Finite element discretization of the Stokes and Navier-Stokes equations with boundary conditions on the pressure

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by Christine Bernardi[1], Tomás Chacón Rebollo[1,2] and Driss Yakoubi[3]

Abstract: We consider the Stokes and Navier–Stokes equations with boundary conditions of Dirichlet type on the velocity on one part of the boundary and involving the pressure on the rest of the boundary. We write the variational formulations of such problems. Next we propose a finite element discretization of them and perform the a priori and a posteriori analysis of the discrete problem. Some numerical experiments confirm the interest of this discretization.

Résumé: Nous considérons les équations de Stokes et de Navier–Stokes munies de conditions aux limites de Dirichlet sur la vitesse sur une partie de la frontière et faisant appel à la pression sur le reste. Nous écrivons la formulation variationnelle de ces problèmes. Puis nous en proposons une discrétisation par éléments finis et effectuons l'analyse a priori et a posteriori du problème discret. Quelques expériences numériques confirment l'intérêt de la discrétisation.

[1] Laboratoire Jacques-Louis Lions, C.N.R.S. & Université Pierre et Marie Curie,
B.C. 187, 4 place Jussieu, 75252 Paris Cedex 05, France.

[2] Departamento de Ecuaciones Diferenciales y Anlisis Numrico
& Instituto de Matemática de la Universidad de Sevilla (IMUS),
Universidad de Sevilla, Tarfia s/n. 41012 Sevilla, Spain.

[3] GIREF, Département de Mathématiques et de Statistique,
Pavillon Vachon, 1045 avenue de la Médecine, Université de Laval, Québec, Canada.
e-mail addresses: bernardi@ann.jussieu.fr, chacon@us.es, yakoubi@giref.ulaval.ca

1. Introduction.

Most works concerning the Stokes or Navier–Stokes equations deal with Dirichlet boundary conditions on the velocity (also called no-slip conditions), see for instance [20] or [33]. However, other types of boundary conditions were suggested in the pioneering paper [4], which was followed by a large number of works on this subject. Among them, the conditions on the normal component of the velocity and the vorticity were thoroughly studied and led to the so-called vorticity–velocity–pressure formulation, introduced in [31] and studied in several other papers, see [17], [18] and [7] for instance. Their extension to mixed boundary conditions was performed in [8]. However it seems that the conditions on the tangential components of the velocity and the pressure have less been studied, we refer to [29] and [15] for first works on these topics and also to [5] in the case of a simple geometry and of the linear Stokes problem. Recent papers deal either with the analysis of the equations [3] [26] or with their discretization [23] [24] [28] [32]. Unfortunately this discretization most often relies on finite difference schemes.

We wish here to propose a discretization in the case of mixed boundary conditions, Dirichlet conditions on the velocity in part of the boundary, conditions on the tangential components of the velocity and on the pressure on another part, both for the Stokes and Navier–Stokes equations. We first write the variational formulation of these problems and recall their main properties. It can be noted that all conditions on the velocity are prescribed in an essential way while the boundary condition on the pressure is treated in a natural way. Next, we consider a finite element discretization: In view of the variational formulation, we decide to use the same finite elements as for standard boundary conditions, more precisely the Taylor–Hood element [22]. We perform the numerical analysis of the discrete problem: Optimal a priori estimates and quasi optimal a posteriori error estimates are derived, both in the linear and nonlinear cases. The arguments are similar to those for standard boundary conditions but require small extensions. In a final step, we present numerical experiments that confirm the interest of our discretization.

The outline of this article is as follows.

- In Section 2, we present the variational formulation of the full system and investigate its well-posedness.
- Section 3 is devoted to the description and a priori and a posteriori error analysis of the discretization of the Stokes problem.
- The a priori and a posteriori analysis of the discretization applied to the Navier-Stokes equations are the object of Section 4.
- In Section 5, we present some numerical experiments.

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2. The continuous problem and its well-posedness.

Let Ω be a bounded connected domain in $\mathbb{R}^d$, $d=2$ or 3, with a Lipschitz-continuous and connected boundary $\partial\Omega$. We assume that this boundary admits a partition without overlap into two parts

$$\partial\Omega=\overline{\Gamma}_1\cup\overline{\Gamma}_2,\qquad \Gamma_1\cap\Gamma_2=\emptyset,$$

where both Γ_1 and Γ_2 have a finite number of connected components. From now on, we also assume that both Γ_1 and Γ_2 have a positive measure in $\partial\Omega$. We denote by $\boldsymbol{n}$ the unit vector normal to $\partial\Omega$ and exterior to Ω.

