Isoperimetry and stability properties of balls with respect to nonlocal energies

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ISOPERIMETRY AND STABILITY PROPERTIES OF BALLS WITH RESPECT TO NONLOCAL ENERGIES

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ABSTRACT. We obtain a sharp quantitative isoperimetric inequality for nonlocal s-perimeters, uniform with respect to s bounded away from 0. This allows us to address local and global minimality properties of balls with respect to the volume-constrained minimization of a free energy consisting of a nonlocal s-perimeter plus a non-local repulsive interaction term. In the particular case $s = 1$ the s-perimeter coincides with the classical perimeter, and our results improve the ones of Knüpfer and Muratov [25, 26] concerning minimality of balls of small volume in isoperimetric problems with a competition between perimeter and a nonlocal potential term. More precisely, their result is extended to its maximal range of validity concerning the type of nonlocal potentials considered, and is also generalized to the case where local perimeters are replaced by their nonlocal counterparts.

1. INTRODUCTION

In the recent paper [7], Caffarelli, Roquejoffre, and Savin have initiated the study of Plateau-type problems with respect to a family of nonlocal perimeter functionals. A regularity theory for such nonlocal minimal surfaces has been developed by several authors [10, 4, 18, 35, 12], while the relation of nonlocal perimeters with their local counterpart has been investigated in [8, 3]. The isoperimetry of balls in nonlocal isoperimetric problems has been addressed in [19]. Precisely, given $s \in (0,1)$ and $n \geq 2$, one defines the s-perimeter of a set $E \subset \mathbb{R}^n$ as

$$P_s(E) := \int_E \int_{E^c} \frac{dx\,dy}{|x-y|^{n+s}} \in [0,\infty]\,.$$

As proved in [19], if $0 < |E| < \infty$ then we have the nonlocal isoperimetric inequality

$$P_s(E) \geq \frac{P_s(B)}{|B|^{(n-s)/n}}\,|E|^{(n-s)/n}\,, \tag{1.1}$$

where $B_r := \{x \in \mathbb{R}^n : |x| < r\}$, $B := B_1$, and $|E|$ is the Lebesgue measure of E. Notice that the right-hand side of (1.1) is equal to $P_s(B_{r_E})$, the s-perimeter of a ball of radius $r_E = (|E|/|B|)^{1/n}$ – so that $|E| = |B_{r_E}|$. Moreover, again in [19] it is shown that equality holds in (1.1) if and only if $E = x + B_{r_E}$ for some $x \in \mathbb{R}^n$. In [24] the following stronger form of (1.1) was proved:

$$P_s(E) \geq \frac{P_s(B)}{|B|^{(n-s)/n}}\,|E|^{(n-s)/n}\Big\{1 + \frac{A(E)^{4/s}}{C(n,s)}\Big\}\,, \tag{1.2}$$

where $C(n,s)$ is a non-explicit positive constant depending on n and s only, while

$$A(E) := \inf\Big\{\frac{|E\Delta(x+B_{r_E})|}{|E|} : x \in \mathbb{R}^n\Big\} \tag{1.3}$$

measures the L^1-distance of E from the set of balls of volume $|E|$ and is commonly known as the *Fraenkel asymmetry of E* (recall that, given two sets E and F, $|E\Delta F| := |E \setminus F| + |F \setminus E|$). Our first main result improves (1.2) by providing the sharp decay rate for $A(E)$ in (1.4). Moreover, we control the constant $C(n,s)$ appearing in (1.2) and make sure it does not degenerate as long as s stays away from 0.

Theorem 1.1. *For every $n \geq 2$ and $s_0 \in (0,1)$ there exists a positive constant $C(n,s_0)$ such that*

$$P_s(E) \geq \frac{P_s(B)}{|B|^{(n-s)/n}}\,|E|^{(n-s)/n}\left\{1+\frac{A(E)^2}{C(n,s_0)}\right\}, \tag{1.4}$$

whenever $s \in [s_0,1]$ and $0<|E|<\infty$.

Remark 1.2. The constant $C(n,s_0)$ we obtain in (1.4) is not explicit. It is natural to conjecture that $C(n,s_0) \approx 1/s_0$ as $s_0 \to 0^+$, see (4.3) below. Letting $s \to 1$ we recover the sharp stability result for the classical perimeter, that was first proved in [22] by symmetrization methods and later extended to anisotropic perimeters in [17] by mass transportation. The latter approach yields an explicit constant $C(n)$ in (1.4) when $s=1$, that grows polynomially in n. It remains an open problem to prove (1.4) with an explicit constant $C(n,s)$.

We next turn to consider nonlocal isoperimetric problems in presence of nonlocal repulsive interaction terms. The starting point is provided by Gamow model for the nucleus, which consists in the volume constraint minimization of the energy $P(E)+V_\alpha(E)$, where $P(E)$ is the (distributional) perimeter of $E \subset \mathbb{R}^n$ defines as

$$P(E) := \sup\Big\{\int_E \mathrm{div} X(x)\,dx : X \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ |X| \leq 1\Big\},$$

while, given $\alpha \in (0,n)$, $V_\alpha(E)$ is the Riesz potential

$$V_\alpha(E) := \int_E \int_E \frac{dx\,dy}{|x-y|^{n-\alpha}}\,. \tag{1.5}$$

By minimizing $P(E)+V_\alpha(E)$ with $|E|=m$ fixed, we observe a competition between the perimeter term, that tries to round up candidate minimizers into balls, and the Riesz potential, that tries to smear them around. (Notice also that, by Riesz inequality, balls are actually the volume constrained *maximizers* of V_α.)

It was recently proved by Knüpfer and Muratov that:

(a) If $n=2$ and $\alpha \in (0,2)$, then there exists $m_0=m_0(n,\alpha)$ such that Euclidean balls of volume $m \leq m_0$ are the only minimizers of $P(E)+V_\alpha(E)$ under the volume constraint $|E|=m$ [25] .

(b) If $n=2$ and α is sufficiently close to 2, then balls are the unique minimizers for $m \leq m_0$ while for $m > m_0$ there are no minimizers [25].

(c) If $3 \leq n \leq 7$ and $\alpha \in (1,n)$, then the result in (a) holds [26].

In [6], Bonacini and Cristoferi have recently extended both (b) and (c) above to the case $n \geq 3$, and have also shown that balls of volume m are volume-constrained L^1-local minimizers of $P(E)+V_\alpha(E)$ if $m < m_\star(n,\alpha)$, while they are never volume-constrained L^1-local minimizers if $m > m_\star(n,\alpha)$. The constant $m_\star(n,\alpha)$ is characterized in terms of a minimization problem, that is explicitly solved in the case $n=3$ (in particular, in the physically relevant case $n=3$, $s=1$, and $\alpha=2$ (Coulomb kernel), one finds $m_\star(3,1,2)=5$, a result that was actually already known in the physics literature since the 30's [5, 14, 21]). Let us also mention that, in addition to (b), further nonexistence results are contained in [26, 30].

We stress that, apart from the special case $n=2$, all these results are limited to the case $\alpha \in (1,n)$, named the far-field dominated regime by Knüpfer and Muratov to mark its contrast to the near-field dominated regime $\alpha \in (0,1]$. Our second and third main results extend (a) and (c) above in two directions: first, by covering the full range $\alpha \in (0,n)$ for all $n \geq 3$, and second, by including the possibility for the dominant perimeter term to be a nonlocal s-perimeter. The global minimality threshold m_0 is shown to be uniformly positive with respect to s and α provided they both stay away from zero.

The local minimality threshold $m_\star(n,s,\alpha)$ is characterized in terms of a minimization problem. In order to include the classical perimeter as a limiting case when $s\to 1$, we recall that, by combining [8, Theorem 1] with [3, Lemma 9 and Lemma 14], one finds that

$$\lim_{s\to 1^-}(1-s)\,P_s(E)=\omega_{n-1}\,P(E) \tag{1.6}$$

whenever E is an open set with $C^{1,\gamma}$-boundary for some $\gamma>0$ (from now on, ω_n denotes the volume of the n-dimensional ball of radius 1). Hence, to recover the classical perimeter we need to suitably renormalize the s-perimeter.

Theorem 1.3. *For every $n\ge 2$, $s_0\in(0,1)$, and $\alpha_0\in(0,n)$, there exists $m_0=m_0(n,s_0,\alpha_0)>0$ such that, if $m\in(0,m_0)$, $s\in(s_0,1)$, and $\alpha\in(\alpha_0,n)$, then the variational problems*

$$\inf\left\{\frac{1-s}{\omega_{n-1}}\,P_s(E)+V_\alpha(E):|E|=m\right\},$$
$$\inf\Big\{P(E)+V_\alpha(E):|E|=m\Big\},$$

admit balls of volume m as their (unique up to translations) minimizers.

Remark 1.4. An important open problem is, of course, to provide explicit lower bounds on m_0.

Let us now define a positive constant $m_\star$ by setting

$$m_\star(n,s,\alpha):=\begin{cases}\omega_n\left(\dfrac{n+s}{n-\alpha}\,\dfrac{s\,(1-s)\,P_s(B)}{\omega_{n-1}\,\alpha\,V_\alpha(B)}\right)^{n/(\alpha+s)}, & \text{if } s\in(0,1)\,,\\[2ex] \omega_n\left(\dfrac{n+1}{n-\alpha}\,\dfrac{P(B)}{\alpha\,V_\alpha(B)}\right)^{n/(\alpha+1)}, & \text{if } s=1\,.\end{cases} \tag{1.7}$$

The constant $m_\star(n,s,\alpha)$ is the threshold for volume-constrained L^1-local minimality of balls with respect to the functional $\frac{1-s}{\omega_{n-1}}\,P_s+V_\alpha$, as shown in the next theorem:

Theorem 1.5. *For every $n\ge 2$, $s\in(0,1)$, and $\alpha\in(0,n)$, let $m_\star=m_\star(n,s,\alpha)$ be as in (1.7). For every $m\in(0,m_\star)$ there exists $\varepsilon_\star=\varepsilon_\star(n,s,\alpha,m)>0$ such that, if $B[m]$ denotes a ball of volume m, then*

$$\frac{1-s}{\omega_{n-1}}\,P_s(B[m])+V_\alpha(B[m])\le\frac{1-s}{\omega_{n-1}}\,P_s(E)+V_\alpha(E)\,, \tag{1.8}$$

whenever $|E|=m$ and $|E\Delta B[m]|\le\varepsilon_\star\,m$. Moreover, if $m>m_\star$, then there exists a sequence of sets $\{E_h\}_{h\in\mathbb{N}}$ with $|E_h|=m$ and $|E_h\Delta B[m]|\to 0$ as $h\to\infty$ such that (1.8) fails with $E=E_h$ for every $h\in\mathbb{N}$.

Both Theorem 1.1 and Theorem 1.3 are obtained by combining a Taylor's expansion of nonlocal perimeters near balls, discussed in section 2, with a uniform version of the regularity theory developed in [7, 10], presented in section 3. In the case of Theorem 1.1, these two tools are combined in section 4 through a suitable version of Ekeland's variational principle. We implement this approach, that was introduced in the case $s=1$ by Cicalese and Leonardi [11], through a penalization argument closer to the one adopted in [1]. Due to the nonlocality of s-perimeters, the implementation itself will not be straightforward, and will require to develop some lemmas of independent interest, like the nucleation lemma (Lemma 4.3) and the truncation lemma (Lemma 4.5).

Concerning Theorem 1.3, our proof is inspired by the strategy used in [16] (see also [15] for a related argument) to show the isoperimetry of balls in isoperimetric problems with log-convex densities. Starting from the results in sections 2 and 3, the proof of Theorem 1.3 is given in section 5.

Finally, the proof of Theorem 1.5 is based on some second variation formulae for nonlocal functionals (discussed in section 6), which are then exploited to characterize the threshold for volume-constrained stability (in the sense of second variation) of balls in section 7. The passage from stability to L^1-local minimality is finally addressed in section 8. The proof of this last result is pretty delicate since we do not know that the ball is a global minimizer, a fact that usually plays a crucial role in this kind of arguments.

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2. A Fuglede-type result for the fractional perimeter

In this section we are going to prove Theorem 1.1 on *nearly spherical sets.* Precisely, we shall consider bounded open sets E with $|E| = |B|$, $\int_E x\,dx = 0$, and whose boundary satisfies

$$\partial E = \{(1+u(x))x : x \in \partial B\}, \qquad \text{where } u \in C^1(\partial B), \tag{2.1}$$

for some u with $\|u\|_{C^1(\partial B)}$ small. We correspondingly seek for a control on some fractional Sobolev norm of u in terms of $P_s(E) - P_s(B)$. More precisely, we shall control

$$[u]^2_{\frac{1+s}{2}} := [u]^2_{H^{\frac{1+s}{2}}(\partial B)} = \iint_{\partial B\times\partial B} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, d\mathcal{H}^{n-1}_x\, d\mathcal{H}^{n-1}_y\,,$$

as well as the L^2-norm of u. This kind of result is well-known in the local case (see Fuglede [23, Theorem 1.2]), and takes the following form in the nonlocal case.

Theorem 2.1. *There exist constants $\varepsilon_0 \in (0,1/2)$ and $c_0 > 0$, depending only on n, with the following property: If E is a nearly spherical set as in* (2.1)*, with $|E| = |B|$, $\int_E x\,dx = 0$, and $\|u\|_{C^1(\partial B)} < \varepsilon_0$, then*

$$P_s(E) - P_s(B) \ge c_0 \left([u]^2_{\frac{1+s}{2}} + s\,P_s(B)\,\|u\|^2_{L^2(\partial B)} \right), \qquad \forall s \in (0,1)\,. \tag{2.2}$$

Remark 2.2. If we multiply by $1-s$ in (2.2) and then take the limit $s \to 1^-$, then by (1.6) and (8.4) we get $P(E) - P(B) \ge c(n)\,\|u\|^2_{H^1}$ whenever $u \in C^{1,\gamma}(\partial B)$ for some $\gamma \in (0,1)$ (thus, on every Lipschitz function $u : \partial B \to \mathbb{R}$ by density). Thus Theorem 2.1 implies [23, Theorem 1.2(4)].

In order to prove Theorem 2.1, we need to premise some facts concerning hypersingular Riesz operators on the sphere. Following [34, pp. 159–160], one defines the *hypersingular Riesz operator on the sphere of order* $\gamma \in (0,1)\cup(1,2)$ as

$$\mathcal{D}^\gamma u(x) := \frac{\gamma\, 2^{\gamma-1}}{\pi^{\frac{n-1}{2}}} \frac{\Gamma(\frac{n-1+\gamma}{2})}{\Gamma(1-\frac{\gamma}{2})}\ \text{p.v.} \left(\int_{\partial B} \frac{u(x)-u(y)}{|x-y|^{n-1+\gamma}}\, d\mathcal{H}^{n-1}_y \right), \qquad x \in \partial B\,, \tag{2.3}$$

cf. [34, Equations (6.22) and (6.47)]. (Here, Γ denotes the usual Euler's Gamma function, and the symbol p.v. means that the integral is taken in the Cauchy principal value sense.) By [34, Lemma 6.26], the k-th eigenvalue of $\mathcal{D}^\gamma$ is given by

$$\lambda^*_k(\gamma) := \frac{\Gamma(k + \frac{n-1+\gamma}{2})}{\Gamma(k + \frac{n-1-\gamma}{2})} - \frac{\Gamma(\frac{n-1+\gamma}{2})}{\Gamma(\frac{n-1-\gamma}{2})}\,, \qquad k \in \mathbb{N}\cup\{0\}\,, \tag{2.4}$$

(so that $\lambda_k^*(\gamma) \geq 0$, $\lambda_k^*(\gamma)$ is strictly increasing in k, and $\lambda_k^*(\gamma) \uparrow \infty$ as $k \to \infty$). Moreover, if we denote by $\mathcal{S}_k$ the finite dimensional subspace of spherical harmonics of degree k, and by $\{Y_k^i\}_{i=1}^{d(k)}$ an orthonormal basis for $\mathcal{S}_k$ in $L^2(\partial B)$, then

$$\mathcal{D}^\gamma Y_k = \lambda_k^*(\gamma)\, Y_k\,, \qquad \forall k \in \mathbb{N} \cup \{0\}\,. \tag{2.5}$$

When no confusion arises, we shall often denote by Y_k a generic element in $\mathcal{S}_k$. Given $s \in (0,1)$, let us now introduce the operator

$$\mathscr{I}_s u(x) := 2\,\mathrm{p.v.}\left(\int_{\partial B} \frac{u(x)-u(y)}{|x-y|^{n+s}}\, d\mathcal{H}^{n-1}_y\right)\,, \qquad u \in C^2(\partial B)\,, \tag{2.6}$$

so that, for every $u \in C^2(\partial B)$,

$$\mathscr{I}_s u = \frac{2^{1-s}\,\pi^{\frac{n-1}{2}}}{1+s}\,\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n+s}{2})}\,\mathcal{D}^{1+s} u\,, \tag{2.7}$$

and

$$[u]^2_{\frac{1+s}{2}} = \int_{\partial B} u\,\mathscr{I}_s u\, d\mathcal{H}^{n-1}\,. \tag{2.8}$$

Let us denote by λ_k^s the k-th eigenvalue of $\mathscr{I}_s$. By (2.4), (2.5), and (2.7) we find that λ_k^s satisfies

$$\lambda_0^s = 0\,, \qquad \lambda_{k+1}^s > \lambda_k^s\,, \qquad \mathscr{I}_s Y_k = \lambda_k^s\, Y_k\,, \qquad \forall k \in \mathbb{N}\cup\{0\}\,, \tag{2.9}$$

and $\lambda_k^s \uparrow \infty$ as $k \to \infty$. If we denote by

$$a_k^i(u) := \int_{\partial B} u\, Y_k^i\, d\mathcal{H}^{n-1}$$

the Fourier coefficient of u corresponding to Y_k^i, then we obtain

$$[u]^2_{\frac{1+s}{2}} = \sum_{k=0}^{\infty}\sum_{i=1}^{d(k)} \lambda_k^s\, a_k^i(u)^2\,. \tag{2.10}$$

Concerning the value of λ_1^s and λ_2^s, we shall need the following proposition.

Proposition 2.3. *One has*

$$\lambda_1^s = s(n-s)\frac{P_s(B)}{P(B)}\,. \tag{2.11}$$

$$\lambda_2^s = \frac{2n}{n-s}\lambda_1^s\,. \tag{2.12}$$

Proof. Since each coordinate function x_i, $i = 1,\dots,n$, belongs to $\mathcal{S}_1$, we have $\mathscr{I}_s x_i = \lambda_1^s x_i$. Hence, inserting x_i in (2.8) and adding up over i, yields

$$\lambda_1^s = \frac{1}{P(B)}\iint_{\partial B\times\partial B} \frac{d\mathcal{H}^{n-1}_x\, d\mathcal{H}^{n-1}_y}{|x-y|^{n+s-2}}\,. \tag{2.13}$$

For $z \in \mathbb{R}^n \setminus \{0\}$, we now set

$$\mathcal{K}(z) := -\frac{1}{n+s-2}\frac{1}{|z|^{n+s-2}}\,.$$

Splitting $\nabla\mathcal{K}$ into its tangential and normal components to ∂B, we compute for $y \notin \overline{B}$ the integral

$$
\begin{aligned}
L(y) &:= \int_{\partial B} \frac{(x-y)\cdot(x-y)}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}_x \qquad (2.14)\\
&= \int_{\partial B} \nabla_x\mathcal{K}(x-y)\cdot x\,d\mathcal{H}^{n-1}_x - \int_{\partial B} \nabla_x\mathcal{K}(x-y)\cdot y\,d\mathcal{H}^{n-1}_x\\
&= \int_{\partial B} (1-x\cdot y)\frac{\partial\mathcal{K}}{\partial\nu(x)}(x-y)\,d\mathcal{H}^{n-1}_x - \int_{\partial B} \nabla_\tau\mathcal{K}(x-y)\nabla_\tau(x\cdot y)\,d\mathcal{H}^{n-1}_x\\
&=: \mathcal{A}(y) - \mathcal{B}(y)\,.
\end{aligned}
$$

We now evaluate separately $\mathcal{A}(y)$ and $\mathcal{B}(y)$. Noticing that $\Delta\mathcal{K}(z) = -s/|z|^{n+s}$, we first integrate $\mathcal{A}(y)$ by parts to obtain

$$
\begin{aligned}
\mathcal{A}(y) &= \int_B \Delta_x\mathcal{K}(x-y)(1-x\cdot y)\,dx + \int_B \nabla_x\mathcal{K}(x-y)\nabla_x(1-x\cdot y)\,dx\\
&= -s\int_B \frac{1-x\cdot y}{|x-y|^{n+s}}\,dx + \int_B \frac{|y|^2 - x\cdot y}{|x-y|^{n+s}}\,dx\\
&= (1-s)\int_B \frac{1-x\cdot y}{|x-y|^{n+s}}\,dx + \int_B \frac{|y|^2-1}{|x-y|^{n+s}}\,dx\,.
\end{aligned}
$$

We now denote by $\Delta_{\mathbb{S}^{n-1}}$ the standard Laplace–Beltrami operator on the sphere and recall that $-\Delta_{\mathbb{S}^{n-1}}x_i = (n-1)x_i$ for $i = 1,\dots,n$. Integrating $\mathcal{B}(y)$ by parts leads to

$$
\begin{aligned}
\mathcal{B}(y) &= -\int_{\partial B} \mathcal{K}(x-y)\Delta_{\mathbb{S}^{n-1}}(x\cdot y)\,d\mathcal{H}^{n-1}_x = (n-1)\int_{\partial B} \mathcal{K}(x-y)x\cdot y\,d\mathcal{H}^{n-1}_x\\
&= -\frac{n-1}{n+s-2}\int_{\partial B} \frac{x\cdot y}{|x-y|^{n+s-2}}\,d\mathcal{H}^{n-1}_x\,.
\end{aligned}
$$

From the above expressions of $\mathcal{A}$ and $\mathcal{B}$, we can let y converge to a point on ∂B to find

$$
L(y) = (1-s)\int_B \frac{1-x\cdot y}{|x-y|^{n+s}}\,dx + \frac{n-1}{n+s-2}\int_{\partial B} \frac{x\cdot y}{|x-y|^{n+s-2}}\,d\mathcal{H}^{n-1}_x\,, \quad y\in\partial B\,. \qquad (2.15)
$$

Integrating over ∂B the first integral on the right hand side of the previous equality, and using the divergence theorem again, we get

$$
\begin{aligned}
\int_B dx \int_{\partial B} \frac{1-x\cdot y}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}_y &= \int_B dx\int_{\partial B} \frac{(y-x)\cdot y}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}_y\,dx\\
&= \int_B dx\int_{\partial B} \frac{\partial\mathcal{K}}{\partial\nu}(y-x)\,d\mathcal{H}^{n-1}_y = -\int_B dx\int_{B^c} \Delta_y\mathcal{K}(y-x)\,dy\\
&= s\int_B\int_{B^c} \frac{1}{|x-y|^{n+s}}\,dx\,dy = sP_s(B)\,.
\end{aligned}
$$

From this formula, integrating both sides of (2.15) and recalling (2.13) and (2.14), we obtain

$$
\lambda_1^s = s(1-s)\frac{P_s(B)}{P(B)} + \frac{n-1}{(n+s-2)P(B)}\iint_{\partial B\times\partial B} \frac{x\cdot y}{|x-y|^{n+s-2}}\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,. \qquad (2.16)
$$

To deal with the last integral of the previous equality we need to rewrite $P_s(B)$ as follows

$$
\begin{aligned}
P_s(B) &= \int_{B^c} dy\int_B \frac{(x-y)\cdot(x-y)}{|x-y|^{n+s+2}}\,dx = -\frac{1}{n+s}\int_{B^c} dy\int_B \nabla_x\Big(\frac{1}{|x-y|^{n+s}}\Big)\cdot(x-y)\,dx\\
&= -\frac{1}{n+s}\int_{B^c}\bigg(-n\int_B \frac{dx}{|x-y|^{n+s}} + \int_{\partial B} \frac{(x-y)\cdot x}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}_x\bigg)\,dy\\
&= \frac{n}{n+s}P_s(B) - \frac{1}{n+s}\int_{B^c} dy\int_{\partial B} \frac{(x-y)\cdot x}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}_x\,.
\end{aligned}
$$

Therefore

$$\begin{aligned}
P_s(B) &= \frac{1}{s}\int_{\partial B} d\mathcal{H}^{n-1}_x \int_{B^c} \frac{(y-x)\cdot x}{|x-y|^{n+s}}\,dy \\
&= -\frac{1}{s(n+s-2)}\int_{\partial B} d\mathcal{H}^{n-1}_x \int_{B^c} \nabla_y\Big(\frac{1}{|x-y|^{n+s-2}}\Big)\cdot x\,dy \\
&= \frac{1}{s(n+s-2)}\iint_{\partial B\times\partial B} \frac{x\cdot y}{|x-y|^{n+s-2}}\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,.
\end{aligned}$$

Combining this last equality with (2.16) leads to the proof of (2.11).

Finally, using (2.4) and exploiting the factorial property of the Gamma function $\Gamma(z+1) = \Gamma(z)\,z$ for every $z \in \mathbb{C}\setminus\{-k : k\in\mathbb{N}\cup\{0\}\}$, we see that

$$\lambda_1^*(\alpha) = \frac{\alpha}{\kappa}\,\frac{\Gamma(\alpha+\kappa)}{\Gamma(\kappa)}\,,\qquad \lambda_2^*(\alpha) = \frac{1+\alpha+2\kappa}{1+\kappa}\,\lambda_1^*(\alpha)\,,\qquad \kappa := \frac{n-1-\alpha}{2}\,. \tag{2.17}$$

Since $\alpha = 1+s$, we infer from (2.7) and (2.17) that $\lambda_2^s/\lambda_1^s = \lambda_2^*(\alpha)/\lambda_1^*(\alpha) = \frac{2n}{n-s}$ which is precisely identity (2.12). $\square$

Proof of Theorem 2.1. Step 1. We start by slightly rephrasing the assumption. Precisely, we consider a function $u \in C^1(\partial B)$ with $\|u\|_{C^1(\partial B)} \le 1/2$ such that there exists $t \in (0, 2\varepsilon_0)$ with the property that the bounded open set F_t whose boundary is given by

$$\partial F_t = \{(1+tu(x))x : \, x \in \partial B\}\,,$$

satisfies

$$|F_t| = |B|\,,\qquad \int_{F_t} x\,dx = 0\,.$$

We thus aim to prove that, if ε_0 and c_0 are small enough, then

$$P_s(F_t) - P_s(B) \ge c_0\,t^2\left([u]^2_{\frac{1+s}{2}} + s\,P_s(B)\,\|u\|^2_{L^2}\right),\qquad \forall s\in(0,1)\,. \tag{2.18}$$

Changing to polar coordinates, we first rewrite

$$P_s(F_t) = \iint_{\partial B\times\partial B}\left(\int_0^{1+tu(x)}\int_{1+tu(y)}^{+\infty} \frac{r^{n-1}\varrho^{n-1}}{(|r-\varrho|^2 + r\varrho|x-y|^2)^{\frac{n+s}{2}}}\,dr\,d\varrho\right) d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,.$$

Then, symmetrizing this formula leads to

$$\begin{aligned}
P_s(F_t) = \frac{1}{2}\iint_{\partial B\times\partial B}\Bigg(&\int_0^{1+tu(x)}\int_{1+tu(y)}^{+\infty} f_{|x-y|}(r,\varrho)\,dr\,d\varrho \\
&+ \int_0^{1+tu(y)}\int_{1+tu(x)}^{+\infty} f_{|x-y|}(r,\varrho)\,dr\,d\varrho\Bigg)\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,,
\end{aligned}$$

where, for $r, \varrho, \theta > 0$, we have set

$$f_\theta(r,\varrho) := \frac{r^{n-1}\varrho^{n-1}}{(|r-\varrho|^2 + r\varrho\,\theta^2)^{\frac{n+s}{2}}}\,.$$

Using the convention $\int_a^b = -\int_b^a$, we formally have

$$\int_0^b\int_a^{+\infty} + \int_0^a\int_b^{+\infty} = \int_a^b\int_a^b + \int_0^a\int_a^{+\infty} + \int_0^b\int_b^{+\infty}\,,$$

so that

$$P_s(F_t)=\frac{1}{2}\iint_{\partial B\times\partial B}\Big(\int_{1+tu(y)}^{1+tu(x)}\int_{1+tu(y)}^{1+tu(x)} f_{|x-y|}(r,\varrho)\,dr\,d\varrho\Big)\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y$$
$$+\iint_{\partial B\times\partial B}\Big(\int_0^{1+tu(x)}\int_{1+tu(x)}^{+\infty} f_{|x-y|}(r,\varrho)\,dr\,d\varrho\Big)\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,. \quad (2.19)$$

Rescaling variables, we find that

$$\int_{\partial B}\Big(\int_0^{1+tu(x)}\int_{1+tu(x)}^{+\infty} f_{|x-y|}(r,\varrho)\,dr\,d\varrho\Big)\,d\mathcal{H}^{n-1}_y$$
$$=(1+tu(x))^{n-s}\int_{\partial B}\int_0^1\int_1^{+\infty} f_{|x-y|}(r,\varrho)\,dr\,d\varrho\,d\mathcal{H}^{n-1}_y\,,\qquad \forall x\in\partial B\,.$$

By symmetry, the triple integral on the right hand side of this identity does not depend on $x\in\partial B$. Its constant value is easily deduced by evaluating (2.19) at $t=0$ and yields

$$P_s(B)=P(B)\int_{\partial B}\int_0^1\int_1^{+\infty} f_{|x-y|}(r,\varrho)\,dr\,d\varrho\,d\mathcal{H}^{n-1}_y\,,\qquad \forall x\in\partial B\,.$$

Combining the last two identities with (2.19), we conclude that

$$P_s(F_t)=\frac{1}{2}\iint_{\partial B\times\partial B}\Big(\int_{1+tu(y)}^{1+tu(x)}\int_{1+tu(y)}^{1+tu(x)} f_{|x-y|}(r,\varrho)\,dr\,d\varrho\Big)\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y$$
$$+\frac{P_s(B)}{P(B)}\int_{\partial B}(1+tu(x))^{n-s}\,d\mathcal{H}^{n-1}_x\,.$$

With a last change of variable in the first term on the right hand side of this identity, we reach the following formula for $P_s(F_t)$:

$$P_s(F_t)=\frac{t^2}{2}\,g(t)+\frac{P_s(B)}{P(B)}\,h(t)\,, \quad (2.20)$$

where we have set

$$g(t):=\iint_{\partial B\times\partial B}\Big(\int_{u(y)}^{u(x)}\int_{u(y)}^{u(x)} f_{|x-y|}(1+tr,1+t\varrho)\,dr\,d\varrho\Big)\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,,$$

and

$$h(t):=\int_{\partial B}(1+tu(x))^{n-s}\,d\mathcal{H}^{n-1}_x\,.$$

Since g depends smoothly on t, we can find $\tau\in(0,t)$ such that $g(t)=g(0)+t\,g'(\tau)$. In addition, observing that

$$\Big|r\,\frac{\partial f_\theta}{\partial r}(1+\tau\,r,1+\tau\,\varrho)+\varrho\,\frac{\partial f_\theta}{\partial\varrho}(1+\tau\,r,1+\tau\,\varrho)\Big|\le\frac{C(n)}{\theta^{n+s}}\,,\qquad \forall r,\varrho\in\Big(-\frac{1}{2},\frac{1}{2}\Big)\,,$$

for a suitable dimensional constant $C(n)$ (whose value is allowed to change from line to line), one can estimate

$$|g'(\tau)|\le C(n)\iint_{\partial B\times\partial B}\frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y=C(n)\,[u]^2_{\frac{1+s}{2}}\,.$$

