on $[-T,T]$.
If $\tilde{\Psi}_0 \in E^k$ satisfies the same bounds as Ψ_0 and if $\tilde{\Psi}$ denotes the corresponding solution of the Cauchy problem, then we have the continuity estimate

$$\sup_{|t|\leq T} d^k(\Psi(t), \tilde{\Psi}(t)) \leq C(R,T)d^k(\Psi_0, \tilde{\Psi}_0)$$

where the constant $C(R,T) > 0$ depends only on R and T.
Finally, this unique solution is globally defined ($T = +\infty$).

We will complete the result of th previous theorem by establishing the continuity of the flow at a fixed time t with respect to the metric d^0_{loc}:

Proposition 1.4 *Let $(\Psi_0^n)_{n\in\mathbb{N}}$ be some sequence in E^0 such that $\Psi_0^n \to \Psi_0$ in (E^0, d^0_{loc}) when $n \to +\infty$. Then for all $t \in \mathbb{R}$, $\Psi^n(t,\cdot) \to \Psi(t,\cdot)$ in (E^0, d^0_{loc}), where Ψ^n and Ψ denote the global solutions of the corresponding Cauchy problems.*

2 Cauchy problem for GP and continuity properties

The main purpose of this section is to present the proofs of Theorem 1.3 and Proposition 1.4. In [4], we find many existence results of solutions to the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + f(|u|^2)u = 0,$$

where $u : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$ and $f : \mathbb{R}^+ \to \mathbb{R}$ is some smooth function. These results are established in dimension $N = 1$ of space and for non-vanishing solutions when $|x| \to +\infty$ with regularity $Z^k(\mathbb{R})$, $k \in \mathbb{N}$, in space, where

$$Z^k(\mathbb{R}) = \left\{u \in L^\infty(\mathbb{R}), \text{such that} \quad u' \in H^{k-1}(\mathbb{R})\right\}$$

is equipped with the norm

$$\|u\|_{Z^k} = \|u\|_{L^\infty(\mathbb{R})} + \sum_{j=1}^{k} \|u^{(j)}\|_{L^2(\mathbb{R})}, \quad \forall u \in Z^k(\mathbb{R}).$$

Equation (GP) corresponds to the choice $f(s) = 1-s$, but the topology of space $Z^k(\mathbb{R})$ is too weak for our needs. In 2006, P. Gérard [2] established a well-posedness result for (GP) in dimension $N = 2, 3$ in the space $\mathcal{C}(\mathbb{R}, E)$ with

$$E = \{u \in H^1_{loc}(\mathbb{R}^N) \quad \text{such that} \quad \nabla u \in L^2(\mathbb{R}^N) \quad \text{and} \quad 1 - |u|^2 \in L^2(\mathbb{R}^N)\},$$

equipped with the distance

$$d(u, \tilde{u}) = \|u - \tilde{u}\|_{Z^1+H^1} + \||u|^2 - |\tilde{u}|^2\|_{L^2}.$$

Recall that for two Banach spaces X and Y, if we endow the space $X+Y$ with the norm

$$\|v\|_{X+Y} = \inf\{\|v_1\|_X + \|v_2\|_Y \quad \text{such that} \quad v = v_1 + v_2 \quad \text{and} \quad (v_1, v_2) \in X \times Y\},$$

then the space $(X \times Y, \|.\|_{X+Y})$ is also a Banach space. We will follow a plan similar to the one adopted in [2]. Remark first that for $N = 1$, we have $E^0 = E$, but $d^0 \geq d$. In fact, for some k, it is possible to endow the space E^k with the distance

$$d_k(u, \tilde{u}) = \|u - \tilde{u}\|_{Z^{k+1}+H^{k+1}} + \||u|^2 - |\tilde{u}|^2\|_{L^2},$$

which generalizes the distance d, but our choice for the distance d^k or even d^k_{loc} seems more adapted to describe the link with the solutions of (HGP). This specificity creates additional difficulties related to possible slow windings of phase at infinity, By contrast, working in dimension one in space simplifies greatly the treatment of the nonlinearity through the Sobolev embeddings.

We prove now some important properties of the space (E^k, d^k) and the Schrödinger operator $S_t = e^{it\Delta}, t \in \mathbb{R}$. We start by a property of S_t on the Zhidkov space $Z^k(\mathbb{R})$.

Lemma 2.1 *i) Let $k \in \mathbb{N}$. There exists $C > 0$ such that, for all $f \in Z^k(\mathbb{R})$ and $t \in \mathbb{R}$, we have $S_t f - f \in H^k(\mathbb{R})$ and*

$$\|S_t f - f\|_{H^k} \leq C(1 + |t|)^{\frac{1}{2}} \|f\|_{Z^k}.$$

Furthermore,

$$\lim_{t \to 0} \|S_t f - f\|_{H^k} = 0.$$

ii) There exists $C > 0$ such that, for all $f \in E^0$, we have

$$\|f\|_{L^\infty(\mathbb{R})} \leq C(1 + \sqrt{\Sigma(f)}). \tag{2.1}$$

Proof. The operator $S_t : Z^k \to Z^k$ is defined as the integral operator

$$\begin{cases} S_t \phi = \int_{\mathbb{R}} K(t, x - y)\phi(y)dy, \quad t \neq 0, \\ S_0 \phi = \phi, \end{cases} \tag{2.2}$$

with[1] $K(t, x) = \frac{\exp(\frac{ix^2}{4t})}{\sqrt{4\pi i t}}$. We also have, for $t \neq 0$,

$$S_t \phi = K(t, .) \star \phi,$$

where $\star$ is the convolution product. Let $\mathcal{F}$ denote the Fourier transform. We have

$$\mathcal{F} S_t \phi = (\mathcal{F} K(t, .)).(\mathcal{F} \phi),$$

[1] $K(t, x)$ is the fundamental solution for the operator $L = i\partial_t + \partial_x^2$.

and, for almost every $\xi \in \mathbb{R}$, we also have

$$\begin{aligned}
\mathcal{F}K(t,.)(\xi) &= \frac{1}{\sqrt{4\pi i t}} \int_{\mathbb{R}} \exp\left(\frac{i}{4t}(x^2 - 4t\xi x)\right) dx \\
&= \frac{\exp(-it\xi^2)}{\sqrt{4\pi i t}} \int_{\mathbb{R}} \exp\left(\frac{i}{4t}(x - 2t\xi)^2\right) dx \\
&= \exp(-it\xi^2).
\end{aligned}$$

Let now $\chi \in \mathcal{D}(\mathbb{R})$ be such that $\chi(x) = 1$ when $|x| \leq 1$. Since

$$\exp(-it\xi^2) - 1 = g(t,\xi)\xi, \tag{2.3}$$

with $g(t,\xi) = -it\chi(t\xi^2)\xi \int_0^1 e^{-its\xi^2} ds + \frac{\exp(-it\xi^2)-1}{\xi}(1 - \chi(t\xi^2))$, we have for all $f \in Z^k(\mathbb{R})$,

$$\begin{aligned}
S_t f - f &= \mathcal{F}^{-1}\mathcal{F}(S_t f - f) \\
&= \mathcal{F}^{-1}((e^{-it\xi^2} - 1)\mathcal{F}f) \\
&= \mathcal{F}^{-1}(g(t,\xi)\xi\mathcal{F}f) \\
&= (\mathcal{F}^{-1}g(t,\xi)) \star \partial_x f \quad \in L^2(\mathbb{R}).
\end{aligned}$$

If $1 \leq j \leq k$, we have $\partial_x^{(j)}(S_t f - f) = (S_t - 1)\partial_x^{(j)} f \in L^2(\mathbb{R})$, so that $S_t f - f \in H^k(\mathbb{R})$. To prove the inequality $\|S_t f - f\|_{H^k} \leq C(1 + |t|)^{\frac{1}{2}} \|f\|_{Z^k}$, notice that we have on the one hand

$$\|S_t f - f\|_{L^2} = \|\mathcal{F}(S_t f - f)\|_{L^2} \leq \|g(t,.)\|_{L^\infty} \|\partial_x f\|_{L^2}. \tag{2.4}$$

On the other hand,[2] there exists $C_1 > 0$ such that $\|g(t,.)\|_{L^\infty} \leq C_1\sqrt{|t|}$, and, if $1 \leq j \leq k$, we have

$$\|\partial_x^{(j)}(S_t f - f)\|_{L^2} = \|\mathcal{F}\partial_x^{(j)}(S_t f - f)\|_{L^2} \leq 2\|\mathcal{F}\partial_x^{(j)} f\|_{L^2} = 2\|\partial_x^{(j)} f\|_{L^2}.$$

This proves the existence of $C > 0$ such that

$$\|S_t f - f\|_{H^k} \leq C(1 + |t|)^{\frac{1}{2}} \|f\|_{Z^k}.$$

The fact that $\lim_{t\to 0} \|S_t f - f\|_{H^k} = 0$ follows from (2.4) and from the continuity of the application $t \mapsto S_t \partial_x^{(j)} f : \mathbb{R} \to L^2(\mathbb{R})$ when $j \geq 1$.

To prove the inequality (2.1), let $\chi \in \mathcal{D}(\mathbb{C},\mathbb{R})$ be such that $0 \leq \chi \leq 1$, $\chi(z) = 1$ when $|z| \leq 1$ and $\chi(z) = 0$ when $|z| \geq 3$. We set

$$f = f_1 + f_2,$$

with $f_1 = \chi(f)f$ and $f_2 = (1 - \chi(f))f$. Clearly, we have

$$\|f\|_{L^\infty(\mathbb{R})} \leq \|f_1\|_{L^\infty(\mathbb{R})} + \|f_2\|_{L^\infty(\mathbb{R})}.$$

We also have

$$\|f_1\|_{L^\infty(\mathbb{R})} \leq 3. \tag{2.5}$$

Let us now show that $f_2 \in H^1(\mathbb{R})$. since

$$f_2 = (1 - \chi(f))f\mathbf{1}_{\{x, \quad |f(x)| \geq 2\}},$$

we have

$$|f_2| \leq |f|(1 - \chi(f)) \leq (1 + |f|)(|f| - 1) = |f|^2 - 1 \in L^2(\mathbb{R}).$$

[2] See the variations of the real function $\xi \mapsto \left|\frac{\exp(-it\xi^2)-1}{\xi}\right|$.

On the other hand, we also have

$$f_2' = (1-\chi(f) - \partial_f\chi(f))f' \in L^2(\mathbb{R}).$$

Hence, $f_2' \in H^1(\mathbb{R})$ and there exists $C_1>0$ and $C_2=C_2(\chi)>0$ such that

$$\|f_2\|_{L^\infty(\mathbb{R})} \leq C_1\|f_2\|_{H^1(\mathbb{R})} \quad \text{et} \quad \|f_2\|_{H^1(\mathbb{R})} \leq C_2\sqrt{\Sigma(f)}. \tag{2.6}$$

Combining (2.5) and (2.6), we deduce that there exists $C>0$ such that

$$\|f\|_{L^\infty(\mathbb{R})} \leq C(1+\sqrt{\Sigma(f)}).$$

■

Lemma 2.2 *For every $k\in\mathbb{N}$, we have*

$$E^k + H^{k+1}(\mathbb{R}) \subset E^k. \tag{2.7}$$

Moreover, there exists $C_1, C_2 > 0$ such that, for every $(v,w),(\tilde{v},\tilde{w}) \in E^k\times H^{k+1}(\mathbb{R})$, we have

$$\begin{aligned} d^k(v+w,\tilde{v}+\tilde{w}) \quad &\leq \quad C_1\left(k+\|v\|_{L^\infty}+\|\tilde{v}\|_{L^\infty}+\|w\|_{L^2}+\|\tilde{w}\|_{L^2}+\right)\|w-\tilde{w}\|_{H^{k+1}} \\ &\quad +(1+\|\tilde{w}\|_{L^2}+\|w\|_{L^2})d^k(v,\tilde{v}), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} d^k(v+w,\tilde{v}+\tilde{w}) \quad &\leq \quad C_2\left(k+1+\|w\|_{L^2}+\|\tilde{w}\|_{L^2}+\sqrt{\Sigma(v)}+\sqrt{\Sigma(\tilde{v})}\right)\|w-\tilde{w}\|_{H^{k+1}} \\ &\quad +(1+\|\tilde{w}\|_{L^2}+\|w\|_{L^2})d^k(v,\tilde{v}). \end{aligned} \tag{2.9}$$

Proof. Let $(v,w)\in E^k\times H^{k+1}$. Since

$$|v+w|^2-1 = |v|^2-1+2Re(w\bar{v})+|w|^2,$$

and since $(v,w)\in L^\infty(\mathbb{R})\times H^1(\mathbb{R})$, we have $Re(w\bar{v}), |w|^2 \in L^2(\mathbb{R})$. This proves (2.7). Let now $(v,w),(\tilde{v},\tilde{w}) \in E^k\times H^{k+1}(\mathbb{R})$. There exists $C>0$ such that

$$\begin{aligned} d^k(v+w,\tilde{v}+\tilde{w}) \quad &\leq \quad \|v-\tilde{v}\|_{L^\infty} + C\|w-\tilde{w}\|_{H^{k+1}} + \||v+w|^2-|\tilde{v}+\tilde{w}|^2\|_{L^2} \\ &\quad +\sum_{j=1}^{k+1}\|(v-\tilde{v})^{(j)}\|_{L^2}+\sum_{j=1}^{k+1}\|(w-\tilde{w})^{(j)}\|_{L^2}. \end{aligned} \tag{2.10}$$