From now on, we use the notation of the three-dimensional case and sometimes explain the modification in dimension $d=2$. Thus, we consider the problems, for $\varepsilon=0$ and $\varepsilon=1$,

$$\begin{cases} -\nu\,\Delta\boldsymbol{u}+\varepsilon\,(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\mathbf{grad}\,p=\boldsymbol{f} & \text{in }\ \Omega,\\ \operatorname{div}\boldsymbol{u}=0 & \text{in }\Omega,\\ \boldsymbol{u}=\boldsymbol{u}_1 & \text{on }\ \Gamma_1,\\ \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{u}_2\times\boldsymbol{n} & \text{on }\Gamma_2,\\ p+\frac{\varepsilon}{2}\,|\boldsymbol{u}|^2=p_2 & \text{on }\ \Gamma_2, \end{cases}\tag{2.1}$$

(in dimension $d=2$, the third component of $\boldsymbol{n}$ is zero, so that $\boldsymbol{u}\times\boldsymbol{n}$ and $\boldsymbol{u}_2\times\boldsymbol{n}$ mean the tangential component of $\boldsymbol{u}$ and $\boldsymbol{u}_2$, respectively, which is scalar). Indeed, the first two lines correspond to the standard Stokes model for $\varepsilon=0$, to the Navier–Stokes equations for $\varepsilon=1$. The unknowns are the velocity $\boldsymbol{u}$ and the pressure p of the fluid, while the quantity $p+\frac{1}{2}\,|\boldsymbol{u}|^2$ represents the dynamical pressure. The data are a density of forces $\boldsymbol{f}$ on the whole domain and the boundary data $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and p_2, while the viscosity ν is a positive constant.

We write a variational formulation of problem (2.1), next prove the existence of a solution first for $\varepsilon=0$, second for $\varepsilon=1$.

2.1. The variational formulation.

With standard notation for the Sobolev spaces $H^s(\Omega)$ and $H^s_0(\Omega)$ (see [1, chap. 3] for details), we introduce the domains of the divergence and curl operators

$$H(\operatorname{div};\Omega)=\big\{\boldsymbol{v}\in L^2(\Omega)^d;\ \operatorname{div}\boldsymbol{v}\in L^2(\Omega)\big\},$$
$$H(\mathbf{curl};\Omega)=\big\{\boldsymbol{v}\in L^2(\Omega)^d;\ \mathbf{curl}\,\boldsymbol{v}\in L^2(\Omega)^{\frac{d(d-1)}{2}}\big\}.$$

We recall from [20, chap. I, sections 2.2 & 2.3] that the normal trace operator: $\boldsymbol{v}\mapsto\boldsymbol{v}\cdot\boldsymbol{n}$ is continuous from $H(\operatorname{div};\Omega)$ into $H^{-\frac12}(\partial\Omega)$ and that the tangential trace operator: $\boldsymbol{v}\mapsto\boldsymbol{v}\times\boldsymbol{n}$ is continuous from $H(\mathbf{curl};\Omega)$ into $H^{-\frac12}(\partial\Omega)^{\frac{d(d-1)}{2}}$. So, we introduce our variational space

$$\mathbb{X}=\Big\{\boldsymbol{v}\in H(\operatorname{div};\Omega)\cap H(\mathbf{curl};\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n}=\mathbf{0}\text{ on }\Gamma_1\text{ and }\boldsymbol{v}\times\boldsymbol{n}=\mathbf{0}\text{ on }\partial\Omega\Big\}.\tag{2.2}$$

Obviously, the trace operator: $\boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n}$ is continuous from $\mathbb{X}$ onto the dual space of $H_{00}^{\frac{1}{2}}(\Gamma_2)$ (see [25, Chap. 1, Section 11.3] for the definition of the space $H_{00}^{\frac{1}{2}}(\Gamma_2)$). So, we denote by $H_{00}^{-\frac{1}{2}}(\Gamma_2)$ its dual space. and by $\langle \cdot, \cdot \rangle_{\Gamma_2}$ the corresponding duality pairing.