Taking into account that $g(0)=[u]^2_{\frac{1+s}{2}}$ and $h(0)=P(B)$, we then infer from (2.20) that

$$P_s(F_t)-P_s(B)\ge\frac{t^2}{2}[u]^2_{\frac{1+s}{2}}+\frac{P_s(B)}{P(B)}\big(h(t)-h(0)\big)-C(n)\,t^3\,[u]^2_{\frac{1+s}{2}}\,. \quad (2.21)$$

We now exploit the volume constraint $|F_t| = |B|$ to deduce that

$$\int_{\partial B} (1 + t\,u)^n \, d\mathcal{H}^{n-1} = n\,|F_t| = n\,|B| = P(B) = h(0)\,,$$

so that

$$h(t) - h(0) = \int_{\partial B} (1 + t\,u)^n \big((1 + t\,u)^{-s} - 1\big)\, d\mathcal{H}^{n-1}_x\,.$$

By a Taylor expansion, we find that for every $|z| \le 1/2$,

$$\big((1+z)^{-s} - 1\big)(1+z)^n = \Big(-sz + \frac{s(s+1)}{2} z^2 + sR_1(z)\Big)\Big(1 + nz + \frac{n(n-1)}{2} z^2 + R_2(z)\Big)\,,$$

with $|R_1(z)| + |R_2(z)| \le C(n)|z|^3$. Thus

$$h(t) - h(0) \ge -s \int_{\partial B} \Big[t\,u + \Big(n - \frac{s+1}{2}\Big) t^2\, u^2\Big]\, d\mathcal{H}^{n-1} - C(n) s\, t^3 \|u\|^2_{L^2}\,. \tag{2.22}$$

Exploiting the volume constraint again, i.e., $\int_{\partial B} \big((1 + t\,u)^n - 1\big) = 0$, and expanding the term $(1 + t\,u)^n$, we get

$$-\int_{\partial B} t\,u\, d\mathcal{H}^{n-1} \ge \frac{(n-1)}{2} \int_{\partial B} t^2\, u^2\, d\mathcal{H}^{n-1} - C(n)\, t^3\, \|u\|^2_{L^2}\,. \tag{2.23}$$

We may now combine (2.23) with (2.22) and (2.11) to obtain

$$\begin{aligned}
\frac{P_s(B)}{P(B)}\big(h(t) - h(0)\big) &\ge -\frac{t^2}{2}\, \frac{s(n-s)P_s(B)}{P(B)} \int_{\partial B} u^2\, d\mathcal{H}^{n-1} - C(n)\, \frac{s\, P_s(B)}{P(B)}\, t^3 \|u\|^2_{L^2} \\
&= -\frac{t^2}{2}\, \lambda_1^s \int_{\partial B} u^2\, d\mathcal{H}^{n-1} - \frac{C(n)}{n-s}\, \lambda_1^s\, t^3 \|u\|^2_{L^2}\,.
\end{aligned}$$

We plug this last inequality into (2.21) to find that

$$P_s(F_t) - P_s(B) \;\ge\; \frac{t^2}{2}\Big([u]^2_{\frac{1+s}{2}} - \lambda_1^s\, \|u\|^2_{L^2}\Big) - C(n)\, t^3\, \Big([u]^2_{\frac{1+s}{2}} + \lambda_1^s\, \|u\|^2_{L^2}\Big)\,. \tag{2.24}$$

Setting for brevity $a_k^i := a_k^i(u)$, we now apply (2.10) to deduce that, for every $\eta \in (0,1)$,

$$\begin{aligned}
[u]^2_{\frac{1+s}{2}} - \lambda_1^s\, \|u\|^2_{L^2} &\ge \sum_{k=1}^{\infty} \sum_{i=1}^{d(k)} \lambda_k^s |a_k^i|^2 - \lambda_1^s \sum_{k=0}^{\infty} \sum_{i=1}^{d(k)} |a_k^i|^2 \\
&= \frac{1}{4} \sum_{k=2}^{\infty} \sum_{i=1}^{d(k)} \lambda_k^s |a_k^i|^2 + \sum_{k=2}^{\infty} \sum_{i=1}^{d(k)} \Big(\frac{3}{4}\lambda_k^s - \lambda_1^s\Big) |a_k^i|^2 - \lambda_1^s |a_0|^2 \\
&\ge \frac{1}{4}\, [u]^2_{\frac{1+s}{2}} + \sum_{k=2}^{\infty} \sum_{i=1}^{d(k)} \Big(\frac{3}{4}\lambda_k^s - \lambda_1^s\Big) |a_k^i|^2 - \lambda_1^s \sum_{i=1}^{n} |a_1^i|^2 - \lambda_1^s |a_0|^2\,.
\end{aligned}$$

Thanks to (2.9) and (2.12), $\frac{3}{4}\lambda_k^s - \lambda_1^s \ge \lambda_1^s/2$ for every $k \ge 2$. Hence,

$$[u]^2_{\frac{1+s}{2}} - \lambda_1^s\, \|u\|^2_{L^2} \ge \frac{1}{4}\, [u]^2_{\frac{1+s}{2}} + \lambda_1^s \Big(\frac{1}{2} \sum_{k=2}^{\infty} \sum_{i=1}^{d(k)} |a_k^i|^2 - \sum_{i=1}^{n} |a_1^i|^2 - |a_0|^2\Big)\,. \tag{2.25}$$

Using the volume constraint again and taking into account that $a_0 = P(B)^{-1/2} \int_{\partial B} u$, one easily estimates for a suitably small value of ε_0,

$$|a_0| \le C(n)\, t\, \|u\|^2_{L^2}\,. \tag{2.26}$$

Similarly, the barycenter constraint $0 = \int_{\partial B} x_i\, (1 + t\,u)^{n+1}\, d\mathcal{H}^{n-1}$ yields

$$\Big|\int_{\partial B} x_i\, u\, d\mathcal{H}^{n-1}\Big| \le C(n)\, t\, \|u\|^2_{L^2}\,,$$

so that, taking into account that $Y_1^i = c(n)\, x_i$ for some constant $c(n)$ depending on n only,

$$|a_1^i| \le C(n)\, t \|u\|_2^2\,, \qquad i = 1, ..., n\,. \tag{2.27}$$

We can now combine (2.26) and (2.27) with $\|u\|_{L^2}^2 = \sum_{k=0}^\infty \sum_{i=1}^{d(k)} |a_k^i|^2$, to conclude that

$$|a_0|^2 + \sum_{i=1}^n |a_1^i|^2 \le C(n)\, t \sum_{k=2}^\infty \sum_{i=1}^{d(k)} |a_k^i|^2\,.$$

This last inequality implies of course that, for ε_0 small,

$$\frac{1}{2} \sum_{k=2}^\infty \sum_{i=1}^{d(k)} |a_k^i|^2 - \sum_{i=1}^n |a_1^i|^2 - |a_0|^2 \ge \frac{\|u\|_{L^2}^2}{4}\,. \tag{2.28}$$

By (2.24), (2.25), and (2.28) we thus find

$$\begin{aligned} P_s(F_t) - P_s(B) &\ge \frac{t^2}{8}\Big([u]^2_{\frac{1+s}{2}} + \lambda_1^s\, \|u\|^2_{L^2}\Big) - C(n)\, t^3\, \Big([u]^2_{\frac{1+s}{2}} + \lambda_1^s\, \|u\|^2_{L^2}\Big) \\ &\ge \frac{t^2}{16}\Big([u]^2_{\frac{1+s}{2}} + \lambda_1^s\, \|u\|^2_{L^2}\Big)\,, \end{aligned}$$

provided ε_0, hence t, is small enough with respect to n. Since $\lambda_1^s \ge s\, P_s(B)$, we have completed the proof of (2.18), thus of Theorem 2.1. □

3. Uniform estimates for almost-minimizers of nonlocal perimeters

A crucial step in our proof of Theorem 1.1 and Theorem 1.3 is the application of the regularity theory for nonlocal perimeter minimizers: indeed, this is the step where we reduce to consider small normal deformations of balls, and thus become able to apply Theorem 2.1. The parts of the regularity theory for nonlocal perimeter minimizers that are relevant to us have been developed in [7, 10] with the parameter s fixed. In other words, there is no explicit discussion on how the regularity estimates should behave as s approaches the limit values 0 or 1, although it is pretty clear [8, 3, 13] that they should degenerate when $s \to 0^+$, and that they should be stable, after scaling s-perimeter by the factor $(1-s)$, in the limit $s \to 1^-$. Since we shall need to exploit these natural uniformity properties, in this section we explain how to deduce these results from the results contained in [7, 10], with the aim of proving Corollary 3.6 below. In order to minimize the amount of technicalities, we shall discuss these issues working with a rather special notion of almost-minimality, that we now introduce. It goes without saying, the results we present should hold true in the more general class of almost-minimizers considered in [10].

We thus introduce the special class of almost-minimizers we shall consider. Given $\Lambda \ge 0$, $s \in (0,1)$, and a *bounded* Borel set $E \subset \mathbb{R}^n$, we say that E is a (global) Λ-minimizer of the s-perimeter if

$$P_s(E) \le P_s(F) + \frac{\Lambda}{1-s}\, |E \Delta F|\,, \tag{3.1}$$

for every *bounded* set $F \subset \mathbb{R}^n$. Since the validity of (3.1) is not affected if we replace E with some E' with $|E\Delta E'| = 0$, we shall always assume that a Λ-minimizer of the s-perimeter has been normalized so that

$$E \text{ is Borel, with } \partial E = \Big\{x \in \mathbb{R}^n : 0 < |E \cap B(x,r)| < \omega_n\, r^n \text{ for every } r > 0\Big\} \tag{3.2}$$

(as show for instance in [31, Proposition 12.19, step two], this can always be done). As explained, we shall need some regularity estimates for Λ-minimizers of the s-perimeter to be uniform with respect to $s \in [s_0, 1]$, for $s_0 \in (0,1)$ fixed. We start with the following uniform density estimates.

(The proof is classical, compare with [31, Theorem 21.11] for the local case, and with [7, Theorem 4.1] for the nonlocal case, but we give the details here in order to keep track of the constants.)

Lemma 3.1. *If $s \in (0,1)$, $\Lambda \ge 0$, and E satisfies the minimality property* (3.1) *and the normalization condition* (3.2), *then we have*

$$|B|\,(1-c_0)\,r^n \ge |E \cap B(x_0,r)| \ge |B|\,c_0\,r^n\,, \tag{3.3}$$

whenever $x_0 \in \partial E$ and $r \le r_0$, where

$$c_0 = \Big(\frac{s}{8\,|B|\,2^{n/s}}\frac{(1-s)P_s(B)}{P(B)}\Big)^{n/s}\,, \qquad r_0 = \Big(\frac{(1-s)\,P_s(B)}{2\,\Lambda\,|B|}\Big)^{1/s}\,.$$

The following elementary lemma (De Giorgi iteration) is needed in the proof.

Lemma 3.2. *Let $\alpha \in (0,1)$, $N > 1$, $M > 0$, and $\{u_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that*

$$u_{k+1}^{1-\alpha} \le N^k\,M\,u_k\,, \qquad \forall k \in \mathbb{N}\,. \tag{3.4}$$

If

$$u_0 \le \frac{1}{N^{(1-\alpha)/\alpha^2}\,M^{1/\alpha}}\,, \tag{3.5}$$

then $u_k \to 0$ as $k \to \infty$.

Proof of Lemma 3.2. By (3.4) and (3.5), induction proves that $u_k \le N^{-k/\alpha}\,u_0$ for every $k \in \mathbb{N}$. □

Proof of Lemma 3.1. Being the two proofs analogous, we only prove the lower bound in (3.3). Up to a translation we may also assume that $x_0 = 0$. We fix $r > 0$, set $u(r) := |E \cap B_r|$, and apply (3.1) with $F = E \setminus B_r$ to find

$$(1-s)\int_E\int_{E^c}\frac{dx\,dy}{|x-y|^{n+s}} \le (1-s)\int_{E\setminus B_r}\int_{E^c\cup(E\cap B_r)}\frac{dx\,dy}{|x-y|^{n+s}} + \Lambda u(r)\,,$$

As a consequence

$$(1-s)\int_{E\cap B_r}\int_{E^c}\frac{dx\,dy}{|x-y|^{n+s}} \le (1-s)\int_{E\setminus B_r}\int_{E\cap B_r}\frac{dx\,dy}{|x-y|^{n+s}} + \Lambda u(r)\,,$$

hence, by adding up $(1-s)\int_{E\setminus B_r}\int_{E\cap B_r}\frac{dx\,dy}{|x-y|^{n+s}}$ to both sides we immediately get, for every $r > 0$,

$$P_s(E\cap B_r) \le 2\int_{E\setminus B_r}\int_{E\cap B_r}\frac{dx\,dy}{|x-y|^{n+s}} + \frac{\Lambda}{1-s}\,u(r)\,. \tag{3.6}$$

On the one hand, $P_s(E\cap B_r) \ge P_s(B)\,(u(r)/|B|)^{(n-s)/n}$ by the isoperimetric inequality (1.1); on the other hand, by the coarea formula

$$\begin{aligned}\int_{E\setminus B_r}\int_{E\cap B_r}\frac{dx\,dy}{|x-y|^{n+s}} &\le \int_{E\cap B_r}dx\int_{B(x,r-|x|)^c}\frac{dy}{|x-y|^{n+s}}\\ &= \frac{P(B)}{s}\int_{E\cap B_r}\frac{dx}{(r-|x|)^s} = \frac{P(B)}{s}\int_0^r\frac{u'(t)}{(r-t)^s}\,dt\,,\end{aligned} \tag{3.7}$$

where we have also taken into account that $u'(t) = \mathcal{H}^{n-1}(E\cap\partial B_t)$ for a.e. $t > 0$. By combining these two facts with (3.6) we find

$$\frac{P_s(B)}{|B|^{(n-s)/n}}\,u(r)^{(n-s)/n} \le \frac{2\,P(B)}{s}\int_0^r\frac{u'(t)}{(r-t)^s}\,dt + \frac{\Lambda}{1-s}\,u(r)\,, \qquad \forall r > 0\,. \tag{3.8}$$

Since $u(r) \le |B|\,r^n$ for every $r > 0$, our choice of r_0 implies that

$$\frac{\Lambda}{1-s}\,u(r) \le \frac{P_s(B)}{2\,|B|^{(n-s)/n}}\,u(r)^{(n-s)/n}\,, \qquad \forall r \le r_0\,,$$

and enables us to deduce from (3.8) that

$$u(r)^{(n-s)/n} \le \frac{4\,P(B)\,|B|^{(n-s)/n}}{s\,P_s(B)} \int_0^r \frac{u'(t)}{(r-t)^s}\,dt\,, \qquad \forall r \le r_0\,. \tag{3.9}$$

By integrating (3.9) on $(0,\ell) \subset (0,r_0)$ and by Fubini's theorem, we thus obtain

$$\int_0^\ell u(r)^{(n-s)/n}\,dr \le \frac{4\,P(B)\,|B|^{(n-s)/n}}{s\,(1-s)\,P_s(B)}\,\ell^{1-s}\,u(\ell), \qquad \forall \ell \le r_0\,. \tag{3.10}$$

We now argue by contradiction, and assume the existence of $\ell_0 \le r_0$ such that $u(\ell_0) \le c_0\,|B|\,\ell_0^n$. Correspondingly we set

$$\ell_k := \frac{\ell_0}{2} + \frac{\ell_0}{2^{k+1}}\,, \qquad u_k := u(\ell_k)\,, \qquad C_1 := \frac{4\,P(B)\,|B|^{(n-s)/n}}{s\,(1-s)\,P_s(B)}\,,$$

and notice that (3.10) implies

$$\frac{\ell_0}{2^{k+2}}\,u_{k+1}^{(n-s)/n} = (\ell_k - \ell_{k+1})u_{k+1}^{(n-s)/n} \le \int_{\ell_{k+1}}^{\ell_k} u^{(n-s)/n} \le C_1\,\ell_k^{1-s}\,u_k \le C_1 \ell_0^{1-s}\,u_k\,,$$

that is, $u_{k+1}^{1-\alpha} \le 2^k\,M\,u_k$ for $M := 4\,C_1\,\ell_0^{-s}$ and $\alpha = s/n$. Since $u_k \to u(\ell_0/2) = |E \cap B_{\ell_0/2}| > 0$ (indeed, $0 \in \partial E$ and (3.2) is in force), by Lemma 3.2 we deduce that

$$u(\ell_0) = u_0 > \frac{1}{2^{(1-\alpha)/\alpha^2}\,M^{1/\alpha}} = \frac{2^{n/s}\,\ell_0^n}{2^{(n/s)^2}\,(4\,C_1)^{n/s}} = c_0\,|B|\,\ell_0^n\,.$$

However, this is a contradiction to $u(\ell_0) \le c_0\,|B|\,\ell_0^n$, and the lemma is proved. □

Introducing a further bit of special terminology, we say that a bounded Borel set $E \subset \mathbb{R}^n$ is a Λ-minimizer of the 1-perimeter if

$$P(E) \le P(F) + \frac{\Lambda}{\omega_{n-1}}\,|E \Delta F|\,,$$

for every bounded $F \subset \mathbb{R}^n$, and if (3.2) holds true. We have the following compactness theorem.

Theorem 3.3. *If $R > 0$, $s_0 \in (0,1)$, and E_h ($h \in \mathbb{N}$) is a Λ-minimizer of the s_h-perimeter with $s_h \in [s_0,1)$ and $E_h \subset B_R$ for every $h \in \mathbb{N}$, then there exist $s_* \in [s_0,1]$ and a Λ-minimizer of the s_*-perimeter E such that, up to extracting subsequences, $s_h \to s_*$, $|E_h \Delta E| \to 0$ and ∂E_h converges to ∂E in Hausdorff distance as $h \to \infty$.*

Proof. Up to extracting subsequences we may obviously assume that $s_h \to s_*$ as $h \to \infty$, where $s_* \in [s_0,1]$. By exploiting (3.1) with $F = B_R$ we see that

$$\sup_{h\in\mathbb{N}} (1-s_h)\,P_{s_h}(E_h) \le 2\Lambda\,|B_R| + \sup_{h\in\mathbb{N}} (1-s_h)\,P_{s_h}(B_R) < \infty\,, \tag{3.11}$$

where we have used the fact that $(1-s)\,P_s(B) \to \omega_{n-1} P(B)$ as $s \to 1^+$ (recall (1.6)).

Step one: We prove the theorem in the case $s_* = 1$. By (3.11) and by [3, Theorem 1], we find that, up to extracting subsequences, $|E_h \Delta E| \to 0$ as $h \to \infty$ for some set $E \subset B_R$ with finite perimeter. By [3, Theorem 2],

$$\omega_{n-1}\,P(E) \le \liminf_{h\to\infty} (1-s_h)\,P_{s_h}(E_h)\,,$$

and, if $F \subset \mathbb{R}^n$ is bounded, then we can find bounded set F_h ($h \in \mathbb{N}$) such that $|F_h \Delta F| \to 0$ as $h \to \infty$ and

$$\omega_{n-1}\,P(F) = \liminf_{h\to\infty} (1-s_h)\,P_{s_h}(F_h)\,.$$

By (3.1), $(1-s_h)\,P_{s_h}(E_h) \le (1-s_h)\,P_{s_h}(F_h) + \Lambda\,|E_h \Delta F_h|$; by letting $h \to \infty$, we find that E is a Λ-minimizer of the 1-perimeter. The fact that ∂E_h converges to ∂E in Hausdorff distance as $h \to \infty$ is now a standard consequences of the uniform density estimates proved in Lemma 3.1.

Step two: We address the case $s_* < 1$. In this case we may notice that (3.11) together with the assumption that $E_h \subset B_R$ allows us to say that $\{P_s(E_h)\}_{h\in\mathbb{N}}$ is bounded in $\mathbb{R}$ for some $s \in (0,1)$. By compactness of the embedding of $H^{s/2}$ in L^1_{loc} and by the assumption $E_h \subset B_R$ we find a set $E \subset B_R$ such that, up to extracting subsequences, $|E_h \Delta E| \to 0$ as $h \to \infty$. If we pick any bounded set $F \subset \mathbb{R}^n$, then by Appendix A there exists a sequence of bounded sets $\{F_h\}_{h\in\mathbb{N}}$ such that

$$\lim_{h\to\infty} |F_h \Delta F| = 0\,, \qquad \limsup_{h\to\infty} P_{s_h}(F_h) \le P_{s_*}(F)\,. \tag{3.12}$$

By applying (3.1) to E_h and F_h, and then by letting $h \to \infty$, we find that

$$P_{s_*}(E) \le \liminf_{h\to\infty} P_{s_h}(E_h) \le \limsup_{h\to\infty} P_{s_h}(F) + \frac{\Lambda}{1-s_h}\,|E_h \Delta F_h| \le P_{s_*}(F) + \frac{\Lambda}{1-s_*}\,|E \Delta F|\,,$$

where the first inequality follows by Fatou's lemma, and the last one by (3.12). Since the Hausdorff convergence of ∂E_h to ∂E is again consequence of Lemma 3.1, the proof is complete. □

The next result is a uniform (with respect to s) version of the classical "improvement of flatness" statement.

Theorem 3.4. *Given $n \ge 2$, $\Lambda \ge 0$, and $s_0 \in (0,1)$, there exist $\tau, \eta, q \in (0,1)$, depending on n, Λ and s_0 only, with the following property: If E is a Λ-minimizer of the s-perimeter for some $s \in [s_0, 1]$ with $0 \in \partial E$ and*

$$B \cap \partial E \subset \Big\{ y \in \mathbb{R}^n : |(y-x)\cdot e| < \tau \Big\}$$

for some $e \in S^{n-1}$, then there exists $e_0 \in S^{n-1}$ such that

$$B_\eta \cap \partial E \subset \Big\{ y \in \mathbb{R}^n : |(y-x)\cdot e_0| < q\,\tau\,\eta \Big\}\,.$$

Proof. Step one: We prove that if $\bar s \in (0,1]$, then there exist $\delta > 0$ and $\bar\tau, \bar\eta, \bar q \in (0,1)$ (depending on n, $\bar s$ and Λ only), such that if $s \in (\bar s - \delta, \bar s + \delta) \cap (0,1]$ and E is a Λ-minimizer of the s-perimeter with $0 \in \partial E$ and

$$B \cap \partial E \subset \Big\{ y \in \mathbb{R}^n : |(y-x)\cdot e| < \bar\tau \Big\}$$

for some $e \in S^{n-1}$, then there exists $e_0 \in S^{n-1}$ such that

$$B_{\bar\eta} \cap \partial E \subset \Big\{ y \in \mathbb{R}^n : |(y-x)\cdot e_0| < \bar q\,\bar\tau\,\bar\eta \Big\}\,.$$

Indeed, it follows from [31, Theorems 24.1 and 26.3] in the case $\bar s = 1$, and from [10, Theorem 1.1] if $\bar s < 1$, that there exist $\bar\tau, \bar\eta, \bar q \in (0,1/2)$ (depending on n, $\bar s$ and Λ only) such that if F is a Λ-minimizer of the $\bar s$-perimeter with

$$0 \in \partial F\,, \qquad B \cap \partial F \subset \Big\{ y \in \mathbb{R}^n : |(y-x)\cdot e| < 2\,\bar\tau \Big\} \tag{3.13}$$

for some $e \in S^{n-1}$, then there exists $e_0 \in S^{n-1}$ such that

$$B_{\bar\eta} \cap \partial F \subset \Big\{ y \in \mathbb{R}^n : |(y-x)\cdot e_0| < \frac{\bar q}{4}\,(2\,\bar\tau)\,\bar\eta \Big\}\,. \tag{3.14}$$

Let us now assume by contradiction that our claim is false. Then we can find a sequence $s_h \to \bar s$ as $h \to \infty$, and, for every $h \in \mathbb{N}$, E_h Λ-minimizer of the s_h-perimeter such that, for some $e_h \in S^{n-1}$,

$$0 \in \partial E_h\,, \qquad B \cap \partial E_h \subset \Big\{ y \in \mathbb{R}^n : |(y-x)\cdot e_h| < \bar\tau \Big\}\,, \qquad \forall h \in \mathbb{N}\,, \tag{3.15}$$

but

$$B_{\bar\eta} \cap \partial E_h \not\subset \Big\{ y \in \mathbb{R}^n : |(y-x)\cdot e_0| < \bar q\,\bar\tau\,\bar\eta \Big\}\,, \qquad \forall h \in \mathbb{N}\,, \forall e_0 \in S^{n-1}\,. \tag{3.16}$$

By the compactness theorem, there exists a Λ-minimizer of the $\bar s$-perimeter F such that ∂E_h converges to ∂F with respect to the Hausdorff distance on compact sets. By the latter information we have $0 \in \partial F$, and we find from (3.15) that F is a Λ-minimizer of the $\bar{s}$-perimeter such that (3.13) holds true. In particular, there exists $e_0 \in S^{n-1}$ such that (3.14) holds true. By exploiting the local Hausdorff convergence of ∂E_h to ∂F one more time, we thus find that, if h is large enough, then

$$B_{\bar{\eta}} \cap \partial E_h \subset \Big\{ y \in \mathbb{R}^n : |(y-x)\cdot e_0| < \bar{q}\,\bar{\tau}\,\bar{\eta} \Big\}\,,$$

a contradiction to (3.16). We have completed the proof of step one.

Step two: We complete the proof of the theorem by covering $[s_0, 1]$ with a finite number of intervals $(\bar{s}_i - \delta_i, \bar{s}_i + \delta_i)$ of the form constructed in step one. □

Improvement of flatness implies $C^{1,\alpha}$-regularity by a standard argument. By exploiting the uniformity of the constants obtained in Theorem 3.4 one thus gets the following uniform regularity criterion.

Corollary 3.5. *If $n \geq 2$, $\Lambda \geq 0$ and $s_0 \in (0,1)$, then there exist positive constants $\varepsilon_0 < 1$, $C_0 > 0$, and $\alpha < 1$, depending on n, Λ and s_0 only, with the following property: If E is a Λ-minimizer of the s-perimeter for some $s \in [s_0, 1)$ and*

$$0 \in \partial E\,, \qquad B \cap \partial E \subset \Big\{ y \in \mathbb{R}^n : |(y-x)\cdot e| < \varepsilon_0 \Big\} \tag{3.17}$$

for some $e \in S^{n-1}$, then $B_{1/2} \cap \partial E$ is the graph of a function with $C^{1,\alpha}$-norm bounded by C_0.

Finally, by Hausdorff convergence of sequences of minimizers, we can exploit the regularity criterion (3.17) and the smoothness of the limit set B via a standard argument (see, e.g., [31, Theorem 26.6]) in order to obtain the following result, that plays a crucial role in the proof of our main results.

Corollary 3.6. *If $n \geq 2$, $\Lambda \geq 0$, $s_0 \in (0,1)$, E_h ($h \in \mathbb{N}$) is a Λ-minimizer of the s_h-perimeter for some $s_h \in [s_0, 1)$, and E_h converges in measure to B, then there exists a bounded sequence $\{u_h\}_{h\in\mathbb{N}} \subset C^{1,\alpha}(\partial B)$ (for some $\alpha \in (0,1)$ independent of h) such that*

$$\partial E_h = \Big\{ (1+u_h(x))x : x \in \partial B \Big\}\,, \qquad \lim_{h\to\infty} \|u_h\|_{C^1(\partial B)} = 0\,.$$

4. Proof of Theorem 1.1

Given $s \in (0,1]$, we introduce the *fractional isoperimetric gap* of $E \subset \mathbb{R}^n$ (with $0 < |E| < \infty$)

$$D_s(E) := \frac{P_s(E)}{P_s(B_{r_E})} - 1\,,$$

where $r_E = (|E|/|B|)^{1/n}$ and $P_1(E) = P(E)$ denotes the distributional perimeter of E. We shall also set

$$\delta_{s_0}(E) := \inf_{s_0 \leq s < 1} D_s(E)\,.$$

With this notation at hand, the quantitative isoperimetric inequality (1.4) takes the form

$$A(E)^2 \leq C(n, s_0)\, \delta_{s_0}(E)\,. \tag{4.1}$$

We begin by noticing that we can easily obtain (4.1) in the case of nearly spherical sets as a consequence of Theorem 2.1.

Remark 4.1. Starting from Corollary 4.2, we shall coherently enumerate the constants appearing in the various statements of this section. For example, thorough this section, the symbol C_0 will always denote the constant appearing in (4.2). No confusion will arise as we shall not need to refer to constants defined in other sections of the paper. Symbols like $C(n,s)$ shall be used to denote generic constants (depending on n and s only) whose precise value shall be inessential to us.

Corollary 4.2. *For every $n \geq 2$ there exist positive constants $C_0(n)$ and $\varepsilon_0(n)$ such that*

$$\frac{C_0(n)}{s}\, D_s(E) \geq A(E)^2 \tag{4.2}$$

whenever $s \in (0,1)$ and E is a nearly spherical set as in (2.1), with $|E| = |B|$, $\int_E x dx = 0$, and $\|u\|_{C^1(\partial B)} \leq \varepsilon_0(n)$. In particular, under these assumptions on E, we have that

$$\frac{C_0(n)}{s_0}\, \delta_{s_0}(E) \geq A(E)^2\,, \qquad \forall s_0 \in (0,1)\,. \tag{4.3}$$

Proof. This follows immediately by (2.2) since

$$A(E) \leq C(n) \int_{\partial B} |u|\, d\mathcal{H}^{n-1} \leq C(n) \sqrt{\int_{\partial B} |u|^2\, d\mathcal{H}^{n-1}}\,.$$

□

The proof of Theorem 1.1 is thus based on a reduction argument to the case considered in Corollary 4.2, much as in the spirit of what done [11] in the case $s = 1$. To this end, we argue by contradiction and assume (4.1) to fail. This gives us a sequence $\{E_h\}_{h\in\mathbb{N}}$ of almost-isoperimetric sets (that is, $D_{s_h}(E_h) \to 0$ as $h \to \infty$ for some $s_h \in [s_0, 1)$) with $|E_h| = |B|$ such that $D_{s_h}(E_h) < M\, A(E_h)^2$, for a constant M as large as we want. By Lemma 4.4 below, the first information allows us to deduce that, up to translations, $|E_h \Delta B| \to 0$ as $h \to \infty$. We next "round-up" our sets E_h by solving a penalized isoperimetric problem, see Lemma 4.6, to obtain a new sequence $\{F_h\}_{h\in\mathbb{N}}$ – with the same properties of $\{E_h\}_{h\in\mathbb{N}}$ concerning isoperimetric gaps and asymmetry – but with the additional feature of being nearly spherical sets associated to functions $\{u_h\}_{h\in\mathbb{N}} \subset C^1(\partial B)$ with $\|u_h\|_{C^1(\partial B)} \to 0$ as $h \to \infty$. By (4.3) this means that $C_0(n)/s_0 \geq M$, which gives a contradiction if we started the argument with M large enough.