Using the identity

$$|v+w|^2-|\tilde{v}+\tilde{w}|^2 = |v|^2-|\tilde{v}|^2+2Re(w\bar{v}-\tilde{w}\bar{\tilde{v}})+|w|^2-|\tilde{w}|^2,$$

and the relations

$$\begin{cases} w\bar{v}-\tilde{w}\bar{\tilde{v}} = (w-\tilde{w})\bar{v}+\tilde{w}\overline{(v-\tilde{v})}, \\ w\bar{v}-\tilde{w}\bar{\tilde{v}} = (w-\tilde{w})\bar{\tilde{v}}+w\overline{(v-\tilde{v})}, \\ |w|^2-|\tilde{w}|^2 = (|w|+|\tilde{w}|)(|w|-|\tilde{w}|), \end{cases}$$

we get

$$\begin{aligned} \||v+w|^2-|\tilde{v}+\tilde{w}|^2\|_{L^2} \quad &\leq \quad \||v|^2-|\tilde{v}|^2\|_{L^2} + (\|w\|_{L^2}+\|\tilde{w}\|_{L^2}+\|v\|_{L^\infty}+\|\tilde{v}\|_{L^\infty})\|w-\tilde{w}\|_{L^2} \\ &\quad +(\|w\|_{L^2}+\|\tilde{w}\|_{L^2})\|v-\tilde{v}\|_{L^\infty}. \end{aligned}$$

The conclusion follows by combining this inequality with (2.10). ■

The previous lemmas allow us to deduce that E^k is kept invariant by S_t. Moreover, we have the following two continuity properties:

Lemma 2.3 *Let $t \in \mathbb{R}$ and $k \in \mathbb{N}$. Then $S_t(E^k) \subset E^k$ and, for each $\Psi_0 \in E^k$, the application*

$$t \in \mathbb{R} \mapsto S_t\Psi_0 \in (E^k, d^k),$$

is continuous. Moreover, for each $(\Psi_0, \tilde{\Psi}_0) \in (E^k)^2$, there exists $C = C\left(k, |t|, \sqrt{\Sigma(\Psi_0)}, \sqrt{\Sigma(\tilde{\Psi}_0)}\right)$ such that

$$d^k(S_t\Psi_0, S_t\tilde{\Psi}_0) \leq Cd^k(\Psi_0, \tilde{\Psi}_0). \tag{2.11}$$

Proof. Let $t \in \mathbb{R}, k \in \mathbb{N}$, and $\Psi_0 \in E^k \subset Z^{k+1}$. Writing $S_t\Psi_0 = \Psi_0 + S_t\Psi_0 - \Psi_0$ and using Lemmas 2.1 and 2.2, we find that $S_t\Psi_0 \in E^k$.
To show the continuity of the application $t \in \mathbb{R} \mapsto S_t\Psi_0 \in (E^k, d^k)$, we just need to show that

$$\lim_{t \to 0} d^k(S_t\Psi_0, \Psi_0) = 0.$$

For this aim, we use Lemma 2.2 and deduce that

$$\begin{aligned} d^k(S_t\Psi_0, \Psi_0) &= d^k(\Psi_0 + S_t\Psi_0 - \Psi_0, \Psi_0) \\ &\leq C\left(k + 1 + \|S_t\Psi_0 - \Psi_0\|_{L^2} + 2\sqrt{\Sigma(\Psi_0)}\right)\|S_t\Psi_0 - \Psi_0\|_{H^{k+1}}; \end{aligned}$$

hence we conclude by using Lemma 2.1.
Let now $(\Psi_0, \tilde{\Psi}_0) \in (E^k)^2 \subset (Z^{k+1})^2$. In view of Lemma 2.1, we have $(S_t\Psi_0 - \Psi_0, S_t\tilde{\Psi}_0 - \tilde{\Psi}_0) \in (H^{k+1})^2$. Using Lemma 2.2, we can write

$$\begin{aligned} d^k(S_t\Psi_0, S_t\tilde{\Psi}_0) &= d^k(\Psi_0 + S_t\Psi_0 - \Psi_0, \tilde{\Psi}_0 + S_t\tilde{\Psi}_0 - \tilde{\Psi}_0) \\ &\leq \left(1 + \|S_t\tilde{\Psi}_0 - \tilde{\Psi}_0\|_{L^2} + \|S_t\Psi_0 - \Psi_0\|_{L^2}\right) d^k(\Psi_0, \tilde{\Psi}_0) \\ &\quad + C\left(k + 1 + \|S_t\tilde{\Psi}_0 - \tilde{\Psi}_0\|_{L^2} + \|S_t\Psi_0 - \Psi_0\|_{L^2} + \sqrt{\Sigma(\Psi_0)} + \sqrt{\Sigma(\tilde{\Psi}_0)}\right) \times \\ &\quad \|(S_t - 1)(\Psi_0 - \tilde{\Psi}_0)\|_{H^{k+1}}. \end{aligned}$$

Furthermore, in view of Lemma 2.1, there exists $C_1 > 0$ such that

$$\begin{aligned} d^k(S_t\Psi_0, S_t\tilde{\Psi}_0) &\leq \left(1 + C_1(1 + |t|)^{\frac{1}{2}}(\|\tilde{\Psi}_0\|_{Z^1} + \|\Psi_0\|_{Z^1})\right) d^k(\Psi_0, \tilde{\Psi}_0) \\ &\quad + CC_1(1 + |t|)^{\frac{1}{2}}\left(k + 1 + C_1(1 + |t|)^{\frac{1}{2}}(\|\tilde{\Psi}_0\|_{Z^1} + \|\Psi_0\|_{Z^1}) + \sqrt{\Sigma(\Psi_0)} + \sqrt{\Sigma(\tilde{\Psi}_0)}\right) \times \\ &\quad \|\Psi_0 - \tilde{\Psi}_0\|_{Z^{k+1}}. \end{aligned}$$

Thus we conclude by using the inequality $\|\Psi_0 - \tilde{\Psi}_0\|_{Z^{k+1}} \leq d^k(\Psi_0, \tilde{\Psi}_0)$. ■

Let $\Psi_0 \in E^k$. The Duhamel formula for (GP) reads

$$\Psi(t) = S_t\Psi_0 + i\int_0^t S_{t-\tau}[(1 - |\Psi|^2)\Psi(\tau)]d\tau.$$

Thus it will be useful to study the application

$$\Psi \mapsto \int_0^t S_{t-\tau}[(1 - |\Psi|^2)\Psi(\tau)]d\tau, \forall \Psi \in \mathcal{C}([-T, T], E^k),$$

with $T > 0$ and $k \in \mathbb{N}$.

Lemma 2.4 *Let $k \in \mathbb{N}$ and $T > 0$. Then the application*

$$\Psi \mapsto G(\Psi) = \int_0^t S_{t-\tau}[(1-|\Psi|^2)\Psi(\tau)]d\tau$$

is defined from $\mathcal{C}([-T,T], E^k)$ to $\mathcal{C}([-T,T], H^{k+1})$. Moreover, for every $R > 0$ there exists $C = C(k,R)$ such that, for all $(\Psi, \tilde{\Psi}) \in \big(\mathcal{C}([-T,T], E^k)\big)^2$ with

$$\sup_{|t|\leq T}\Big(\|\partial_x\Psi(t)\|_{H^k} + \sqrt{\Sigma(\Psi(t))}\Big) \leq R, \quad \textit{and} \quad \sup_{|t|\leq T}\Big(\|\partial_x\tilde{\Psi}(t)\|_{H^k} + \sqrt{\Sigma(\tilde{\Psi}(t))}\Big) \leq R,$$

we have

$$\sup_{|t|\leq T}\|G(\Psi) - G(\tilde{\Psi})\|_{H^{k+1}} \leq TC(k,R)\sup_{|t|\leq T} d^k(\Psi(t), \tilde{\Psi}(t)). \tag{2.12}$$

Proof. Let $k \in \mathbb{N}$. Let us first show that the application $\Psi \mapsto \Gamma(\Psi) = (1-|\Psi|^2)\Psi$ is locally Lipschitz-continuous from (E^k, d^k) to H^{k+1}. Let $(\Psi_1, \Psi_2) \in (E^k)^2$ and $j \leq k+1$. We have

$$(\Gamma(\Psi_1) - \Gamma(\Psi_2))^{(j)} = \sum_{l=0}^{j} C_l^j(|\Psi_1|^2 - |\Psi_2|^2)^{(l)}\Psi_1^{(j-l)} + \sum_{l=0}^{j} C_l^j(\Psi_1 - \Psi_2)^{(j-l)}(|\Psi_2|^2 - 1)^{(l)}. \tag{2.13}$$

We now find an upper bound for the L^2-norm of the first term of the right-hand of (2.13). If $(j \neq 0) \wedge (l \neq 0)$, we have

$$\begin{aligned}(|\Psi_1|^2 - |\Psi_2|^2)^{(l)}\Psi_1^{(j-l)} \quad &= \quad \sum_{s=0}^{l} C_s^l \langle \Psi_1^{(l-s)}, (\Psi_1 - \Psi_2)^{(s)}\rangle_{\mathbb{C}}\Psi_1^{(j-l)} \\ &\quad + \sum_{s=0}^{l} C_s^l \langle \Psi_2^{(l-s)}, (\Psi_1 - \Psi_2)^{(s)}\rangle_{\mathbb{C}}\Psi_1^{(j-l)}.\end{aligned}$$

Hence there exists $C > 0$ such that

$$\|\langle \Psi_1^{(l-s)}, (\Psi_1 - \Psi_2)^{(s)}\rangle_{\mathbb{C}}\Psi_1^{(j-l)}\|_{L^2} \leq C\|\Psi_1'\|^2_{H^{j-1}} d^k(\Psi_1, \Psi_2),$$

which implies that

$$\|(|\Psi_1|^2 - |\Psi_2|^2)^{(l)}\Psi_1^{(j-l)}\|_{L^2} \leq C2^l\|\Psi_1'\|_{H^{j-1}}(\|\Psi_1'\|_{H^{j-1}} + \|\Psi_2'\|_{H^{j-1}})d^k(\Psi_1, \Psi_2).$$

If $(j \neq 0) \wedge (l = 0)$, we have

$$\|(|\Psi_1|^2 - |\Psi_2|^2)\Psi_1^{(j)}\|_{L^2} \leq (\|\Psi_1\|_{L^\infty} + \|\Psi_2\|_{L^\infty})\|\Psi_1'\|_{H^{j-1}} d^k(\Psi_1, \Psi_2).$$

If $(j = 0) \wedge (l = 0)$, we have

$$\|(|\Psi_1|^2 - |\Psi_2|^2)\Psi_1\|_{L^2} \leq \|\Psi_1\|_{L^\infty} d^k(\Psi_1, \Psi_2).$$

Thus, using (2.1) to estimate $\|\Psi_1\|_{L^\infty}$ and $\|\Psi_2\|_{L^\infty}$, we find that there exists $C_1 = C_1\Big(j, \|\Psi_1'\|_{H^j}, \|\Psi_2'\|_{H^j}, \sqrt{\Sigma(\Psi_1)}, \sqrt{\Sigma(\Psi_2)}\Big)$ such that

$$\|\sum_{l=0}^{j} C_l^j(|\Psi_1|^2 - |\Psi_2|^2)^{(l)}\Psi_1^{(j-l)}\|_{L^2} \leq C_1 d^k(\Psi_1, \Psi_2).$$

A similar argument allows us to obtain the same estimate for the L^2-norm of the second term of the right-hand side of (2.13). This allows to prove the existence of
$C = C\left(k, \|\Psi_1'\|_{H^k}, \|\Psi_2'\|_{H^k}, \sqrt{\Sigma(\Psi_1)}, \sqrt{\Sigma(\Psi_2)}\right)$ such that

$$\|\Gamma(\Psi_1) - \Gamma(\Psi_2)\|_{H^{k+1}} \leq C d^k(\Psi_1, \Psi_2).$$

It follows that for all $R > 0$, there exists $C(k, R) > 0$ such that, for every $(\Psi_1, \Psi_2) \in (E^k)^2$ with

$$\|\Psi_1'\|_{H^k} + \sqrt{\Sigma(\Psi_1)} \leq R, \quad \text{and} \quad \|\Psi_2'\|_{H^k} + \sqrt{\Sigma(\Psi_2)} \leq R,$$

we have

$$\|\Gamma(\Psi_1) - \Gamma(\Psi_2)\|_{H^{k+1}} \leq C(k, R) d^k(\Psi_1, \Psi_2),$$

and the application Γ is locally Lipschitz. Thus the inequality (2.12) is a consequence of $G(\Psi) = \int_0^t S_{t-\tau}\Gamma(\Psi(\tau))d\tau$ and of

$$\partial_x^{(j)} G(\Psi) = \int_0^t S_{t-\tau}\partial_x^{(j)}\Gamma(\Psi(\tau))d\tau \quad \forall j \leq k+1.$$

■

2.1 Proof of Theorem 1.3

Let $T > 0$. We define the set E_T by

$$E_T = \left\{ \Psi \in \mathcal{C}([-T, T], E^k), \quad \sup_{|t| \leq T} \left(\sum_{j=2}^{k+1} \|\partial_x^{(j)}\Psi\|_{L^2} + \sqrt{\Sigma(\Psi)} \right) \leq 2R \right\}.$$

In what follows, we prove that for T small enough, the function

$$\Psi \mapsto F(\Psi) = S_t\Psi_0 + iG(\Psi),$$

is a contraction on E_T. Let $\Psi \in \mathcal{C}([-T, T], E^k)$. Lemmas 2.4 and 2.2 imply that $G(\Psi) \in \mathcal{C}([-T, T], H^{k+1})$ and $F(\Psi) = S_t\Psi_0 + iG(\Psi) \in \mathcal{C}([-T, T], E^k)$. On the other hand, for $\Phi = F(\Psi)$ with $\Psi \in E_T$, and $t \in [-T, T]$, we have

$$\begin{aligned} d^k(\Phi, \Psi_0) &= d^k(S_t\Psi_0 + iG(\Psi), \Psi_0) \\ &\leq (1 + \|G(\Psi)\|_{L^2}) d^k(S_t\Psi_0, \Psi_0) + C\left(k + \|S_t\Psi_0\|_{L^\infty} + \|G(\Psi)\|_{L^2}\right) \|G(\Psi)\|_{H^{k+1}} \\ &\leq C_1(R) d^k(S_t\Psi_0, \Psi_0) + TC_2(k, R), \end{aligned}$$

hence

$$\sup_{|t| \leq T} d^k(\Phi, \Psi_0) \leq C_1(R) \sup_{|t| \leq T} d^k(S_t\Psi_0, \Psi_0) + TC_2(k, R).$$

In view of Lemma 2.3, we have $\lim_{t \to 0} d^k(S_t\Psi_0, \Psi_0) = 0$. Hence there exists $T_1 > 0$ such that

$$\sup_{|t| \leq T_1} d^k(\Phi, \Psi_0) \leq (2 - \sqrt{2})R,$$

so that

$$\begin{aligned} \sup_{|t| \leq T_1} \left(\sum_{j=2}^{k+1} \|\partial_x^{(j)}\Phi\|_{L^2} + \sqrt{\Sigma(\Phi)} \right) &\leq \sup_{|t| \leq T_1} d^k(\Phi, \Psi_0) + \sum_{j=2}^{k+1} \|\Psi_0^{(j)}\|_{L^2} + \sqrt{2\Sigma(\Psi_0)} \\ &\leq 2R, \end{aligned}$$

and $F(M_{T_1}) \subset M_{T_1}$. In view of Lemmas 2.4 and 2.2, there exists $C > 0$ such that, for $\Psi, \tilde{\Psi} \in M_T$, we have

$$\begin{aligned} d^k(F(\Psi), F(\tilde{\Psi})) &= d^k(S_t\Psi_0 + iG(\Psi), S_t\Psi_0 + iG(\tilde{\Psi})) \\ &\quad + C\left(k + 2\|S_t\Psi_0\|_{L^\infty} + \|G(\Psi)\|_{L^2} + \|G(\tilde{\Psi})\|_{L^2}\right)\|G(\Psi) - G(\tilde{\Psi})\|_{H^{k+1}} \\ &\leq TC_1(k,R)\sup_{|t|\leq T} d^k(\Psi, \tilde{\Psi}). \end{aligned}$$

Thus proves the existence of $T = T(k,R)$ with $T_1 > T > 0$ such that F is contraction on M_T. Thus F has a unique fixed point $\Psi \in \mathcal{C}([-T,T], E^k,)$ which is the unique solution of (GP). The proof of the Lipschitz estimate is similar.