Remark 2.1. Let Ω^* be any domain included in Ω such that $\partial\Omega^* \cap \partial\Omega$ is contained in Γ_1. The restrictions of functions of $\mathbb{X}$ to Ω^* belong to $H^1(\Omega^*)^d$, see [2, thm 2.5] for instance. On the other hand, when Γ_2 is of class $\mathcal{C}^{1,1}$ or convex (where "convex" means that there exists a convex neighbourhood of Γ_2 in Ω), it can be proven [2, thms 2.12 & 2.17] that $\mathbb{X}$ is imbedded in $H^1(\Omega)^d$. Unfortunately, when Γ_2 has re-entrant corners or edges, it is only imbedded in $H^{\frac{1}{2}}(\Omega)^d$, see [16].

The aim of the space $\mathbb{X}$ is of course to take into account the boundary conditions on the velocity (we recall that, in dimension $d = 2$, $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ means that the tangential component of $\boldsymbol{v}$ vanishes). Next we define the bilinear forms

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu \int_\Omega (\mathbf{curl}\, \boldsymbol{u})(\boldsymbol{x}) \cdot (\mathbf{curl}\, \boldsymbol{v})(\boldsymbol{x})\, d\boldsymbol{x}, \qquad b(\boldsymbol{v}, q) = -\int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x}, \quad (2.3)$$

together with the trilinear form

$$N(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{w})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} - \frac{1}{2} \int_\Omega (\boldsymbol{u} \cdot \boldsymbol{w})(\boldsymbol{x}) (\operatorname{div} \boldsymbol{v})(\boldsymbol{x})\, d\boldsymbol{x}. \quad (2.4)$$

Note that, in dimension $d = 2$, $\mathbf{curl}\, \boldsymbol{u}$ is a scalar function, so that $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{w}$ means the vector function with components $(\mathbf{curl}\, \boldsymbol{u}) w_y$ and $-(\mathbf{curl}\, \boldsymbol{u}) w_x$. With this notation, we consider the problem

Find $(\boldsymbol{u}, p)$ *in* $\big(H(\operatorname{div}; \Omega) \cap H(\mathbf{curl}; \Omega)\big) \times L^2(\Omega)$ *such that*

$$\boldsymbol{u} = \boldsymbol{u}_1 \text{ on } \Gamma_1 \qquad \text{and} \qquad \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{u}_2 \times \boldsymbol{n} \text{ on } \Gamma_2, \quad (2.5)$$

and

$$\begin{aligned} &\forall \boldsymbol{v} \in \mathbb{X}, \quad a(\boldsymbol{u}, \boldsymbol{v}) + \varepsilon\, N(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} - \langle p_2, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\Gamma_2}, \\ &\forall q \in L^2(\Omega), \quad b(\boldsymbol{u}, q) = 0. \end{aligned} \quad (2.6)$$

Indeed, we have the following result.

Proposition 2.2. *Any solution* $(\boldsymbol{u}, p)$ *of the variational problem* (2.5) – (2.6) *such that* p *belongs to* $H^1(\Omega)$ *is a solution of problem* (2.1) *(in the distribution sense). Conversely, any solution* $(\boldsymbol{u}, p)$ *of problem* (2.1) *which belongs to* $\mathcal{C}^2(\Omega)^d \times \mathcal{C}^1(\Omega)$ *and also to* $\mathcal{C}^0(\overline{\Omega})^d \times \mathcal{C}^0(\overline{\Omega})$ *is a solution of the variational problem* (2.5) – (2.6).

Proof: The third and fourth lines in (2.1) are obviously equivalent to (2.5). On the other han, taking q equal to $\operatorname{div} \boldsymbol{u}$ in (2.6) yields the second line in (2.1). Finally, we recall that, by integration by parts and for a function $\boldsymbol{v}$ in $\mathcal{D}(\overline{\Omega})^d \cap \mathbb{X}$ (note that such a function has its trace $\boldsymbol{v} \times \boldsymbol{n}$ equal to zero on all the boundary $\partial\Omega$ and that a weak regularity of p is needed for the last line)

$$
\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v}) &= \nu \int_\Omega \mathbf{curl}(\mathbf{curl}\, \boldsymbol{u})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}, \\
N(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega \big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\big)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} - \frac{1}{2} \int_{\Gamma_2} |\boldsymbol{u}|^2(\tau)(\boldsymbol{v} \cdot \boldsymbol{n})(\tau)\, d\tau, \\
b(\boldsymbol{v}, p) &= \int_\Omega \boldsymbol{v}(\boldsymbol{x}) \cdot \mathbf{grad}\, p(\boldsymbol{x})\, d\boldsymbol{x} - \langle p, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\Gamma_2},
\end{aligned}
$$