In order to make this argument rigorous we need to premise a series of remarks that seem interesting in their own. The first one is a nucleation lemma for nonlocal perimeters in the spirit of [2, VI.13], see also [31, Lemma 29.10]. Here, $E^{(1)}$ stands for the set of points of density 1 of a measurable set E.

Lemma 4.3. *If $n \geq 2$, $s \in (0,1)$, $P_s(E) < \infty$, and $0 < |E| < \infty$, then there exists $x \in E^{(1)}$ such that*

$$|E \cap B(x,1)| \geq \min\Big\{ \frac{\chi_1\, |E|}{(1-s)\, P_s(E)}, \frac{1}{\chi_2} \Big\}^{n/s}\,, \tag{4.4}$$

where

$$\chi_1(n,s) := \frac{(1-s)\, P_s(B)}{4\, |B|^{(n-s)/n}\, \xi(n)}\,,$$

$$\chi_2(n,s) := \frac{2^{3+(n/s)}\, |B|^{(n-s)/n}\, P(B)}{s(1-s)\, P_s(B)}\,,$$

and where $\xi(n)$ is Besicovitch's covering constant (see for instance [31, Theorem 5.1]*). In particular, $0 < \inf\{\chi_1(n,s), \chi_2(n,s)^{-1} : s \in [s_0,1)\} < \infty$ for every $s_0 \in (0,1)$.*

Proof. Step one: We show that if $x \in E^{(1)}$ with

$$|E \cap B(x,1)| \leq \Big(\frac{(1-s)\, P_s(B)}{2\, |B|^{(n-s)/n}\, \alpha} \Big)^{n/s} \tag{4.5}$$

for some α satisfying

$$\alpha \geq \frac{2^{2+(n/s)}\, P(B)}{s}\,, \tag{4.6}$$

then there exists $r_x \in (0,1]$ such that

$$|E\cap B(x,r_x)| \le \frac{(1-s)}{\alpha}\int_{E\cap B(x,r_x)}\int_{E^c}\frac{dz\,dy}{|z-y|^{n+s}}\,. \tag{4.7}$$

Indeed, if not, setting for brevity $u(r) := |E\cap B(x,r)|$ we have $(1-s)\int_{E\cap B(x,r)}\int_{E^c}\frac{dz\,dy}{|z-y|^{n+s}} \le \alpha\, u(r)$ for every $r\le 1$. By adding up $(1-s)\int_{E\setminus B(x,r)}\int_{E\cap B(x,r)}\frac{dz\,dy}{|z-y|^{n+s}}$ to both sides, we get

$$P_s(E\cap B(x,r)) \le \int_{E\setminus B(x,r)}\int_{E\cap B(x,r)}\frac{dz\,dy}{|z-y|^{n+s}} + \frac{\alpha}{1-s}\,u(r)$$

for every $r\le 1$ so that, arguing as in the proof of Lemma 3.1, we get

$$\frac{P_s(B)}{|B|^{(n-s)/n}}\,u(r)^{(n-s)/n} \le \frac{P(B)}{s}\int_0^r\frac{u'(t)}{(r-t)^s}\,dt + \frac{\alpha}{1-s}\,u(r)\,,\qquad \forall r\le 1\,, \tag{4.8}$$

cf. with (3.8). By (4.5) we have

$$\frac{\alpha}{1-s}\,u(r) \le \frac{\alpha}{1-s}\,u(1)^{s/n}\,u(r)^{(n-s)/n} \le \frac{P_s(B)}{2\,|B|^{(n-s)/n}}\,u(r)^{(n-s)/n}\,,$$

so that (4.8) gives

$$u(r)^{(n-s)/n} \le \frac{2\,P(B)\,|B|^{(n-s)/n}}{s\,P_s(B)}\int_0^r\frac{u'(t)}{(r-t)^s}\,dt\,,\qquad \forall r\le 1\,. \tag{4.9}$$

Notice that (4.9) implies (3.9) with 1 in place of r_0. Moreover, (4.6) implies that $u(1)\le c_0|B|$, where

$$c_0 = \Big(\frac{s}{8\,|B|\,2^{n/s}}\,\frac{(1-s)P_s(B)}{P(B)}\Big)^{n/s}\,,$$

is the constant defined in Lemma 3.1. Therefore, by repeating the very same iteration argument seen in the proof of Lemma 3.1 (notice that $u(r)>0$ for every $r>0$ since $x\in E^{(1)}$), we see that $u(1)>c_0|B|$, and thus find a contradiction. This completes the proof of step one.

Step two: We complete the proof of the lemma. We argue by contradiction, and assume that for every $x\in E^{(1)}$ we have

$$|E\cap B(x,1)| \le \min\Big\{\frac{\chi_1\,|E|}{(1-s)\,P_s(E)},\frac{1}{\chi_2}\Big\}^{n/s}\,. \tag{4.10}$$

If we set

$$\alpha := \frac{(1-s)\,P_s(B)}{2\,|B|^{(n-s)/n}}\,\min\Big\{\frac{\chi_1\,|E|}{(1-s)\,P_s(E)},\frac{1}{\chi_2}\Big\}^{-1}\,, \tag{4.11}$$

then (4.10) takes the form of (4.5) for a value of α that (by definition of χ_2) satisfies (4.6). Hence, by step one, for every $x\in E^{(1)}$ there exists $r_x\in(0,1]$ such that (4.7) holds true with α as in (4.11). By applying Besicovitch covering theorem, see [31, Corollary 5.2], we find a countable disjoint family of balls $\{B(x_h,r_h)\}_{h\in\mathbb{N}}$ such that $x_h\in E^{(1)}$, $r_h=r_{x_h}$ is such that (4.7) holds true with $x=x_h$, and thus

$$\begin{aligned}|E| &\le \xi(n)\sum_{h\in\mathbb{N}}|E\cap B(x_h,r_h)| \le \frac{\xi(n)(1-s)}{\alpha}\sum_{h\in\mathbb{N}}\int_{E\cap B(x_h,r_h)}\int_{E^c}\frac{dz\,dy}{|z-y|^{n+s}}\\ &\le \frac{\xi(n)(1-s)P_s(E)}{\alpha} \le \chi_1\,\frac{\xi(n)\,2\,|B|^{(n-s)/n}}{(1-s)\,P_s(B)}\,|E| = \frac{|E|}{2}\,,\end{aligned}$$

by definition of χ_1. This is a contradiction, and the lemma is proved. □

Next, we prove the following "soft" stability lemma. An analogous statement was proved in [24, Lemma 3.1] in the case one works with $D_{s_0}(E)$ in place of $\delta_{s_0}(E)$, and under the additional assumption that $A(E) \le 3/2$. This last assumption was not a real restriction in [24], as the soft stability lemma was applied to sets enjoying certain symmetry properties that, in turn, were granting that $A(E) \le 3/2$. We avoid here the use of symmetrization arguments by exploiting the more general tool provided us by the nucleation lemma, Lemma 4.3.

Lemma 4.4. *If $n \ge 2$ and $s_0 \in (0,1)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ (depending on n, s_0, and ε) such that if $\delta_{s_0}(E) < \delta$ then $A(E) < \varepsilon$.*

Proof. By contradiction, we assume the existence of a sequence of sets $E_h \subset \mathbb{R}^n$, $h \in \mathbb{N}$, such that

$$|E_h| = |B|\,, \qquad A(E_h) \ge \varepsilon\,, \qquad \lim_{h\to\infty} \delta_{s_0}(E_h) = 0\,, \tag{4.12}$$

where ε is a positive constant. In particular there exist $s_h \in [s_0, 1)$, $h \in \mathbb{N}$, such that

$$\lim_{h\to\infty} \frac{P_{s_h}(E_h)}{P_{s_h}(B)} = 1\,. \tag{4.13}$$

Without loss of generality, we assume that $s_h \to s_* \in [s_0, 1]$ as $h \to \infty$. Since $(1-s)\,P_s(B) \to \omega_{n-1}\,P(B)$ as $s \to 1^-$, we find that

$$\sup_{h\in\mathbb{N}} (1-s_h)\,P_{s_h}(E_h) < \infty\,. \tag{4.14}$$

By Lemma 4.3, see (4.4), we find that, up to translations,

$$|E_h \cap B| \ge \min\left\{ \frac{\chi_1(n,s_h)\,|B|}{(1-s_h)\,P_{s_h}(E_h)}, \frac{1}{\chi_2(n,s_h)} \right\}^{n/s_h} \ge \kappa_*\,, \tag{4.15}$$

for some positive constant κ_* independent of h. By compactness of the embedding of $H^{s/2}(\mathbb{R}^n)$ into $L^1_{loc}(\mathbb{R}^n)$ when $s_* < 1$, or by [3, Theorem 1] in case $s_* = 1$, we exploit (4.14) to deduce that, up to extracting subsequences, there exists a measurable set E such that for every $K \subset\subset \mathbb{R}^n$ we have $|(E_h \Delta E) \cap K| \to 0$ as $h \to \infty$. By local convergence of E_h to E and by (4.12), we have $|E| \le |B|$. If $s_* = 1$, then by [3, Theorem 2] and by (4.13) we find

$$\omega_{n-1}\,P(E) \le \liminf_{h\to\infty} (1-s_h)\,P_{s_h}(E_h) = \liminf_{h\to\infty} (1-s_h)\,P_{s_h}(B) = \omega_{n-1}\,P(B)\,,$$

that is, $P(E) \le P(B)$. If, instead, $s_* < 1$, then (4.13) gives

$$P_{s_*}(B) = \lim_{h\to\infty} P_{s_h}(E_h) = \lim_{h\to\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1_{E_h}(x) 1_{E_h^c}(y)}{|x-y|^{n+s_h}}\,dxdy \ge P_{s_*}(E)\,,$$

where the last inequality follows by Fatou's lemma. In both cases, $P_{s_*}(E) \le P_{s_*}(B)$. Should it be $|E| = |B|$, then, by the (nonlocal, if $s_* < 1$) isoperimetric theorem, we would be able to conclude that $A(E) = 0$, against $A(E_h) \ge \varepsilon$ for every $h \in \mathbb{N}$. Should it be $|E| = 0$, then we would get a contradiction with (4.15). Therefore, it must be $0 < |E| < |B|$. By a standard application of the concentration-compactness lemma (see, e.g., [24, Lemma 3.1]), $0 < |E| < |B|$ can happen only if there exists $\lambda \in (0,1)$ such that for every $\sigma > 0$ and h large enough there exist $F_h^\sigma, G_h^\sigma \subset E_h$ with the property that

$$|E_h \setminus (F_h^\sigma \cup G_h^\sigma)| < \sigma\,, \quad ||F_h^\sigma| - \lambda\,|B|| < \sigma\,, \quad ||G_h^\sigma| - (1-\lambda)\,|B|| < \sigma\,,$$

and $\mathrm{dist}(F_h^\sigma, G_h^\sigma) \to +\infty$ as $h \to \infty$. Let us now set

$$K_{s,\eta}(z) := \frac{1_{\{\eta<|z|<\eta^{-1}\}}}{|z|^{n+s}} + \frac{1_{\{|z|<\eta\}}}{\eta^{n+s}}\,, \qquad z \in \mathbb{R}^n\,,$$

so that $K_{s,\eta}(x-y)\le |x-y|^{-(n+s)}$, and thus

$$\begin{aligned}
P_{s_h}(E_h) &\geq \int_{F_h^\sigma}\int_{E_h^c} K_{s_h,\eta}(x-y)\,dxdy+\int_{G_h^\sigma}\int_{E_h^c} K_{s_h,\eta}(x-y)\,dxdy\\
&\geq \int_{F_h^\sigma}\int_{(F_h^\sigma)^c} K_{s_h,\eta}(x-y)\,dxdy+\int_{G_h^\sigma}\int_{(G_h^\sigma)^c} K_{s_h,\eta}(x-y)\,dxdy-\frac{C(n)\sigma}{\eta^{n+s_h}}\\
&\geq \int_{B_{a_h^\sigma}}\int_{(B_{a_h^\sigma})^c} K_{s_h,\eta}(x-y)\,dxdy+\int_{B_{b_h^\sigma}}\int_{(B_{b_h^\sigma})^c} K_{s_h,\eta}(x-y)\,dxdy-\frac{C(n)\sigma}{\eta^{n+s_h}}\,,
\end{aligned}$$

where in the last inequality we have used [20, Lemma A.2] and we have chosen $a_h^\sigma, b_h^\sigma>0$ in such a way that $|B_{a_h^\sigma}|=|F_h^\sigma|$ and $|B_{b_h^\sigma}|=|G_h^\sigma|$. We now *first* let $\sigma\to 0^+$, to obtain

$$P_{s_h}(E_h)\geq \int_{B_a}\int_{(B_a)^c} K_{s_h,\eta}(x-y)\,dxdy+\int_{B_b}\int_{(B_b)^c} K_{s_h,\eta}(x-y)\,dxdy\,,$$

where a and b are such that $|B_a|=\lambda|B|$ and $|B_b|=(1-\lambda)|B|$. Next we let $\eta\to 0^+$, divide by $P_{s_h}(B)$, and then let $h\to\infty$ to reach the contradiction

$$1\geq \frac{P_{s_*}(B_a)}{P_{s_*}(B)}+\frac{P_{s_*}(B_b)}{P_{s_*}(B)}=\lambda^{(n-s_*)/n}+(1-\lambda)^{(n-s_*)/n}>1\,.$$

This completes the proof of the lemma. □

Next, we introduce the variational problems with penalization needed to round-up the nearly-isoperimetric sets E_h into nearly-spherical sets F_h. Precisely, we shall consider the problems

$$\inf\Big\{(1-s)\,P_s(E)+\Lambda\,\big||E|-|B|\big|+|\boldsymbol{\alpha}(E)-\boldsymbol{\alpha}|:E\subset\mathbb{R}^n\Big\}\,,\tag{4.16}$$

where $s\in(0,1)$, $\Lambda\geq 0$, $\boldsymbol{\alpha}>0$, and

$$\boldsymbol{\alpha}(E):=\inf\Big\{|E\Delta(x+B)|:x\in\mathbb{R}^n\Big\}\,,\qquad E\subset\mathbb{R}^n\,.$$

Notice that the existence of minimizers in (4.16) is a non-trivial issue. Indeed, minimizing sequences, in general, are compact only with respect to local convergence in measure, with respect to which $\Lambda\,\big||E|-|B|\big|$ is just upper semicontinuous if $|E|\leq|B|$. In addition, we cannot obtain global convergence through the isoperimetric argument used in the proof of Lemma 4.4, since (as we shall see in the proof of Lemma 4.6) a minimizing sequence in (4.16) will not be in general a sequence with vanishing isoperimetric gap (because $\boldsymbol{\alpha}(E)$ has to stay close to $\boldsymbol{\alpha}$). Therefore we have to resort to a finer argument, and show how to modify an arbitrary minimizing sequence into a uniformly bounded minimizing sequence. We base our argument on the following truncation lemma: the proof by contradiction is inspired by [2, VI.14], see also [31, Lemma 29.12].

Lemma 4.5. *Let $n\geq 2$, $s\in(0,1)$, and $E\subset\mathbb{R}^n$. If $|E\setminus B|\leq\eta<1$, then there exists $1\leq r_E\leq 1+C_1(n,s)\,\eta^{1/n}$ such that*

$$(1-s)\,P_s(E\cap B_{r_E})\leq(1-s)\,P_s(E)-\frac{|E\setminus B_{r_E}|}{C_2(n,s)\,\eta^{s/n}}\,,\tag{4.17}$$

where

$$C_1(n,s):=2^{1+(n-s)/s}\Big(\frac{4\,|B|^{(n-s)/n}\,P(B)}{s\,(1-s)\,P_s(B)}\Big)^{1/s}\,,\qquad C_2(n,s):=\frac{2|B|^{(n-s)/n}}{(1-s)\,P_s(B)}\,.\tag{4.18}$$

In particular, $\sup\{C_1(n,s)+C_2(n,s):s_0\leq s<1\}<\infty$.

Proof. Without loss of generality we consider a set E with $|E\setminus B|\le\eta<1$ and $|E\setminus B_{1+C_1\,\eta^{1/n}}|>0$. Correspondingly, if we set $u(r):=|E\setminus B_r|$, $r>0$, then u is a decreasing function with

$$[0,1+c_1\,\eta^{1/n}]\subset\operatorname{spt}u\qquad u(1)\le\eta\,,\qquad u'(r)=-\mathcal{H}^{n-1}(E\cap\partial B_r)\quad\text{for a.e. }r>0\,. \tag{4.19}$$

Arguing by contradiction, we now assume that

$$(1-s)\,P_s(E)\le(1-s)\,P_s(E\cap B_r)+\frac{u(r)}{C_2\,\eta^{s/n}}\,,\qquad\forall r\in(1,1+C_1\,\eta^{1/n})\,. \tag{4.20}$$

First, we notice that we have the identity

$$P_s(E\cap B_r)-P_s(E)=2\int_{E\cap B_r}\int_{E\cap B_r^c}\frac{dx\,dy}{|x-y|^{n+s}}-P_s(E\setminus B_r)\,,\qquad\forall r>0\,;$$

second, by arguing as in the proof of (3.7), and by (4.19), we see that

$$\int_{E\cap B_r}\int_{E\cap B_r^c}\frac{dx\,dy}{|x-y|^{n+s}}\le\frac{P(B)}{s}\int_r^\infty\frac{-u'(t)}{(t-r)^s}\,dt\,,\qquad\forall r>0\,;$$

finally, by (1.1), $P_s(E\setminus B_r)\ge P_s(B)|B|^{(s-n)/n}\,u(r)^{(n-s)/n}$. We may thus combine these three remarks with (4.20) to conclude that, if $r\in(1,1+C_1\,\eta^{1/n})$, then

$$\begin{aligned}0\quad&\le\quad\frac{2\,P(B)}{s}\int_r^\infty\frac{-u'(t)}{(t-r)^s}\,dt-\frac{P_s(B)}{|B|^{(n-s)/n}}\,u(r)^{(n-s)/n}+\frac{u(r)}{(1-s)\,C_2\,\eta^{s/n}}\\&\le\quad\frac{2\,P(B)}{s}\int_r^\infty\frac{-u'(t)}{(t-r)^s}\,dt-\frac{P_s(B)}{2|B|^{(n-s)/n}}\,u(r)^{(n-s)/n}\,,\end{aligned} \tag{4.21}$$

where in the last inequality we have used our choice of C_2 and the fact that $u(r)\le\eta$ for every $r>1$. We rewrite (4.21) in the more convenient form

$$u(r)^{(n-s)/n}\le C_3\int_r^\infty\frac{-u'(t)}{(t-r)^s}\,dt\,,\qquad\forall r\in(1,1+c\,\eta^{1/n})\,, \tag{4.22}$$

where we have set

$$C_3(n,s):=\frac{4\,|B|^{(n-s)/n}\,P(B)}{s\,P_s(B)}\,.$$

Let us set $r_k:=1+(1-2^{-k})\,C_1\,\eta^{1/n}$, so that $r_0=1$, $r_k<r_{k+1}$, and $r_\infty=1+C_1\,\eta^{1/n}$. Correspondingly, if we set $u_k=u(r_k)$, then by (4.19) we find that $u_0\le\eta$, $u_k\ge u_{k+1}$, and $u_\infty=\lim_{k\to\infty}u_k>0$. We are now going to show that (4.22) implies $u_\infty=0$, thus obtaining a contradiction and proving the lemma. Indeed, if we integrate (4.22) on (r_k,r_{k+1}) we get

$$\begin{aligned}(r_{k+1}-r_k)\,u_{k+1}^{(n-s)/n}\quad&\le\quad C_3\int_{r_k}^{r_{k+1}}dr\int_r^\infty\frac{-u'(t)}{(t-r)^s}\,dt\\&=\quad C_3\int_{r_k}^{r_{k+1}}(-u'(t))\,dt\int_{r_k}^t\frac{dr}{(t-r)^s}+C_3\int_{r_{k+1}}^\infty(-u'(t))\,dt\int_{r_k}^{r_{k+1}}\frac{dr}{(t-r)^s}\,.\end{aligned} \tag{4.23}$$

On the one hand we easily find that

$$\int_{r_k}^{r_{k+1}}(-u'(t))\,dt\int_{r_k}^t\frac{dr}{(t-r)^s}\le\frac{(r_{k+1}-r_k)^{1-s}}{1-s}\,(u_k-u_{k+1})\,; \tag{4.24}$$

on the other hand we notice that, for every $t>r_{k+1}$, since $|b^{1-s}-a^{1-s}|\le|b-a|^{1-s}$ for $a,b\ge0$,

$$\int_{r_k}^{r_{k+1}}\frac{dr}{(t-r)^s}=\frac{(t-r_k)^{1-s}-(t-r_{k+1})^{1-s}}{1-s}\le\frac{(r_{k+1}-r_k)^{1-s}}{1-s}\,.$$

Hence, since $|E|<\infty$ implies $\lim_{r\to\infty}u(r)=0$,

$$\int_{r_{k+1}}^\infty(-u'(t))\,dt\int_{r_k}^{r_{k+1}}\frac{dr}{(t-r)^s}\le\frac{(r_{k+1}-r_k)^{1-s}}{1-s}\,u_{k+1}\,. \tag{4.25}$$

We combine (4.23), (4.24), and (4.25) to find

$$(r_{k+1}-r_k)\,u_{k+1}^{(n-s)/n}\le\frac{C_3}{1-s}\,(r_{k+1}-r_k)^{1-s}\,u_k\,.$$

Since $r_{k+1}-r_k=C_1\,\eta^{1/n}\,2^{-k-1}$, we conclude that $u_{k+1}^{1-\alpha}\le N^k\,M\,u_k$, where

$$\alpha=\frac{s}{n}\,,\qquad N=2^s\,,\qquad M=\Big(\frac{2}{C_1\,\eta^{1/n}}\Big)^s\,\frac{C_3}{1-s}\,.$$

We notice that, since $u_0\le\eta<1$, we have $u_0\le(N^{(1-\alpha)/a^2}M^{1/\alpha})^{-1}$ thanks to our choice of C_1. We are thus in the position to apply Lemma 3.2 to get $u_\infty=0$ and obtain the required contradiction. □

Given $n\ge 2$, $s\in(0,1)$, $\alpha>0$, and $E\subset\mathbb{R}^n$, let us set for the sake of brevity

$$\mathcal{F}_{s,\Lambda,\boldsymbol{\alpha}}(E):=(1-s)\,P_s(E)+\Lambda\,\Big||E|-|B|\Big|+|\boldsymbol{\alpha}(E)-\boldsymbol{\alpha}|\,.$$

We now prove the existence of global minimizers of $\mathcal{F}_{s,\Lambda,\boldsymbol{\alpha}}$.

Lemma 4.6. *If $n\ge 2$, $s\in(0,1)$, $\Lambda>\Lambda_0(n,s)$ and $\boldsymbol{\alpha}<\varepsilon_1(n,s)$, then there exists a minimizer E in the variational problem (4.16), that is, $\mathcal{F}_{s,\Lambda,\boldsymbol{\alpha}}(E)\le\mathcal{F}_{s,\Lambda,\boldsymbol{\alpha}}(F)$ for every $F\subset\mathbb{R}^n$. Moreover, up to a translation, this minimizer satisfies*

$$E\subset B_{C_4(n,s)}\,.$$

Here we have set

$$\begin{aligned}\Lambda_0(n,s)&:=\frac{(1-s)\,P_s(B)}{|B|}\,,\\ \varepsilon_1(n,s)&:=\frac{1}{2}\min\Big\{1,\Big(\frac{1}{(\Lambda+1)C_2(n,s)}\Big)^{n/s},4|B|\Big\}\,,\\ C_4(n,s)&:=1+C_1(n,s)\,(2\varepsilon_1(n,s))^{1/n}\,.\end{aligned}$$

In particular, $\inf\{\varepsilon_1(n,s):s_0\le s<1\}>0$ *and* $\sup\{\Lambda_0(n,s)+C_4(n,s):s_0\le s<1\}<\infty$.

Proof. Step one: We first show that, since $s\in(0,1)$ and $\Lambda>(1-s)\,P_s(B)/|B|$, then the unit ball B is the unique solution, up to a translation, of the minimization problem

$$\min\big\{(1-s)\,P_s(E)+\Lambda\big||E|-|B|\big|\,:\,E\subset\mathbb{R}^n\big\}\,. \tag{4.26}$$

Indeed, by comparing any set E with a ball having its same volume and thanks to (1.1), we immediately reduce the competition class in (4.26) to the family of balls in $\mathbb{R}^n$. Note that, if $r>1$, then $P_s(B)<P_s(B_r)$, so that only balls with radius $r\le 1$ have to be considered. At the same time, if $\Lambda>(1-s)\,P_s(B)/\omega_n$, then one immediately gets that

$$(1-s)\,P_s(B_r)+\Lambda\big||B_r|-|B|\big|=r^{n-s}(1-s)\,P_s(B)+\Lambda\omega_n(1-r^n)$$

as a function of $r\in[0,1]$ attains its minimum at $r=1$.

Step two: Let us denote by γ the infimum value in (4.16), and let us consider sets E_h ($h\in\mathbb{N}$) with $\mathcal{F}_{s,\Lambda,\boldsymbol{\alpha}}(E_h)\le\gamma+h^{-1}\,\boldsymbol{\alpha}$. Since $\boldsymbol{\alpha}<\varepsilon_1\le 2|B|$, we immediately get that $\gamma\le(1-s)P_s(B)$. Therefore, since by step one $(1-s)\,P_s(B)\le(1-s)\,P_s(E_h)+\Lambda\,||E_h|-|B||$, we conclude that $|\boldsymbol{\alpha}(E_h)-\boldsymbol{\alpha}|\le h^{-1}\boldsymbol{\alpha}$. Hence, up to translations, we obtain that

$$|E_h\setminus B|\le|E_h\Delta B|\le 2\,\boldsymbol{\alpha}<2\varepsilon_1<1\,,\qquad\forall h\in\mathbb{N}\,.$$

If we set $\eta:=2\,\boldsymbol{\alpha}$, then by Lemma 4.5 we can find $1\le r_h\le 1+C_1(n,s)\,\eta^{1/n}$ such that

$$(1-s)\,P_s(E_h\cap B_{r_h})\le(1-s)\,P_s(E_h)-\frac{|E_h\setminus B_{r_h}|}{C_2(n,s)\,\eta^{s/n}}\,. \tag{4.27}$$

Since $|\boldsymbol{\alpha}(I)-\boldsymbol{\alpha}(J)|\le |I\Delta J|$ for every $I,J\subset\mathbb{R}^n$, if we set $F_h:=E_h\cap B_{r_h}$ then

$$\Lambda\,||F_h|-|B||+|\boldsymbol{\alpha}(F_h)-\boldsymbol{\alpha}|\le \Lambda\,||E_h|-|B||+|\boldsymbol{\alpha}(E_h)-\boldsymbol{\alpha}|+(\Lambda+1)\,|E_h\setminus B_{r_h}|\,,$$

so that (4.27) implies (by our choice of $\varepsilon_1>\eta/2$)

$$\mathcal{F}_{s,\Lambda,\boldsymbol{\alpha}}(F_h)\le \mathcal{F}_{s,\Lambda,\boldsymbol{\alpha}}(E_h)+\left((\Lambda+1)-\frac{1}{C_2(n,s)\,\eta^{s/n}}\right)|E_h\setminus B_{r_h}|\le \mathcal{F}_{s,\Lambda,\boldsymbol{\alpha}}(E_h)\,.$$

From this we conclude that $\mathcal{F}_{s,\Lambda,\boldsymbol{\alpha}}(F_h)\to\gamma$ as $h\to\infty$, that is, $\{F_h\}_{h\in\mathbb{N}}$ is still a minimizing sequence for (4.16) with the additional feature that, by construction,

$$F_h\subset B_{1+C_1\,(2\varepsilon_1)^{1/n}}\,,\qquad \forall h\in\mathbb{N}\,.$$

It is now easy to prove the existence of a minimizer in (4.16). $\square$

Proof of Theorem 1.1. Since both sides of (4.1) are scaling invariant, we may assume that $|E|=|B|$. We want to show the existence of $\delta_0=\delta_0(n,s_0)>0$ such that, if $M>0$ is large enough, then

$$A(E)^2\le M\,\delta_{s_0}(E)\,,\qquad \text{whenever } \delta_{s_0}(E)\le\delta_0\,. \tag{4.28}$$

(Notice that, since we always have $A(E)\le 2$, then $A(E)^2\le(4/\delta_0)\delta_{s_0}(E)$ whenever $\delta_{s_0}(E)\ge\delta_0$: in other words, (4.28) immediately implies (4.1).) To prove (4.28) we argue by contradiction, assuming that there exists a sequence E_h of sets with $|E_h|=|B|$, $\delta_{s_0}(E_h)\to 0$ as $h\to\infty$, but

$$\delta_{s_0}(E_h)<\frac{A(E_h)^2}{M}\,. \tag{4.29}$$

By Lemma 4.4 (and since $|E_h|=|B|$) we can thus find $s_h\in[s_0,1)$ and $h\in\mathbb{N}$ such that

$$\lim_{h\to\infty}\frac{P_{s_h}(E_h)}{P_{s_h}(B)}=1\,,\qquad D_{s_h}(E_h)\le\frac{|E_h\Delta B|^2}{M|B|^2}\,,\qquad \lim_{h\to\infty}\boldsymbol{\alpha}(E_h)=0\,. \tag{4.30}$$

We set $\boldsymbol{\alpha}_h:=\boldsymbol{\alpha}(E_h)$ (so that, up to translations, $\boldsymbol{\alpha}_h=|E_h\Delta B|$) and consider the minimization problems

$$\inf\Big\{(1-s_h)\,P_{s_h}(E)+\Lambda\big||E|-|B|\big|+|\boldsymbol{\alpha}(E)-\boldsymbol{\alpha}_h|\,:\,E\subset\mathbb{R}^n\Big\}\,, \tag{4.31}$$

where Λ is chosen so that

$$\Lambda>\sup_{s\in[s_0,1)}\frac{(1-s)\,P_s(B)}{|B|}\,; \tag{4.32}$$

notice that the right-hand side of (4.32) is finite since $(1-s)\,P_s(B)\to\omega_{n-1}\,P(B)$ as $s\to 1^-$. For the same reason, $\inf_{s\in[s_0,1)}\varepsilon_1(n,s)>0$, and thus for every h large enough we may entail that

$$\boldsymbol{\alpha}_h<\inf_{s\in[s_0,1)}\varepsilon_1(n,s)\,.$$

We can thus apply Lemma 4.6 to prove the existence of minimizers F_h in (4.31) with

$$F_h\subset B_{C_4(n,s_h)}\subset B_{C_5(n,s_0)}\,,\qquad \text{with}\quad C_5(n,s_0):=\sup_{s\in[s_0,1)}C_4(n,s)<\infty\,. \tag{4.33}$$