Equation (GP) is invariant with respect to the change of variable $t \mapsto t + c$ (c is some constant). Then a similar result to that of Theorem 1.3 can be proved if we replace the initial condition at $t_0 = 0$ by an initial condition at $t_0 \neq 0$. This proves the existence of maximal $T^\star, T_\star > 0$ such that the solution can be continued on the interval $]-T_\star, T^\star[$.

2.1.1 Global well-posedness and conservation of energy in E^k, $k \geq 1$

We start with the case $k = 1$. We already proved that (GP) had a maximal solution in the space $\mathcal{C}(]-T_\star, T^\star[, E^1)$. In what follows, we will show that $]-T_\star, T^\star[= \mathbb{R}$. Since $\partial_t\Psi \in L^\infty(]-T_\star, T^\star[, L^2(\mathbb{R}))$, we can write

$$\frac{d}{dt}\Sigma(\Psi(t)) = \int_{\mathbb{R}} \left(\langle\partial_x\Psi, \partial_x\partial_t\Psi\rangle_{\mathbb{C}} + (|\Psi|^2 - 1)\langle\Psi, \partial_t\Psi\rangle_{\mathbb{C}}\right) dx = 0. \tag{2.14}$$

Thus, we have $\Sigma(\Psi(t)) = \Sigma(\Psi_0)$ on $]-T_\star, T^\star[$. Next we prove the existence of $T_1 > 0$, depending only on $\Sigma(\Psi_0)$, such that $[-T_1, T_1] \subset]-T_\star, T^\star[$ and that $\partial_x^2\Psi(t,.)$ stays bounded in $L^2(\mathbb{R})$ for $|t| \leq T_1$. For all $t \in]-T_\star, T^\star[$, we have

$$|\partial_x^2((|\Psi|^2 - 1)\Psi)| \leq (3|\Psi|^2 + 1)|\partial_x^2\Psi| + 2|\Psi||\partial_x\Psi|^2,$$

which next implies that

$$\|\partial_x^2((|\Psi|^2 - 1)\Psi)\|_{L^2} \leq (3\|\Psi\|_{L^\infty}^2 + 1)\|\partial_x^2\Psi\|_{L^2} + 2\|\Psi\|_{L^\infty}\|\partial_x\Psi\|_{L^4}^2.$$

On the other hand, using Lemma 2.1 and Gagliardo-Nirenberg inequality [3], we find that there exist C_1, C_2 such that

$$\begin{cases} \|\Psi\|_{L^\infty} \leq C_1(1 + \sqrt{\Sigma(\Psi)}), \\ \|\partial_x\Psi\|_{L^4} \leq C_2\|\partial_x\Psi\|_{L^2}^{\frac{3}{4}}\|\partial_x^2\Psi\|_{L^2}^{\frac{1}{4}}. \end{cases}$$

Then there exist $D_1, D_2 > 0$ depending on $\Sigma(\Psi_0)$ such that for all $T' > 0$ with $[-T', T'] \subset]-T_\star, T^\star[$, we have

$$\|\partial_x^2((|\Psi|^2 - 1)\Psi)\|_{L^\infty([-T',T'],L^2)} \leq D_1\|\partial_x^2\Psi\|_{L^\infty([-T',T'],L^2)} + D_2.$$

Combined with Duhamel formula

$$\Psi(t) = S_t\Psi_0 - i\int_0^t S_{t-\tau}((|\Psi|^2 - 1)\Psi(\tau))d\tau, \tag{2.15}$$

this shows that

$$\|\partial_x^2\Psi\|_{L^\infty([-T',T'],L^2)} \leq \|\partial_x^2\Psi_0\|_{L^2} + 2T'\left(D_1\|\partial_x^2\Psi\|_{L^\infty([-T',T'],L^2)} + D_2\right).$$

This proves the existence of $T_1 = T_1(\Sigma(\Psi_0))$ such that $[-T_1, T_1] \subset]-T_\star, T^\star[$ and that $\partial_x^2\Psi(t,.)$ stays bounded in $L^2(\mathbb{R})$ for $|t| \leq T_1$. By iterating this argument, we find that Ψ can be extended into a global solution with $\Psi \in \mathcal{C}(\mathbb{R}, E^1)$ and $\Sigma(\Psi(t)) = \Sigma(\Psi_0)$ for all $t \in \mathbb{R}$.

The general case $k \geq 1$ can be treated as above. Indeed, for $2 \leq j \leq k+1$, we have

$$((|\Psi|^2-1)\Psi)^{(j)} = \sum_{l=1}^{j}\sum_{s=0}^{l} C_l^j C_s^l \langle \Psi^{(s)}, \Psi^{(l-s)}\rangle_{\mathbb{C}} \Psi^{(j-l)} + (|\Psi|^2-1)\Psi^{(j)}, \tag{2.16}$$

from which we get

$$\|((|\Psi|^2-1)\Psi)^{(j)}\|_{L^2} \leq \sum_{l=1}^{j}\sum_{s=0}^{l} C_l^j C_s^l \|\Psi^{(s)}\Psi^{(l-s)}\Psi^{(j-l)}\|_{L^2} + (\|\Psi\|_{L^\infty}^2+1)\|\Psi^{(j)}\|_{L^2}.$$

Thus to upper-bound the term $\|\Psi^{(s)}\Psi^{(l-s)}\Psi^{(j-l)}\|_{L^2}$, we discuss essentially the following two cases:

1. The three indexes $s, l-s, j-l$ are mutually distinct. In this case we denote $l_3 = \max(s, l-s, j-l)$ and l_1, l_2 the others two. Then $\max(l_1, l_2) < l_3 < j$ and we have
$$\begin{aligned}\|\Psi^{(s)}\Psi^{(l-s)}\Psi^{(j-l)}\|_{L^2} &\leq \|\Psi^{(l_1)}\Psi^{(l_2)}\|_{L^\infty}\|\Psi^{(l_3)}\|_{L^2}\\ &\leq C\|\Psi^{(l_1)}\Psi^{(l_2)}\|_{H^1}\|\Psi^{(l_3)}\|_{L^2}\end{aligned}$$

2. There are two indexes $l_1 = l_2$. Let l_3 be the other one, then
   (a) Either $l_1 = l_2 = 0$, then $l_3 = j$ and we have
   $$\|\Psi^{(s)}\Psi^{(l-s)}\Psi^{(j-l)}\|_{L^2} \leq \|\Psi\|_{L^\infty}^2\|\Psi^{(j)}\|_{L^2}.$$
   (b) Or $l_1 = l_2 \neq 0$, then by using Gagliardo-Nirenberg inequality [3], there exists C_1 such that
   $$\begin{aligned}\|\Psi^{(s)}\Psi^{(l-s)}\Psi^{(j-l)}\|_{L^2} &\leq \|\Psi^{(l_3)}\|_{L^\infty}\|\Psi^{(l_1)}\|_{L^4}^2\\ &\leq C_1\|\Psi^{(l_3)}\|_{L^\infty}\|\Psi^{(l_1)}\|_{L^2}^{\frac{3}{2}}\|\Psi^{(l_1+1)}\|_{L^2}^{\frac{1}{2}}\\ &\leq \frac{C_1}{2}\|\Psi^{(l_3)}\|_{L^\infty}\left(\|\Psi^{(l_1)}\|_{L^2}^3 + \|\Psi^{(l_1+1)}\|_{L^2}\right).\end{aligned}$$

Thus, there exist $D_1, D_2 > 0$ depending on $\|\Psi\|_{L^\infty}$ and the L^2-norm of the derivatives of Ψ of order $< j$, such that

$$\|((|\Psi|^2-1)\Psi)^{(j)}\|_{L^2} \leq D_1\|\Psi^{(j)}\|_{L^2} + D_2.$$

Then for every $T' > 0$ such that $[-T', T'] \subset]-T_\star, T^\star[$, we find using (2.15) that there exist $E_1, E_2 > 0$ depending on $\|\Psi\|_{L^\infty}$ and the L^2-norm of the derivatives of Ψ of order $< j$ such that

$$\|\Psi^{(j)}\|_{L^\infty([-T',T'],L^2)} \leq \|\Psi_0^{(j)}\|_{L^2} + 2T'\left(E_1\|\Psi^{(j)}\|_{L^\infty([-T',T'],L^2)} + E_2\right).$$

This allows, by induction on j, to prove the existence of $T_j > 0$ depending only on $\Sigma(\Psi_0)$ such that $[-T_j, T_j] \subset]-T_\star, T^\star[$ and $\Psi^{(j)}(t,.)$ stays bounded in $L^2(\mathbb{R})$ for $|t| \leq T_j$. It follows that the quantity $\sum_{j=1}^{k+1}\|\Psi^{(j)}(t,.)\|_{L^2} + \|1-|\Psi(t,.)|^2\|_{L^2}$ is bounded in $[-T_{k+1}, T_{k+1}]$ and, by iterating the previous argument, it can not blow up on $]-T_\star, T^\star[$.

2.1.2 Global well-posedness and conservation of energy in E^0

The idea is to use the energy conservation to move from local to global. To establish the latter, in view of (2.14), we proceed by regularizing the initial datum so that it belongs to E^1. More specifically, we approximate Ψ_0 in the sens of the distance d^0 by a sequence Ψ_0^ϵ of elements from E^1. By Proposition 1.3, we have

$$\sup_{|t|\leq T} d^0(\Psi^\epsilon(t), \Psi(t)) \to 0,$$

when $\epsilon \to 0$, where Ψ^ϵ is the solution of (GP) with initial datum Ψ_0^ϵ. Then for all t, we have

$$\Sigma(\Psi^\epsilon(t)) \to \Sigma(\Psi(t)),$$

when $\epsilon \to 0$. Hence the energy conservation of Ψ^ϵ implies that of Ψ.

2.2 Proof of Proposition 1.4

We start by a weak convergence result for which the Gross-Pitaevskii flow is continuous.

Proposition 2.5 *Let $(\Psi_{n,0})_{n\in\mathbb{N}} \in (E^0)^{\mathbb{N}}$ and $\Psi_0 \in E^0$ such that*

$$\begin{cases} \Psi'_{n,0} \rightharpoonup \Psi'_0 \quad dans \quad L^2(\mathbb{R}), \\ 1-|\Psi_{n,0}|^2 \rightharpoonup 1-|\Psi_0|^2 \quad dans \quad L^2(\mathbb{R}), \end{cases} \tag{2.17}$$

and, for all compact set $K \subset \mathbb{R}$,

$$\Psi_{n,0} \to \Psi_0 \quad dans \quad L^\infty(K). \tag{2.18}$$

We denote by Ψ_n and Ψ the global solutions for (GP) corresponding to initial datum $\Psi_{n,0}$ and Ψ_0, respectively. Then for all $t \in \mathbb{R}$ and for all compact set $K \subset \mathbb{R}$, we have

$$\begin{cases} \partial_x\Psi_n(t,.) \rightharpoonup \partial_x\Psi(t,.) \quad dans \quad L^2(\mathbb{R}), \\ 1-|\Psi_n(t,.)|^2 \rightharpoonup 1-|\Psi(t,.)|^2 \quad dans \quad L^2(\mathbb{R}), \\ \Psi_n(t,.) \rightharpoonup \Psi(t,.) \quad dans \quad L^\infty(K). \end{cases} \tag{2.19}$$