We shall assume (as we can do up to translations) that

$$\int_{F_h}x\,dx=0\,,\qquad\forall h\in\mathbb{N}\,. \tag{4.34}$$

By the minimality of each F_h, recalling (4.29) and (4.30) we have that

$$\begin{aligned}\mathcal{F}_{s_h,\Lambda,\boldsymbol{\alpha}_h}(F_h)&\le F_{s_h,\Lambda,\boldsymbol{\alpha}_h}(E_h)=(1-s_h)\,P_{s_h}(E_h)\le(1-s_h)\,P_{s_h}(B)+\frac{(1-s_h)\boldsymbol{\alpha}_h^2P_{s_h}(B)}{M|B|^2}\\&\le(1-s_h)\,P_{s_h}(F_h)+\Lambda\big||F_h|-|B|\big|+\frac{(1-s_h)\boldsymbol{\alpha}_h^2P_{s_h}(B)}{M|B|^2}\,,\end{aligned} \tag{4.35}$$

where in the last inequality we used step one in the proof of Lemma 4.6. Since $\boldsymbol{\alpha}_h \to 0$, we infer that $\boldsymbol{\alpha}(F_h)/\boldsymbol{\alpha}_h \to 1$ as $h \to \infty$. By taking into account (4.34), this implies in particular that

$$\lim_{h\to\infty} |F_h \Delta B| = 0\,. \tag{4.36}$$

If we now exploit the minimality property of each F_h together with the Lipschitz properties of $t \mapsto |t-|B||$, $t \mapsto |t-\boldsymbol{\alpha}_h|$, and the inequality $|\boldsymbol{\alpha}(I)-\boldsymbol{\alpha}(J)| \le |I\Delta J|$ for every $I, J \subset \mathbb{R}^n$, then we find that each F_h enjoy a uniform global almost-minimality property of the form

$$(1-s_h)P_{s_h}(F_h) \le (1-s_h)P_{s_h}(G) + (\Lambda+1)|F_h \triangle G|\,, \qquad \forall G \subset \mathbb{R}^n\,. \tag{4.37}$$

By (4.33), (4.36), (4.37), and Corollary 3.6, we find that F_h is nearly spherical, in the sense that $\partial F_h = \{x\,(1+u_h(x)) : x \in \partial B\}$, where $\|u_h\|_{C^1(\partial B)} \to 0$ as $h \to \infty$. Let now $\lambda_h > 0$ be such that $|\lambda_h\, F_h| = |B|$, and set $G_h = \lambda_h\, F_h$. We notice that, by (4.35),

$$\begin{aligned}(1-s_h)\big(P_{s_h}(G_h)-P_{s_h}(B)\big) &= (1-s_h)\,P_{s_h}(F_h)\,(\lambda_h^{n-s}-1) + (1-s_h)\big(P_{s_h}(F_h)-P_{s_h}(B)\big)\\ &\le (1-s_h)\,P_{s_h}(F_h)\,(\lambda_h^{n-s}-1) - \Lambda\,||F_h|-|B|| + \frac{(1-s_h)\boldsymbol{\alpha}_h^2 P_{s_h}(B)}{M|B|^2}\,.\end{aligned}$$

Again by (4.35), we have $(1-s_h)\,P_{s_h}(F_h) \le (1-s_h)\,P_{s_h}(B) + (1-s_h)\boldsymbol{\alpha}_h^2 P_{s_h}(B)/(M|B|^2) \le C_6$, provided we set

$$C_6(n,s_0) := \sup_{s\in[s_0,1)} (1-s)\,P_s(B)\big(1+|B|^{-2}\inf_{s\in[s_0,1)}\varepsilon_1(n,s)^2\big)\,,$$

and assume $M \ge 1$. Thus, by taking into account that $\lambda^{n-s}-1 \le |\lambda^n - 1|$ for every $\lambda > 0$ and that $\lambda_h \to 1$, we get

$$\begin{aligned}(1-s_h)\big(P_{s_h}(G_h)-P_{s_h}(B)\big) &\le C_6\,(\lambda_h^{n-s}-1) - \frac{\Lambda}{2}\,|B|\,|\lambda_h^n-1| + \frac{(1-s_h)\boldsymbol{\alpha}_h^2 P_{s_h}(B)}{M|B|^2}\\ &\le \Big(C_6 - \frac{\Lambda}{2}\,|B|\Big)|\lambda_h^n-1| + \frac{(1-s_h)\boldsymbol{\alpha}_h^2 P_{s_h}(B)}{M|B|^2}\,.\end{aligned}$$

We thus strengthen (4.32) into $\Lambda > C_6/|B|$ to find that $P_{s_h}(G_h)-P_{s_h}(B) \le \boldsymbol{\alpha}_h^2 P_{s_h}(B)/(M|B|^2)$, that is

$$D_{s_h}(G_h) \le \frac{\boldsymbol{\alpha}_h^2}{M|B|^2}\,,$$

that we combine with Corollary 4.2 to get

$$A(G_h)^2 \le \frac{C_0(n)}{s_0}\,D_{s_h}(G_h) \le \frac{C_0}{s_0\,M|B|^2}\,\boldsymbol{\alpha}_h^2\,.$$

Now, by scale invariance $A(G_h) = A(F_h)$; moreover, by (4.36), $|F_h| \to |B|$ as $h \to \infty$, and thus $A(F_h)^2 \ge \boldsymbol{\alpha}(F_h)^2/(2|B|^2)$ for h large enough; finally, as noticed in proving (4.36), $\boldsymbol{\alpha}(F_h)/\boldsymbol{\alpha}_h \to 1$ as $h \to \infty$, so that $A(F_h)^2 \ge \boldsymbol{\alpha}_h/(4|B|^2)$ for every h large enough, and we conclude that

$$\frac{\boldsymbol{\alpha}_h^2}{4} \le \frac{C_0}{s_0\,M}\,\boldsymbol{\alpha}_h^2\,.$$

We may thus choose

$$M > \max\Big\{1, \frac{4\,C_0(n)}{s_0}\Big\}\,,$$

in order to find a contradiction. This completes the proof of Theorem 1.1. $\square$

5. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. We shall continue the enumeration of constants that we started in section 4, working with the same convention set in Remark 4.1. We begin with an existence result. In the following, given a set $E \subset \mathbb{R}^n$ we shall set

$$\mathrm{Per}_s(E) := \begin{cases} \frac{1-s}{\omega_{n-1}} P_s(E)\,, & \text{if } s \in (0,1)\,, \\ P(E)\,, & \text{if } s = 1\,. \end{cases}$$

Notice that, by (1.6), at least on smooth sets Per_s is continuous as a function of $s \in (0,1]$. Recall that V_α denotes the Riesz potential defined in (1.5).

Lemma 5.1. *If $n \geq 2$, $s \in (0,1]$, and $\alpha \in (0,n)$, then there exist positive constants $m_1(n,\alpha,s)$ and $R_0(n,s)$ with the following property: For every $m < m_1$, the variational problem*

$$\inf\Big\{\mathrm{Per}_s(E) + V_\alpha(E) : |E| = m\Big\} \tag{5.1}$$

admits minimizers, and every minimizer E in (5.1) *satisfies (up to a translation) the uniform bound*

$$E \subset B_{(m/|B|)^{1/n}\,R_0}\,.$$

Moreover,

$$\sup\Big\{\frac{1}{m_1(n,\alpha,s)} + R_0(n,s) : \alpha \in [\alpha_0,n)\,, s \in [s_0,1]\Big\} < \infty\,, \qquad \forall s_0 \in (0,1),\ \alpha_0 \in (0,n)\,. \tag{5.2}$$

Proof of Lemma 5.1. We first notice that, as expected, the truncation lemma for nonlocal perimeters, namely Lemma 4.5, holds true as well for classical perimeters. This can be seen either by adapting the argument of Lemma 4.5 to the local case, or can be inferred as a particular case of [31, Lemma 29.12]. Either ways, one ends up showing that if $n \geq 2$ and $E \subset \mathbb{R}^n$ is such that $|E \setminus B| \leq \eta < 1$, then there exists $1 \leq r_E \leq 1 + C_1^*\,\eta^{1/n}$ such that

$$P(E \cap B_{r_E}) \leq P(E) - \frac{|E \setminus B_{r_E}|}{C_2^*\,\eta^{1/n}}\,,$$

where C_1^* and C_2^* are positive constants that depend on the dimension n only. We then extend the definition of $C_1(n,s)$ and $C_2(n,s)$ given in (4.18) to the case $s = 1$ by setting $C_1(n,1) = C_1^*$ and $C_2(n,1) = C_2^*$. In conclusion, this shows that for every $n \geq 2$, $s \in (0,1]$ and $E \subset \mathbb{R}^n$ is such that $|E \setminus B| \leq \eta < 1$, there exists $1 \leq r_E \leq 1 + C_1(n,s)\,\eta^{1/n}$ such that

$$\mathrm{Per}_s(E \cap B_{r_E}) \leq \mathrm{Per}_s(E) - \frac{|E \setminus B_{r_E}|}{C_2(n,s)\,\eta^{1/n}}\,,$$

where $C_1(n,s)$ a $C_2(n,s)$ are such that

$$\sup\Big\{C_1(n,s) + C_2(n,s) : s \in [s_0,1]\Big\} < \infty\,, \qquad \forall s_0 \in (0,1)\,.$$

With this tool at hand, we now pick $n \geq 2$, $\alpha \in (0,n)$, $s \in (0,1]$, and denote by γ the infimum in (5.1). We claim that for every $m < m_1$,

$$\gamma = \inf\Big\{\mathrm{Per}_s(E) + V_\alpha(E) : |E| = m\,, E \subset B_{(m/|B|)^{1/n}\,R_0}\Big\}\,, \tag{5.3}$$

where

$$m_1 = m_1(n,s,\alpha) := |B|\,\min\left\{1, \frac{\mathrm{Per}_s(B)}{8|B|^2 C(n,s)\,V_\alpha(B)}, \frac{\mathrm{Per}_s(B)}{2|B|^2 C(n,s)\,V_\alpha(B)}\left(\frac{|B|}{8\,C_2\,C_7}\right)^{2n/s}\right\}^{n/(\alpha+s)}\,,$$

$$R_0(n,s) := 3(1 + C_1)\,,$$

$C(n,s)$ is a constant such that (1.4) holds, and C_7 is defined as

$$C_7(n,s,\alpha) := 2\Big(\mathrm{Per}_s(B) + V_\alpha(B)\Big).$$

(Note that (5.2) follows immediately from $(1-s)\,P_s(B) \to \omega_{n-1}\,P(B)$ as $s \to 1^+$ and from the fact that $C(n,s) \le C(n,s_0)$ if $s \ge s_0$.) We start noting that if $B[m]$ denotes the ball of volume m then, since $m \le |B|$,

$$\begin{aligned} \gamma &\le \mathrm{Per}_s(B[m]) + V_\alpha(B[m]) \\ &= \Big(\frac{m}{|B|}\Big)^{(n-s)/n}\mathrm{Per}_s(B) + \Big(\frac{m}{|B|}\Big)^{(n+\alpha)/n} V_\alpha(B) \\ &\le C_7\Big(\frac{m}{|B|}\Big)^{(n-s)/n}, \end{aligned} \tag{5.4}$$

where in the last inequality we have used the definition of C_7. If E is a generic set with

$$|E| = m\,, \qquad \mathrm{Per}_s(E) + V_\alpha(E) \le \gamma + V_\alpha(B)\Big(\frac{m}{|B|}\Big)^{(n+\alpha)/n}, \tag{5.5}$$

then by (5.4) we find

$$D_s(E) \le \frac{2\,(m/|B|)^{(n+\alpha)/n}\,V_\alpha(B)}{(m/|B|)^{(n-s)/n}\,\mathrm{Per}_s(B)} = \frac{2\,V_\alpha(B)}{\mathrm{Per}_s(B)}\Big(\frac{m}{|B|}\Big)^{(\alpha+s)/n}. \tag{5.6}$$

Let us set $E_* := \lambda\,E$ where $\lambda := (|B|/m)^{1/n}$, so that $|E_*| = |B|$. Since $D_s(E) = D_s(E_*)$, up to a translation we have, recalling (1.4),

$$|E_*\Delta B| \le |B|\bigg(C(n,s)\Big(\frac{m}{|B|}\Big)^{(\alpha+s)/n}\frac{2\,V_\alpha(B)}{\mathrm{Per}_s(B)}\bigg)^{1/2} =: \eta\,.$$

By Lemma 4.5 we can find $r_* \le 1 + C_1\,\eta^{1/n}$ such that

$$\mathrm{Per}_s(E_*\cap B_{r_*}) \le \mathrm{Per}_s(E_*) - \frac{|E_*\setminus B_{r_*}|}{C_2\,\eta^{s/n}}\,.$$

In particular, scaling back to E and setting $r_m = r_*/\lambda$, we find

$$\mathrm{Per}_s(E\cap B_{r_m}) \le \mathrm{Per}_s(E) - \Big(\frac{m}{|B|}\Big)^{(n-s)/n}\frac{|B|}{C_2\,\eta^{s/n}}\,\frac{|E\setminus B_{r_m}|}{m}\,.$$

Since trivially $V_\alpha(E\cap B_{r_m}) \le V_\alpha(E)$, we conclude that

$$\mathrm{Per}_s(E\cap B_{r_m}) + V_\alpha(E\cap B_{r_m}) \le \mathrm{Per}_s(E) + V_\alpha(E) - \Big(\frac{m}{|B|}\Big)^{(n-s)/n}\frac{u|B|}{C_2\,\eta^{s/n}}\,, \tag{5.7}$$

where we have set $u := |E\setminus B_{r_m}|/m$. Let us now consider $F := \mu(E\cap B_{r_m})$ for $\mu > 0$ such that $|F| = m$. Since $\mu = (1-u)^{-1/n}$ with $u < \eta$, if we assume that $\eta \le 1/2$, and take into account that

$$\frac{1}{(1-u)^p} \le 1 + 2^{p+1}\,u \qquad \forall u \in [0,1/2]\,,$$

then, by $\max\{\mu^{n-s},\mu^{n+\alpha}\} = \mu^{n+\alpha} \le 1 + 8\,u$ and by (5.7), we conclude that

$$\begin{aligned} \mathrm{Per}_s(F) + V_\alpha(F) &= \mu^{n-s}\mathrm{Per}_s(E\cap B_{r_m}) + \mu^{n+\alpha}\,V_\alpha(E\cap B_{r_m}) \\ &\le (1+8\,u)\Big(\mathrm{Per}_s(E\cap B_{r_m}) + V_\alpha(E\cap B_{r_m})\Big) \\ &\le \mathrm{Per}_s(E) + V_\alpha(E) + \Big(8\,C_7 - \frac{|B|}{C_2\,\eta^{s/n}}\Big)\Big(\frac{m}{|B|}\Big)^{(n-s)/n} u\,, \end{aligned}$$

where we have also taken into account that, by (5.7), (5.5), (5.4), and $m \le |B|$,

$$\begin{aligned} \mathrm{Per}_s(E \cap B_{r_m}) + V_\alpha(E \cap B_{r_m}) &\le \Big(\frac{m}{|B|}\Big)^{(n-s)/n} \mathrm{Per}_s(B) + 2\Big(\frac{m}{|B|}\Big)^{(n+\alpha)/n} V_\alpha(B) \\ &\le C_7 \Big(\frac{m}{|B|}\Big)^{(n-s)/n} . \end{aligned}$$

Since the definition of m_1 implies that $\eta^{s/n} \le |B|/(8\,C_2\,C_7)$, we have proved that for every set E as in (5.5) we can find a set F with $|F| = m$ and $F \subset B_{\mu r_m}$ such that $\mathrm{Per}_s(F) + V_\alpha(F) \le \mathrm{Per}_s(E) + V_\alpha(E)$. This implies (5.3) and completes the proof of the lemma by observing that $\mu \le 1 + 2^{1+1/n} u < 3$ and $r_m = r_*/\lambda \le (1 + C_1)(m/|B|)^{\frac{1}{n}}$. □

Next, we want to show that minimizers in (5.1), once rescaled to have the volume of the unit ball, are Λ-minimizers of the s-perimeter for some uniform value of Λ.

Lemma 5.2. *If $n \ge 2$, $s \in (0,1]$, $\alpha \in (0,n)$, E is a minimizer in* (5.1) *for $m < m_1$, and $E_* = \lambda\, E$ for $\lambda > 0$ such that $|E_*| = |B|$, then $E_* \subset B_{R_0}$ and*

$$\mathrm{Per}_s(E_*) \le \mathrm{Per}_s(F) + \Lambda_1\, |E_* \Delta F| , \tag{5.8}$$

for every $F \subset \mathbb{R}^n$. Here,

$$\begin{aligned} \Lambda_1(n,\alpha,s) &:= \frac{4\,C_7}{|B|} + \frac{6\,|B|\,(1+C_8)\,C_8^{\alpha/n}}{\alpha} , \\ C_8(n,\alpha,s) &:= \Big(1 + \frac{V_\alpha(B)}{\mathrm{Per}_s(B)}\Big)^{n/(n-s)} . \end{aligned}$$

In particular,

$$\sup_{s\in[s_0,1],\alpha\in[\alpha_0,n)} \Lambda_1(n,s,\alpha) < \infty , \qquad \forall s_0 \in (0,1)\,, \alpha_0 \in (0,n) .$$

Proof. We first notice that, if $F, G \subset \mathbb{R}^n$ with $|F| < \infty$, then

$$V_\alpha(F) - V_\alpha(G) \le \frac{2\,P(B)}{\alpha} \Big(\frac{|F|}{|B|}\Big)^{\alpha/n} |F \setminus G| . \tag{5.9}$$

(This is a more precise version of [33, Lemma 5.2.1].) Indeed, if $r_F = (|F|/|B|)^{1/n}$ is the radius of the ball of volume $|F|$, then

$$V_\alpha(F) - V_\alpha(G) \le 2 \int_F \int_{F\setminus G} \frac{dx\,dy}{|x-y|^{n-\alpha}} = 2 \int_{F\setminus G} dx \int_F \frac{dy}{|x-y|^{n-\alpha}} \le 2|F\setminus G| \int_{B_{r_F}} \frac{dz}{|z|^{n-\alpha}} ,$$

that is (5.9). We now prove that E_* satisfies (5.8). Of course, we may directly assume that $\mathrm{Per}_s(F) \le \mathrm{Per}_s(E_*)$. We also claim that we can reduce to prove (5.8) in the case that

$$\frac{1}{2} \le \frac{|F|}{|B|} \le C_8 . \tag{5.10}$$

Indeed, if we compare E with a ball of volume m (see (5.4)) and then multiply the resulting inequality by λ^{n-s}, we find

$$\mathrm{Per}_s(E_*) + \frac{V_\alpha(E_*)}{\lambda^{\alpha+s}} \le \mathrm{Per}_s(B) + \frac{V_\alpha(B)}{\lambda^{\alpha+s}} \le \mathrm{Per}_s(B) + V_\alpha(B) , \tag{5.11}$$

where in the last inequality we have taken into account that $\lambda \ge 1$ (because $m \le m_1 \le |B|$). If now F is such that $|F| \le |B|/2$, then $|E_* \Delta F| \ge |B|/2$, and thus (5.8) trivially holds true by (5.11) and our definition of Λ_1. If instead the upper bound in (5.10) does not hold, then we obtain a contradiction by combining $\mathrm{Per}_s(F) \le \mathrm{Per}_s(E_*)$, (1.1) (or the classical isoperimetric inequality if $s = 1$), and (5.11). We have thus reduced to prove (5.8) in the case that (5.10) holds true. If we now set $\mu = (m/|F|)^{1/n}$, then $|\mu\, F| = m$, and by minimality of E in (5.1) and by (5.9) we find that

$$\begin{aligned}\mathrm{Per}_s(E) &\le \mathrm{Per}_s(\mu\, F) + V_\alpha(\mu\, F) - V_\alpha(E)\\ &= \mathrm{Per}_s(\mu\, F) + \mu^{n+\alpha}\Big(V_\alpha(F) - V_\alpha(E_*)\Big) + \Big((\lambda\,\mu)^{n+\alpha} - 1\Big)\, V_\alpha(E)\,,\end{aligned}$$

where in the last identity we have added and subtracted $V_\alpha(\lambda\,\mu\, E)$. We multiply this inequality by λ^{n-s}, apply (5.9) and (5.10) to the second term on the right-hand side, and take into account that $\lambda^{n-s}\, V_\alpha(E) = \lambda^{-s-\alpha}\, V_\alpha(E_*)$, to find that

$$\begin{aligned}\mathrm{Per}_s(E_*) &\le (\lambda\,\mu)^{n-s}\,\mathrm{Per}_s(F) + \lambda^{n-s}\mu^{n+\alpha}\,\frac{2\,P(B)\,C_8^{\alpha/n}}{\alpha}\,|F\setminus E_*|\\ &\quad+\Big((\lambda\,\mu)^{n+\alpha} - 1\Big)\,\frac{V_\alpha(E_*)}{\lambda^{\alpha+s}}\,.\end{aligned} \tag{5.12}$$

We now estimate the various terms on the right-hand side of (5.12). Since $|F| \ge |B|/2$ and $|B| - |F| = |E_*| - |F| \le |E_*\Delta F|$ give

$$(\lambda\,\mu)^{n-s} = \Big(1 + \frac{|B| - |F|}{|F|}\Big)^{(n-s)/n} \le 1 + \frac{n-s}{n}\,\frac{|E_*\Delta F|}{|B|/2} \le 1 + \frac{2}{|B|}|E_*\Delta F|\,, \tag{5.13}$$

by $\mathrm{Per}_s(F) \le \mathrm{Per}_s(E_*)$ and (5.11) we find

$$(\lambda\,\mu)^{n-s}\,\mathrm{Per}_s(F) \le \mathrm{Per}_s(F) + \frac{C_7}{|B|}\,|E_*\Delta F|\,. \tag{5.14}$$

Since (5.10) also gives $|E_*\Delta F| \le (1 + C_8)\,|B|$, by (5.13) and $m \le |B|$ we have

$$\begin{aligned}\lambda^{n-s}\,\mu^{n+\alpha} &= \mu^{\alpha+s}\,(\lambda\mu)^{n-s} \le \Big(\frac{m}{|B|}\Big)^{(\alpha+s)/n}\Big(1 + \frac{2(n-s)}{P(B)}\,|E_*\Delta F|\Big)\\ &\le 1 + \frac{2(n-s)}{n}\,(1 + C_8) \le 3\,(1 + C_8)\,.\end{aligned} \tag{5.15}$$

Finally, by $|F| \ge |B|/2$ we find that

$$(\lambda\,\mu)^{n+\alpha} = \Big(1 + \frac{|B| - |F|}{|F|}\Big)^{1+(\alpha/n)} \le 1 + (2^{1+(\alpha/n)} - 1)\,\Big|\frac{|B| - |F|}{|F|}\Big| \le 1 + \frac{6}{|B|}\,|E_*\Delta F|\,, \tag{5.16}$$

that combined with (5.11) gives

$$\Big((\lambda\,\mu)^{n+\alpha} - 1\Big)\,\frac{V_\alpha(E_*)}{\lambda^{\alpha+s}} \le \frac{3\,C_7}{|B|}\,|E_*\Delta F|\,.$$

We now plug (5.14), (5.15), (5.16), and (5.11) into (5.12) to complete the proof of (5.8). □

Proof of Theorem 1.3. Let us fix $s_0 \in (0,1)$ and $\alpha_0 \in (0,n)$, and let

$$\bar m_1 := \inf\Big\{m_1(n,\alpha,s) : \alpha\in[\alpha_0,n)\,, s\in[s_0,1)\Big\}\,,$$

so that, by Lemma 5.1 and Lemma 5.2, $\bar m_1 > 0$ and for every $m < \bar m_1$, $\alpha \in [\alpha_0, n)$, and $s \in [s_0, 1)$, there exists a minimizer $E_{m,\alpha,s}$ of

$$\inf\Big\{\mathrm{Per}_s(E) + V_\alpha(E) : |E| = m\Big\}$$

such that

$$\mathrm{Per}_s(E_{m,\alpha,s}) \le \mathrm{Per}_s(F) + \bar\Lambda_1\,|E_{m,\alpha,s}\Delta F|\,, \qquad \forall F\subset\mathbb{R}^n\,,$$

where

$$\bar\Lambda_1 := \sup\Big\{\Lambda_1(n,\alpha,s) : \alpha\in[\alpha_0,n)\,, s\in[s_0,1)\Big\} < \infty\,.$$

We now want to show the existence of $m_0 \le \bar m_1$ such that $A(E_{m,\alpha,s}) = 0$ for $m < m_0$, which implies that $E_{m,\alpha,s}$ is a ball (recall (1.3)).

We argue by contradiction and construct sequences $\{s_h\}_{h\in\mathbb{N}}\subset[s_0,1]$, $\{\alpha_h\}_{h\in\mathbb{N}}\subset[\alpha_0,n)$, and $\{E_h\}_{h\in\mathbb{N}}$ minimizers of $\mathrm{Per}_{s_h}+V_{\alpha_h}$ at volume m_h, such that $m_h\to 0^+$ as $h\to\infty$ and, if we set $\lambda_h=(|B|/m_h)^{1/n}$, then $E_{h,*}=\lambda_h\,E_h$ is a $\bar\Lambda_1$-minimizers of the s_h-perimeter with

$$|E_{h,*}|=|B|\,,\qquad A(E_{h,*})=A(E_h)>0\,,\qquad \forall h\in\mathbb{N}\,.$$

By (5.6) and either by Theorem 1.1 if $s_h<1$, or by [22, Theorem 1.1] in the case $s_h=1$, we have that, for a suitable positive constant $C(n,s_0)$,

$$\frac{A(E_h)^2}{C_0(n,s_0)}\le D_{s_h}(E_h)\le\frac{2\,V_{\alpha_h}(B)}{\mathrm{Per}_{s_h}(B)}\Big(\frac{m_h}{|B|}\Big)^{(\alpha_h+s_h)/n}\,,$$

so that

$$A(E_{h,*})\le C(n,s_0,\alpha_0)\,m_h^{(\alpha_0+s_0)/2n}\,,\qquad \forall h\in\mathbb{N}\,.$$

Up to translations, we may thus assume

$$\lim_{h\to\infty}|E_{h,*}\Delta B|=0\,.$$

By Corollary 3.6, we thus have

$$\partial E_{h,*}=\Big\{(1+u_h(x))\,x:x\in\partial B\Big\}\,,\qquad u_h\in C^1(\partial B)\,,\qquad \forall h\in\mathbb{N}\,,$$

where $\|u_h\|_{C^1(\partial B)}\to 0$ as $h\to\infty$. Since $|E_{h,*}|=|B|$, by Lemma 5.3 below we find that

$$V_\alpha(B)-V_\alpha(E_{h,*})\le C(n)\,\Big([u_h]^2_{\frac{1-\alpha}{2}}+\|u_h\|^2_{L^2(\partial B)}\Big)\,,\qquad \forall\alpha\in(0,n)\,,$$

where

$$[u]^2_{\frac{1-\alpha}{2}}:=\iint_{\partial B\times\partial B}\frac{|u(x)-u(y)|^2}{|x-y|^{n-\alpha}}d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,.$$

Notice, in particular, that

$$[u]^2_{\frac{1-\alpha}{2}}\le 2^{\alpha+s}\,[u]^2_{\frac{1+s}{2}}\,,\qquad \forall\alpha\in(0,n)\,,s\in(0,1)\,.\tag{5.17}$$

At the same time, by $\mathrm{Per}_{s_h}(E_h)+V_{\alpha_h}(E_h)\le\mathrm{Per}_{s_h}(B_{r_h})+V_{\alpha_h}(B_{r_h})$, where $|B_{r_h}|=m_h$, we have

$$\begin{aligned}\delta_{s_0}(E_h)\quad&\le\quad D_{s_h}(E_h)\le\frac{V_{\alpha_h}(B_{r_h})-V_{\alpha_h}(E_h)}{\mathrm{Per}_{s_h}(B_{r_h})}\\&\le\quad m_h^{(\alpha_h+s_h)/n}\,\frac{C(n)\,\Big([u_h]^2_{\frac{1-\alpha}{2}}+\|u_h\|^2_{L^2(\partial B)}\Big)}{\inf_{s\in[s_0,1)}\mathrm{Per}_s(B)}\\&\le\quad C(n,s_0)\,m_h^{(\alpha_h+s_h)/n}\,\Big([u_h]^2_{\frac{1+s_0}{2}}+\|u_h\|^2_{L^2(\partial B)}\Big)\,,\end{aligned}$$

where we used (5.17). On the other hand, by Theorem 2.1 (notice that we can assume without loss of generality that $\int_{E_h}x\,dx=0$ for every $h\in\mathbb{N}$)

$$\delta_{s_0}(E_h)\ge\frac{s_0}{C(n)}\,\Big([u_h]^2_{\frac{1+s_0}{2}}+\|u_h\|^2_{L^2(\partial B)}\Big)\,.$$

We have thus proved

$$\frac{s_0}{C(n)}\le C(n,s_0)\,m_h^{(\alpha_h+s_h)/n}\,,$$

and since $\alpha_h\ge\alpha_0$, $s_h\ge s_0$, and $m_h\to 0$, this inequality leads to a contradiction for h sufficiently large. $\square$

Let us recall that, by Riesz's rearrangement inequality, for every $\alpha \in (0,n)$

$$V_\alpha(B) \ge V_\alpha(E) \qquad \text{whenever } |E| = |B|\,, \tag{5.18}$$

with equality if and only if $E = x + B$ for some $x \in \mathbb{R}^n$. (Indeed, the radial convolution kernel $|z|^{\alpha-n}$ is strictly decreasing.) Due to the maximality property of balls expressed in (5.18), one expect the quantity V_α to satisfy an estimate of the form $V_\alpha(E) \ge V_\alpha(B) - C(n,\alpha)\,\|u\|^2$ on nearly spherical sets of volume $|B|$, for some suitable norm $\|\cdot\|$. This is exactly the content of the following lemma.

Lemma 5.3. *There exist positive constants ε_0 and C_0, depending on n only, with the following property: If $E \subset \mathbb{R}^n$ is an open set such that $|E| = |B|$ and*

$$\partial E = \Big\{(1+u(x))\,x : x \in \partial B\Big\}\,,$$

for some function $u \in C^1(\partial B)$ with $\|u\|_{C^1(\partial B)} \le \varepsilon_0$, then

$$V_\alpha(B) - V_\alpha(E) \le C_0\left([u]^2_{\frac{1-\alpha}{2}} + \alpha V_\alpha(B)\|u\|^2_{L^2(\partial B)}\right), \qquad \forall \alpha \in (0,n)\,.$$

Proof. The proof of this result is very similar to the one of Theorem 2.1.

As in that proof, we slightly change notation and assume that E_t is an open set with $|E_t| = |B|$ and

$$E_t = \Big\{(1+t\,u(x))\,x : x \in \partial B\Big\}\,, \qquad \|u\|_{C^1(\partial B)} \le \frac{1}{2}\,, \qquad t \in (0, 2\varepsilon_0)\,.$$

Given $r, \rho, \theta \ge 0$ we now set

$$f_\theta(r,\rho) := \frac{r^{n-1}\,\rho^{n-1}}{(|r-\rho|^2 + r\,\rho\,\theta^2)^{(n-\alpha)/2}}\,,$$

so that

$$V_\alpha(E_t) = \int_{\partial B} d\mathcal{H}^{n-1}_x \int_{\partial B} d\mathcal{H}^{n-1}_y \int_0^{1+t\,u(x)} dr \int_0^{1+t\,u(y)} f_{|x-y|}(r,\rho)\,d\rho\,.$$

By exploiting the identity

$$2\int_0^a\int_0^b = \int_0^a\int_0^a + \int_0^b\int_0^b - \int_a^b\int_a^b\,, \qquad a, b \in \mathbb{R}\,,$$

we find that

$$\begin{aligned} V_\alpha(E_t) \quad &= \quad \int_{\partial B} d\mathcal{H}^{n-1}_x \int_{\partial B} d\mathcal{H}^{n-1}_y \int_0^{1+t\,u(x)} dr \int_0^{1+t\,u(x)} f_{|x-y|}(r,\rho)\,d\rho \\ &\quad -\frac{1}{2}\int_{\partial B} d\mathcal{H}^{n-1}_x \int_{\partial B} d\mathcal{H}^{n-1}_y \int_{1+t\,u(y)}^{1+t\,u(x)} dr \int_{1+t\,u(y)}^{1+t\,u(x)} f_{|x-y|}(r,\rho)\,d\rho\,. \end{aligned} \tag{5.19}$$

By a change of variable, for every $x \in \partial B$ we find

$$\begin{aligned} &\int_{\partial B} d\mathcal{H}^{n-1}_y \int_0^{1+t\,u(x)} dr \int_0^{1+t\,u(x)} f_{|x-y|}(r,\rho)\,d\rho \\ = \quad &(1+t\,u(x))^{n+\alpha} \int_{\partial B} d\mathcal{H}^{n-1}_y \int_0^1 dr \int_0^1 f_{|x-y|}(r,\rho)\,d\rho = (1+t\,u(x))^{n+\alpha}\,\frac{V_\alpha(B)}{P(B)}\,, \end{aligned}$$

where in the last identity we have used (5.19) with $u = 0$. Hence,

$$\begin{aligned} V_\alpha(E_t) \quad &= \quad -\frac{1}{2}\int_{\partial B} d\mathcal{H}^{n-1}_x \int_{\partial B} d\mathcal{H}^{n-1}_y \int_{1+t\,u(y)}^{1+t\,u(x)} dr \int_{1+t\,u(y)}^{1+t\,u(x)} f_{|x-y|}(r,\rho)\,d\rho \\ &\quad +\frac{V_\alpha(B)}{P(B)} \int_{\partial B} (1+t\,u)^{n+\alpha}\,d\mathcal{H}^{n-1}\,, \end{aligned}$$

from which we conclude that

$$V_\alpha(B)-V_\alpha(E_t)=\frac{t^2}{2}\,g(t)+\frac{V_\alpha(B)}{P(B)}\,(h(0)-h(t))\,,$$

provided we set $h(t):=\int_{\partial B}(1+t\,u)^{n+\alpha}\,d\mathcal{H}^{n-1}$ and

$$g(t):=\int_{\partial B}d\mathcal{H}^{n-1}_x\int_{\partial B}d\mathcal{H}^{n-1}_y\int_{u(y)}^{u(x)}dr\int_{u(y)}^{u(x)}f_{|x-y|}(1+t\,r,1+t\,\rho)\,d\rho\,.$$

Since $|B|=|E_t|$ implies $\int_{\partial B}(1+t\,u)^n=n\,|E_t|=n\,|B|=P(B)=h(0)$, we get

$$\begin{aligned}h(0)-h(t)&=\int_{\partial B}(1+tu)^n\big(1-(1+tu)^\alpha\big)\,d\mathcal{H}^{n-1}\\&\le-\alpha\,t\int_{\partial B}u\,d\mathcal{H}^{n-1}-\alpha\,(2n+\alpha-1)\,\frac{t^2}{2}\int_{\partial B}u^2\,d\mathcal{H}^{n-1}+C(n)\,\alpha\,t^3\,\|u\|^2_{L^2}\,.\end{aligned}$$

In addition, because $|B|=|E_t|$ also gives $0=\int_{\partial B}\big((1+t\,u)^n-1\big)$, we can likewise deduce that

$$-t\int_{\partial B}u\,d\mathcal{H}^{n-1}\le(n-1)\frac{t^2}{2}\int_{\partial B}u^2\,d\mathcal{H}^{n-1}+C(n)\,t^3\,\|u\|^2_{L^2}\,,$$

therefore

$$h(0)-h(t)\le-\alpha\,(n+\alpha)\,\frac{t^2}{2}\int_{\partial B}u^2\,d\mathcal{H}^{n-1}+\alpha\,C(n)\,t^3\,\|u\|^2_{L^2}\,.$$

Furthermore, we notice that

$$g(0)=\iint_{\partial B\times\partial B}\frac{|u(x)-u(y)|^2}{|x-y|^{n-\alpha}}d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y=[u]^2_{\frac{1-\alpha}{2}}\,.$$

Arguing as in the proof of Theorem 2.1, we infer that $g(t)=g(0)+t\,g'(\tau)$ for some $\tau\in(0,t)$ and with $|g'(\tau)|\le C(n)\,g(0)$. Hence,

$$\begin{aligned}V_\alpha(B)-V_\alpha(E_t)\le\frac{t^2}{2}&\left([u]^2_{\frac{1-\alpha}{2}}-\alpha(n+\alpha)\frac{V_\alpha(B)}{P(B)}\,\|u\|^2_{L^2}\right)\\&+C(n)\,t^3\Big([u]^2_{\frac{1-\alpha}{2}}+\alpha V_\alpha(B)\|u\|^2_{L^2}\Big)\,.\end{aligned}\tag{5.20}$$

This last estimate obviously implies the announced result. □

6. First and second variation formulae and local minimizers

In this section we provide first and second variation formulae for the functionals P_s (compare with [12, Section 4]) and V_α, and actually for generic nonlocal functionals behaving like P_s and V_α. Before introducing our precise setting, let us recall what is the situation in the case of the classical perimeter functional (see, e.g., [36, Section 9], [27, Chapter 10] or [31, Sections 17.3 and 17.6]), and set some useful terminology.

Given an open set Ω and a vector field $X\in C^\infty_c(\Omega;\mathbb{R}^n)$, we denote by $\{\Phi_t\}_{t\in\mathbb{R}}$ the *flow induced by* X, that is the smooth map $(t,x)\in\mathbb{R}\times\mathbb{R}^n\mapsto\Phi_t(x)\in\mathbb{R}^n$ defined by solving the family of ODEs (parameterized by $x\in\mathbb{R}^n$)

$$\begin{cases}\partial_t\Phi_t(x)=X(\Phi_t(x))\,, & t\in\mathbb{R}\,,\\ \Phi_0(x)=x\,.\end{cases}\tag{6.1}$$

By the implicit function theorem, there always exists $\varepsilon>0$ such that $\{\Phi_t\}_{|t|<\varepsilon}$ is a smooth family of diffeomorphisms. Given $E\subset\mathbb{R}^n$ with $|E|<\infty$, one says that X *induces a volume-preserving flow on* E if $|\Phi_t(E)|=|E|$ for every $|t|<\varepsilon$.

If E is a set of finite perimeter in Ω and $E_t:=\Phi_t(E)$, then $\{E_t\}_{|t|<\varepsilon}$ is a family of sets of finite perimeter in Ω, $t\mapsto P(E_t;\Omega)$ is a smooth function on $|t|<\varepsilon$ (thanks to the area formula for rectifiable sets), and it makes sense to define the first and second variations of the perimeter at E along X (or, more precisely, along the flow induced by X via (6.1)) as

$$\delta P(E;\Omega)[X] := \frac{d}{dt}P(E_t;\Omega)_{\big|t=0}\,, \qquad \delta^2 P(E;\Omega)[X] := \frac{d^2}{dt^2}P(E_t;\Omega)_{\big|t=0}\,.$$

One says that E is a *volume-constrained stationary set for the perimeter in* Ω if $\delta P(E;\Omega)[X]=0$ whenever X induces a volume-preserving flow on E; if in addition $\delta^2 P(E;\Omega)[X]\geq 0$ for every X inducing a volume-preserving flow on E, then E is said to be a *volume-constrained stable set for the perimeter in* Ω. The interest into these properties stems from the immediate fact that if E is a *local volume-constrained perimeter minimizer in* Ω, that is, if $P(E;\Omega)<\infty$ and, for some $\delta>0$,

$$P(E;\Omega)\leq P(F;\Omega)\,, \qquad \forall F\subset\Omega\,, \quad |E|=|F|\,, \quad |E\Delta F|<\delta\,, \tag{6.2}$$

then E is automatically a volume-constrained stable set for the perimeter in Ω. In order to effectively exploit stability one needs explicit formulas for $\delta P(E;\Omega)[X]$ and $\delta^2 P(E;\Omega)[X]$ in terms of X. When $\partial E\cap\Omega$ is a C^2-hypersurface one can obtain such formulas by using the area formula, Taylor's expansions, and the divergence theorem on $\partial E\cap\Omega$. Denoting by $\mathrm{H}_{\partial E}$ the scalar mean curvature of $\partial E\cap\Omega$ (with respect to the orientation induced by the outer unit normal ν_E to E), by $\mathrm{c}^2_{\partial E}$ the sum of the squares of the principal curvatures of $\partial E\cap\Omega$, and setting $\zeta=X\cdot\nu_E$ for the normal component of X with respect to ν_E, one gets the classical formulae

$$\delta P(E;\Omega)[X] \;=\; \int_{\partial E\cap\Omega}\mathrm{H}_{\partial E}\,\zeta\,d\mathcal{H}^{n-1}\,, \tag{6.3}$$

$$\begin{aligned}\delta^2 P(E;\Omega)[X] \;=\; &\int_{\partial E}|\nabla_\tau\zeta|^2-\mathrm{c}^2_{\partial E}\,\zeta^2\,d\mathcal{H}^{n-1}\\ &+\int_{\partial E}\mathrm{H}_{\partial E}\big((\mathrm{div}X)\,\zeta-\mathrm{div}_\tau\big(\zeta\,X_\tau\big)\big)\,d\mathcal{H}^{n-1}\,.\end{aligned} \tag{6.4}$$

(Here, $X_\tau=X-\zeta\,\nu_E$ is the tangential projection of X along ∂E, while ∇_τ and div_τ denote the tangential gradient and the tangential divergence operators to ∂E.) If E is a volume-constrained stationary set for the perimeter in Ω, then $\mathrm{H}_{\partial E}$ is constant on $\partial E\cap\Omega$ and

$$\delta^2 P(E;\Omega)[X]=\int_{\partial E}|\nabla_\tau\zeta|^2-\mathrm{c}^2_{\partial E}\,\zeta^2\,d\mathcal{H}^{n-1} \tag{6.5}$$

whenever X induces a volume-preserving flow on E. Indeed, $|E_t|=|E|$ for every $|t|<\varepsilon$ implies

$$0=\frac{d}{dt}|E_t|_{\big|t=0}=\int_{\partial E}\zeta\,d\mathcal{H}^{n-1}\,, \qquad 0=\frac{d^2}{dt^2}|E_t|_{\big|t=0}=\int_{\partial E}(\mathrm{div}X)\,\zeta\,d\mathcal{H}^{n-1}\,. \tag{6.6}$$

By combining the first condition in (6.6) with $\delta P(E;\Omega)[X]=0$ and (6.3), one finds that $\mathrm{H}_{\partial E}$ is constant on $\partial E\cap\Omega$. By combining (6.4), the second condition in (6.6), the fact that $\mathrm{H}_{\partial E}$ is constant on $\partial E\cap\Omega$, and the identity $\int_{\partial E}\mathrm{div}_\tau\big(\zeta\,X_\tau\big)\,d\mathcal{H}^{n-1}=0$ (which follows by the tangential divergence theorem), one deduces (6.5).

We now want to obtain these kind of variation formulas for the nonlocal functionals considered in this paper. We shall actually work in a broader framework. Precisely, given $s\in(0,1)$ and $\alpha\in(0,n)$, we fix thorough this section two convolution kernels $K,G\in C^1(\mathbb{R}^n\setminus\{0\};[0,\infty))$ which are symmetric by the origin (i.e., $K(-z)=K(z)$ and $G(-z)=G(z)$ for every $z\in\mathbb{R}^n\setminus\{0\}$) and satisfy the pointwise bounds

$$K(z)\leq\frac{C_K}{|z|^{n+s}}\,, \qquad G(z)\leq\frac{C_G}{|z|^{n-\alpha}}\,, \qquad \forall z\in\mathbb{R}^n\setminus\{0\}\,, \tag{6.7}$$

for some constants C_K and C_G. Correspondingly, given $E\subset\mathbb{R}^n$, we consider the nonlocal functionals (defined in $[0,\infty]$)

$$P_K(E)=\iint_{E\times E^c}K(x-y)\,dx\,dy\,,\qquad V_G(E)=\iint_{E\times E}G(x-y)\,dx\,dy\,.$$

Notice that the two functionals are substantially different only in presence of the singularities allowed in (6.7). Indeed, by virtue of (6.7), K is possibly singular only close to the origin, while G is possibly singular only at infinity (in the sense that the integral of G may diverge at infinity). When no singularity is present, then the two functionals are essentially equivalent in the sense that one has

$$P_K(E)=|E|\,\|K\|_{L^1(\mathbb{R}^n)}-V_K(E)\,,\qquad \text{if } K\in L^1(\mathbb{R}^n) \text{ and } |E|<\infty. \tag{6.8}$$

We next introduce the restrictions of P_K and V_G to a given open set Ω. Following [7], we set

$$\begin{aligned}
P_K(E,\Omega)\ &:=\ \int_{E\cap\Omega}\int_{E^c\cap\Omega}K(x-y)\,dx\,dy+\int_{E\cap\Omega}\int_{E^c\setminus\Omega}K(x-y)\,dx\,dy\\
&\qquad+\int_{E\setminus\Omega}\int_{E^c\cap\Omega}K(x-y)\,dx\,dy\,,\\
V_G(E,\Omega)\ &:=\ \int_{E\cap\Omega}\int_{E\cap\Omega}G(x-y)\,dx\,dy+2\int_{E\cap\Omega}\int_{E\setminus\Omega}G(x-y)\,dx\,dy\,.
\end{aligned}$$

If $P_K(E;\Omega)<\infty$, $X\in C^\infty_c(\Omega;\mathbb{R}^n)$, and $E_t=\Phi_t(E)$ as before, then one finds from the area formula that $t\mapsto P_K(E_t;\Omega)$ is a smooth function for $|t|<\varepsilon$, and correspondingly is able to define the first and second variations of $P_K(\cdot,\Omega)$ at E along X as

$$\delta P_K(E;\Omega)[X]=\frac{d}{dt}P_K(E_t;\Omega)\Big|_{t=0}\,,\qquad \delta^2P_K(E;\Omega)[X]=\frac{d^2}{dt^2}P_K(E_t;\Omega)\Big|_{t=0}\,.$$

Identical definitions are adopted when V_G is considered in place of P_K and E is such that $V_G(E;\Omega)<\infty$ (as it is the case, for example, whenever E is bounded).

Having set our terminology, we now turn to the problem of expressing first and second variations along X in terms of boundary integrals involving X and its derivatives, in the spirit of (6.3) and (6.4). These formulas involve some "nonlocal" variants of the quantities $\mathrm{H}_{\partial E}$ and $\mathrm{c}^2_{\partial E}$, that are introduced as follows. Given $E\subset\mathbb{R}^n$, $x\in\mathbb{R}^n$, and a non-negative Borel function J on $\mathbb{R}^n$, we define (as elements of $[-\infty,\infty]$)

$$\begin{aligned}
\mathrm{H}_{J,\partial E}(x)\ &:=\ \text{p.v.}\left(\int_{\mathbb{R}^n}\big(\chi_{E^c}(y)-\chi_E(y)\big)\,J(x-y)\,dy\right) &&(6.9)\\
&=\ \limsup_{\varepsilon\to0^+}\int_{\mathbb{R}^n\setminus B(x,\varepsilon)}\big(\chi_{E^c}(y)-\chi_E(y)\big)\,J(x-y)\,dy\,,\\
\mathrm{H}^*_{J,\partial E}(x)\ &:=\ 2\int_E J(x-y)\,dy\,. &&(6.10)
\end{aligned}$$

Moreover, given an orientable hypersurface M of class C^1 in $\mathbb{R}^n$, and denoting by ν_M an orientation of M, we define $\mathrm{c}^2_{J,M}:M\to[0,\infty]$ by setting

$$\mathrm{c}^2_{J,M}(x)\ :=\ \int_M J(x-y)|\nu_M(x)-\nu_M(y)|^2\,d\mathcal{H}^{n-1}_y\,,\qquad \forall x\in M\,. \tag{6.11}$$

The functions $\mathrm{H}_{J,\partial E}$ and $\mathrm{H}^*_{J,\partial E}$ will play the role of nonlocal mean curvatures for P_K when $J=K$ and for V_G when $J=G$, respectively. As it turns out, if $J\in L^1(\mathbb{R}^n)$ then the two quantities are equivalent up to a constant and a change of sign, that is,

$$\mathrm{H}_{J,\partial E}(x)=\|J\|_{L^1(\mathbb{R}^n)}-\mathrm{H}^*_{J,\partial E}(x)\,,\qquad \forall x\in\mathbb{R}^n\,,$$

a result that, of course, is in accord with (6.8). We are now in the position to the state the main theorem of this section.

Theorem 6.1. *Let $K, G \in C^1(\mathbb{R}^n\setminus\{0\};[0,\infty))$ be even functions satisfying (6.7) for some $s \in (0,1)$ and $\alpha \in (0,n)$, let Ω be an open set in $\mathbb{R}^n$, let $E \subset \mathbb{R}^n$ be an open set with $C^{1,1}$-boundary such that $\partial E \cap \Omega$ is a C^2-hypersurface, and, given $X \in C_c^\infty(\Omega;\mathbb{R}^n)$, set $\zeta = X \cdot \nu_E$. If $P_K(E;\Omega) < \infty$ and $\int_{\partial E}(1+|z|)^{-n-s}\,d\mathcal{H}^{n-1}_z < \infty$, then*

$$\delta P_K(E;\Omega)[X] = \int_{\partial E} \mathrm{H}_{K,\partial E}\,\zeta\,d\mathcal{H}^{n-1}\,, \tag{6.12}$$

$$\begin{aligned}\delta^2 P_K(E;\Omega)[X] &= \iint_{\partial E\times\partial E} K(x-y)|\zeta(x)-\zeta(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y - \int_{\partial E} \mathrm{c}^2_{K,\partial E}\,\zeta^2\,d\mathcal{H}^{n-1}\\ &\quad+\int_{\partial E}\mathrm{H}_{K,\partial E}\Big((\mathrm{div}X)\,\zeta - \mathrm{div}_\tau\big(\zeta\,X_\tau\big)\Big)\,d\mathcal{H}^{n-1}\,. \end{aligned}\tag{6.13}$$

If $V_G(E;\Omega) < \infty$ and $\int_E |z|^{-n+\alpha}\,dz < \infty$, then

$$\begin{aligned}\delta V_G(E;\Omega)[X] &= \int_{\partial E}\mathrm{H}^*_{G,\partial E}\,\zeta\,d\mathcal{H}^{n-1}\,.\\ \delta^2 V_G(E;\Omega)[X] &= -\iint_{\partial E\times\partial E} G(x-y)|\zeta(x)-\zeta(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y + \int_{\partial E}\mathrm{c}^2_{G,\partial E}\,\zeta^2\,d\mathcal{H}^{n-1}\\ &\quad+\int_{\partial E}\mathrm{H}^*_{G,\partial E}\Big((\mathrm{div}X)\,\zeta - \mathrm{div}_\tau\big(\zeta\,X_\tau\big)\Big)\,d\mathcal{H}^{n-1}\,. \end{aligned}\tag{6.14}$$

Remark 6.2. Let E be as in Theorem 6.1. By arguing as in the deduction of (6.5) from (6.3) and (6.4), we see that if E is a volume-constrained stationary set for P_K, then

$$\delta^2 P_K(E;\Omega)[X] = \iint_{\partial E\times\partial E} K(x-y)|\zeta(x)-\zeta(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y - \int_{\partial E}\mathrm{c}^2_{K,\partial E}\,\zeta^2\,d\mathcal{H}^{n-1}\,.$$

whenever X is volume-preserving on E. Similarly, if E is a volume-constrained stationary set for V_G, then

$$\delta^2 V_G(E;\Omega)[X] = -\iint_{\partial E\times\partial E} G(x-y)|\zeta(x)-\zeta(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y + \int_{\partial E}\mathrm{c}^2_{G,\partial E}\,\zeta^2\,d\mathcal{H}^{n-1}\,,$$

whenever X is volume-preserving on E.

The fact that ∂E is of class $C^{1,1}$ guarantees that $\mathrm{c}^2_{K,\partial E}(x) \in \mathbb{R}$ for every $x \in \partial E$. It also implies that $\zeta = X\cdot\nu_E$ is a Lipschitz function, which in turn guarantees that the first-integral on the right-hand side of (6.13) converge. The convergence of $\mathrm{c}^2_{G,\partial E}$ and of the first integral on the right-hand side of (6.14) is trivial. In the next two propositions we address the continuity properties of $\mathrm{H}_{K,\partial E}$ and $\mathrm{H}^*_{G,\partial E}$.

Proposition 6.3. *If $s \in (0,1)$, $K \in C^1(\mathbb{R}^n\setminus\{0\};[0,\infty))$ is even and satisfies $K(z) \le C_K/|z|^{n+s}$ for every $z \in \mathbb{R}^n\setminus\{0\}$, Ω and E are open sets, and $\partial E\cap\Omega$ is an hypersurface of class $C^{1,\sigma}$ for some $\sigma \in (s,1)$, then (6.9) defines a continuous real-valued function $\mathrm{H}_{K,\partial E}$ on $\partial E \cap \Omega$.*

Proof. Given $\delta \in [0,1/2)$, let $\eta_\delta \in C^\infty([0,\infty);[0,1])$ be such that $\eta_\delta = 1$ on $[0,\delta)\cup(1/\delta,\infty)$, $\eta_\delta = 0$ on $[2\delta, 1/2\delta)$, and $|\eta'_\delta| \le 2/\delta$ on $[0,\infty)$, and $\eta_\delta(s)\downarrow 0$ for every $s > 0$ as $\delta \to 0^+$. If we set $K_\delta(z) = (1-\eta_\delta(|z|))\,K(z)$, $z \in \mathbb{R}^n$, then $K_\delta \in C^1_c(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, so that

$$\mathrm{H}_{K_\delta,\partial E}(x) = \int_{E^c} K_\delta(x-y)\,dy - \int_E K_\delta(x-y)\,dy\,,\qquad \forall x\in\mathbb{R}^n\,,$$

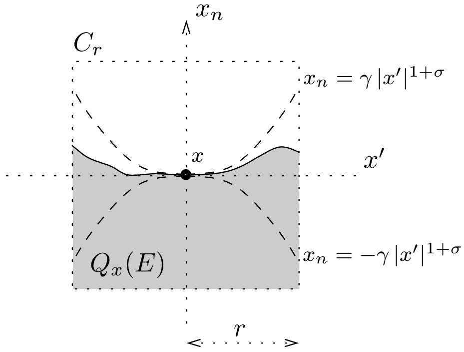

FIGURE 1. The sets defined in (6.16). The region $P_{r,\gamma}$ is that part of C_r encolosed by the graphs $x_n = \pm\gamma|x'|^{1+\sigma}$.

and thus $\mathrm{H}_{K_\delta,\partial E}$ is a continuous function on $\mathbb{R}^n$ for every $\delta > 0$. In fact, we notice for future use that $\mathrm{H}_{K_\delta,\partial E} \in C^1(\mathbb{R}^n)$, with

$$\nabla \mathrm{H}_{K_\delta,\partial E}(x) = \int_{E^c} \nabla K_\delta(x-y)\,dy - \int_E \nabla K_\delta(x-y)\,dy\,, \qquad \forall x \in \mathbb{R}^n\,. \tag{6.15}$$

Let us now decompose $x \in \mathbb{R}^n$ as $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and set

$$C_r = \Big\{x \in \mathbb{R}^n : |x'| < r\,, |x_n| < r\Big\}\,, \qquad P_{r,\gamma} = \Big\{x \in C_r : \gamma\,|x'|^{1+\sigma} < x_n\Big\}\,,$$

for $r > 0$ and $\gamma > 0$. If $\Omega' \subset\subset \Omega$, then we can find $r > 0$ and $\gamma > 0$ such that for every $x \in \partial E \cap \Omega'$ there exists a rotation around the origin followed by a translation, denoted by Q_x, such that

$$\Big(C_r \setminus P_{r,\gamma}\Big) \cap \{x_n > 0\} \subset Q_x(E^c)\,, \qquad \Big(C_r \setminus P_{r,\gamma}\Big) \cap \{x_n < 0\} \subset Q_x(E)\,, \tag{6.16}$$

see Figure 1. Provided $\varepsilon < \delta < 2\delta < r$, we thus find that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} (\chi_{E^c}(y) - \chi_E(y))\,K(x-y)\,dy - \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} (\chi_{E^c}(y) - \chi_E(y))\,K_\delta(x-y)\,dy\Big| \\
= \;&\Big|\int_{E^c\setminus B(x,\varepsilon)} \eta_\delta(|x-y|)\,K(x-y)\,dy - \int_{E\setminus B(x,\varepsilon)} \eta_\delta(|x-y|)\,K(x-y)\,dy\Big| \\
\le \;&\Big|\int_{(C_r\cap E^c)\setminus B(x,\varepsilon)} \eta_\delta(|x-y|)\,K(x-y)\,dy - \int_{(C_r\cap E)\setminus B(x,\varepsilon)} \eta_\delta(|x-y|)\,K(x-y)\,dy\Big| \\
&+2\int_{\mathbb{R}^n\setminus B_{1/2\delta}} K(z)\,dz \\
\le \;&\int_{Q_x^{-1}(P_{r,\gamma})\setminus B(x,\varepsilon)} \eta_\delta(|x-y|)\,K(x-y)\,dy + 2\int_{\mathbb{R}^n\setminus B_{1/2\delta}} K(z)\,dz\,,
\end{aligned}$$

where in the last inequality we have used (6.16) and the symmetry of K to cancel out opposite contributions from the points in E^c and in E lying in $Q_x^{-1}(C_r \setminus P_{r,\gamma})$. We now notice that

$$\begin{aligned}\omega(\delta) &:= \int_{Q_x^{-1}(P_{r,\gamma})\setminus B(x,\varepsilon)} \eta_\delta(|x-y|)\, K(x-y)\, dy \\ &\le \int_{Q_x^{-1}(P_{r,\gamma})} \eta_\delta(|x-y|)\, K(x-y)\, dy \\ &= \int_{|z'|<r} dz' \int_{-\gamma\,|z'|^{1+\sigma}}^{\gamma\,|z'|^{1+\sigma}} \eta_\delta(|z|)\, K(z)\, dz_n \\ &\le C_K \int_{|z'|<r} dz' \int_{-\gamma\,|z'|^{1+\sigma}}^{\gamma\,|z'|^{1+\sigma}} \frac{dz_n}{(|z'|^2+|z_n|^2)^{(n+s)/2}}\,.\end{aligned}$$

Since $\eta_\delta(z) \to 0$ for every $z \in \mathbb{R}^n \setminus \{0\}$ as $\delta \to 0^+$, and since

$$\int_{|z'|<r} dz' \int_{-\gamma\,|z'|^{1+\sigma}}^{\gamma\,|z'|^{1+\sigma}} \frac{dz_n}{(|z'|^2+|z_n|^2)^{(n+s)/2}} < \infty\,,$$

we conclude that $\omega(\delta) \to 0$ as $\delta \to 0$ (with a velocity that depends on C_K, s, r, γ and σ only). Since $\int_{\mathbb{R}^n\setminus B_{1/2\delta}} K(z)\, dz \to 0$ as $\delta \to 0^+$ (with a velocity that depends on C_K and s only), we conclude that, if $\omega_0(\delta) = \omega(\delta) + 2\int_{\mathbb{R}^n\setminus B_{1/2\delta}} K(z)\, dz$, then

$$\Big|\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} (\chi_{E^c}(y) - \chi_E(y))\, K(x-y)\, dy - \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} (\chi_{E^c}(y) - \chi_E(y))\, K_\delta(x-y)\, dy\Big| \le \omega_0(\delta)\,,$$

for every $x \in \partial E \cap \Omega'$ and every $\varepsilon < \delta < 2\delta < r$. We thus conclude that $\mathrm{H}_{K,\partial E}(x) \in \mathbb{R}$ for every $x \in \partial E \cap \Omega'$, and that $\mathrm{H}_{K_\delta,\partial E} \to \mathrm{H}_{K,\partial E}$ uniformly on $\partial E \cap \Omega'$. In particular, $\mathrm{H}_{K,\partial E}$ is real-valued and continuous on $\partial E \cap \Omega$. □

Since the function $z \mapsto |z|^{-n+\alpha}$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we also have the following result:

Proposition 6.4. *If $G \in C^1(\mathbb{R}^n \setminus \{0\}; [0,\infty))$ is even and satisfies* (6.7) *for some $\alpha \in (0,n)$ and $\int_E |z|^{-n+\alpha}\, dz < \infty$ (this is the case for instance if E is bounded), then* (6.10) *defines a continuous real-valued function $\mathrm{H}^*_{G,\partial E}$ on $\mathbb{R}^n$.*

Proof of Theorem 6.1. We shall detail the proof of the theorem only in the case of P_K, being the discussion for V_G similar. We denote by ε the positive number such that $\{\Phi_t\}_{|t|<\varepsilon}$ is a smooth family of diffeomorphisms of $\mathbb{R}^n$.