Proof. We denote $\eta_n = 1-|\Psi_n|^2$. The weak convergence in (2.17) implies the existence of a constant $M>0$ such that

$$\Sigma(\Psi_{n,0}) \le M^2 \quad \forall n \in \mathbb{N}.$$

Since the energy Σ is conserved along the flow, we also have

$$\|\partial_x\Psi_n(t,.)\|_{L^2(\mathbb{R})} \le \sqrt{2}M \quad \text{and} \quad \|\eta_n(t,.)\|_{L^2(\mathbb{R})} \le 2M, \tag{2.20}$$

for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Then Lemma 2.1 implies the existence of a constant $C>0$ such that $\|\Psi_n(t,.)\|_{L^\infty(\mathbb{R})} \le C(1+\sqrt{\Sigma(\Psi_n(t,.))})$ for all $t \in \mathbb{R}$.
Since $\|\partial_x\eta_n(t,.)\|_{L^2(\mathbb{R})} \le 2\|\Psi_n(t,.)\|_{L^\infty(\mathbb{R})}\|\partial_x\Psi_n(t,.)\|_{L^2(\mathbb{R})}$, there exists two constants $K_M, L_M > 0$ depending on M, such that

$$\|\partial_x\eta_n(t,.)\|_{L^2(\mathbb{R})} \le K_M \quad \text{and} \quad \|\Psi_n(t,.)\|_{L^\infty(\mathbb{R})} \le L_M, \tag{2.21}$$

for all $t \in \mathbb{R}$. In particular, for some $T>0$, we deduce that

$$\int_0^T\int_{\mathbb{R}} |\partial_x\Psi_n(t,x)|^2 dxdt \le M^2T \quad \text{and} \quad \int_0^T\int_{\mathbb{R}} \eta_n^2(t,x)^2 dxdt \le M^2T. \tag{2.22}$$

Inequalities (2.21) and (2.22) will allow us to construct weak limits for $\partial_x\Psi_n$ and η_n. In view of (2.22), there exist two functions $\Phi_1 \in L^2([0,T]\times\mathbb{R})$ and $N \in L^2([0,T]\times\mathbb{R})$ such that up to a further subsequence,

$$\partial_x\Psi_n \rightharpoonup \Phi_1 \quad \text{in} \quad L^2([0,T]\times\mathbb{R}) \quad \text{and} \quad \eta_n \rightharpoonup N \quad \text{in} \quad L^2([0,T]\times\mathbb{R}), \tag{2.23}$$

when $n \to +\infty$. Similarly, (2.21) proves the existence of $\Phi \in L^\infty([0,T]\times\mathbb{R})$ such that, up to a further subsequence,

$$\Psi_n \overset{*}{\rightharpoonup} \Phi \quad \text{in} \quad L^\infty([0,T]\times\mathbb{R}), \tag{2.24}$$

when $n \to +\infty$. Combined with (2.23), this shows that $\Phi_1 = \partial_x\Phi$ in the sense of distributions. Our goal now is to check that the function Φ is a solution to (GP). This requires to improve the convergences in (2.23) and (2.24). With this goal in mind, we define the function $\chi_p = \chi(./p)$ when $p \in \mathbb{N}$ and $\chi \in \mathcal{D}(\mathbb{R})$ with $\chi \equiv 1$ on $[-1,1]$ and $\chi \equiv 0$ on $]-\infty,2]\cup[2,+\infty[$. Inequalities (2.21) and (2.22) prove that the sequence $(\chi_p\Psi_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}([0,T],H^1(\mathbb{R}))$. By the Rellich-Kondrachov theorem, the sets $\{\ \chi_p\Psi_n(t,.), n \in \mathbb{N}\}$ are relatively compacts in $H^{-1}(\mathbb{R})$ for any fixed $t \in [0,T]$. On the other hand, the function Ψ_n is solution to (GP), so that $\partial_t \Psi_n \in \mathcal{C}([0,T], H^{-1}(\mathbb{R}))$, and we also have

$$\begin{aligned}\|\partial_t \Psi_n(t,.)\|_{H^{-1}(\mathbb{R})} &\leq \|\partial_x \Psi_n(t,.)\|_{L^2(\mathbb{R})} + \|\Psi_n(t,.)\|_{L^\infty(\mathbb{R})} \|\eta_n(t,.)\|_{L^2(\mathbb{R})} \\ &\leq M(\sqrt{2} + 2L_M).\end{aligned}$$

As a consequence, the functions $\chi_p \Psi_n$ are équicontinuous in $\mathcal{C}([0,T], H^{-1}(\mathbb{R}))$. Applying the Arzela-Ascoli theorem and using the Cantor diagonal argument, we can find a further sub-sequence (independent of p) such that for any $p \in \mathbb{N}^*$

$$\chi_p \Psi_n \to \chi_p \Phi \quad \text{in} \quad \mathcal{C}([0,T], H^{-1}(\mathbb{R})), \tag{2.25}$$

when $n \to +\infty$. Recalling that the functions $\chi_p \Psi_n$ are uniformly bounded in $\mathcal{C}([0,T], H^1(\mathbb{R}))$, we deduce that the convergence in (2.18) also holds in the spaces $\mathcal{C}([0,T], H^s(\mathbb{R}))$ for any $s < 1$. In particular, by the Sobolev embedding theorem, we obtain

$$\chi_p \Psi_n \to \chi_p \Phi \quad \text{in} \quad \mathcal{C}([0,T], \mathcal{C}(\mathbb{R})). \tag{2.26}$$

Such convergences are enough to establish that Φ is solution to (GP). Let $h \in \mathcal{D}(\mathbb{R})$. Since the functions $\chi_p \Psi_n$ are uniformly bounded in $\mathcal{C}([0,T], \mathcal{C}(\mathbb{R}))$, for $p \in \mathbb{N}$ such that $supp(h) \subset [-p,p]$, we get

$$h\eta_n(t,.) = h(1 - \chi_p^2 |\Psi_n(t,.))|^2) \to h(1 - \chi_p^2 |\Phi(t,.))|^2) = h(1 - |\Phi(t,.))|^2) \quad \text{in} \quad \mathcal{C}(\mathbb{R}), \tag{2.27}$$

when $n \to +\infty$. Since this convergence is uniform with respect to $t \in [0,T]$, (2.23) implies that $N = 1 - |\Phi|^2$. Similarly,

$$h\Psi_n(t,.) = h\chi_p \Psi_n(t,.) \to h\chi_p \Phi(t,.) = h\Phi(t,.) \quad \text{in} \quad \mathcal{C}(\mathbb{R}). \tag{2.28}$$

In view of (2.23), we deduce that

$$h\eta_n \Psi_n \to h(1 - |\Phi|^2)\Phi \quad \text{in} \quad L^2([0,T] \times \mathbb{R}).$$

Going back to (2.23) and (2.24), we recall that

$$i\partial_t \Psi_n \to i\partial_t \Phi \quad \text{in} \quad \mathcal{D}'([0,T] \times \mathbb{R}) \quad \text{and} \quad \partial_x^2 \Psi_n \to \partial_x^2 \Phi \quad \text{in} \quad \mathcal{D}'([0,T] \times \mathbb{R}),$$

when $n \to +\infty$, so that it remains to take the limit, when $n \to +\infty$ in the expression

$$\langle i\partial_t \Psi_n + \partial_x^2 \Psi_n + \eta_n \Psi_n, h \rangle_{\mathcal{D}' \times \mathcal{D}},$$

where $h \in \mathcal{D}([0,T] \times \mathbb{R})$, to deduce that Φ is solution to (GP) in the sense of distributions. Moreover, we infer from the convergence in any compact set $K \subset \mathbb{R}$ and from relation (2.28) that $\Phi(0,.) = \Psi_0$. We now prove that $\Phi \in \mathcal{C}([0,T], Z^s(\mathbb{R}))$ for any $\frac{1}{2} < s < 1$, with

$$Z^s = \{u \in L^\infty(\mathbb{R}), u' \in H^{s-1}(\mathbb{R})\}.$$

Let $t \in [0,T]$. Up to a subsequence (depending on t), we deduce from (2.20), (2.18) and (2.28) that

$$\partial_x \Psi_n(t,.) \rightharpoonup \partial_x \Phi(t,.) \quad \text{in} \quad L^2(\mathbb{R}) \quad \text{and} \quad \eta_n(t,.) \rightharpoonup 1 - |\Phi(t,.)|^2 \quad \text{in} \quad L^2(\mathbb{R}), \tag{2.29}$$

when $n \to +\infty$. On the other hand, we know that

$$\int_{\mathbb{R}} |\partial_x \Phi(t,.)|^2 \leq M^2 \quad \text{and} \quad \int_{\mathbb{R}} (1 - |\Phi(t,.)|^2)^2 \leq M^2. \tag{2.30}$$

Arguing as in the proof of (2.22), we find that $\Phi(t,x)$ is uniformly bounded with respect to $x \in \mathbb{R}$ and $t \in [0,T]$. In particular, $\partial_x\Phi \in L^\infty([0,T], L^2(\mathbb{R}))$ et $1-|\Phi|^2 \in L^\infty([0,T], H^1(\mathbb{R}))$. Since

$$i\partial_t(\partial_x\Phi) = -\partial_x^3\Phi - \partial_x(\eta\Phi),$$

we have $\partial_x\Phi \in W^{1,\infty}([0,T], H^{-2}(\mathbb{R})) \subset \mathcal{C}([0,T], H^{-2}(\mathbb{R}))$. Hence, $\partial_x\Phi$ is continuous with values into $H^s(\mathbb{R})$ for any $-2 < s < 0$. Similarly, η_n is solution to the equation

$$\partial_t\eta_n = 2\partial_x(\langle i\partial_x\Psi_n, \Psi_n\rangle_{\mathbb{C}}). \tag{2.31}$$

In view of the convergence established in (2.28) and (2.23), we have

$$h\langle i\partial_x\Psi_n, \Psi_n\rangle_{\mathbb{C}} \to h\langle i\partial_x\Phi, \Phi\rangle_{\mathbb{C}} \quad \text{in} \quad L^2([0,T]\times\mathbb{R}),$$

for any $h \in \mathcal{D}(\mathbb{R})$. Using (2.27) and taking the limit when $n \to +\infty$ in (2.31), we find that

$$\partial_t(1-|\Phi|^2) = 2\partial_x(\langle i\partial_x\Phi, \Phi\rangle_{\mathbb{C}}),$$

in the sense of distributions. We deduce as above that $1-|\Phi|^2 \in W^{1,\infty}([0,T], H^{-1}(\mathbb{R})) \subset \mathcal{C}([0,T], H^{-1}(\mathbb{R}))$. Moreover, $1-|\Phi|^2$ is continuous from $[0,T]$ into $H^s(\mathbb{R})$ for all $-1 \le s < 1$. It remains to apply the Sobolev embedding theorem to guarantee that $\Phi \in \mathcal{C}([0,T], L^\infty(\mathbb{R}))$, so that $\Phi \in \mathcal{C}([0,T], Z^s(\mathbb{R}))$ for $\frac{1}{2} < s < 1$. The two functions Φ and Ψ are two solutions to (GP) in $\mathcal{C}([0,T], Z^s(\mathbb{R}))$ with the same initial data Ψ_0. To conclude, we need the following result of Cauchy problem for (GP) in the space $Z^s(\mathbb{R}) \supsetneq E^0$, equipped with the norm

$$\|\psi\|_{Z^s} = \|\psi\|_{L^\infty(\mathbb{R})} + \|\psi\|_{H^{s-1}(\mathbb{R})},$$

with $\frac{1}{2} < s < 1$.

Proposition 2.6 *Let $\frac{1}{2} < s < 1$ et $\psi_0 \in Z^s(\mathbb{R})$. There exists a unique maximal solution $\psi \in \mathcal{C}(]T_{min}, T_{max}[, Z^s(\mathbb{R}))$ to (GP) with $\psi(0,.) = \psi_0$.*

Proof. We refer to [1] for the proof. ■

In view of Proposition 2.6, the two solutions Φ and Ψ are equal. We have just proved that for any $t \in [0,T]$ and up to a subsequence (independent of t),

$$\partial_x\Psi_n(t,.) \rightharpoonup \partial_x\Psi(t,.) \quad \text{and} \quad \eta_n(t,.) \rightharpoonup 1-|\Psi(t,.)|^2 \quad \text{in} \quad L^2(\mathbb{R}), \tag{2.32}$$

and that for any compact set $K \subset \mathbb{R}$,

$$\Psi_n(t,.) \to \Psi(t,.) \quad \text{in} \quad L^\infty(K),$$

when $n \to +\infty$. To complete the proof, we argue by contradiction. Assume that there exists $T > 0, h \in L^2(\mathbb{R})$, and $\delta > 0$, such that for a further subsequence $(\Psi_{\phi(n)})_{n\in\mathbb{N}}$,

$$\Big|\int_{\mathbb{R}}(\partial_x\Psi_{\phi(n)}(T,x) - \partial_x\Psi(T,x))\bar{h}(x)dx\Big| > \delta.$$

Up to the choice of a further subsequence (possibly depending on T),this is in contradiction with (2.32). A similar argument proves the weak convergence of $\{\eta_n\}_{n\in\mathbb{N}}$ and the uniform convergence of $(\Psi_n)_{n\in\mathbb{N}}$ on any compact set $K \subset \mathbb{R}$. Since the proof extends with no change to the case where T is negative, this concludes the proof. ■

Proposition 2.5 together with the conservation of the energy Σ along the flow $\Psi(t,.)$ yield the following result of strong convergence

Corollary 2.7 *Let the sequence* $(\Psi_{n,0})_{n\in\mathbb{N}} \in (E^0)^{\mathbb{N}}$ *and* $\Psi_0 \in E^0$ *satisfy*

$$\lim_{n\to\infty} d^0_{loc}(\Psi_{n,0}, \Psi_0) = 0. \tag{2.33}$$

Then the solutions Ψ_n *and* Ψ *of (GP) with initial data* $\Psi_{n,0}$ *and* Ψ_0*, respectively, satisfy*

$$\lim_{n\to\infty} d^0_{loc}(\Psi_n(t,.), \Psi(t,.)) = 0 \quad \forall t \in \mathbb{R}.$$

Proof. The condition (2.33) implies (2.17) and (2.18). Indeed, if $K \subset \mathbb{R}$ is a compact set, for any $x \in K$ we have

$$\Psi_{n,0}(x) - \Psi_0(x) = \Psi_{n,0}(0) - \Psi_0(0) + \int_0^x (\Psi'_{n,0}(z) - \Psi'_0(z))dz,$$

which implies in turn that

$$\|\Psi_{n,0} - \Psi_0\|_{L^\infty(K)} \le \|\Psi_{n,0} - \Psi_0\|_{L^\infty(-1,1)} + \sqrt{|K| + dist(0,K)}\|\Psi'_{n,0} - \Psi'_0\|_{L^2(\mathbb{R})}.$$