Step one: Given $\delta \ge 0$, we define K_δ as in the proof of Proposition 6.3. Our goal here is proving (6.12) and (6.13) with K_δ in place of K. We first claim that $\mathrm{H}_{K_\delta,\partial E} \in C^1(\mathbb{R}^n)$, and that $\nabla \mathrm{H}_{K_\delta,\partial E}$ can be expressed both as in (6.15) and as in (6.17) below. Since E is an open set with Lipschitz boundary and $K_\delta \in C^1_c(\mathbb{R}^n)$, by the Gauss–Green theorem, the symmetry of K_δ, and (6.15), we find that

$$\nabla \mathrm{H}_{K_\delta,\partial E}(x) = 2\int_{\partial E} K_\delta(y-x)\, \nu_E(y)\, d\mathcal{H}^{n-1}_y\,, \qquad \forall x \in \mathbb{R}^n\,. \tag{6.17}$$

We now notice that, since $E_{t+h} = \Phi_h(E_t)$, by the area formula we get, whenever $|t| < \varepsilon$ and $|t+h| < \varepsilon$,

$$\begin{aligned}P_{K_\delta}(E_{t+h}, \Omega) &= \int_{E_t\cap\Omega}\int_{E_t^c\cap\Omega} K_\delta(\Phi_h(x) - \Phi_h(y)) J_{\Phi_h}(x) J_{\Phi_h}(y)\, dx\, dy \\ &\quad + \int_{E_t\cap\Omega}\int_{E_t^c\setminus\Omega} K_\delta(\Phi_h(x) - y) J_{\Phi_h}(x)\, dx\, dy + \int_{E_t\setminus\Omega}\int_{E_t^c\cap\Omega} K_\delta(x - \Phi_h(y)) J_{\Phi_h}(y)\, dx\, dy\,,\end{aligned}$$

where J_{Φ_h} stands for the Jacobian of the map Φ_h. Since $\Phi_h = \mathrm{Id} + h\,X + O(h^2)$ and $J_{\Phi_h} = 1 + h\,\mathrm{div}X + O(h^2)$ uniformly on $\mathbb{R}^n$ as $h \to 0$, we deduce from

$$\frac{d}{dt}P_{K_\delta}(E_t,\Omega) = \frac{d}{dh}P_{K_\delta}(E_{t+h},\Omega)_{\big|h=0}\,,$$

and by the smoothness of K_δ that

$$\begin{aligned}\frac{d}{dt}P_{K_\delta}(E_t,\Omega) = &\int_{E_t\cap\Omega}\int_{E_t^c\cap\Omega}\nabla K_\delta(x-y)\cdot(X(x)-X(y))\,dx\,dy\\&+\int_{E_t\cap\Omega}\int_{E_t^c\cap\Omega}K_\delta(x-y)(\mathrm{div}X(x)+\mathrm{div}X(y))\,dx\,dy\\&+\int_{E_t\cap\Omega}\int_{E_t^c\setminus\Omega}\Big(\nabla K_\delta(x-y)\cdot X(x)+K_\delta(x-y)\mathrm{div}X(x)\Big)\,dx\,dy\\&+\int_{E_t\setminus\Omega}\int_{E_t^c\cap\Omega}\Big(\nabla K_\delta(x-y)\cdot X(y)+K_\delta(x-y)\mathrm{div}X(y)\Big)\,dx\,dy\\=&\ I_1+I_2+I_3+I_4\,.\end{aligned}$$

By symmetry of K_δ and by the divergence theorem, we find

$$\begin{aligned}I_1 &= \int_{E_t^c\cap\Omega}\left(\int_{E_t}\nabla K_\delta(x-y)\cdot X(x)\,dx\right)dy+\int_{E_t\cap\Omega}\left(\int_{E_t^c}\nabla K_\delta(y-x)\cdot X(y)\,dy\right)dx\\&=-\int_{E_t^c\cap\Omega}\left(\int_{E_t}K_\delta(x-y)\mathrm{div}X(x)\,dx\right)dy+\int_{E_t^c\cap\Omega}\left(\int_{\partial E_t}K_\delta(x-y)X(x)\cdot\nu_{E_t}(x)\,d\mathcal{H}^{n-1}_x\right)dy\\&\quad-\int_{E_t\cap\Omega}\left(\int_{E_t^c}K_\delta(x-y)\mathrm{div}X(y)\,dy\right)dx-\int_{E_t\cap\Omega}\left(\int_{\partial E_t}K_\delta(x-y)X(y)\cdot\nu_{E_t}(y)\,d\mathcal{H}^{n-1}_y\right)dx\,,\end{aligned}$$

which leads to

$$I_1+I_2=\int_{E_t^c\cap\Omega}\left(\int_{\partial E_t}K_\delta(x-y)X(x)\cdot\nu_{E_t}(x)d\mathcal{H}^{n-1}_x\right)dy-\int_{E_t\cap\Omega}\left(\int_{\partial E_t}K_\delta(x-y)X(y)\cdot\nu_{E_t}(y)d\mathcal{H}^{n-1}_y\right)dx\,.$$

Similarly, we get that

$$\begin{aligned}I_3 &= \int_{E_t^c\setminus\Omega}\left(\int_{\partial E_t}K_\delta(x-y)X(x)\cdot\nu_{E_t}(x)\,d\mathcal{H}^{n-1}_x\right)dy,\\I_4 &= -\int_{E_t\setminus\Omega}\left(\int_{\partial E_t}K_\delta(x-y)X(y)\cdot\nu_{E_t}(y)\,d\mathcal{H}^{n-1}_y\right)dx\,.\end{aligned}$$

By exploiting once more the symmetry of K_δ we thus conclude that (for every t small enough)

$$\frac{d}{dt}P_{K_\delta}(E_t,\Omega)=\int_{\partial E_t}\mathrm{H}_{K_\delta,\partial E_t}\,(X\cdot\nu_{E_t})\,d\mathcal{H}^{n-1}\,, \tag{6.18}$$

which of course implies (6.12) with K_δ in place of K by setting $t=0$. Having in mind to differentiate (6.18), we now notice that, by the area formula,

$$\int_{\partial E_t}\mathrm{H}_{K_\delta,\partial E_t}\,(X\cdot\nu_{E_t})\,d\mathcal{H}^{n-1}=\int_{\partial E}\mathrm{H}_{K_\delta,\partial E_t}(\Phi_t)\,(X(\Phi_t)\cdot\nu_{E_t}(\Phi_t))\,J^{\partial E}_{\Phi_t}\,d\mathcal{H}^{n-1}\,,$$

where $J^{\partial E}_{\Phi_t}$ denotes the tangential Jacobian of Φ_t with respect to ∂E. Therefore,

$$
\begin{aligned}
\frac{d^2}{dt^2}P_{K_\delta}(E_t,\Omega)\Big|_{t=0} &= \frac{d}{dt}\Big(\int_{\partial E_t}\mathrm{H}_{K_\delta,\partial E_t}\,(X\cdot\nu_{E_t})\,d\mathcal{H}^{n-1}\Big)\Big|_{t=0} \qquad (6.19)\\
&= \int_{\partial E}\frac{d}{dt}\big(\mathrm{H}_{K_\delta,\partial E_t}(\Phi_t)\big)\Big|_{t=0}(X\cdot\nu_E)\,d\mathcal{H}^{n-1}\\
&\quad+\int_{\partial E}\mathrm{H}_{K_\delta,\partial E}\,\frac{d}{dt}\big(X(\Phi_t)\cdot\nu_{E_t}(\Phi_t))\,J^{\partial E_t}_{\Phi_t}\big)\Big|_{t=0}\,d\mathcal{H}^{n-1}\\
&= J_1+J_2\,.
\end{aligned}
$$

In order to compute J_1 we begin noticing that, by the area formula and since $K_\delta\in L^1(\mathbb{R}^n)$,

$$
\mathrm{H}_{K_\delta,\partial E_t}(\Phi_t(x))=\int_{\mathbb{R}^n}\big(\chi_{E^c}(y)-\chi_E(y)\big)K_\delta(\Phi_t(x)-\Phi_t(y))\,J_{\Phi_t}(y)\,dy\,.
$$

By symmetry and smoothness of K_δ, by the Taylor's expansions in t of Φ_t and J_{Φ_t} mentioned above, by recalling that $\mathrm{H}_{K_\delta,\partial E}\in C^1(\mathbb{R}^n)$ and (6.15), and by the divergence theorem, we get

$$
\begin{aligned}
\frac{d}{dt}\big(\mathrm{H}_{K_\delta,\partial E_t}(\Phi_t(x))\big)\Big|_{t=0} &= \int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))\nabla K_\delta(x-y)\cdot(X(x)-X(y))\,dy\\
&\quad+\int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))K_\delta(x-y)\mathrm{div}X(y)\,dy\\
&= \nabla\mathrm{H}_{K_\delta,\partial E}(x)\cdot X(x)+\int_{E^c}\nabla K_\delta(y-x)\cdot X(y)\,dy-\int_E\nabla K_\delta(y-x)\cdot X(y)\,dy\\
&\quad+\int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))K_\delta(x-y)\mathrm{div}X(y)\,dy\\
&= \nabla\mathrm{H}_{K_\delta,\partial E}(x)\cdot X(x)-2\int_{\partial E}K_\delta(x-y)X(y)\cdot\nu_E(y)\,dy\,.
\end{aligned}
$$

By this last identity and by the symmetry of K_δ, setting $\zeta=X\cdot\nu_E$ we find that

$$
\begin{aligned}
J_1 &= -2\iint_{\partial E\times\partial E}K_\delta(x-y)\,\zeta(x)\,\zeta(y)\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y+\int_{\partial E}\big(\nabla\mathrm{H}_{K_\delta,\partial E}\cdot X\big)\,\zeta\,d\mathcal{H}^{n-1}\\
&= \iint_{\partial E\times\partial E}K_\delta(x-y)|\zeta(x)-\zeta(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y-2\iint_{\partial E\times\partial E}K_\delta(x-y)\,\zeta(x)^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\\
&\quad+\int_{\partial E}\big(\nabla\mathrm{H}_{K_\delta,\partial E}\cdot\nu_E\big)\,\zeta^2\,d\mathcal{H}^{n-1}+\int_{\partial E}\big(\nabla_\tau\mathrm{H}_{K_\delta,\partial E}\cdot X_\tau\big)\,\zeta\,d\mathcal{H}^{n-1}\,, \qquad (6.20)
\end{aligned}
$$

where in the last identities we have simply completed a square and used the identity $X=\zeta\,\nu+X_\tau$. By (6.17) we also get

$$
\begin{aligned}
\nabla\mathrm{H}_{K_\delta,\partial E}(x)\cdot\nu_E(x) &= -\int_{\partial E}K_\delta(x-y)|\nu_E(x)-\nu_E(y)|^2\,d\mathcal{H}^{n-1}_y+2\int_{\partial E}K_\delta(x-y)\,d\mathcal{H}^{n-1}_y\\
&= -\mathrm{c}^2_{K_\delta,\partial E}(x)+2\int_{\partial E}K_\delta(x-y)\,d\mathcal{H}^{n-1}_y\,,
\end{aligned}
$$

and thus we conclude from (6.20) that

$$
\begin{aligned}
J_1 &= \iint_{\partial E\times\partial E}K_\delta(x-y)\,|\zeta(x)-\zeta(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y-\int_{\partial E}\mathrm{c}^2_{K_\delta,\partial E}\,\zeta^2\,d\mathcal{H}^{n-1}\\
&\quad+\int_{\partial E}\big(\nabla_\tau\mathrm{H}_{K_\delta,\partial E}\cdot X_\tau\big)\,\zeta\,d\mathcal{H}^{n-1}\,. \qquad (6.21)
\end{aligned}
$$

In order to compute J_2, we notice that, by arguing as in [9, Step three, proof of Proposition 3.9] (see also [36, Section 9]), one finds

$$\frac{d}{dt}\Big(X(\Phi_t)\cdot\nu_{E_t}(\Phi_t)J^{\partial E}_{\Phi_t}\Big)_{\big|t=0} = Z\cdot\nu_E - 2X_\tau\cdot\nabla_\tau\zeta + \mathrm{B}_{\partial E}[X_\tau,X_\tau] + \mathrm{div}_\tau\big(\zeta\,X\big)\,,$$

where Z is the vector field defined by

$$Z(x) = \partial^2_{tt}\Phi_t(x)_{\big|t=0}\,,\qquad x\in\mathbb{R}^n\,,$$

and where $\mathrm{B}_{\partial E}$ denotes the second fundamental form of ∂E. Hence,

$$\begin{aligned} J_2 &= \int_{\partial E}\mathrm{H}_{K_\delta,\partial E}\Big(Z\cdot\nu_E - 2X_\tau\cdot\nabla_\tau\zeta + \mathrm{B}_{\partial E}[X_\tau,X_\tau]\Big)\,d\mathcal{H}^{n-1} \\ &\quad + \int_{\partial E}\mathrm{H}_{K_\delta,\partial E}\,\mathrm{div}_\tau\big(\zeta\,X\big)\,d\mathcal{H}^{n-1}\,. \end{aligned} \tag{6.22}$$

By the tangential divergence theorem

$$\int_{\partial E}\mathrm{div}_\tau Y\,d\mathcal{H}^{n-1} = \int_{\partial E} Y\cdot\nu_E\,\mathrm{H}_E\,d\mathcal{H}^{n-1}\qquad\forall Y\in C^1_c(\Omega;\mathbb{R}^n)$$

(recall that $\mathrm{H}_{\partial E}$ denotes the scalar mean curvature of ∂E taken with respect to ν_E), so that the sum of the second lines of (6.21) and (6.22) is equal to

$$\begin{aligned} &\int_{\partial E}\mathrm{H}_{K_\delta,\partial E}\,\mathrm{div}_\tau\big(\zeta\,X\big)\,d\mathcal{H}^{n-1} + \int_{\partial E}\big(\nabla_\tau\mathrm{H}_{K_\delta,\partial E}\cdot X_\tau\big)\,\zeta\,d\mathcal{H}^{n-1} \\ &\qquad = \int_{\partial E}\mathrm{div}_\tau\big(\mathrm{H}_{K_\delta,\partial E}\zeta\,X\big)\,d\mathcal{H}^{n-1} = \int_{\partial E}\mathrm{H}_{K_\delta,\partial E}\,\mathrm{H}_{\partial E}\,\zeta^2\,d\mathcal{H}^{n-1}\,. \end{aligned}$$

We thus deduce from (6.19), (6.21), and (6.22), that

$$\begin{aligned} \frac{d^2}{dt^2}P_{K_\delta}(E_t,\Omega)_{\big|t=0} &= \iint_{\partial E\times\partial E}K_\delta(x-y)\,|\zeta(x)-\zeta(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y - \int_{\partial E}\mathrm{c}^2_{K_\delta,\partial E}\,\zeta^2\,d\mathcal{H}^{n-1} \\ &\quad + \int_{\partial E}\mathrm{H}_{K_\delta,\partial E}\Big(Z\cdot\nu_E - 2X_\tau\cdot\nabla_\tau\zeta + \mathrm{B}_{\partial E}[X_\tau,X_\tau] + \mathrm{H}_{\partial E}\,\zeta^2\Big)\,d\mathcal{H}^{n-1}\,. \end{aligned}$$

By exploiting the identity

$$Z\cdot\nu_E - 2X_\tau\cdot\nabla_\tau\zeta + \mathrm{B}_{\partial E}[X_\tau,X_\tau] + \mathrm{H}_{\partial E}\zeta^2 = -\mathrm{div}_\tau\big(\zeta\,X_\tau\big) + (\mathrm{div}X)\,\zeta$$

(see, for example, [1, Proof of Theorem 3.1]), we thus come to prove (6.13) with K_δ in place of K.

Step two: We now prove (6.12) and (6.13) by taking the limit as $\delta\to0^+$ in (6.12) and (6.13) with K_δ in place of K. Let us set $\varphi_\delta(t) := P_{K_\delta}(E_t;\Omega)$ and $\varphi(t) := P_K(E_t;\Omega)$, so that φ_δ and φ are smooth functions on $(-\varepsilon,\varepsilon)$ with

$$\lim_{\delta\to0^+}\varphi_\delta(t) = \varphi(t)\,,\qquad\forall|t|<\varepsilon\,. \tag{6.23}$$

(This follows by monotone convergence, as $\eta_\delta\downarrow0^+$ as $\delta\to0^+$ on $(0,\infty)$.) Let $\Omega'\subset\subset\Omega$ be an open set such that $\mathrm{spt}X\subset\subset\Omega'$. Thanks to the smoothness of $\{\Phi_t\}_{|t|<\varepsilon}$, the argument in the proof of Proposition 6.3 can be repeated for every set $E_t=\Phi_t(E)$ corresponding to $|t|<\varepsilon$ with the same constants r and γ, thus showing that

$$\lim_{\delta\to0^+}\sup_{|t|<\varepsilon}\sup_{\partial E_t\cap\Omega'}|\mathrm{H}_{K_\delta,\partial E_t}-\mathrm{H}_{K,\partial E_t}| = 0\,. \tag{6.24}$$

At the same time, by step one,

$$\varphi'_\delta(t) = \int_{\partial E_t}\mathrm{H}_{K_\delta,\partial E_t}\,\zeta\,d\mathcal{H}^{n-1}\,,\qquad\forall|t|<\varepsilon\,, \tag{6.25}$$

so that (6.24) and (6.25) imply that

$$\lim_{\delta\to 0^+}\sup_{|t|<\varepsilon}\left|\varphi_\delta'(t)-\int_{\partial E_t}\mathrm{H}_{K,\partial E_t}\,\zeta\,d\mathcal{H}^{n-1}\right|=0\,. \tag{6.26}$$

By the mean value theorem, (6.23) and (6.26) give

$$\varphi'(t)=\int_{\partial E_t}\mathrm{H}_{K,\partial E_t}\,\zeta\,d\mathcal{H}^{n-1}\,,\qquad \forall |t|<\varepsilon\,,$$

which implies (6.12) for $t=0$. In order to prove (6.13), we first notice that, by step one,

$$\begin{aligned}\varphi_\delta''(t) &= \iint_{\partial E_t\times\partial E_t}K_\delta(x-y)|\zeta(x)-\zeta(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y-\int_{\partial E_t}\mathrm{c}^2_{K_\delta,\partial E_t}\,\zeta^2\,d\mathcal{H}^{n-1}\\ &\quad+\int_{\partial E_t}\mathrm{H}_{K_\delta,\partial E_t}\Big((\mathrm{div}X)\,\zeta-\mathrm{div}_\tau\big(\zeta\,X_\tau\big)\Big)\,d\mathcal{H}^{n-1}\,,\qquad \forall |t|<\varepsilon\,.\end{aligned} \tag{6.27}$$

Let $A_1(t,\delta)$, $A_2(t,\delta)$ and $A_3(t,\delta)$ denote the three integrals on the right-hand side of (6.27), and let $A_1(t)$, $A_2(t)$ and $A_3(t)$ stand for the corresponding integrals obtained by replacing K_δ with K. By arguing as above, we just need to prove that for $i=1,2,3$ we have $A_i(t,\delta)\to A_i(t)$ uniformly on $|t|<\varepsilon$ as $\delta\to 0^+$. The fact that $A_3(t,\delta)\to A_3(t)$ uniformly on $|t|<\varepsilon$ as $\delta\to 0^+$ follows from (6.24) and of the smoothness of X. Finally, when $i=1,2$, the uniform convergence of $A_i(t,\delta)\to A_i(t)$ for $|t|<\varepsilon$ as $\delta\to 0^+$ is a simple consequence of the fact that ζ is Lipschitz and compactly supported in Ω', and that $\{\Omega'\cap\partial E_t\}_{|t|<\varepsilon}$ is a uniform family of C^2-hypersurfaces. This completes the proof of the theorem. □

7. The stability threshold

In this section we consider the family of functionals $\mathrm{Per}_s+\beta\,V_\alpha$ ($\beta>0$) and discuss in terms of the value of β the volume-constrained stability of $\mathrm{Per}_s+\beta\,V_\alpha$ around the unit ball B. Our interest in this problem lies in the fact that, as we shall prove in section 8, stability is actually a necessary and sufficient condition for volume-constrained local minimality. Therefore the analysis carried on in this section will provide the basis for the proof of Theorem 1.5. We set

$$\beta_\star(n,s,\alpha):=\begin{cases}\dfrac{1-s}{\omega_{n-1}}\inf\limits_{k\ge 2}\dfrac{\lambda_k^s-\lambda_1^s}{\mu_k^\alpha-\mu_1^\alpha}\,, & \text{if } s\in(0,1)\,,\\[2ex] \inf\limits_{k\ge 2}\dfrac{\lambda_k^1-\lambda_1^1}{\mu_k^\alpha-\mu_1^\alpha}\,, & \text{if } s=1\,,\end{cases} \tag{7.1}$$

where, for every $k\in\mathbb{N}\cup\{0\}$,

$$\lambda_k^1 = k(k+n-2)\,, \tag{7.2}$$

$$\lambda_k^s = \frac{2^{1-s}\,\pi^{\frac{n-1}{2}}}{1+s}\,\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n+s}{2})}\left(\frac{\Gamma(k+\frac{n+s}{2})}{\Gamma(k+\frac{n-2-s}{2})}-\frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n-2-s}{2})}\right),\qquad s\in(0,1)\,, \tag{7.3}$$

$$\mu_k^\alpha = \frac{2^{1+\alpha}\,\pi^{\frac{n-1}{2}}}{1-\alpha}\,\frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}\left(\frac{\Gamma(k+\frac{n-\alpha}{2})}{\Gamma(k+\frac{n-2+\alpha}{2})}-\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{n-2+\alpha}{2})}\right),\qquad \alpha\in(0,1)\,, \tag{7.4}$$

$$\mu_k^\alpha = 2^\alpha\,\pi^{\frac{n-1}{2}}\,\frac{\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{n-\alpha}{2})}\left(\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{n-2+\alpha}{2})}-\frac{\Gamma(k+\frac{n-\alpha}{2})}{\Gamma(k+\frac{n-2+\alpha}{2})}\right),\qquad \alpha\in(1,n)\,, \tag{7.5}$$

$$\mu_k^1 = \frac{4\,\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\left(\frac{\Gamma'(k+\frac{n-1}{2})}{\Gamma(k+\frac{n-1}{2})}-\frac{\Gamma'(\frac{n-1}{2})}{\Gamma(\frac{n-1}{2})}\right). \tag{7.6}$$

Here Γ denotes the Euler's Gamma function, while Γ' is the derivative of Γ, so that Γ'/Γ is the digamma function. By exploiting basic properties of the Gamma function, it is straightforward to check that $\lambda_k^s/\mu_k^\alpha\to\infty$ as $k\to\infty$, so that the infimum in (7.1) is achieved, and $\beta_\star>0$. We shall actually prove that the infimum is always achieved at $k = 2$ and the formula for $\beta_\star$ considerably simplifies (see Proposition 7.4).

Theorem 7.1. *The unit ball B is a volume-constrained stable set for* $\mathrm{Per}_s + \beta\, V_\alpha$ *if and only if* $\beta \in (0, \beta_\star]$.

Let us first of all explain the origin of the formula (7.1) for $\beta_\star$. Since B is a volume-constrained stationary set for P, P_s, and V_α (indeed, B is a global volume-constrained minimizer of P and P_s, and a global volume-constrained maximizer of V_α), by Remark 6.2 we find that (setting $K_s(z) = |z|^{-(n+s)}$ and $G_\alpha(z) = |z|^{-(n-\alpha)}$ for every $z \in \mathbb{R}^n \setminus \{0\}$)

$$\delta^2 P(B)[X] \;=\; \iint_{\partial B} |\nabla_\tau \zeta|^2 d\mathcal{H}^{n-1} - \int_{\partial B} \mathrm{c}^2_{\partial B}\, \zeta^2\, d\mathcal{H}^{n-1}\,, \tag{7.7}$$

$$\delta^2 P_s(B)[X] \;=\; \iint_{\partial B\times\partial B} \frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^{n+s}}\, d\mathcal{H}^{n-1}_x\, d\mathcal{H}^{n-1}_y - \int_{\partial B} \mathrm{c}^2_{K_s,\partial B}\, \zeta^2\, d\mathcal{H}^{n-1}\,, \tag{7.8}$$

$$\delta^2 V_\alpha(B)[X] \;=\; -\iint_{\partial B\times\partial B} \frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^{n-\alpha}}\, d\mathcal{H}^{n-1}_x\, d\mathcal{H}^{n-1}_y + \int_{\partial B} \mathrm{c}^2_{G_\alpha,\partial B}\, \zeta^2\, d\mathcal{H}^{n-1}\,, \tag{7.9}$$

for every X inducing a volume-preserving flow on B (here, $\zeta = X \cdot \nu_B$). The reason why we are able to discuss the volume-constrained stability of $\mathrm{Per}_s + \beta\, V_\alpha$ at B is that the Sobolev semi-norms $[u]_{H^1(\partial B)}$, $[u]_{H^{(1+s)/2}(\partial B)}$, and $[u]_{H^{(1-\alpha)/2}(\partial B)}$, can all be decomposed in terms of the Fourier coefficients of u with respect to a orthonormal basis of spherical harmonics.

Indeed, recalling our notation $\{Y^i_k\}_{i=1}^{d(k)}$ for an orthonormal basis in $L^2(\partial B)$ of the space $\mathcal{S}_k$ of spherical harmonics of degree k, we have proved in (2.10) that

$$\iint_{\partial B\times\partial B} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, d\mathcal{H}^{n-1}_x\, d\mathcal{H}^{n-1}_y = \sum_{k=0}^{\infty}\sum_{i=1}^{d(k)} \lambda^s_k\, a^i_k(u)^2\,, \tag{7.10}$$

where $a^i_k(u) = \int_{\partial B} u\, Y^i_k\, d\mathcal{H}^{n-1}$. Similarly, it is well-known that

$$\int_{\partial B} |\nabla_\tau u|^2\, d\mathcal{H}^{n-1} = \sum_{k=0}^{\infty}\sum_{i=1}^{d(k)} \lambda^1_k\, a^i_k(u)^2\,, \tag{7.11}$$

with λ^1_k defined as in (7.2); see, for example, [32]. We finally claim that for every $\alpha \in (0, n)$ we have

$$\iint_{\partial B\times\partial B} \frac{|u(x)-u(y)|^2}{|x-y|^{n-\alpha}}\, d\mathcal{H}^{n-1}_x \mathcal{H}^{n-1}_y = \sum_{k=0}^{\infty}\sum_{i=1}^{d(k)} \mu^\alpha_k\, a^i_k(u)^2\,, \tag{7.12}$$

for μ^α_k defined as in (7.4), (7.5), and (7.6). Indeed, following [34, p. 151], one defines the *Riesz operator on the sphere of order* $\gamma \in (0, n-1)$ as

$$\mathcal{R}^\gamma u(x) := \frac{1}{2^\gamma\, \pi^{\frac{n-1}{2}}}\, \frac{\Gamma(\frac{n-1-\gamma}{2})}{\Gamma(\frac{\gamma}{2})} \int_{\partial B} \frac{u(y)}{|x-y|^{n-1-\gamma}}\, d\mathcal{H}^{n-1}_y\,, \qquad x \in \partial B\,.$$

By [34, Lemma 6.14], the k-th eigenvalue of $\mathcal{R}^\gamma$ is given by

$$\mu^*_k(\gamma) = \frac{\Gamma(k + \frac{n-1-\gamma}{2})}{\Gamma(k + \frac{n-1+\gamma}{2})}\,, \qquad k \in \mathbb{N}\cup\{0\}\,, \tag{7.13}$$

so that $\mu^*_k(\gamma) > 0$, $\mu^*_k(\gamma)$ is strictly decreasing in k, and $\mu^*_k(\gamma) \downarrow 0$ as $k \to \infty$. Moreover

$$\mathcal{R}^\gamma Y_k = \mu^*_k(\gamma)\, Y_k\,, \qquad \forall k \in \mathbb{N}\cup\{0\}\,, \tag{7.14}$$

where Y_k denotes a generic spherical harmonic of degree k. In particular

$$\frac{1}{2^\gamma\, \pi^{\frac{n-1}{2}}}\, \frac{\Gamma(\frac{n-1-\gamma}{2})}{\Gamma(\frac{\gamma}{2})} \int_{\partial B} \frac{d\mathcal{H}^{n-1}_y}{|x-y|^{n-1-\gamma}} = \mu^*_0(\gamma) \qquad \text{for every } x \in \partial B\,. \tag{7.15}$$

Next, similarly to what we have done in section 2, we introduce for every $\alpha \in (0,n)$ the operator

$$\mathscr{R}_\alpha u(x) := 2\int_{\partial B} \frac{u(x)-u(y)}{|x-y|^{n-\alpha}}\, d\mathcal{H}^{n-1}_y\,, \qquad u \in C^1(\partial B)\,,$$

so that, for every $u \in C^1(\partial B)$,

$$[u]^2_{\frac{1-\alpha}{2}} = \iint_{\partial B\times\partial B} \frac{|u(x)-u(y)|^2}{|x-y|^{n-\alpha}}\, d\mathcal{H}^{n-1}_x \mathcal{H}^{n-1}_y = \int_{\partial B} u\,\mathscr{R}_\alpha u\, d\mathcal{H}^{n-1}\,. \tag{7.16}$$

If $\alpha \in (1,n)$ then $\gamma = \alpha - 1 \in (0, n-1)$, and thus we can deduce from (7.15) and (7.16) that

$$\mathscr{R}_\alpha = 2^\alpha\, \pi^{\frac{n-1}{2}}\, \frac{\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{n-\alpha}{2})} \Big(\mu_0^*(\alpha-1)\mathrm{Id} - \mathcal{R}^{\alpha-1}\Big)\,, \qquad \alpha \in (1,n)\,.$$

In particular, we deduce from (7.13) and (7.14) that (7.12) holds true with μ_k^α defined as in (7.5) whenever $\alpha \in (1,n)$. If $\alpha \in (0,1)$, then $\mathscr{R}_\alpha$ becomes singular and by applying (2.3) with $\gamma = 1-\alpha \in (0,1)$ we have

$$\mathscr{R}_\alpha = \frac{2^{1+\alpha}\pi^{\frac{n-1}{2}}}{1-\alpha}\, \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}\, \mathcal{D}^{1-\alpha}\,, \qquad \alpha \in (0,1)\,.$$

In particular, it follows from (2.4) and (2.5) that (7.12) holds true with μ_k^α defined as in (7.4). Finally, to prove (7.12) in the case $\alpha = 1$, it just suffice to notice that $\mathscr{R}_\alpha Y \to \mathscr{R}_1 Y$ as $\alpha \to 1$ for every spherical harmonic Y: therefore the eigenvalue μ_k^1 of $\mathscr{R}_1$ can be simply computed by taking the limit of μ_k^α as $\alpha \to 1^+$ in (7.5) or as $\alpha \to 1^-$ in (7.4). In both ways one verifies the validity of (7.12) with $\alpha = 1$ and with μ_k^1 defined as in (7.6).

As a last preparatory remark to the proof of Theorem 7.1, let us notice that by (7.4), (7.5), and (7.6) (and by exploiting some classical properties of the Gamma and digamma functions), one has

$$\mu_0^\alpha = 0\,, \qquad \mu_{k+1}^\alpha > \mu_k^\alpha\,, \qquad \mathscr{R}_\alpha Y_k = \mu_k^\alpha\, Y_k\,, \qquad \forall k \in \mathbb{N}\cup\{0\}\,, \quad \forall \alpha \in (0,n)\,. \tag{7.17}$$

In addition, $\{\mu_k^\alpha\}$ is bounded for $\alpha \in (1,n)$, and $\mu_k^\alpha \uparrow \infty$ as $k \to \infty$ for $\alpha \in (0,1]$. Finally, we notice that since the coordinate functions x_i, $i = 1,\dots,n$, belong to $\mathcal{S}_1$, we have $\mathscr{R}_\alpha x_i = \mu_1^\alpha x_i$ by (7.17). Inserting x_i in (7.16) and adding up over i, yields

$$\mu_1^\alpha = \frac{1}{P(B)} \iint_{\partial B\times\partial B} \frac{d\mathcal{H}^{n-1}_x\, d\mathcal{H}^{n-1}_y}{|x-y|^{n-2-\alpha}} = \int_{\partial B} \frac{d\mathcal{H}^{n-1}_y}{|z-y|^{n-2-\alpha}}\,, \qquad \forall z \in \partial B\,. \tag{7.18}$$

We can thus conclude that

$$\mathrm{c}^2_{\partial B} = n-1\,, \qquad \mathrm{c}^2_{K_s,\partial B} = \lambda_1^s\,, \qquad \mathrm{c}^2_{V_\alpha,\partial B} = \mu_1^\alpha\,,$$

for every $s \in (0,1)$ and $\alpha \in (0,n)$: indeed, the first identity is trivial, while the second and the third one follow from (6.11), (2.13), and (7.18).

Starting from the above considerations, given $s \in (0,1]$ and $\alpha \in (0,n)$ we are led to consider the following quadratic functionals

$$\begin{aligned}
\mathcal{QP}_1(u) &:= \int_{\partial B} |\nabla_\tau u|^2\, d\mathcal{H}^{n-1} - (n-1)\int_{\partial B} u^2\, d\mathcal{H}^{n-1}\,,\\
\mathcal{QP}_s(u) &:= \frac{1-s}{\omega_{n-1}}\bigg(\iint_{\partial B\times\partial B} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, d\mathcal{H}^{n-1}_x\mathcal{H}^{n-1}_y - \lambda_1^s\int_{\partial B} u^2\, d\mathcal{H}^{n-1}\bigg)\,,\\
\mathcal{QV}_\alpha(u) &:= \iint_{\partial B\times\partial B} \frac{|u(x)-u(y)|^2}{|x-y|^{n-\alpha}}\, d\mathcal{H}^{n-1}_x\mathcal{H}^{n-1}_y - \mu_1^\alpha\int_{\partial B} u^2\, d\mathcal{H}^{n-1}\,.
\end{aligned}$$

We set

$$\widetilde{H}^{\frac{1+s}{2}}(\partial B) := \Big\{u \in H^{\frac{1+s}{2}}(\partial B) : \int_{\partial B} u\, d\mathcal{H}^{n-1} = 0\Big\}\,,$$

and notice the validity of the following proposition.

Proposition 7.2. *If $s \in (0,1]$, $\alpha \in (0,n)$, and $\beta > 0$, then*

$$\mathcal{QP}_s(u) - \beta\,\mathcal{QV}_\alpha(u) = \begin{cases} \displaystyle\sum_{k=2}^{\infty}\sum_{i=1}^{d(k)} \Big(\frac{1-s}{\omega_{n-1}}(\lambda_k^s - \lambda_1^s) - \beta(\mu_k^\alpha - \mu_1^\alpha)\Big)\, a_k^i(u)^2\,, & \text{if } s \in (0,1)\,,\\ \displaystyle\sum_{k=2}^{\infty}\sum_{i=1}^{d(k)} \Big((\lambda_k^1 - \lambda_1^1) - \beta(\mu_k^\alpha - \mu_1^\alpha)\Big)\, a_k^i(u)^2\,, & \text{if } s = 1\,. \end{cases}$$

for every $u \in \widetilde{H}^{\frac{1+s}{2}}(\partial B)$. In particular, $\mathcal{QP}_s - \beta\,\mathcal{QV}_\alpha \geq 0$ on $\widetilde{H}^{\frac{1+s}{2}}(\partial B)$ if and only if $\beta \in (0,\beta_\star]$.

Proof. This is immediate from the definition of $\beta_\star$ and from (7.10), (7.11), and (7.12), once one takes into account that $a_0(u) = 0$ for every $u \in L^2(\partial B)$ with $\int_{\partial B} u\, d\mathcal{H}^{n-1} = 0$. (Indeed, $\mathcal{S}_0$ is the space of constant functions on ∂B.) □

We premise a final lemma to the proof of Theorem 7.1.

Lemma 7.3. *Given $n \geq 2$, there exist positive constants C_0 and δ_0, depending on n only, with the following property: If $v \in C^\infty(\partial B)$ and $\|v\|_{C^1(\partial B)} \leq \delta_0$, then there exists $X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that*

(i) $\mathrm{div}X = 0$ *on $B_2 \setminus B_{1/2}$;*
(ii) *the flow Φ_t induced by X satisfies $\Phi_1(x) = (1+v(x))x$ for every $x \in \partial B$;*
(iii) $\|X\cdot\nu_B - v\|_{C^1(\partial B)} \leq C_0\,\|v\|^2_{C^1(\partial B)}$.

If in addition $|\Phi_1(B)| = |B|$, then $|\Phi_t(B)| = |B|$ for every $t \in (-1,1)$.

Proof. Let $\chi : [0,\infty) \to [0,1]$ be a smooth cut-off function such that $\chi(r) = 1$ for $r \in [1/2,2]$ and $\chi(r) = 0$ for $r \in [0,1/4] \cup [3,\infty)$, and define $X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ by setting

$$X(x) = \frac{\chi(|x|)}{n}\bigg(\Big(1+v\Big(\frac{x}{|x|}\Big)\Big)^n - 1\bigg)\frac{x}{|x|^n}\,, \qquad x \in \mathbb{R}^n\,.$$

Direct computations show the validity of (i) and (iii) (the latter with a constant C_0 that depends on δ_0). Up to further decrease the value of δ_0 we can ensure that Φ_t is a diffeomorphism for every $|t| \leq 1$. By a direct computation we see that

$$\Phi_t(x) = \Big(1 + t\big(\big(1+v(x)\big)^n - 1\big)\Big)^{\frac{1}{n}} x\,,$$

for every $x \in \partial B$ and $|t| \leq 1$. In particular, (ii) holds true. By (6.6) and by (i) we infer that

$$\frac{d^2}{dt^2}|E_t| = \int_{\partial E_t} (\mathrm{div}X)(X\cdot\nu_{E_t})\, d\mathcal{H}^{n-1} = 0 \qquad \forall |t| \leq 1\,,$$

that is, $t \mapsto |E_t|$ is affine on $[-1,1]$. In particular, if $|E_1| = |B| = |E_0|$, then $|E_t| = |B|$ for every $t \in [-1,1]$. □

Proof of Theorem 7.1. We fix $\beta > 0$ and claim that B is a volume-constrained stable set for $\mathrm{Per}_s + \beta\,V_\alpha$ if and only if

$$\mathcal{QP}_s(u) - \beta\,\mathcal{QV}_\alpha(u) \geq 0\,, \qquad \forall u \in C^\infty(\partial B) \text{ with } \textstyle\int_{\partial B} u\, d\mathcal{H}^{n-1} = 0\,; \tag{7.19}$$

the theorem will then follow by a standard density argument and by Proposition 7.2. By (7.7), (7.8), and (7.9), we see that B is a volume-constrained stable set for $\mathrm{Per}_s + \beta\,V_\alpha$ if and only if

$$\mathcal{QP}_s(X\cdot\nu_B) - \beta\,\mathcal{QV}_\alpha(X\cdot\nu_B) \geq 0\,, \qquad \forall X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ inducing a volume-preserving flow on } B\,. \tag{7.20}$$

Now, the fact that (7.19) implies (7.20) is obvious: indeed, recall (6.6), $u = X \cdot \nu_B$ satisfies $\int_{\partial B} u\, d\mathcal{H}^{n-1} = 0$ whenever X induces a volume-preserving flow on B. To prove the reverse implication, let us fix $u \in C^\infty(\partial B)$ with $\int_{\partial B} u\, d\mathcal{H}^{n-1} = 0$, and consider the open sets

$$E_\delta = \Big\{(1+\delta\, u(x))\, x : x \in \partial B\Big\}\,, \qquad \delta \in (0,1)\,.$$

Since $\int_{\partial B} u\, d\mathcal{H}^{n-1} = 0$, we have that $\big||E_\delta| - |B|\big| \le C\delta^2$ for some constant C depending on u only. Therefore, if $F_\delta = (|B|/|E_\delta|)^{1/n}\, E_\delta$, then we have

$$F_\delta = \Big\{(1+v_\delta(x))\, x : x \in \partial B\Big\}\,, \qquad \delta \in (0,1)\,,$$

for some $v_\delta \in C^\infty(\partial B)$ with $\|v_\delta\|_{C^1(\partial B)} \le C\,\delta$ and $\|v_\delta - \delta\, u\|_{C^1(\partial B)} \le C\,\delta^2$ (again, the constant C does not depend on δ). Provided δ is small enough we can apply Lemma 7.3 to find a vector field $X_\delta \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ inducing a volume-preserving flow on B, and with the property that

$$\|X_\delta \cdot \nu_B - v_\delta\|_{C^1(\partial B)} \le C\,\|v_\delta\|^2_{C^1(\partial B)} \le C\,\delta^2\,.$$

In particular, $\|X_\delta \cdot \nu_B - \delta\, u\|_{C^1(\partial B)} \le C\,\delta^2$, and thus by (7.20) we have (recall that $\mathcal{QP}$ and $\mathcal{QV}$ are quadratic forms)

$$0 \le \mathcal{QP}_s(X_\delta \cdot \nu_B) - \beta\,\mathcal{QV}_\alpha(X_\delta \cdot \nu_B) \le \mathcal{QP}_s(\delta\, u) - \beta\,\mathcal{QV}_\alpha(\delta\, u) + C\,\delta^3\,.$$

We divide by δ^2 and let $\delta \to 0^+$ to find that $\mathcal{QP}_s(u) - \beta\,\mathcal{QV}_\alpha(u) \ge 0$. This shows that (7.20) implies (7.19), and thus completes the proof of the theorem. □

We close this section with the following result.

Proposition 7.4. *For every $n \ge 2$, $s \in (0,1]$ and $\alpha \in (0,n)$ one has*

$$\beta_\star(n,s,\alpha) = \begin{cases} \dfrac{n+s}{n-\alpha}\,\dfrac{s\,(1-s)\,P_s(B)}{\alpha\,\omega_{n-1}V_\alpha(B)}\,, & \textit{if } s \in (0,1)\,, \\[2ex] \dfrac{n+1}{n-\alpha}\,\dfrac{P(B)}{\alpha\,V_\alpha(B)}\,, & \textit{if } s = 1\,. \end{cases} \tag{7.21}$$

Proof. By appendix C

$$\beta_\star(n,s,\alpha) = \begin{cases} (1-s)\,\dfrac{\lambda_2^s - \lambda_1^s}{\mu_2^\alpha - \mu_1^\alpha}\,, & \text{if } s \in (0,1)\,, \\[2ex] \dfrac{\lambda_2^1 - \lambda_1^1}{\mu_2^\alpha - \mu_1^\alpha}\,, & \text{if } s = 1\,, \end{cases}$$

We then find (7.21) by Proposition 2.3 and by Proposition 8.4 below. □

8. Proof of Theorem 1.5

We are now in the position of proving Theorem 1.5. We begin with the following result, which extends Theorem 2.1 to the family of functionals $\mathrm{Per}_s + \beta\, V_\alpha$ with $\beta \in (0,\beta_\star)$.

Theorem 8.1. *For every $s \in (0,1)$, $\alpha \in (0,n)$, and $\beta \in (0,\beta_\star(n,s,\alpha))$, there exist positive constants $c_0 = c_0(n)$ and $\varepsilon_\beta = \varepsilon_\beta(n,s,\alpha)$ with the following property: If E is a nearly spherical set as in (2.1) with $|E| = |B|$, $\int_E x\, dx = 0$, and $\|u\|_{C^1(\partial B)} < \varepsilon_\beta$, then*

$$\big(\mathrm{Per}_s + \beta V_\alpha\big)(E) - \big(\mathrm{Per}_s + \beta V_\alpha\big)(B) \ge c_0\Big(1 - \frac{\beta}{\beta_\star}\Big)\Big((1-s)[u]^2_{\frac{1+s}{2}} + \|u\|^2_{L^2(\partial B)}\Big)\,. \tag{8.1}$$

Moreover, we can take ε_β of the form

$$\varepsilon_\beta = \Big(1 - \frac{\beta}{\beta_\star}\Big)\,\varepsilon_0(n)\,, \tag{8.2}$$

for a suitable positive constant $\varepsilon_0(n)$.

Remark 8.2. If $\beta \in (0, \beta_\star(n,1,\alpha))$ and u satisfies the assumptions of Theorem 8.1, then

$$\big(P+\beta V_\alpha\big)(E)-\big(P+\beta V_\alpha\big)(B) \geq c_0\Big(1-\frac{\beta}{\beta_\star(n,1,\alpha)}\Big)\,\|u\|^2_{H^1(\partial B)}\,. \tag{8.3}$$

To prove this observe that, by a standard approximation argument, it suffices to consider the case when $u \in C^{1,\gamma}(\partial B)$ for some $\gamma \in (0,1)$, and thus $\mathrm{Per}_s(E) \to P(E)$ as $s \to 1^-$ by (1.6). By (7.21) and again by (1.6), $\beta_\star(n,s,\alpha) \to \beta_\star(n,1,\alpha)$ as $s \to 1^-$. In particular, we can find $\tau > 0$ such that $\beta < \beta_\star(n,s,\alpha)$ and $\varepsilon_\beta(n,1,\alpha) < \varepsilon_\beta(n,s,\alpha)$ for every $s \in (1-\tau,1)$. We may thus apply (8.1) with $s \in (1-\tau,1)$ and then let $\tau \to 0^+$, to find that

$$\big(P+\beta V_\alpha\big)(E)-\big(P+\beta V_\alpha\big)(B) \geq c_0\Big(1-\frac{\beta}{\beta_\star(n,1,\alpha)}\Big)\,\limsup_{s\to 1^{-1}}(1-s)[u]^2_{\frac{1+s}{2}}\,.$$

Finally, by (7.2) and (7.3) we find that $\lambda_k^s \to \omega_{n-1}\lambda_k^1$ as $s \to 1^-$, hence recalling (7.10) and (7.11) we get

$$\lim_{s\to 1^-}(1-s)[u]^2_{\frac{1+s}{2}} = \omega_{n-1}\int_{\partial B}|\nabla_\tau u|^2 \tag{8.4}$$

and (8.3) is proved.

Remark 8.3. Theorem 2.1 follows from Theorem 8.1 by letting $\alpha \to n^-$ in (8.1). Indeed, denoting by C a generic constant depending on n only, we notice that (7.4), (7.5), and (7.6) give $\mu_k^\alpha - \mu_1^\alpha \leq C\,(n-\alpha)$ for all $k \geq 2$. At the same time, by exploiting (2.4), (2.7), and (2.9) we find that

$$(1-s)\,\lambda_1^s \geq \frac{1}{C}\,,\qquad \forall s \in (0,1)\,,$$

so that by Proposition 2.3, again for every $k \geq 2$,

$$(1-s)(\lambda_k^s-\lambda_1^s) \geq (1-s)(\lambda_2^s-\lambda_1^s) = \frac{n+s}{n-s}(1-s)\lambda_1^s \geq \frac{1}{C}\,.$$

We thus conclude from (7.1) that

$$\beta_\star(n,s,\alpha) \geq \frac{c(n)}{n-\alpha}\,,$$

for a suitable positive constant $c(n)$. In particular, $\beta_\star(n,s,\alpha) \to \infty$ as $\alpha \to n^-$ uniformly with respect to $s \in (0,1)$, and (2.2) follows by letting $\alpha \to n^-$ in (8.1).

Before discussing the proof of Theorem 8.1 we need the following observation, which parallels Proposition 2.3.

Proposition 8.4. *For every $\alpha \in (0,n)$, one has*

$$\mu_1^\alpha = \alpha(n+\alpha)\frac{V_\alpha(B)}{P(B)}\,, \tag{8.5}$$

$$\mu_2^\alpha = \frac{2n}{n+\alpha}\,\mu_1^\alpha\,. \tag{8.6}$$

Proof. By scaling, $V_\alpha(B_r) = r^{n+\alpha}V_\alpha(B)$. Hence,

$$(n+\alpha)V_\alpha(B) = \frac{d}{dr}\Big|_{r=1}V_\alpha(B_r) = 2\int_B dx\int_{\partial B}\frac{d\mathcal{H}^{n-1}_y}{|x-y|^{n-\alpha}}\,.$$

Since

$$\frac{1}{|x-y|^{n-\alpha}} = \frac{1}{\alpha}\,\mathrm{div}_x\Big(\frac{x-y}{|x-y|^{n-\alpha}}\Big)$$

by the divergence theorem we get

$$\alpha(n+\alpha)V_\alpha(B) = 2\iint_{\partial B\times\partial B}\frac{(x-y)\cdot x}{|x-y|^{n-\alpha}}\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,.$$

By symmetry, the right-hand side of the last identity is equal to

$$\iint_{\partial B\times\partial B}\frac{(x-y)\cdot x}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y+\iint_{\partial B\times\partial B}\frac{(y-x)\cdot y}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}_y\,d\mathcal{H}^{n-1}_x$$
$$=\iint_{\partial B\times\partial B}\frac{d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y}{|x-y|^{n+s-2}}\,,$$

so that (8.5) follows from (7.18). One can deduce (8.6) from (7.4), (7.6), and (7.5) (depending on whether $\alpha\in(1,n)$, $\alpha=1$ or $\alpha\in(0,1)$) by exploiting the factorial property of the Gamma function. Since a similar argument was presented in Proposition 2.3, we omit the details. □

Proof of Theorem 8.1. We consider $u\in C^1(\partial B)$ with $\|u\|_{C^1(\partial B)}\le 1/2$ and assume the existence of $t\in(0,2\,\varepsilon_\beta)$ such that the open set E_t whose boundary is given by

$$\partial E_t=\Big\{(1+t\,u(x))\,x: x\in\partial B\Big\}$$

satisfies $|E_t|=|B|$ and $\int_{E_t}x\,dx=0$. If ε_β is small enough then (2.24), (5.20), and (8.5) imply that

$$\big(\mathrm{Per}_s+\beta V_\alpha\big)(E_t)-\big(\mathrm{Per}_s+\beta V_\alpha\big)(B)\ge\frac{t^2}{2}\Big(\mathcal{QP}_s(u)-\beta\mathcal{QV}_\alpha(u)\Big)$$
$$-C(n)t^3\bigg(\frac{1-s}{\omega_{n-1}}\big([u]^2_{\frac{1+s}{2}}+\lambda_1^s\|u\|^2_{L^2}\big)+\beta\big([u]^2_{\frac{1-\alpha}{2}}+\mu_1^\alpha\|u\|^2_{L^2}\big)\bigg)\,. \tag{8.7}$$

By Proposition 7.2 and by definition of $\beta_\star$ we have

$$\begin{aligned}\mathcal{QP}_s(u)-\beta\mathcal{QV}_\alpha(u)&=\sum_{k=2}^{\infty}\sum_{i=1}^{d(k)}\bigg(\frac{1-s}{\omega_{n-1}}(\lambda_k^s-\lambda_1^s)-\beta(\mu_k^\alpha-\mu_1^\alpha)\bigg)\,|a_k^i|^2\\&\ge\frac{1-s}{\omega_{n-1}}\Big(1-\frac{\beta}{\beta_\star}\Big)\sum_{k=2}^{\infty}\sum_{i=1}^{d(k)}(\lambda_k^s-\lambda_1^s)|a_k^i|^2\\&=\frac{1-s}{\omega_{n-1}}\Big(1-\frac{\beta}{\beta_\star}\Big)\Big([u]^2_{\frac{1+s}{2}}-\lambda_1^s\|u\|^2_{L^2}\Big)\,,\end{aligned}$$

thus using (2.25) and (2.28) we find

$$\mathcal{QP}_s(u)-\beta\mathcal{QV}_\alpha(u)\ge\frac{1-s}{4}\Big(1-\frac{\beta}{\beta_\star}\Big)\Big([u]^2_{\frac{1+s}{2}}+\lambda_1^s\|u\|^2_{L^2}\Big)\,. \tag{8.8}$$

Choosing ε_β small enough, we can apply (2.28) and (8.6) to estimate

$$\mu_1^\alpha\|u\|^2_{L^2}\le 2\mu_1^\alpha\sum_{k=2}^{\infty}\sum_{i=1}^{d(k)}|a_k^i|^2\le\frac{2(n+\alpha)}{n-\alpha}\sum_{k=2}^{\infty}\sum_{i=1}^{d(k)}(\mu_k^\alpha-\mu_1^\alpha)|a_k^i|^2\le C(n)\mathcal{QV}_\alpha(u)\,, \tag{8.9}$$

where in the last inequality we have used the temporary assumption that

$$\alpha\le n-\frac{1}{2}\,. \tag{8.10}$$

By (8.9) and by (8.8) (which gives, in particular, $\mathcal{QP}_s(u)\ge\beta\mathcal{QV}_\alpha(u)$), we find

$$\beta\Big([u]^2_{\frac{1-\alpha}{2}}+\mu_1^\alpha\|u\|^2_{L^2}\Big)=\beta\mathcal{QV}_\alpha(u)+2\beta\mu_1^\alpha\|u\|^2_{L^2}\le C(n)\beta\,\mathcal{QV}_\alpha(u)\le C(n)\mathcal{QP}_s(u)\,. \tag{8.11}$$

By gathering (8.7), (8.8), and (8.11) we end up with

$$\big(\mathrm{Per}_s+\beta V_\alpha\big)(E_t)-\big(\mathrm{Per}_s+\beta V_\alpha\big)(B)\ge\frac{1-s}{\omega_{n-1}}\Big(\frac{t^2}{8}\Big(1-\frac{\beta}{\beta_\star}\Big)-C(n)t^3\Big)\Big([u]^2_{\frac{1+s}{2}}+\lambda_1^s\|u\|^2_{L^2}\Big).$$

By choosing $\varepsilon_0(n)$ suitably small in (8.2), and by exploiting (2.4), (2.7), and (2.9) to deduce that $(1-s)\,\lambda_1^s \geq c(n) > 0$ for a suitable positive constant $c(n)$, we deduce that

$$\big(\mathrm{Per}_s + \beta V_\alpha\big)(E_t) - \big(\mathrm{Per}_s + \beta V_\alpha\big)(B) \geq c_0 t^2 \Big(1 - \frac{\beta}{\beta_\star}\Big)\Big((1-s)[u]^2_{\frac{1+s}{2}} + \|u\|^2_{L^2}\Big)\,,$$

for a constant c_0 which only depends on n. This completes the proof of the theorem in the case (8.10) holds true. Let us now assume that $\alpha \in (n-1/2, n)$, and prove a stronger version of (5.20). Since $|E_t| = |B|$, we can write

$$V_\alpha(B) - V_\alpha(E_t) = \big(V_\alpha(B) - |B|^2\big) - \big(V_\alpha(E_t) - |E_t|^2\big)\,.$$

If we set

$$f_\theta(r,\rho) := \frac{r^{n-1}\varrho^{n-1}}{(|r-\varrho|^2 + r\,\varrho\,\theta^2)^{\frac{n-\alpha}{2}}} - r^{n-1}\rho^{n-1}\,, \qquad r,\rho,\theta \geq 0\,,$$

then we find

$$V_\alpha(E_t) - |E_t|^2 = \iint_{\partial B\times\partial B} \Big(\int_0^{1+tu(x)} \int_0^{1+tu(y)} f_{|x-y|}(r,\rho)\,dr\,d\varrho\Big)\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,,$$

Arguing as in the proof of Lemma 5.3, we derive that

$$\begin{aligned}
V_\alpha(E_t) - |E_t|^2 &= -\frac{t^2}{2}\tilde g(t) + \frac{V_\alpha(B)}{P(B)}\int_{\partial B}(1+tu)^{n+\alpha}\,d\mathcal{H}^{n-1} - \frac{|B|^2}{P(B)}\int_{\partial B}(1+tu)^{2n}\,d\mathcal{H}^{n-1}\\
&= -\frac{t^2}{2}\tilde g(t) + \frac{V_\alpha(B)-|B|^2}{P(B)}\int_{\partial B}(1+tu)^{n+\alpha}\,d\mathcal{H}^{n-1}\\
&\qquad - \frac{|B|^2}{P(B)}\int_{\partial B}(1+tu)^{2n}\Big(1-(1+tu)^{\alpha-n}\Big)\,d\mathcal{H}^{n-1}\,,
\end{aligned}$$

with

$$\tilde g(t) := \iint_{\partial B\times\partial B}\left(\int_{u(y)}^{u(x)}\int_{u(y)}^{u(x)} f_{|x-y|}(1+tr, 1+t\rho)\,drd\rho\right)\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,.$$

Setting $h(t) := \int_{\partial B}(1+tu)^{n+\alpha}$ and

$$\ell(t) := \int_{\partial B}(1+tu)^{2n}\Big(1-(1+tu)^{\alpha-n}\Big)\,d\mathcal{H}^{n-1}\,,$$

we conclude that

$$V_\alpha(B) - V_\alpha(E_t) = \frac{t^2}{2}\tilde g(t) + \frac{V_\alpha(B)-|B|^2}{P(B)}\big(h(0)-h(t)\big) + \frac{|B|^2}{P(B)}\,\ell(t)\,.$$

In the proof of Lemma 5.3 we showed that

$$h(0) - h(t) \leq -\alpha\,(n+\alpha)\,\frac{t^2}{2}\int_{\partial B} u^2\,d\mathcal{H}^{n-1} + C(n)\,t^3\,\|u\|^2_{L^2}\,. \tag{8.12}$$

In the same way (using Taylor expansion and $|E_t| = |B|$) we obtain that

$$\ell(t) \leq (n-\alpha)(2n+\alpha)\frac{t^2}{2}\|u\|^2_{L^2} + (n-\alpha)C(n)t^3\|u\|^2_{L^2}\,. \tag{8.13}$$

Then, noticing that

$$\alpha(n+\alpha) = 2n^2 - (n-\alpha)(2n+\alpha)$$

and using (8.5), we compute

$$\begin{aligned}
\frac{V_\alpha(B)-|B|^2}{P(B)} &= \frac{1}{\alpha(n+\alpha)}\Big(\mu_1^\alpha - \alpha(n+\alpha)\frac{|B|^2}{P(B)}\Big)\\
&= \frac{1}{\alpha(n+\alpha)}\Big(\mu_1^\alpha - 2n^2\frac{|B|^2}{P(B)}\Big) + (n-\alpha)\frac{(2n+\alpha)|B|^2}{\alpha(n+\alpha)P(B)}\,.
\end{aligned} \tag{8.14}$$

On the other hand, (8.5) implies

$$\mu_1^\alpha \underset{\alpha\to n}{\longrightarrow} 2n^2\frac{|B|^2}{P(B)} =: \mu_1^n\,.$$

From the explicit value of μ_1^α given by (7.5), we easily infer that $|\mu_1^\alpha - \mu_1^n| \le (n-\alpha)C(n)$. Hence,

$$\Big|\frac{V_\alpha(B)-|B|^2}{P(B)}\Big| \le (n-\alpha)C(n)\,. \tag{8.15}$$

Gathering (8.12), (8.13), (8.14), and (8.15), we are led to

$$\frac{V_\alpha(B)-|B|^2}{P(B)}\big(h(0)-h(t)\big) + \frac{|B|^2}{P(B)}\,\ell(t) \le -(\mu_1^\alpha-\mu_1^n)\frac{t^2}{2}\|u\|^2_{L^2} + (n-\alpha)C(n)t^3\|u\|^2_{L^2}\,.$$

Next, from the smooth dependence $\tilde g$ on t, we can find $\tau\in(0,t)$ such that $\tilde g(t) = \tilde g(0) + t\,\tilde g'(\tau)$. Since $\alpha\in(n-1/2,n)$, we have the estimate

$$\Big|r\,\frac{\partial f_\theta}{\partial r}(1+\tau\, r, 1+\tau\,\varrho) + \varrho\,\frac{\partial f_\theta}{\partial \varrho}(1+\tau\, r, 1+\tau\,\varrho)\Big| \le (n-\alpha)\frac{C(n)}{\theta^{n-\alpha}}\big(1+|\log(\theta)|\big) \le (n-\alpha)\frac{C(n)}{\theta^{3/4}}\,,$$

for all $r,\varrho\in(-\frac12,\frac12)$, all $\theta\in(0,2]$, and a suitable constant $C(n)$. In turn, the sequence $\{\mu_k^{n-3/4}\}$ is bounded and one can estimate

$$|g'(\tau)| \le (n-\alpha)C(n)\iint_{\partial B\times\partial B}\frac{|u(x)-u(y)|^2}{|x-y|^{3/4}}\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y \le (n-\alpha)C(n)\,\|u\|^2_{L^2}\,,$$

therefore

$$V_\alpha(B)-V_\alpha(E_t) \le \frac{t^2}{2}\tilde g(0) - (\mu_1^\alpha-\mu_1^n)\frac{t^2}{2}\|u\|^2_{L^2} + (n-\alpha)C(n)t^3\|u\|^2_{L^2}\,.$$

Then, we notice that

$$\tilde g(0) = [u]^2_{\frac{1-\alpha}{2}} - \iint_{\partial B\times\partial B}|u(x)-u(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,.$$

Also, from (7.5) we infer that

$$\lim_{\alpha\to n}\mu_k^\alpha = \mu_1^n\,,\qquad \forall k\ge 1\,.$$

Hence, by dominated convergence we have

$$[u]^2_{\frac{1-\alpha}{2}} = \sum_{k=1}^\infty\sum_{i=1}^{d(k)}\mu_k^\alpha|a_k^i|^2 \underset{\alpha\to n}{\longrightarrow} \mu_1^n\sum_{k=1}^\infty\sum_{i=1}^{d(k)}|a_k^i|^2\,.$$

Since we obviously have

$$[u]^2_{\frac{1-\alpha}{2}} \underset{\alpha\to n}{\longrightarrow} \iint_{\partial B\times\partial B}|u(x)-u(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\,,$$

we have thus proved that

$$\iint_{\partial B\times\partial B}|u(x)-u(y)|^2\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y = \mu_1^n\sum_{k=1}^\infty\sum_{i=1}^{d(k)}|a_k^i|^2\,.$$

As a consequence,

$$V_\alpha(B)-V_\alpha(E_t) \le \frac{t^2}{2}\sum_{k=2}^\infty\sum_{i=1}^{d(k)}(\mu_k^\alpha-\mu_1^\alpha)|a_k^i|^2 - (\mu_1^\alpha-\mu_1^n)\frac{t^2}{2}|a_0|^2 + (n-\alpha)C(n)t^3\|u\|^2_{L^2}\,.$$

Recalling (2.26) and the fact that $|\mu_1^\alpha-\mu_1^n|\le(n-\alpha)C(n)$, we conclude that

$$V_\alpha(B)-V_\alpha(E_t) \le \frac{t^2}{2}\mathcal{Q}\mathcal{V}_\alpha(u) + (n-\alpha)C(n)t^3\|u\|^2_{L^2}\,, \tag{8.16}$$

that is the required strengthening of (5.20). Now we can apply (2.24) together with (8.16) to find that

$$\big(\mathrm{Per}_s+\beta V_\alpha\big)(E_t)-\big(\mathrm{Per}_s+\beta V_\alpha\big)(B)\geq \frac{t^2}{2}\Big(\mathcal{QP}_s(u)-\beta\mathcal{QV}_\alpha(u)\Big)$$
$$-\,C(n)t^3\bigg(\frac{1-s}{\omega_{n-1}}\big([u]^2_{\frac{1+s}{2}}+\lambda_1^s\|u\|^2_{L^2}\big)+(n-\alpha)\beta\|u\|^2_{L^2}\bigg)\,. \tag{8.17}$$

Arguing as in the previous case, it yields

$$\big(\mathrm{Per}_s+\beta V_\alpha\big)(E_t)-\big(\mathrm{Per}_s+\beta V_\alpha\big)(B)\geq \frac{t^2}{8}\Big(1-\frac{\beta}{\beta_\star}\Big)\bigg(\frac{1-s}{\omega_{n-1}}[u]^2_{\frac{1+s}{2}}+(1-s)\lambda_1^s\|u\|^2_{L^2}\bigg)$$
$$-\,C(n)t^3\bigg(\frac{1-s}{\omega_{n-1}}[u]^2_{\frac{1+s}{2}}+(1-s)\lambda_1^s\|u\|^2_{L^2}+(n-\alpha)\beta_\star\|u\|^2_{L^2}\bigg)\,.$$

Since $(n-\alpha)\beta_\star\leq C(n)$ by (7.21), we conclude as in the previous case. □

As a last tool in the proof of Theorem 1.5 we prove the following lemma.