In view of Proposition 2.5, the weak convergence in $L^2(\mathbb{R})$ to $\partial_x\Psi(t,.)$ and of $(\partial_x\Psi_n(t,.))_{n\in\mathbb{N}}$ to $1-|\Psi(t,.)|^2$ hold. Moreover, $(\Psi_n(t,.))_{n\in\mathbb{N}}$ tends to $\Psi(t,.)$ in $L^\infty(-1,1)$ for any $t \in \mathbb{R}$. Thus for any $t \in \mathbb{R}$, we have

$$\begin{cases} \liminf_{n\to\infty} \|\partial_x\Psi_n(t,.)\|_{L^2} \ge \|\partial_x\Psi(t,.)\|_{L^2}, \\ \liminf_{n\to\infty} \|1-|\Psi_n(t,.)|^2\|_{L^2} \ge \|1-|\Psi(t,.)|^2\|_{L^2}. \end{cases}$$

Furthermore, since

$$\lim_{n\to\infty} \Sigma(\Psi_n(t,.)) = \lim_{n\to\infty} \Sigma(\Psi_{n,0}) = \Sigma(\Psi_0) = \Sigma(\Psi(t,.)),$$

we have

$$\begin{aligned} \frac{1}{2}\limsup_{n\to\infty} \|\partial_x\Psi_n\|^2_{L^2} + \frac{1}{4}\liminf_{n\to\infty}\|1-|\Psi_n|^2\|^2_{L^2} &\le \limsup_{n\to\infty} \Sigma(\Psi_n) \\ &= \lim_{n\to\infty} \Sigma(\Psi_n) \\ &= \frac{1}{2}\|\partial_x\Psi\|^2_{L^2} + \frac{1}{4}\|1-|\Psi|^2\|^2_{L^2} \\ &\le \frac{1}{2}\liminf_{n\to\infty}\|\partial_x\Psi_n\|^2_{L^2} \\ &\quad + \frac{1}{4}\liminf_{n\to\infty}\|1-|\Psi_n|^2\|^2_{L^2}. \end{aligned}$$

Thus, for any $t \in \mathbb{R}$, the two sequences $(\|\partial_x\Psi_n(t,.)\|_{L^2})_{n\in\mathbb{N}}$ and $(\|1-|\Psi_n(t,.)|^2\|_{L^2})_{n\in\mathbb{N}}$ converge to $\|\partial_x\Psi(t,.)\|_{L^2}$ and $\|1-|\Psi(t,.)|^2\|_{L^2}$, respectively. The weak convergence together with the convergence of the $L^2(\mathbb{R})-$norm yield the strong convergence. This completes the proof. ∎

3 From the classical formulation to the hydrodynamical one

The main purpose of this section is to present the proofs of Proposition 1.1 and Theorem 1.2.

3.1 Proof of Proposition 1.1

Let $k \in \mathbb{N}$. We define the application Φ_1 by

$$\begin{array}{rrcl} \Phi_1 : & A^k & \longrightarrow & \mathcal{NV}^k(\mathbb{R}) \times \mathbb{R}/(2\pi\mathbb{Z}) \\ & u & \longmapsto & ((\eta(x), v(x)), \theta) := ((1-|u(x)|^2, \langle \frac{i}{\bar{u}(x)}, u'(x)\rangle_{\mathbb{C}}), \arg(u(0))). \end{array}$$

Let $((\eta,v),\theta)\in\mathcal{NV}^k(\mathbb{R})\times\mathbb{R}/(2\pi\mathbb{Z})$. Clearly, we have

$$\Phi_1(\Phi(((\eta,v),\theta)))=((\eta,v),\theta).$$

Let $\Psi\in A^k$ and let $\omega=|\Psi|^{-1}\Psi$. Then $\omega\in E^0$ and we have

$$\omega'-i\langle\frac{i}{\bar{\Psi}},\Psi'\rangle_{\mathbb{C}}\omega=0\quad\text{almost everywhere.}\tag{3.1}$$

This yields

$$\omega=\omega(0)\exp\left(i\int_0^x\langle\frac{i}{\bar{\Psi}},\Psi'\rangle_{\mathbb{C}}\right)=|\Psi|^{-1}\Phi(\Phi_1(\Psi)).$$

It follows that

$$\Phi(\Phi_1(\Psi))=\Psi,$$

so that Φ is a bijection whose inverse is $\Phi^{-1}=\Phi_1$. To establish the continuity properties, We begin by proving the following lemmas

Lemma 3.1 *Let $(g_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ be two sequences with elements in $L^\infty(\mathbb{R})$ and $L^2(\mathbb{R})$ respectively, the sequence $(g_n)_{n\in\mathbb{N}}$ being in addition bounded in $L^\infty(\mathbb{R})$. Assume that there exists $(g,f)\in L^\infty(\mathbb{R})\times L^2(\mathbb{R})$ such that, for any compact set $K\subset\mathbb{R}$, the sequence $(g_n)_{n\in\mathbb{N}}$ converges to g in $L^\infty(K)$ and the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^2(\mathbb{R})$. Then the sequence $(f_ng_n)_{n\in\mathbb{N}}$ converges to fg in $L^2(\mathbb{R})$.*

Proof. We can easily see that for any compact set $K\subset\mathbb{R}$, the sequence $(f_ng_n)_{n\in\mathbb{N}}$ converges to fg in $L^2(K)$. There exists $M>0$ such that for every $n\in\mathbb{N}$ we have $\|g_n\|_{L^\infty(\mathbb{R})}\le M$. Let $\epsilon>0$. There exists $x_0>0$ and $(n_0,n_1)\in(\mathbb{N})^2$ (depending on x_0) such that

$$\begin{cases}\int_{\mathbb{R}\setminus[-x_0,x_0]}|f|^2\le\frac{\epsilon^2}{6(M+\|g\|_{L^\infty(\mathbb{R})})^2}, & \\ \int_{\mathbb{R}\setminus[-x_0,x_0]}|f_n-f|^2\le\frac{\epsilon^2}{6M^2} & \forall n\ge n_0,\\ \int_{-x_0}^{x_0}|f_ng_n-fg|^2\le\frac{\epsilon^2}{3} & \forall n\ge n_1.\end{cases}$$

Then for every $n\ge\max(n_0,n_1)$, we have

$$\begin{aligned}\int_{\mathbb{R}}|f_ng_n-fg|^2 &= \int_{-x_0}^{x_0}|f_ng_n-fg|^2+\int_{\mathbb{R}\setminus[-x_0,x_0]}|(f_n-f)g_n+(g_n-g)f|^2\\ &\le \int_{-x_0}^{x_0}|f_ng_n-fg|^2+2M^2\int_{\mathbb{R}\setminus[-x_0,x_0]}|f_n-f|^2\\ &\quad +2(M+\|g\|_{L^\infty(\mathbb{R})})^2\int_{\mathbb{R}\setminus[-x_0,x_0]}|f|^2\\ &\le \epsilon^2,\end{aligned}$$

which completes the proof. ∎

Lemma 3.2 *Let $k\in\mathbb{N}$ and $\Psi\in A^k$. We set $\eta=1-|\Psi|^2$ and $v=\langle\frac{i}{\bar{\Psi}},\Psi'\rangle_{\mathbb{C}}$. Then the application $\Psi\mapsto((\eta,v),\arg(\Psi(0)))$ is continuous from (A^k,d^k_{loc}) into $(\mathcal{NV}^k(\mathbb{R})\times\mathbb{R}/(2\pi\mathbb{Z}),\|.\|_{X^k}+|.|_{\mathbb{R}/(2\pi\mathbb{Z})})$ and Lipschitz-continuous from (A^k,d^k) into $(\mathcal{NV}^k(\mathbb{R})\times\mathbb{R}/(2\pi\mathbb{Z}),\|.\|_{X^k}+|.|_{\mathbb{R}/(2\pi\mathbb{Z})})$.*

Proof. Let $\Psi\in A^k$ and let $(\Psi_n)_{n\in\mathbb{N}}$ be a sequence in A^k such that $d^k_{loc}(\Psi_n,\Psi)\to 0$ when $n\to 0$. We first prove that the sequence $((\eta_n,v_n))_{n\in\mathbb{N}}$ converges to (η,v) in $X^k(\mathbb{R})$. Let $j\le k$. We have

$$v_n^{(j)}=(\langle\frac{i}{\bar{\Psi}_n},\Psi'_n\rangle_{\mathbb{C}})^{(j)}=\sum_{l=0}^{j}C_l^j\left\langle i\left(\frac{1}{\bar{\Psi}_n}\right)^{(l)},\Psi_n^{(j-l+1)}\right\rangle_{\mathbb{C}}.$$

Using the Faà di Bruno formula for the derivative of two composite functions, we obtain

$$\left(\frac{1}{\Psi_n}\right)^{(l)} = \sum_{\pi\in\Gamma_l}(-1)^{|\pi|}|\pi|!\Psi_n^{-(|\pi|+1)}\prod_{B\in\pi}\Psi_n^{(|B|)}, \quad l\geq 1,$$

where Γ_l is the set of partitions of $\{1,...,l\}$. Then

$$\begin{aligned}(\langle \frac{i}{\bar{\Psi}_n},\Psi'_n\rangle_{\mathbb{C}})^{(j)} &= \sum_{l=1}^{j}\sum_{\pi\in\Gamma_l}C_l^j(-1)^{|\pi|}|\pi|!\left\langle i\bar{\Psi}_n^{-(|\pi|+1)}\prod_{B\in\pi}\bar{\Psi}_n^{(|B|)},\Psi_n^{(j-l+1)}\right\rangle_{\mathbb{C}} \\ &\quad+\left\langle\frac{i}{\bar{\Psi}_n},\Psi_n^{(j+1)}\right\rangle_{\mathbb{C}}. \end{aligned} \tag{3.2}$$

Since $d^k_{loc}(\Psi_n,\Psi)\to 0$ and $\Psi\in A^k$, for any compact set $K\subset\mathbb{R}$, there exists an integer $N_K\in\mathbb{N}$ such that the sequence $\left(\frac{1}{\Psi_n}\right)_{n\geq N_K}$ converges to $\frac{1}{\Psi}$ in $L^\infty(K)$. Moreover, there exists an integer $n_0\geq N_K$ and $M>0$ such that, for any $n\geq n_0$, we have $\|\frac{1}{\Psi_n}\|_{L^\infty(\mathbb{R})}\leq M$. For every $1\leq j\leq k$, the sequence $\left(\prod_{B\in\pi}\bar{\Psi}_n^{(|B|)}\right)_{n\in\mathbb{N}}$ converges to $\prod_{B\in\pi}\bar{\Psi}^{(|B|)}$ in $L^2(\mathbb{R})$, and the sequence $\left(\Psi_n^{(j-l+1)}\right)_{n\in\mathbb{N}}$ converges to $\Psi^{(j-l+1)}$ in $L^\infty(\mathbb{R})$, since $(\Psi'_n)_{n\in\mathbb{N}}$ converges to Ψ' in $H^k(\mathbb{R})$ and $\sum_{B\in\pi}|B|=l$. In view of Lemma 3.1, the two sequences $\left(\left\langle\frac{i}{\bar{\Psi}_n},\Psi_n^{(j+1)}\right\rangle_{\mathbb{C}}\right)_{n\in\mathbb{N}}$ and $\left(\bar{\Psi}_n^{-(|\pi|+1)}\prod_{B\in\pi}\bar{\Psi}_n^{(|B|)}\right)_{n\in\mathbb{N}}$ converge in $L^2(\mathbb{R})$ to $\left\langle\frac{i}{\bar{\Psi}},\Psi^{(j+1)}\right\rangle_{\mathbb{C}}$ and $\bar{\Psi}^{-(|\pi|+1)}\prod_{B\in\pi}\bar{\Psi}^{(|B|)}$, respectively. This proves the convergence of $(v_n^{(j)})_{n\in\mathbb{N}}$ to $v^{(j)}$ in $L^2(\mathbb{R})$.
On the other hand, for $1\leq j\leq k+1$, we have

$$\begin{aligned}\eta_n^{(j)} &= -2\sum_{l=0}^{j}C_l^j\langle\Psi_n^{(l)},\Psi_n^{(j-l)}\rangle_{\mathbb{C}} \\ &= -2\sum_{l=1}^{j-1}C_l^j\langle\Psi_n^{(l)},\Psi_n^{(j-l)}\rangle_{\mathbb{C}}-2\langle\Psi_n,\Psi_n^{(j)}\rangle_{\mathbb{C}}.\end{aligned}\tag{3.3}$$

The sequence $\left(\langle\Psi_n^{(l)},\Psi_n^{(j-l)}\rangle_{\mathbb{C}}\right)_{n\in\mathbb{N}}$ converges to $\langle\Psi^{(l)},\Psi^{(j-l)}\rangle_{\mathbb{C}}$ in $L^2(\mathbb{R})$. In view of Lemma 2.1, there exists $C>0$ such that $\|\Psi_n\|_{L^\infty(\mathbb{R})}\leq C(1+\sqrt{\Sigma(\Psi_n)})$. Then the sequence $(\Psi_n)_{n\in\mathbb{N}}$ is bounded from above in $L^\infty(\mathbb{R})$ and Lemma 3.1 implies the convergence of the sequence $\left(\langle\Psi_n,\Psi_n^{(j)}\rangle_{\mathbb{C}}\right)_{n\in\mathbb{N}}$ to $\langle\Psi,\Psi^{(j)}\rangle_{\mathbb{C}}$ in $L^2(\mathbb{R})$, which proves the convergence of $\left(\eta_n^{(j)}\right)_{n\in\mathbb{N}}$ to $\eta^{(j)}$ in $L^2(\mathbb{R})$.