Lemma 8.5. *Let $s\in(0,1]$ and $\alpha\in(0,n)$. If $\beta<\beta_\star$, then B is a local volume-constrained minimizer of* $\mathrm{Per}_s+\beta\,V_\alpha$*. If $\beta>\beta_\star$, then B is not a local volume-constrained minimizer of* $\mathrm{Per}_s+\beta\,V_\alpha$.

Proof. If B is a local volume-constrained minimizer of $\mathrm{Per}_s+\beta\,V_\alpha$, then B is automatically a volume-constrained stable set for $\mathrm{Per}_s+\beta\,V_\alpha$, and thus $\beta\leq\beta_\star$ by Theorem 7.1. We are thus left to prove that if $\beta<\beta_\star$, then B is a local volume-constrained minimizer of $\mathrm{Per}_s+\beta\,V_\alpha$. To this end, we argue by contradiction and assume the existence of some $\beta<\beta_*$ such that there exists a sequence $\{E_h\}_{h\in\mathbb{N}}$ with

$$|E_h|=|B|\,,\qquad \lim_{h\to\infty}|E_h\Delta B|=0\,,\qquad \mathrm{Per}_s(E_h)+\beta\,V_\alpha(E_h)<\mathrm{Per}_s(B)+\beta\,V_\alpha(B)\,,\quad \forall h\in\mathbb{N}\,. \tag{8.18}$$

We divide the proof in two steps.

Step one: We show the existence of a radius $R>0$ (depending on n, s and α only) such that the sequence E_h in (8.18) can actually be assumed to satisfy the additional constraint

$$E_h\subset B_R\,,\qquad \forall h\in\mathbb{N}\,. \tag{8.19}$$

To show this, let us introduce a parameter $\eta<1$ (whose precise value will be chosen shortly depending on n, s and α) and let us assume without loss of generality and thanks to (8.18) that $|E_h\Delta B|<\eta$ for every $h\in\mathbb{N}$. By Lemma 4.5, see in particular (4.17), there exists a sequence $\{r_h\}_{h\in\mathbb{N}}$ with $1\leq r_h\leq 1+C_1\,\eta^{1/n}$ such that

$$\mathrm{Per}_s(E_h\cap B_{r_h})\leq \mathrm{Per}_s(E_h)-\frac{|E_h\setminus B_{r_h}|}{C_2\,\eta^{1/n}}\,, \tag{8.20}$$

where C_1 and C_2 depend on n and s only. Next, we consider $\mu_h>0$ such that $F_h:=\mu_h\,(E_h\cap B_{r_h})$ satisfies $|F_h|=|B|$. Since $|E_h\Delta B|\to 0$ as $h\to\infty$, it must be that $\mu_h\to 1$ and $|F_h\Delta B|\to 0$ as $h\to\infty$. In particular, we can assume without loss of generality that $F_h\subset B_R$ for every $h\in\mathbb{N}$, provided we set $R:=2+C_1\,\eta^{1/n}$. We finally show that

$$\mathrm{Per}_s(F_h)+\beta\,V_\alpha(F_h)\leq \mathrm{Per}_s(E_h)+\beta\,V_\alpha(E_h)\,. \tag{8.21}$$

Indeed, by setting $u_h:=|E_h\setminus B_{r_h}|$ we find that

$$\begin{aligned}\mathrm{Per}_s(F_h)+\beta\,V_\alpha(F_h)&=\mu_h^{n-s}\mathrm{Per}_s(E_h\cap B_{r_h})+\mu_h^{n+\alpha}\beta\,V_\alpha(E_h\cap B_{r_h})\\&\leq(1+C\,u_h)\Big(\mathrm{Per}_s(E_h\cap B_{r_h})+\beta\,V_\alpha(E_h\cap B_{r_h})\Big)\,,\end{aligned}$$

where $C = C(n,s,\alpha)$. By $V_\alpha(E_h \cap B_{r_h}) \le V_\alpha(E_h)$, (8.20), and (8.18), we conclude that

$$\begin{aligned}\mathrm{Per}_s(F_h) + \beta\, V_\alpha(F_h) \quad &\le \quad \mathrm{Per}_s(E_h) + \beta\, V_\alpha(E_h)\\ &\qquad + \Big(C\,\Big(\mathrm{Per}_s(B) + \beta_\star\, V_\alpha(B)\Big) - \frac{1}{C_2\,\eta^{1/n}}\Big)\, u_h\,,\end{aligned}$$

so that (8.21) follows provided η was suitably chosen in terms of n, s and α only.

Step two: Given $M > 0$ and a sequence E_h satisfying (8.18) and (8.19) we now consider the variational problems

$$\gamma_h := \inf\Big\{\mathrm{Per}_s(E) + \beta\, V_\alpha(E) + M\,|E\Delta E_h| : E \subset \mathbb{R}^n\Big\}\,, \qquad h \in \mathbb{N}\,, \tag{8.22}$$

and prove the existence of minimizers. Indeed, if R is as in (8.19), then $V_\alpha(E \cap B_R) \le V_\alpha(E)$ by set inclusion, $\mathrm{Per}_s(E \cap B_R) \le \mathrm{Per}_s(E)$ by Lemma B.1, while, if we set $F = E \cap B_R$, then by (8.19),

$$\begin{aligned}|F\Delta E_h| \quad &= \quad |F \setminus E_h| + |E_h \setminus F| \le |E \setminus E_h| + |(E_h \cap B_R) \setminus F| + |E_h \setminus B_R|\\ &= \quad |E \setminus E_h| + |(E_h \cap B_R) \setminus E|\\ &\le \quad |E\Delta E_h|\,.\end{aligned}$$

Thus the value of γ_h is not changed if we restrict the minimization class by imposing $E \subset B_R$. By the Direct Method, there exists a minimizer F_h in (8.22) for every $h \in \mathbb{N}$, with $F_h \subset B_R$. We now claim that there exists $\Lambda > 0$ such that

$$\mathrm{Per}_s(F_h) \le \mathrm{Per}_s(E) + \Lambda\,|E\Delta F_h|\,, \qquad \forall E \subset \mathbb{R}^n\,, \tag{8.23}$$

for every $h \in \mathbb{N}$. Indeed, by minimality of F_h in (8.22) and by (5.9), we find that for every bounded set $E \subset \mathbb{R}^n$ one has

$$\mathrm{Per}_s(F_h) - \mathrm{Per}_s(E) \le \frac{2\,P(B)\,\beta}{\alpha}\,\Big(\frac{|E|}{|B|}\Big)^{\alpha/n}\,|E \setminus F_h| + M\,\Big(|E\Delta E_h| - |F_h\Delta E_h|\Big)\,.$$

In particular, (8.23) follows provided

$$\Lambda \ge \frac{2^{1+\alpha}\,P(B)\,\beta\,R^\alpha}{\alpha} + M\,, \tag{8.24}$$

whenever $|E| \le |B_{2R}|$. To address the complementary case, we just notice that, setting for the sake of brevity $\mathcal{F} := \mathrm{Per}_s + \beta\,V_\alpha$, by (8.18) and by minimality of F_h one has

$$\mathcal{F}(B) > \mathcal{F}(E_h) \ge \mathcal{F}(F_h) + M\,|F_h\Delta E_h|\,. \tag{8.25}$$

In particular $\mathrm{Per}_s(F_h) \le \mathcal{F}(B)$ for every $h \in \mathbb{N}$. Hence, if $|E| \ge |B_{2R}|$ then by $F_h \subset B_R$ we have

$$\mathrm{Per}_s(E) + \Lambda\,|E\Delta F_h| \ge \Lambda(|E| - |F_h|) \ge \Lambda\,|B|(2^n - 1)\,R^n \ge \mathrm{Per}_s(F_h)\,,$$

provided

$$\Lambda \ge \frac{\mathcal{F}(B)}{|B|(2^n - 1)\,R^n}\,. \tag{8.26}$$

We choose Λ to be the maximum between the right-hand sides of (8.24) and (8.26), and in this way (8.23) is proved. We now notice that by (8.25), (8.18), and up to discard finitely many h's, we can assume that

$$|F_h\Delta B| \le \frac{2\,\mathcal{F}(B)}{M}\,, \qquad \forall h \in \mathbb{N}\,. \tag{8.27}$$

Let now ε_β be defined as in Theorem 8.1. By Corollary 3.6 there exist $\alpha \in (0,1)$ and $\delta > 0$ (depending on n, s and α only) such that the following holds: If F is a Λ-minimizer of the s-perimeter with $|F\Delta B| < \delta$ (Λ as in (8.23)), then there is $u \in C^{1,\alpha}(\partial B)$ such that

$$\partial F = \Big\{(1 + u(x))\,x : x \in \partial B\Big\}\,, \qquad \|u\|_{C^1(\partial B)} < \varepsilon_\beta\,.$$

Hence, by (8.23) and (8.27), we can choose M large enough (depending on n, s and α) in such a way that, for every $h \in \mathbb{N}$, there exists $u_h \in C^{1,\alpha}(\partial B)$ with

$$\partial F_h = \Big\{(1+u_h(x))\,x : x \in \partial B\Big\}\,, \qquad \|u_h\|_{C^1(\partial B)} < \varepsilon_\beta\,.$$

Let us set $t_h := (|F_h|/|B|)^{1/n}$ and $G_h := x_h + t_h\,F_h$ for x_h such that $\int_{G_h} x\,dx = 0$. By (8.27), we can make $|t_h - 1|$ small enough in terms of ε_β to entail that for every $h \in \mathbb{N}$ there exists $v_h \in C^{1,\alpha}(\partial B)$ with

$$\partial G_h = \Big\{(1+v_h(x))\,x : x \in \partial B\Big\}\,, \qquad \|v_h\|_{C^1(\partial B)} < \varepsilon_\beta\,.$$

By Theorem 8.1 we conclude that

$$\mathcal{F}(B) \le \mathcal{F}(G_h) = t_h^{n-s}\,\mathrm{Per}_s(F_h) + t_h^{n+\alpha}\,\beta\,V_\alpha(F_h) \le \max\{t_h^{n-s}, t_h^{n+\alpha}\}\,\mathcal{F}(F_h)\,, \tag{8.28}$$

which in turn gives, in combination with (8.25),

$$\frac{\mathcal{F}(B)}{\max\{t_h^{n-s}, t_h^{n+\alpha}\}} + M\,|F_h \Delta E_h| \le \mathcal{F}(B)\,. \tag{8.29}$$

If $t_h = 1$ for a value of h, then by (8.29) we find $F_h = E_h$ and thus $\mathcal{F}(F_h) = \mathcal{F}(E_h) < \mathcal{F}(B)$, a contradiction to (8.28). At the same time, since $\mathcal{F}(B) > 0$, (8.29) implies that $t_h \ge 1$ for every $h \in \mathbb{N}$. We may thus assume that $t_h > 1$ for every $h \in \mathbb{N}$. Since $|F_h \Delta E_h| \ge ||F_h| - |B|| = |B|\,(t_h^n - 1)$, by (8.29) we find

$$M\,|B|\,(t_h^n - 1) \le \mathcal{F}(B)\,\Big(1 - \frac{1}{t_h^{n+\alpha}}\Big)\,,$$

where, say, $t_h \in (1, 3/2)$ for every $h \in \mathbb{N}$. However, if M is large enough depending on n, s, and α only, we actually have that

$$M\,|B|\,(t^n - 1) > \mathcal{F}(B)\,\Big(1 - \frac{1}{t^{n+\alpha}}\Big)\,, \qquad \forall t \in (1, 3/2)\,.$$

We thus find a contradiction also in the case that $t_h > 1$ for every $h \in \mathbb{N}$. This completes the proof of the lemma. □

Proof of Theorem 1.5. Given $m > 0$ let us define $\beta > 0$ by setting

$$\beta = \Big(\frac{m}{|B|}\Big)^{(n+\alpha)/n}\Big(\frac{|B|}{m}\Big)^{(n-s)/n} = \Big(\frac{m}{|B|}\Big)^{(s+\alpha)/n}\,.$$

(Notice that $\beta < \beta_\star$ if and only if $m < m_\star$, since by (1.7) and (7.21) we have $m_\star = |B|\,\beta_\star^{n/(s+\alpha)}$.) By exploiting this identity and the scaling properties of Per_s and V_α, and denoting by $B[m]$ a ball of volume m, given $\delta > 0$ we notice that

$$\mathrm{Per}_s(B) + \beta\,V_\alpha(B) \le \mathrm{Per}_s(F) + \beta\,V_\alpha(F)\,, \qquad \text{whenever } |F| = |B| \text{ and } |F \Delta B| < \delta$$

if and only if

$$\mathrm{Per}_s(B[m]) + V_\alpha(B[m]) \le \mathrm{Per}_s(E) + V_\alpha(E)\,, \qquad \text{whenever } |E| = m \text{ and } |E \Delta B[m]| < \tfrac{m}{|B|}\,\delta\,.$$

As a consequence, Theorem 1.5 is equivalent to Lemma 8.5. □

Appendix A. A simple Γ-convergence result

Here we prove the Γ-convergence of P_s to P_{s_*} in the limit $s \to s_*$, with $s_* \in (0,1)$ fixed. Of course, if $|(E_h \Delta E) \cap K| \to 0$ for every $K \subset\subset \mathbb{R}^n$ and $s_h \to s_* \in (0,1)$ as $h \to \infty$, then by Fatou's lemma one easily obtains

$$P_{s_*}(E) \le \liminf_{h\to\infty} P_{s_h}(E_h) \,,$$

that is the Γ-liminf inequality. The proof of the Γ-limsup inequality is only slightly longer. For the sake of simplicity, we shall limit ourselves to work with bounded sets (this is the case we need in the paper). Precisely, given a bounded set $F \subset \mathbb{R}^n$, we want to construct a sequence $\{F_h\}_{h\in\mathbb{N}}$ of bounded sets such that $|F_h \Delta F| \to \infty$ as $h \to \infty$ and

$$\limsup_{h\to\infty} P_{s_h}(F_h) \le P_{s_*}(F) \,. \tag{A.1}$$

We now prove (A.1). We start by recalling the following nonlocal coarea formula due to Visintin [37],

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\, dy = 2\int_0^1 P_s(\{u>t\})\, dt \,, \qquad s \in (0,1)\,, \tag{A.2}$$

that holds true (as an identity in $[0,\infty]$) whenever $u : \mathbb{R}^n \to [0,1]$ is Borel measurable; see [3, Lemma 10]. Next we use [29, Proposition 14.5] to infer that if $P_{s_*}(F) < \infty$ and we set $u_\varepsilon = 1_F \star \rho_\varepsilon$, ρ_ε a standard ε-mollifier, then

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|}{|x-y|^{n+s_*}}\, dy = 2\, P_{s_*}(F) \,. \tag{A.3}$$

Combining (A.2) and (A.3) with a classical argument by De Giorgi, see, e.g. [31, Theorem 13.8], we reduce the proof of (A.1) to the case that F is a bounded, smooth set. This implies that $P_s(F) < \infty$ for every $s \in (0,1)$. In particular, if we let $s_{**} \in (0,1)$ be such that $s_h < s_{**}$ for every $h \in \mathbb{N}$, then we trivially find that, for every $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\frac{1_{F\times F^c}(x,y)}{|x-y|^{n+s_h}} \le 1_{(F\times F^c)\cap\{|x-y|>1\}}(x,y) + \frac{1_{F\times F^c \cap\{|x-y|\le 1\}}(x,y)}{|x-y|^{n+s_{**}}} =: g(x,y)\,,$$

where $g \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ thanks to the fact that $P_{s_{**}}(F) < \infty$. In particular,

$$\lim_{h\to\infty} P_{s_h}(F) = P_{s_*}(F)\,,$$

whenever $s_h \to s_* \in (0,1)$ as $h \to \infty$ and F is a smooth bounded set. This proves (A.1).

Appendix B. A geometric lemma

The following natural fact, which is well-known in the case of the classical perimeter, was used in the proof of Lemma 8.5. We give a proof since it may be useful elsewhere.

Lemma B.1. *If $s \in (0,1]$ and $E \subset \mathbb{R}^n$ is such that $P_s(E) < \infty$, then $P_s(E \cap K) \le P_s(E)$ for every convex set $K \subset \mathbb{R}^n$.*

Proof. The case $s = 1$ being classical, we can assume $s < 1$. Since any convex set can be written as a countable intersections of half-space, it is enough to prove that $P_s(E \cap H) \le P_s(E)$ whenever H is an half-space. By approximation, it suffices to prove this estimate when E is bounded. We now observe that, if we set $F := E \cup H$, using that $E \subset F$, $E \setminus H = F \setminus H$, and $F \cap H = H$, we get

$$\begin{aligned} P_s(E) - P_s(E \cap H) &= \int_E \int_{E^c} \frac{dx\,dy}{|x-y|^{n+s}} - \int_{E\cap H} \int_{(E\cap H)^c} \frac{dx\,dy}{|x-y|^{n+s}} \\ &= \left(\int_{E\cap H} + \int_{E\setminus H} \right) \int_{E^c} \frac{dx\,dy}{|x-y|^{n+s}} - \left(\int_{E^c} + \int_{E\setminus H} \right) \int_{E\cap H} \frac{dx\,dy}{|x-y|^{n+s}} \\ &= \left(\int_{E^c} - \int_{E\cap H} \right) \int_{E\setminus H} \frac{dx\,dy}{|x-y|^{n+s}} \\ &\geq \left(\int_{F^c} - \int_{F\cap H} \right) \int_{F\setminus H} \frac{dx\,dy}{|x-y|^{n+s}}. \end{aligned}$$

We now observe that (just by doing the above steps backward) the last term is formally equal to $P_s(F) - P_s(H)$. However, this does not really make sense as both $P_s(F)$ and $P_s(H)$ are actually infinite. For this reason, we have to consider a local version of P_s: given a set G and a domain A, we define the s-perimeter of G inside A as

$$P_s(G; A) := \left(\int_{G\cap A} \int_{G^c\cap A} + \int_{G\cap A} \int_{G^c\cap A^c} + \int_{G\cap A^c} \int_{G^c\cap A} \right) \frac{dx\,dy}{|x-y|^{n+s}}.$$

With this notation, if B_R is a large ball which contains E (recall that E is bounded), since F is equal to H outside B_R it is easy to check that

$$\left(\int_{F^c} - \int_{F\cap H} \right) \int_{F\setminus H} \frac{dx\,dy}{|x-y|^{n+s}} = P_s(F; B_R) - P_s(H; B_R).$$

Applying [3, Proposition 17] we deduce that $P_s(F; B_R) - P_s(H; B_R) \geq 0$, concluding the proof. $\square$

Appendix C. About the constant $\beta_\star$

We have already noticed that, in order to show the equivalence between the two formulas (1.7) and (7.1) for $\beta_\star$, it suffices to show that, for every $s \in (0, 1]$ and $\alpha \in (0, n)$, one has

$$\frac{\lambda_k^s - \lambda_1^s}{\mu_k^\alpha - \mu_1^\alpha} \geq \frac{\lambda_2^s - \lambda_1^s}{\mu_2^\alpha - \mu_1^\alpha} \qquad \forall k \geq 2\,. \tag{C.1}$$

Proof of (C.1) *in the case that* $s \in (0, 1)$ *and* $\alpha \in (0, 1)$. In this case, the repeated application of the factorial property of the gamma function shows that (C.1) is equivalent in proving that the quantity

$$X_k := \frac{\frac{\prod_{j=1}^k (j + \frac{n+s}{2})}{\prod_{j=1}^k (j + \frac{n-2-s}{2})} - 1}{\frac{\prod_{j=1}^k (j + \frac{n-\alpha}{2})}{\prod_{j=1}^k (j + \frac{n-2+\alpha}{2})} - 1}$$

attains its minimal value on $k \geq 1$ at $k = 1$. To this end it is convenient to rewrite X_k as follows: first, we notice that

$$X_k = \frac{\frac{\prod_{j=1}^k (j + \frac{n-1}{2} + t)}{\prod_{j=1}^k (j + \frac{n-1}{2} - t)} - 1}{\frac{\prod_{j=1}^k (j + \frac{n-1}{2} + \tau)}{\prod_{j=1}^k (j + \frac{n-1}{2} - \tau)} - 1}, \qquad \text{where} \quad t := \frac{1+s}{2}, \quad \tau := \frac{1-\alpha}{2}\,,$$

(and thus, $0 < \tau < t$); second, we set

$$a_k := \prod_{j=2}^k \alpha_j, \qquad b_k := \prod_{j=2}^k \beta_j, \qquad c_k := \prod_{j=2}^k \gamma_j, \qquad d_k := \prod_{j=2}^k \delta_j\,, \tag{C.2}$$

where

$$\alpha_k := k + \frac{n-1}{2} + t, \qquad \beta_k := k + \frac{n-1}{2} - t,$$

$$\gamma_k := k + \frac{n-1}{2} + \tau, \qquad \delta_k := k + \frac{n-1}{2} - \tau.$$

In this way, $X_k \ge X_1$ for every $k \ge 2$ can be rephrased into

$$\frac{\frac{a_k \alpha_1}{b_k \beta_1} - 1}{\frac{c_k \gamma_1}{d_k \delta_1} - 1} \ge \frac{\frac{\alpha_1}{\beta_1} - 1}{\frac{\gamma_1}{\delta_1} - 1}, \qquad \forall k \ge 2. \tag{C.3}$$

It is useful to rearrange the terms in (C.3) and rewrite it as

$$a_k d_k \alpha_1(\gamma_1 - \delta_1) + b_k d_k(\alpha_1 \delta_1 - \beta_1 \gamma_1) + b_k c_k \gamma_1(\beta_1 - \alpha_1) \ge 0, \qquad \forall k \ge 2. \tag{C.4}$$

We now observe that, setting $\ell := (n+1)/2$, we have

$$\alpha_1 = \ell + t, \quad \beta_1 = \ell - t, \quad \gamma_1 = \ell + \tau, \quad \delta_1 = \ell - \tau, \quad \alpha_1\delta_1 - \beta_1\gamma_1 = 2\ell(t - \tau).$$

Hence, substituting these formulas into the above expression we find that

$$\begin{aligned} \text{left-hand side of (C.4)} &= 2a_k d_k(\ell + t)\tau + 2b_k d_k \ell(t - \tau) - 2b_k c_k(\ell + \tau)t \\ &= 2(a_k d_k - b_k c_k)t\tau + 2(a_k - b_k)d_k \ell \tau - 2(c_k - d_k)b_k \ell t. \end{aligned}$$

Therefore (C.4) follows by showing that

$$a_k d_k \ge b_k c_k, \qquad \forall\, k \ge 2, \tag{C.5}$$

$$(a_k - b_k)d_k \tau \ge (c_k - d_k)b_k t, \qquad \forall\, k \ge 2. \tag{C.6}$$

To prove (C.5) it suffices to observe that

$$\alpha_j \delta_j - \beta_j \gamma_j = 2\Big(j + \frac{n-1}{2}\Big)(t - \tau) \ge 0 \qquad \forall\, j \ge 1,$$

so that

$$a_k d_k = \prod_{j=2}^{k} \alpha_j \delta_j \ge \prod_{j=2}^{k} \beta_j \gamma_j = b_k c_k, \qquad \forall k \ge 2,$$

as desired. We now prove (C.6) by induction. A simple manipulation shows that (C.6) in the case $k = 2$ is equivalent to $d_2 \ge b_2$, which is true, so that we directly focus on the inductive hypothesis. By noticing that $a_{k+1} = a_k \alpha_{k+1}$, and that analogous identities hold for β_k, γ_k and δ_k, we can equivalently reformulate the $(k+1)$-case of (C.6) as

$$(a_k \alpha_{k+1} - b_k \beta_{k+1}) d_k \delta_{k+1} \tau \ge (c_k \gamma_{k+1} - d_k \delta_{k+1}) b_k \beta_{k+1} t.$$

This last inequality can be conveniently rewritten as

$$\begin{aligned} &a_k(\alpha_{k+1} - \beta_{k+1}) d_k \delta_{k+1} \tau + \beta_{k+1}\delta_{k+1}(a_k - b_k) d_k \tau \\ &\qquad \ge c_k(\gamma_{k+1} - \delta_{k+1}) b_k \beta_{k+1} t + \beta_{k+1}\delta_{k+1}(c_k - d_k) b_k t. \end{aligned}$$

Indeed, by the inductive hypothesis $(a_k - b_k)d_k\tau \ge (c_k - d_k)b_k t$, it is clear that a sufficient condition for this last inequality (and thus, for (C.6)) to hold true, is that

$$a_k(\alpha_{k+1} - \beta_{k+1}) d_k \delta_{k+1} \tau \ge c_k(\gamma_{k+1} - \delta_{k+1}) b_k \beta_{k+1} t. \tag{C.7}$$

By $\alpha_{k+1} - \beta_{k+1} = 2t$ and $\gamma_{k+1} - \delta_{k+1} = 2\tau$, (C.7) is equivalent to

$$2(a_k d_k \delta_{k+1} - b_k c_k \beta_{k+1}) t\tau \ge 0.$$

Finally, this inequality holds true because of (C.5) and the fact that $\delta_{k+1} \ge \beta_{k+1}$. This complete the proof of (C.6), and thus of (C.1) in the case that $\sigma \in (0,1)$ and $\alpha \in (0,1)$.

Proof of (C.1) *in the case that* $s \in (0,1)$ *and* $\alpha \in (1,n)$. By the factorial property of the gamma function (C.1) is now equivalent in proving that $X_k \geq X_1$ for every $k \geq 2$, where we have now set

$$X_k := \frac{\frac{\prod_{j=1}^k \left(j+\frac{n+s}{2}\right)}{\prod_{j=1}^k \left(j+\frac{n-2-s}{2}\right)} - 1}{1 - \frac{\prod_{j=1}^k \left(j+\frac{n-\alpha}{2}\right)}{\prod_{j=1}^k \left(j+\frac{n-2+\alpha}{2}\right)}} \,.$$

We notice that

$$X_k = \frac{\frac{\prod_{j=1}^k \left(j+\frac{n-1}{2}+t\right)}{\prod_{j=1}^k \left(j+\frac{n-1}{2}-t\right)} - 1}{1 - \frac{\prod_{j=1}^k \left(j+\frac{n-1}{2}-\tau\right)}{\prod_{j=1}^k \left(j+\frac{n-1}{2}+\tau\right)}} \,, \qquad \text{where} \quad t := \frac{1+s}{2}, \quad \tau := \frac{\alpha-1}{2} \,.$$

We next define a_k, b_k, c_k and d_k as in (C.2), with α_k, β_k, γ_k and δ_k given by

$$\alpha_k := k + \frac{n-1}{2} + t, \qquad \beta_k := k + \frac{n-1}{2} - t,$$

$$\gamma_k := k + \frac{n-1}{2} - \tau, \qquad \delta_k := k + \frac{n-1}{2} + \tau.$$

We have thus reformulated (C.1) as

$$\frac{\frac{a_k \alpha_1}{b_k \beta_1} - 1}{1 - \frac{c_k \gamma_1}{d_k \delta_1}} \geq \frac{\frac{\alpha_1}{\beta_1} - 1}{1 - \frac{\gamma_1}{\delta_1}} \,, \qquad \forall k \geq 2 \,,$$

which is in turn equivalent to

$$a_k d_k \alpha_1 (\delta_1 - \gamma_1) + b_k d_k (\beta_1 \gamma_1 - \alpha_1 \delta_1) + b_k c_k \gamma_1 (\alpha_1 - \beta_1) \geq 0 \,, \qquad \forall k \geq 2 \,. \tag{C.8}$$

If we set $\ell = (n+1)/2$, then we find

$$\alpha_1 = \ell + t, \quad \beta_1 = \ell - t, \quad \gamma_1 = \ell - \tau, \quad \delta_1 = \ell + \tau, \quad \alpha_1\delta_1 - \beta_1\gamma_1 = 2\ell(t+\tau) \,,$$

so that

$$\begin{aligned} \text{left-hand side of (C.8)} \quad &= \quad 2a_k d_k (\ell + t)\tau - 2b_k d_k \ell (t+\tau) + 2b_k c_k t(\ell - \tau) \\ &= \quad 2(a_k d_k - b_k c_k) t\tau + 2(a_k - b_k) d_k \ell \tau + 2(c_k - d_k) b_k \ell t \,. \end{aligned}$$

We are thus left to prove that

$$a_k d_k \geq b_k c_k \,, \qquad \forall\, k \geq 2 \,, \tag{C.9}$$

$$(a_k - b_k) d_k \tau \geq (d_k - c_k) b_k t \,, \qquad \forall\, k \geq 2 \,. \tag{C.10}$$

To prove (C.9) it suffices to observe that

$$\alpha_j \delta_j - \beta_j \gamma_j = 2\Big(j + \frac{n-1}{2}\Big)(t+\tau) \geq 0 \qquad \forall\, j \geq 1,$$

where $t > 0$ and $\tau > 0$. To prove (C.10) we argue once again by induction. One easily sees that (C.6) in the case $k = 2$ is equivalent to say that $d_2 \geq b_2$, which is true also in the present case. We now check the inductive hypothesis. The $(k+1)$-case of (C.6) is now equivalent to

$$(a_k \alpha_{k+1} - b_k \beta_{k+1}) d_k \delta_{k+1} \tau \geq (d_k \delta_{k+1} - c_k \gamma_{k+1}) b_k \beta_{k+1} t \,.$$

We reformulate this as

$$\begin{aligned} a_k(\alpha_{k+1} - \beta_{k+1}) d_k \delta_{k+1} \tau + \beta_{k+1} \delta_{k+1} (a_k - b_k) d_k \tau & \\ \geq c_k (\delta_{k+1} - \gamma_{k+1}) b_k \beta_{k+1} t + \beta_{k+1} \delta_{k+1} (d_k - c_k) b_k t \,. & \end{aligned}$$

By the inductive hypothesis $(a_k - b_k) d_k \tau \geq (d_k - c_k) b_k t$, thus we are left to check that

$$a_k (\alpha_{k+1} - \beta_{k+1}) d_k \delta_{k+1} \tau \geq c_k (\delta_{k+1} - \gamma_{k+1}) b_k \beta_{k+1} t \,. \tag{C.11}$$

By $\alpha_{k+1} - \beta_{k+1} = 2t$ and $\delta_{k+1} - \gamma_{k+1} = 2\tau$, (C.11) is equivalent to $2(a_k d_k \delta_{k+1} - b_k c_k \beta_{k+1}) t\tau \geq 0$, which is true thanks to (C.9) and $\delta_{k+1} \geq \beta_{k+1}$. The proof of (C.10), thus of (C.1) in the case that $\sigma \in (0,1)$ and $\alpha \in (1,n)$, is now complete.

Proof of (C.1) *in the remaining cases.* The case that $s \in (0,1)$ and $\alpha = 1$ is covered by taking the limit as $\alpha \to 1^-$ with s fixed in (C.1) for $\alpha \in (0,1)$. This proves (C.1) for every $s \in (0,1)$ and $\alpha \in (0,n)$. The case $s = 1$ is recovered by multiplying (C.1) by $1 - s$ when $s \in (0,1)$ and then taking the limit as $s \to 1^-$ with α fixed. The proof of (C.1) is now complete.

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