Let $\Psi_0,\Psi\in A^k$. The local Lipschitz continuity is obtained by applying formulas (3.2) and (3.3) on Ψ_0 t Ψ, by taking the difference, and by estimating the $L^2(\mathbb{R})-$norms of $\Psi^m\Psi^{(K)}-\Psi_0^m\Psi_0^{(K)}$ with $(m,K)\in\mathbb{Z}^*\times\mathbb{N}$. In the case where m is negative, we just note that

$$\begin{aligned}\|\Psi^m\Psi^{(K)}-\Psi_0^m\Psi_0^{(K)}\|_{L^2(\mathbb{R})} &\leq \frac{\|\Psi_0^{(K)}\|_{L^2(\mathbb{R})}}{\inf_x|\Psi^{-m}(x)\Psi_0^{-m}(x)|}\|\Psi^{-m}-\Psi_0^{-m}\|_{L^\infty(\mathbb{R})} \\ &\quad+\frac{1}{\inf_x|\Psi^{-m}(x)|}\|\Psi^{(K)}-\Psi_0^{(K)}\|_{L^2(\mathbb{R})} \\ &\leq \|\Psi\Psi_0\|^m_{L^\infty(\mathbb{R})}\|\Psi_0^{(K)}\|_{L^2(\mathbb{R})}\|\Psi^{-m}-\Psi_0^{-m}\|_{L^\infty(\mathbb{R})} \\ &\quad+\|\Psi\|^m_{L^\infty(\mathbb{R})}\|\Psi^{(K)}-\Psi_0^{(K)}\|_{L^2(\mathbb{R})}.\end{aligned}$$

Besides, the function $\arg:\mathbb{C}^*\to\mathbb{R}/(2\pi Z)$ is of class $\mathcal{C}^\infty$ (hence locally continuous Lipschitz). Then, by fixing $\Psi_0\in A^k$, we find that there exist two strictly nonnegative constants C and δ depending on

Ψ_0 such that, for all $\Psi \in A^k$ with $d^k(\Psi_0, \Psi) \leq \delta$, we have

$$\|(\eta_0 - \eta, v_0 - v)\|_{X^k} + |\arg(\Psi_0(0)) - \arg(\Psi(0))|_{\mathbb{R}/(2\pi Z)} \leq C d^k(\Psi_0, \Psi),$$

with $\eta = 1 - |\Psi|^2$, $\eta_0 = 1 - |\Psi_0|^2$, $v = \langle \frac{i}{\Psi}, \Psi' \rangle_{\mathbb{C}}$ and $v_0 = \langle \frac{i}{\Psi_0}, \Psi_0' \rangle_{\mathbb{C}}$. ■

The following lemma proves the converse result to that of previous lemma.

Lemma 3.3 *Let $k \in \mathbb{N}$ and $((\eta, v), \theta) \in \mathcal{NV}^k(\mathbb{R}) \times \mathbb{R}/(2\pi\mathbb{Z})$. We set $\Psi = \sqrt{1-\eta}u$ with $u = \exp\big(i(\theta + \int_0^x v(z)dz)\big)$. Then $\Psi \in A^k$ and the application $((\eta, v), \theta) \mapsto \Psi$ is continuous from $(\mathcal{NV}^k(\mathbb{R}) \times \mathbb{R}/(2\pi\mathbb{Z}))$ into (A^k, d^k_{loc}).*

Proof. Let $((\eta, v), \theta) \in \mathcal{NV}^k(\mathbb{R}) \times \mathbb{R}/(2\pi\mathbb{Z})$ and let $((\eta_n, v_n), \theta_n)_{n\in\mathbb{N}}$ be a sequence of elements in $\mathcal{NV}^k(\mathbb{R}) \times \mathbb{R}/(2\pi\mathbb{Z})$ such that

$$\|(\eta_n - \eta, v_n - v)\|_{X^k} + |\theta_n - \theta|_{\mathbb{R}/(2\pi\mathbb{Z})} \to 0 \quad \text{when} \quad n \to 0. \tag{3.4}$$

We have to show that $(\Psi_n = \sqrt{1-\eta_n}u_n)_{n\in\mathbb{N}}$ converges to $\Psi = \sqrt{1-\eta}u$ in (E^k, d^k_{loc}). First, we clearly have $\|\Psi_n - \Psi\|_{L^\infty(-1,1)} \to 0$ and $\||\Psi_n|^2 - |\Psi|^2\|_{L^2} \to 0$ when $n \to 0$. Let $1 \leq j \leq k+1$. We have

$$\Psi_n^{(j)} = (\sqrt{1-\eta_n}u_n)^{(j)} = \sum_{l=0}^{j} C_l^j (\sqrt{1-\eta_n})^{(l)} u_n^{(j-l)},$$

$$u_n^{(l)} = \sum_{\gamma \in \Gamma_{j-l}} i^{|\gamma|} u_n \prod_{D \in \gamma} v_n^{(|D|-1)}, \quad j - l \geq 1,$$

and, by using the Faà di Bruno formula, we obtain

$$(\sqrt{1-\eta_n})^{(l)} = \sum_{\pi \in \Gamma_l} C(|\pi|)(1-\eta_n)^{\frac{1}{2}-|\pi|} \prod_{B \in \pi} \eta_n^{(|B|)}, \quad l \geq 1,$$

where Γ_l is the set of partitions for $\{1, ..., l\}$ and $C(|\pi|) = (-1)^{|\pi|} \prod_{s=0}^{|\pi|-1} (\frac{1}{2} - s)$. In the first sum, we have $\sum_{B\in\pi} |B| = l$ and $\sum_{D\in\gamma} |D| = j-l$, hence the sequence $\Big(\prod_{D\in\gamma, B\in\pi} v_n^{(|D|-1)} \eta_n^{(|B|)}\Big)_{n\in\mathbb{N}}$ converges to $\prod_{D\in\gamma, B\in\pi} v^{(|D|-1)} \eta^{(|B|)}$ in $L^2(\mathbb{R})$. Since $\|(\eta_n - \eta, v_n - v)\|_{X^k} \to 0$ when $n \to 0$, then for any compact set $K \subset \mathbb{R}$, the sequence $(u_n(1-\eta_n)^{\frac{1}{2}-|\pi|})_{n\in\mathbb{N}}$ converges to $u(1-\eta)^{\frac{1}{2}-|\pi|}$ in $L^\infty(K)$. Moreover, there exists $n_0 \in \mathbb{N}$ and $M > 0$ such that, for every $n \geq n_0$, we have $\|u_n(1-\eta_n)^{\frac{1}{2}-|\pi|}\|_{L^\infty(\mathbb{R})} \leq M$. In view of Lemma 3.1, it follows that the sequence $(\Psi^{(j)})_{n\in\mathbb{N}}$ converges to $\Psi^{(j)}$ in $L^2(\mathbb{R})$ for any $1 \leq j \leq k+1$, which finally proves that $d^k_{loc}(\Psi_n, \Psi) \to 0$ and $n \to 0$. ■

Conversely, the following result provide a counterexample to the continuity or the Lipschitz-continuity, which shows the importance of our choice for the topology d^k or d^k_{loc}.

Lemma 3.4 *The application $((\eta, v), \theta) \mapsto \Psi$ is not continuous from $(\mathcal{NV}^k(\mathbb{R}) \times \mathbb{R}/(2\pi\mathbb{Z}))$ in (A^k, d^k) and the application $\Psi \mapsto ((\eta, v), \arg(\Psi(0)))$ is not locally continuous Lipschitz from (A^k, d^k_{loc}) in $(\mathcal{NV}^k(\mathbb{R}) \times \mathbb{R}/(2\pi\mathbb{Z}))$.*

Proof. We provide a counterexample in each of the two cases when $k = 0$; these counterexamples can readily be adapted to the general cases. We define the sequence $((\eta_n, v_n))_{n\in\mathbb{N}^*}$ with elements in $\mathcal{NV}^0(\mathbb{R})$ by

$$((\eta_n(x), v_n(x)), \theta_n) = \left(\left(0, \frac{1}{n(1+|x|)}\right), 0\right).$$

We remark that

$$\lim_{n\to+\infty} \|(\eta_n, v_n)\|_{X^0} = 0.$$

We set

$$\Psi_n(x) = \exp\left(\frac{i}{n}\int_0^x \frac{dz}{(1+|z|)}\right) = \exp\left(\frac{i}{n}\delta(x)\ln(1+|x|)\right),$$

with

$$\delta(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

Then

$$\begin{aligned} d^0(\Psi_n, 1) &= \left\|\exp\left(\frac{i}{n}\delta\ln(1+|.|)\right) - 1\right\|_{L^\infty(\mathbb{R})} + \|v_n\|_{L^2(\mathbb{R})} \\ &= \left\|\exp\left(\frac{i}{n}\ln(1+|e^{n\pi}-1|)\right) - 1\right\|_{L^\infty(\mathbb{R})} + \|v_n\|_{L^2(\mathbb{R})} \\ &\geq 2. \end{aligned}$$

Hence $d^0(\Psi_n, 1) \nrightarrow 0$ and the application $(\eta, v) \mapsto \Psi$ is not continuous. The second counterexample is as follows. Let $\epsilon > 0$ and let $(\Psi, \tilde{\Psi}) \in (E^0)^2$ be defined by

$$\begin{cases} \tilde{\Psi}(x) = \Psi(x) = 1, \quad x \in]-\infty, 0[, \\ \Psi(x) = 1 \quad \text{et} \quad \tilde{\Psi}(x) = \exp(i\pi\epsilon x), \quad x \in [0, \frac{1}{\epsilon}[, \\ \Psi'(x) = \tilde{\Psi}'(x), \quad x \in [\frac{1}{\epsilon}, +\infty[. \end{cases}$$

Notice that Ψ and $\tilde{\Psi}$ depend on ϵ and that $|\Psi| = |\tilde{\Psi}|$. We set

$$\begin{cases} (\eta, v) = \left(1 - |\Psi|^2, \langle \frac{i}{\Psi}, \Psi'\rangle_{\mathbb{C}}\right), \\ (\tilde{\eta}, \tilde{v}) = \left(1 - |\tilde{\Psi}|^2, \langle \frac{i}{\tilde{\Psi}}, \tilde{\Psi}'\rangle_{\mathbb{C}}\right). \end{cases}$$

Then in view of equality $|\Psi| = |\tilde{\Psi}|$, we have on the one hand,

$$v + \tilde{v} = \frac{1}{|\Psi|^2}\langle i(\Psi + \tilde{\Psi}), \Psi'\rangle_{\mathbb{C}} = 0,$$

and

$$\|(\tilde{\eta}, \tilde{v}) - (\eta, v)\|_{X^0}^2 = \pi^2\epsilon + 4\int_{\frac{1}{\epsilon}}^{\infty} v^2.$$

On the other hand, for $\epsilon < 1$, we have

$$\begin{aligned} d^0_{loc}(\Psi, \tilde{\Psi}) &= \|1 - \exp(i\pi\epsilon.)\|_{L^\infty(0,1)} + \pi\sqrt{\epsilon} \\ &= |1 - \exp(i\pi\epsilon)| + \pi\sqrt{\epsilon} \\ &= \frac{\sin(\pi\epsilon)}{\sin(\frac{1}{2}\pi(1-\epsilon))} + \pi\sqrt{\epsilon}. \end{aligned}$$

Assume now that there exists $0 < C = C(\Psi, \tilde{\Psi}) \leq M$ (M is independent of ϵ) such that, for any $(\Psi, \tilde{\Psi}) \in A^0$, we have

$$\|(\tilde{\eta}, \tilde{v}) - (\eta, v)\|_{X^0} \leq C d^0_{loc}(\Psi, \tilde{\Psi}).$$

Then

$$\int_{\frac{1}{\epsilon}}^{\infty} v^2 \leq \frac{1}{4}\left(M\left(\frac{\sin(\pi\epsilon)}{\sin(\frac{1}{2}\pi(1-\epsilon))} + \pi\sqrt{\epsilon}\right) - \pi^2\epsilon\right),$$

and

$$\lim_{\epsilon\to 0}\left(\int_{\frac{1}{\epsilon}}^{\infty} v^2\right) = 0. \tag{3.5}$$

Take for instance

$$\begin{cases} \Psi(x) = 1 + i(x-\epsilon^{-1})e^{-(x-\epsilon^{-1})}, \\ \tilde{\Psi}(x) = -1 + i(x-\epsilon^{-1})e^{-(x-\epsilon^{-1})} \end{cases}$$

when $x \in [\frac{1}{\epsilon}, +\infty[$. Then we easily verify that

$$\int_{\frac{1}{\epsilon}}^{\infty} v^2 = \int_{\frac{1}{\epsilon}}^{\infty} \langle \frac{i}{\bar{\Psi}}, \Psi' \rangle_{\mathbb{C}} = \int_0^{\infty} \frac{(x-1)^2}{(1+x^2e^{-2x})^2 e^{2x}} dx,$$

which is in contraction with (3.5). ■

3.2 Proof of Theorem 1.2

We start by defining the function Ψ_0 by

$$\Psi_0 = \sqrt{1-\eta_0} \exp\left(i(\int_0^x v_0(z)dz) \right).$$

Clearly $\Psi_0 \in E^k$. Then, in view of the study of the Cauchy problem for (GP) which we have done, there exists $\Psi \in \mathcal{C}(\mathbb{R}, E^k)$ satisfying (GP) with $\Psi(0,.) = \Psi_0$. The function Ψ_0 does not vanish and satisfies the property

$$\lim_{|x|\to+\infty} |\Psi_0(x)| = 1.$$

Then there exists δ and $T_1, T_2 > 0$ such that

$$\inf_{t\in[-T_1,T_2], x\in\mathbb{R}} |\Psi(t,x)| > \delta.$$

Proposition 3.5 *Let*

$$\begin{cases} \eta(t,.) = 1 - |\Psi(t,.)|^2, \\ v(t,.) = \langle \frac{i}{\Psi(t,.)}, \partial_x \Psi(t,.) \rangle_{\mathbb{C}}, \quad t \in [-T_1, T_2]. \end{cases}$$

Then the function $(\eta, v) \in \mathcal{C}([-T_1,T_2], \mathcal{NV}^k)$ is solution to (HGP) with $(\eta, v)(0,.) = (\eta_0, v_0)$.

Proof. We treat the more difficult case ($k = 0$). We will show that (η, v) satisfies (HGP) in the sense of distributions on $[-T_1, T_2] \times \mathbb{R}$. To this end, we use the following regularization argument: Let $\rho \in \mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \rho = 1$ and let $\epsilon > 0$. We set $\rho_\epsilon(x) = \frac{1}{\epsilon}\rho(x/\epsilon)$ and $\Psi_\epsilon^0 = \Psi_0 \star \rho_\epsilon$. Let Ψ_ϵ be the solution of (GP) such that $\Psi_\epsilon(0,.) = \Psi_\epsilon^0$. In view of Theorem (1.3), we have

$$\sup_{t\in[-T_1,T_2]} d^0(\Psi_\epsilon(t,.), \Psi(t,.)) \to 0, \tag{3.6}$$

when $\epsilon \to 0$. Let $\xi \in \mathcal{D}([-T_1,T_2] \times \mathbb{R})$. Then

$$\langle \eta, \partial_t \xi \rangle_{\mathcal{D}\times\mathcal{D}'} = \langle |\Psi_\epsilon|^2 - |\Psi|^2, \partial_t \xi \rangle_{\mathcal{D}\times\mathcal{D}'} + 2\langle \langle \partial_t \Psi_\epsilon, \Psi_\epsilon \rangle_{\mathbb{C}}, \xi \rangle_{\mathcal{D}\times\mathcal{D}'},$$

and

$$\begin{aligned} \langle i(\partial_x^2 \Psi_\epsilon + (1-|\Psi_\epsilon|^2)\Psi_\epsilon), \Psi_\epsilon \rangle_{\mathbb{C}} &= \langle i\partial_x^2 \Psi_\epsilon, \Psi_\epsilon \rangle_{\mathbb{C}} \\ &= \partial_x \langle i\partial_x \Psi_\epsilon, \Psi_\epsilon \rangle_{\mathbb{C}} \\ &= -\partial_x \langle i\Psi_\epsilon, \partial_x \Psi_\epsilon \rangle_{\mathbb{C}}. \end{aligned}$$

Relation (3.6) shows that

$$
\begin{aligned}
|\Psi_\epsilon|^2 - |\Psi|^2 \to 0 \quad \text{in} \quad \mathcal{C}([-T_1, T_2], L^2(\mathbb{R})),\\
\langle i\Psi_\epsilon, \partial_x \Psi_\epsilon\rangle_{\mathbb{C}} \to \langle i\Psi, \partial_x\Psi\rangle_{\mathbb{C}} = (1-\eta)v \quad \text{in} \quad \mathcal{C}([-T_1, T_2], L^2(\mathbb{R})),
\end{aligned}
$$

when $\epsilon \to 0$, which means that

$$\partial_t \eta = 2\partial_x((1-\eta)v) \quad \text{in} \quad \mathcal{D}'([-T_1, T_2]\times \mathbb{R}).$$

We treat similarly the equation in $\partial_t v$. First, we set $v_\epsilon = \langle \frac{i}{\bar{\Psi}_\epsilon}, \partial_x \Psi_\epsilon\rangle_{\mathbb{C}}$, $\eta_\epsilon = 1 - |\Psi_\epsilon|^2$ and $\gamma_\epsilon = \eta_\epsilon - v_\epsilon^2 - \frac{\partial_x\eta_\epsilon}{2(1-\eta_\epsilon)} + \frac{(\partial_x\eta_\epsilon)^2}{4(1-\eta_\epsilon)^2}$. Then we get [3]

$$
\begin{aligned}
\partial_t \langle \frac{i}{\bar{\Psi}_\epsilon}, \partial_x\Psi_\epsilon\rangle_{\mathbb{C}} &= \langle \partial_t(\frac{i}{\bar{\Psi}_\epsilon}), \partial_x \Psi_\epsilon\rangle_{\mathbb{C}} + \langle \frac{i}{\bar{\Psi}_\epsilon}, \partial_x\partial_t\Psi_\epsilon\rangle_{\mathbb{C}}\\
&= \langle -\frac{\partial_x^2\bar{\Psi} + \eta_\epsilon\bar{\Psi}}{(\bar{\Psi}_\epsilon)^2}, \partial_x\Psi_\epsilon\rangle_{\mathbb{C}} + \langle \frac{i}{\bar{\Psi}_\epsilon}, i\partial_x(\partial_x^2\Psi_\epsilon + \eta_\epsilon\Psi_\epsilon)\rangle_{\mathbb{C}}\\
&= \langle -\frac{\partial_x\bar{\Psi}_\epsilon}{(\bar{\Psi}_\epsilon)^2}, \partial_x^2\Psi + \eta_\epsilon\Psi\rangle_{\mathbb{C}} + \langle \frac{1}{\bar{\Psi}_\epsilon}, \partial_x(\partial_x^2\Psi_\epsilon + \eta_\epsilon\Psi_\epsilon)\rangle_{\mathbb{C}}\\
&= \partial_x\langle \frac{1}{\bar{\Psi}_\epsilon}, \partial_x^2\Psi_\epsilon + \eta_\epsilon\Psi_\epsilon\rangle_{\mathbb{C}}.
\end{aligned}
$$

On the other hand, we have $\frac{\partial_x\eta_\epsilon}{2(1-\eta_\epsilon)} = -\langle \frac{1}{\bar{\Psi}_\epsilon}, \partial_x\Psi_\epsilon\rangle_{\mathbb{C}}$. Hence

$$
\begin{aligned}
\frac{(\partial_x\eta_\epsilon)^2}{4(1-\eta_\epsilon)^2} - v_\epsilon^2 &= \left(\frac{\partial_x\eta_\epsilon}{2(1-\eta_\epsilon)} - v_\epsilon\right)\left(\frac{\partial_x\eta_\epsilon}{2(1-\eta_\epsilon)} + v_\epsilon\right)\\
&= -\langle \frac{i-1}{\bar{\Psi}_\epsilon}, \partial_x\Psi_\epsilon\rangle_{\mathbb{C}}\langle \frac{1+i}{\bar{\Psi}_\epsilon}, \partial_x\Psi_\epsilon\rangle_{\mathbb{C}}\\
&= -\langle \partial_x(\frac{1}{\bar{\Psi}_\epsilon}), \partial_x\Psi_\epsilon\rangle_{\mathbb{C}}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\gamma_\epsilon &= \langle \frac{1}{\bar{\Psi}_\epsilon}, \eta_\epsilon\Psi_\epsilon\rangle_{\mathbb{C}} - \langle \partial_x(\frac{1}{\bar{\Psi}_\epsilon}), \partial_x\Psi_\epsilon\rangle_{\mathbb{C}} + \partial_x\langle \frac{1}{\bar{\Psi}_\epsilon}, \partial_x\Psi_\epsilon\rangle_{\mathbb{C}}\\
&= \langle \frac{1}{\bar{\Psi}_\epsilon}, \partial_x^2\Psi_\epsilon + \eta_\epsilon\Psi_\epsilon\rangle_{\mathbb{C}},
\end{aligned}
$$

and

$$\partial_t v_\epsilon = \partial_x\gamma_\epsilon.$$

Let now $\xi \in \mathcal{D}([-T_1, T_2]\times\mathbb{R})$. We have on the one hand,

$$\langle v_\epsilon, \partial_t\xi\rangle_{\mathcal{D}\times\mathcal{D}'} = \langle v - v_\epsilon, \partial_t\xi\rangle_{\mathcal{D}\times\mathcal{D}'} - \langle \partial_x\gamma_\epsilon, \xi\rangle_{\mathcal{D}\times\mathcal{D}'}.$$

On the other hand, it follows from (3.6) that

$$
\begin{aligned}
\eta_\epsilon \to \eta \quad &\text{in} \quad \mathcal{C}([-T_1, T_2], H^1(\mathbb{R})),\\
v_\epsilon \to v \quad &\text{in} \quad \mathcal{C}([-T_1, T_2], L^2(\mathbb{R})),\\
\frac{1}{1-\eta_\epsilon} \to \frac{1}{1-\eta} \quad &\text{in} \quad \mathcal{C}([-T_1, T_2], L^\infty(\mathbb{R})),
\end{aligned}
$$

[3] In what follows we use the identity $\langle a, b\rangle_{\mathbb{C}} = \langle 1, \bar{a}b\rangle_{\mathbb{C}}$ for any $(a,b)\in\mathbb{C}^2$

when $\epsilon \to 0$. Then

$$v_\epsilon^2 \to v^2 \quad \text{in} \quad \mathcal{C}([-T_1, T_2], L^1(\mathbb{R})),$$

$$\frac{(\partial_x \eta_\epsilon)^2}{(1-\eta_\epsilon)^2} \to \frac{(\partial_x \eta)^2}{(1-\eta)^2} \quad \text{in} \quad \mathcal{C}([-T_1, T_2], L^1(\mathbb{R})),$$

when $\epsilon \to 0$. Finally, we deduce that

$$\partial_x \gamma_\epsilon \to \eta - v^2 - \frac{\partial_x \eta}{2(1-\eta)} + \frac{(\partial_x \eta)^2}{4(1-\eta)^2} \quad \text{in} \quad \mathcal{C}([-T_1, T_2], H^{-2}(\mathbb{R})),$$

when $\epsilon \to 0$, and

$$\partial_t v = \partial_x \left(\eta - v^2 - \frac{\partial_x \eta}{2(1-\eta)} + \frac{(\partial_x \eta)^2}{4(1-\eta)^2} \right) \quad \text{in} \quad \mathcal{D}'([-T_1, T_2] \times \mathbb{R}).$$

■

Let now $(\eta, v) \in \mathcal{C}(I, \mathcal{NV}^k)$ be a solution of (HGP). We will show in the following proposition that we can reconstruct a solution Ψ of (GP) from (η, v). Such solution will be given by

$$\Psi(t,x) = \sqrt{1-\eta(t,x)} \exp(i\varphi(t,x)),$$

where φ is defined up to a constant $c = c(t)$ by $\partial_x \varphi = v$. It remains to determine c. More specifically, we show that there exists a function $c : I \to \mathbb{R}$ (depending on (η, v)) such that the function φ defined by

$$\varphi = \int_0^x v + c,$$

allows to reconstruct Ψ. The function c represents the temporal evolution of the phase of $\Psi(t, 0)$ and satisfies, for (η, v) smooth enough,

$$c(t) = \int_0^t \partial_t \varphi(\tau, x) d\tau + \int_0^x (v(0,z) - v(t,z)) dz.$$

In this case Ψ will be solution of (GP) if

$$\partial_t \varphi = S,$$

with

$$S = \eta - v^2 - \partial_x \left(\frac{\partial_x \eta}{2(1-\eta)} \right) + \frac{(\partial_x \eta)^2}{4(1-\eta)^2}.$$

This gives us the idea of the choice of c in the following proposition.

Proposition 3.6 *Let $I = [-T_1, T_2]$, with $T_1, T_2 > 0$, let $k \in \mathbb{N}$ and let $(\eta, v) \in \mathcal{C}(I, \mathcal{NV}^k)$ be a solution of (HGP). We set $\varphi_v(t,x) = \int_0^x v(t,z)dz$. There exists a unique function $c \in \mathcal{C}^1(I, \mathbb{R})$ satisfying $c(0) = 0$ such that the function $\Psi \in \mathcal{C}(I, E^k)$, defined by*

$$\Psi(t,x) = \sqrt{1-\eta(t,x)} \exp\left(i(\varphi_v(t,x) + c(t))\right),$$

is a solution of (GP).

Proof. Note that $\Psi \in \mathcal{C}(I, E^k)$ for all $k \in \mathbb{N}$. To show that the function Ψ defined above is solution to (GP), we detail the more difficult case $(k = 0)$. We define $a, S \in \mathcal{D}'(I \times \mathbb{R}, \mathbb{R})$ by

$$S(t,x) = \left(\eta - v^2 - \partial_x \left(\frac{\partial_x \eta}{2(1-\eta)} \right) + \frac{(\partial_x \eta)^2}{4(1-\eta)^2} \right)(t,x),$$

$$a(t,x) = \int_0^x (v(0,z) - v(t,z))dz + \int_0^t S(\tau,x)d\tau.$$

Then

$$\partial_t a = S - \partial_t \varphi_v \quad \text{in} \quad \mathcal{D}'(I \times \mathbb{R}, \mathbb{R}). \tag{3.7}$$

Let $\chi \in \mathcal{D}(\mathbb{R},\mathbb{R})$ be such that $\int_{\mathbb{R}} \chi(z)dz = 1$. We set

$$c(t) = \langle a(t,.), \chi \rangle_{\mathcal{D}'(\mathbb{R}) \times \mathcal{D}(\mathbb{R})}.$$

By construction, we have $c(0) = 0$ and, since $(\eta, v) \in \mathcal{C}(I, \mathcal{NV}^0)$, we also have $c \in \mathcal{C}^1(I, \mathbb{R})$. On the other hand, since (η, v) is solution to (HGP), we have

$$\begin{aligned} \partial_x(\partial_t a) &= -\partial_t v + \partial_x S \\ &= 0. \end{aligned}$$

Let $\xi \in \mathcal{D}(I \times \mathbb{R}, \mathbb{R})$. We set

$$\alpha(t) = \int_{\mathbb{R}} \xi(t,z)dz \in \mathcal{D}(I).$$

Then

$$\begin{aligned} \langle \varphi_v + c, \partial_t \xi \rangle_{\mathcal{D}' \times \mathcal{D}} &= -\langle \partial_t \varphi_v, \xi \rangle_{\mathcal{D}' \times \mathcal{D}} + \langle c, \partial_t \xi \rangle_{\mathcal{D}' \times \mathcal{D}} \\ &= -\langle S, \xi \rangle_{\mathcal{D}' \times \mathcal{D}} + \langle \partial_t a, \xi \rangle_{\mathcal{D}' \times \mathcal{D}} + \langle c, \partial_t \xi \rangle_{\mathcal{D}' \times \mathcal{D}} \\ &= -\langle S, \xi \rangle_{\mathcal{D}' \times \mathcal{D}} + \langle \partial_t a, \alpha \rangle_{\mathcal{D}'(I) \times \mathcal{D}(I)} + \langle c, \partial_t \alpha \rangle_{\mathcal{D}'(I) \times \mathcal{D}(I)} \\ &= -\langle S, \xi \rangle_{\mathcal{D}' \times \mathcal{D}}, \end{aligned} \tag{3.8}$$

which means that $\partial_t(\varphi_v + c) = S$ in $\mathcal{D}'(I \times \mathbb{R}, \mathbb{R})$. We shall prove that the function Ψ, defined by

$$\Psi(t,x) = \sqrt{1 - \eta(t,x)} \exp\left(i(\varphi_v(t,x) + c(t))\right),$$

is solution to (GP) on $I \times \mathbb{R}$. To this end, we need to compute the two derivatives $\partial_t \Psi$ and $\partial_x^2 \Psi$ in the sense of distributions. We use a regularization argument, where $(\eta_\epsilon, v_\epsilon)$ is defined for each $\epsilon > 0$ by

$$\begin{cases} \eta_\epsilon(t,.) = \eta(t,.) \star \rho_\epsilon, \\ v_\epsilon(t,.) = v(t,.) \star \rho_\epsilon \quad t \in I, \end{cases}$$

with $\rho_\epsilon(x) = \frac{1}{\epsilon}\rho(\frac{x}{\epsilon})$, $\rho \in \mathcal{D}(\mathbb{R})$, and $\int_{\mathbb{R}} \rho = 1$. We denote

$$\Psi_\epsilon = \sqrt{1 - \eta_\epsilon} \exp\left(i(\int_0^x v_\epsilon + c) \right).$$

The fact that $\|(\eta_\epsilon - \eta, v_\epsilon - v)(t,.)\|_{X^0} \to 0$ when $\epsilon \to 0$, together with Lemma 3.3, yield $d^0_{loc}(\Psi_\epsilon(t,.), \Psi(t,.)) \to 0$ when $\epsilon \to 0$. Let now $\xi \in \mathcal{D}(I \times \mathbb{R}, \mathbb{C})$. Then

$$\begin{aligned} \langle \Psi, \partial_t \xi \rangle_{\mathcal{D}' \times \mathcal{D}} &= \langle \Psi - \Psi_\epsilon, \partial_t \xi \rangle_{\mathcal{D}' \times \mathcal{D}} - \langle \partial_t \Psi_\epsilon, \xi \rangle_{\mathcal{D}' \times \mathcal{D}} \\ &= \langle \Psi - \Psi_\epsilon, \partial_t \xi \rangle_{\mathcal{D}' \times \mathcal{D}} - \left\langle \left(\frac{\partial_t \eta_\epsilon}{2(1 - \eta_\epsilon)} + i\partial_t(\int_0^x v_\epsilon + c) \right) \Psi_\epsilon, \xi \right\rangle_{\mathcal{D}' \times \mathcal{D}}. \end{aligned} \tag{3.9}$$

Since $\Psi_\epsilon \to \Psi$ in $\mathcal{D}'(I \times \mathbb{R}, \mathbb{C})$ when $\epsilon \to 0$, the first term of the right hand side of (3.9) converges to zero. For the second term, we have $\frac{\partial_t \eta}{1-\eta}\Psi \in \mathcal{C}(I, H^{-1}(\mathbb{R}))$, hence

$$\frac{\partial_t \eta_\epsilon}{2(1 - \eta_\epsilon)} \Psi_\epsilon \to \frac{\partial_t \eta}{1 - \eta} \Psi \quad \text{in} \quad \mathcal{D}'(I \times \mathbb{R}, \mathbb{C}),$$

when $\epsilon \to 0$. A similar argument [4] shows that $\partial_t(\int_0^x v_\epsilon + c)\Psi_\epsilon \to \partial_t(\varphi_v + c)\Psi$ in $\mathcal{D}'(I \times \mathbb{R}, \mathbb{C})$. We have just shown that, in the sense of distributions,

$$i\partial_t\Psi = -\left(i\frac{\partial_t\eta}{2(1-\eta)} + \partial_t(\varphi_v + c)\right)\Psi. \tag{3.10}$$

Similarly, we prove that, in the sense of distributions, we also have

$$\partial_x^2\Psi = \left(S - \eta + i\frac{\partial_x((1-\eta)v)}{1-\eta}\right)\Psi. \tag{3.11}$$

Thus combining (3.10) and (3.11), we find that, in the sense of distributions,

$$i\partial_t\Psi + \partial_x^2\Psi + \Psi(1 - |\Psi|^2) = 0,$$

since $\partial_t(\varphi_v + c) = S$ in $\mathcal{D}'(I \times \mathbb{R}, \mathbb{R})$ and (η, v) is solution to (HGP). Since the derivative of the continuous function c in the sense of distributions is completely determined by S and $\partial_t\varphi_v$, and since $c(0) = 0$, the uniqueness of c follows. ∎

Proposition 3.6 proves that the solution of (HGP) constructed by Proposition 3.5 is unique in the space $\mathcal{C}([-T_1, T_2], \mathcal{NV}^k(\mathbb{R}))$. Indeed, let $(\eta_1, v_1), (\eta_2, v_2) \in \mathcal{C}([-T_1, T_2], \mathcal{NV}^k(\mathbb{R}))$ denote two solutions to the equation (HGP) that satisfy

$$(\eta_1(0,.), v_1(0,.)) = (\eta_2(0,.), v_2(0,.)) = (\eta_0, v_0) \in \mathcal{NV}^k(\mathbb{R}).$$

Then, in view of Proposition 3.6, there exist $c_1, c_2 \in \mathcal{C}^1([-T_1, T_2], \mathbb{R})$ such that the two functions

$$\Psi_1 = \sqrt{1-\eta_1}\exp(i(\int_0^x v_1 + c_1)) \quad \text{et} \quad \Psi_2 = \sqrt{1-\eta_2}\exp(i(\int_0^x v_2 + c_2)),$$

are solutions to (GP) in the space $\mathcal{C}([-T_1, T_2], E^k)$ satisfying

$$\Psi_1(0,.) = \Psi_2(0,.) = \sqrt{1-\eta_0}\exp(i(\int_0^x v_0)).$$

Since the Cauchy problem for (GP) is well-posed in the space $\mathcal{C}([-T_1, T_2], E^k)$, we have in view of Proposition (1.1) that

$$\varPhi^{-1}(\Psi_1(t,.)) = \varPhi^{-1}(\Psi_2(t,.)) \quad \forall t \in [-T_1, T_2].$$

Consequently, we obtain that

$$\eta_1 = \eta_2 \quad \text{and} \quad v_1 = v_2.$$

This proves the well-posedness in $\mathcal{C}([-T_1, T_2], \mathcal{NV}^k(\mathbb{R}))$ of the Cauchy problem of (HGP).

Equation (HGP) is invariant with respect to the change of variable $t \to t + c$. Then a similar result to that of Proposition 3.5 can be proved for an initial data at time $t = t_0 \neq 0$. By using iteratively the two Propositions 3.5 and 3.6, this allows to prove the existence of maximal $T_*, T^* > 0$ such that the solution (η, v) can be extended to the interval $]-T_*, T^*[$, with

$$\lim_{t\to T^*}\max_{x\in\mathbb{R}}\eta(t,x) = 1, \quad \text{when} \quad T^* < +\infty,$$

$$\lim_{t\to -T_*}\max_{x\in\mathbb{R}}\eta(t,x) = 1, \quad \text{when} \quad -T_* > -\infty.$$

[4] We already found that $\partial_t(\varphi_v + c) = S \in \mathcal{C}(I, H^{-1}(\mathbb{R}))$.

3.2.1 Conservation of the quantities H and P

We show in this section that the energy

$$H(\eta, v) = \frac{1}{8}\int_{\mathbb{R}} \frac{(\partial_x \eta)^2}{1-\eta} + \frac{1}{2}\int_{\mathbb{R}} (1-\eta)v^2 + \frac{1}{4}\int_{\mathbb{R}} \eta^2$$

is conserved along $(\eta(t,.), v(t,.))$ when $(\eta, v) \in \mathcal{C}(]-T_*, T^*[, \mathcal{NV}^k)$. In view of Proposition 3.6, for any $[-T_1, T_2] \subset]-T_*, T^*[$, there exists $c \in \mathcal{C}^1([-T_1, T_2])$ such that the function

$$\Psi = \sqrt{1-\eta}\exp(i(\int_0^x v + c))$$

is solution to (GP) on $[-T_1, T_2] \times \mathbb{R}$. We have shown that the Ginzburg-Landau energy

$$\Sigma(\Psi) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x \Psi|^2 + \int_{\mathbb{R}} (1 - |\Psi|^2)^2$$

is conserved along the flow $\Psi(t,.)$. Moreover, we have

$$H(\eta(t,.), v(t,.)) = \Sigma(\Psi(t,.)) \quad \forall t \in [-T_1, T_2].$$

Thus $H(\eta(t,.), v(t,.))$ is constant on $[-T_1, T_2]$.

We now prove the conservation of the momentum P defined by

$$P(\eta, v) = \frac{1}{2}\int_{\mathbb{R}} \eta v.$$

In the case where $k \geq 1$, it suffices to note that if $(\eta, v) \in \mathcal{C}(]-T_*, T^*[, \mathcal{NV}^k)$, then

$$\frac{d}{dt}\int_{\mathbb{R}} \eta v = \int_{\mathbb{R}} \partial_t \eta v + \int_{\mathbb{R}} \eta \partial_t v.$$

Replacing $\partial_t \eta$ and $\partial_t v$ by the right-hand side member of (HGP) gives the desired result. The case $k = 0$ is more difficult. To treat this case, it is useful to show that the application $(\eta_0, v_0) \mapsto (\eta(t,.), v(t,.))$ is continuous from $\mathcal{NV}^0$ to $\mathcal{NV}^0$ in the following sense: For every sequence $((\eta_{0,n}, v_{0,n}))_{n\in\mathbb{N}}$ with elements in $\mathcal{NV}^0$ that converges to (η_0, v_0) in $(\mathcal{NV}^0, \|.\|_{X^0})$, the sequence $((\eta_n, v_n))_{n\in\mathbb{N}}$ of the solution of (HGP) with initial data $(\eta_{0,n}, v_{0,n})$ satisfies, for each $t \in]-T_*, T^*[$,

$$\lim_{n\to+\infty} \|(\eta_n(t,.), v_n(t,.)) - (\eta(t,.), v(t,.))\|_{X^0(\mathbb{R})} = 0.$$

This result is a consequence of Lemmas 3.2 and 3.3, and of Corollary 2.7, via the following diagram

$$\begin{tikzcd}
(\eta_0, v_0) \arrow[r, "3.3"] \arrow[d] & \Psi_0 \arrow[d, "2.7"] \\
(\eta(t,.), v(t,.)) & \Psi(t,.) \arrow[l, "3.2"']
\end{tikzcd}$$

where $\Psi_0 = \sqrt{1-\eta_0}\exp(i(\int_0^x v_0))$ and Ψ is the solution of (HGP) with initial data Ψ_0. Now, in order to prove the conservation of the momentum P, we use the following regularization argument: Let $\rho_\epsilon = \frac{1}{\epsilon}\rho(./\epsilon)$ with $\rho \in \mathcal{D}(\mathbb{R})$ and $\epsilon > 0$ and let

$$\begin{cases} \eta_\epsilon^0 = \eta_0 \star \rho_\epsilon, \\ v_\epsilon^0 = v_0 \star \rho_\epsilon. \end{cases}$$

Let $(\eta_\epsilon, v_\epsilon)$ be the solution of (GP) with initial data $(\eta_\epsilon^0, v_\epsilon^0)$. Noting that

$$\|(\eta_\epsilon^0, v_\epsilon^0) - (\eta_0, v_0)\|_{X^0(\mathbb{R})} \to 0,$$

when $\epsilon \to 0$, the above continuity property shows that, for any $t \in]-T_*, T^*[$,

$$\lim_{\epsilon \to 0} \|(\eta_\epsilon(t,.), v_\epsilon(t,.)) - (\eta(t,.), v(t,.))\|_{X^0(\mathbb{R})} = 0.$$

Thus, for any $t \in]-T_*, T^*[$, we have

$$\lim_{\epsilon \to 0} P(\eta_\epsilon(t,.), v_\epsilon(t,.)) = P(\eta(t,.), v(t,.)).$$

It remains to show that

$$\frac{d}{dt} P(\eta_\epsilon(t,.), v_\epsilon(t,.)) = 0.$$

This follows from

$$\begin{aligned}
\int_{\mathbb{R}} \partial_t v_\epsilon \eta_\epsilon &= -\int_{\mathbb{R}} \left(\eta_\epsilon - v_\epsilon^2 - \partial_x \left(\frac{\partial_x \eta_\epsilon}{2(1-\eta_\epsilon)} \right) + \frac{(\partial_x \eta_\epsilon)^2}{4(1-\eta_\epsilon)^2} \right) \partial_x \eta_\epsilon \\
&= -\frac{1}{2} \int_{\mathbb{R}} \partial_x(\eta_\epsilon^2) + \int_{\mathbb{R}} \partial_x \eta_\epsilon v_\epsilon^2 - \frac{1}{4} \int_{\mathbb{R}} \frac{\partial_x (\partial_x \eta_\epsilon)^2 (1-\eta_\epsilon) + (\partial_x \eta_\epsilon)^3}{(1-\eta_\epsilon)^2} \\
&= \int_{\mathbb{R}} \partial_x \eta_\epsilon v_\epsilon^2 - \frac{1}{4} \int_{\mathbb{R}} \partial_x \left(\frac{(\partial_x \eta_\epsilon)^2}{1-\eta_\epsilon} \right).
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} \partial_t \eta_\epsilon v_\epsilon &= 2 \int_{\mathbb{R}} \partial_x ((1-\eta_\epsilon) v_\epsilon) v_\epsilon \\
&= -\int_{\mathbb{R}} \partial_x \eta_\epsilon v_\epsilon^2 \\
&= -\int_{\mathbb{R}} \partial_t v_\epsilon \eta_\epsilon.
\end{aligned}$$

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