where τ stands for the tangential coordinate(s) on $\partial\Omega$. Then, thanks to the identity

$$
-\Delta \boldsymbol{u} = \mathbf{curl}(\mathbf{curl}\, \boldsymbol{u}) - \mathbf{grad}\, (\operatorname{div} \boldsymbol{u}), \tag{2.7}
$$

taking $\boldsymbol{v}$ in $\mathscr{D}(\Omega)^d$ gives the first equation in (2.1). The fifth equation then follows by taking $\boldsymbol{v}$ in $\mathcal{D}(\overline{\Omega})^d \cap \mathbb{X}$ and looking at the terms on Γ_2 issued from (2.6).
The converse property is proved by the same arguments, together with the regularity of $(\boldsymbol{u}, p)$.

We now prove the existence of a solution for problem (2.5) – (2.6).

2.2. The Stokes problem.

In the case $\varepsilon = 0$ of the Stokes problem, problem (2.5) – (2.6) is of standard saddle-point type. So, its well-posedness requires two inf-sup conditions. The first one is an extension of the usual inf-sup condition for the Stokes problem to our boundary conditions, its proof can be found in [5, proof of thm 2.1] or in [6, lemma 3.1]. The space $\mathbb{X}$ is now provided with the graph norm of $H(\operatorname{div}; \Omega) \cap H(\mathbf{curl}; \Omega)$, i.e.

$$
\|\boldsymbol{v}\|_{\mathbb{X}} = \big(\|\boldsymbol{v}\|^2_{L^2(\Omega)^d} + \|\operatorname{div} \boldsymbol{v}\|^2_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} \big)^{\frac{1}{2}}, \tag{2.8}
$$

which is smaller than $\|\cdot\|_{H^1(\Omega)^d}$.

Lemma 2.3. *There exists a constant $\beta > 0$ such that the following inf-sup condition holds*

$$
\forall q \in L^2(\Omega), \quad \sup_{\boldsymbol{v} \in \mathbb{X}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathbb{X}}} \geq \beta\, \|q\|_{L^2(\Omega)}. \tag{2.9}
$$

The next lemma requires the kernel

$$
\mathbb{V} = \big\{ \boldsymbol{v} \in \mathbb{X};\ \forall q \in L^2(\Omega),\ b(\boldsymbol{v}, q) = 0 \big\},
$$

which is obviously characterized by

$$
\mathbb{V} = \big\{ \boldsymbol{v} \in \mathbb{X};\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \big\}.
$$

Lemma 2.4. *There exists a constant $\alpha > 0$ such that the following ellipticity property holds*

$$\forall \boldsymbol{v} \in \mathbb{V}, \quad a(\boldsymbol{v}, \boldsymbol{v}) \geq \alpha \, \|\boldsymbol{v}\|_{\mathbb{X}}^2. \tag{2.10}$$

Proof: Due to the definition of $\mathbb{V}$, we have for all $\boldsymbol{v}$ in $\mathbb{V}$,

$$a(\boldsymbol{v}, \boldsymbol{v}) = \nu \, \big(\, \|\mathbf{curl}\, \boldsymbol{v}\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \|\mathrm{div}\, \boldsymbol{v}\|^2_{L^2(\Omega)} \big).$$

Since the boundary of Ω is connected, this last quantity is bounded from below by $c \, \|\boldsymbol{v}\|_{\mathbb{X}}^2$, see [2, cor. 3.19], whence the desired ellipticity properry.

We are now in a position to prove the first existence result. For any data $\boldsymbol{u}_1$ on Γ_1 and $\boldsymbol{u}_2$ on Γ_2, we denote by $\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)$ the function equal to $\boldsymbol{u}_1$ on Γ_1 and to $\boldsymbol{u}_2$ on Γ_2.

Theorem 2.5. *Assume that the data $\boldsymbol{f}$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and p_2 satisfy*

$$\boldsymbol{f} \in L^2(\Omega)^d, \qquad \mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2) \in H^{\frac{1}{2}}(\partial\Omega)^d, \qquad p_2 \in H^{\frac{1}{2}}_{00}(\Gamma_2). \tag{2.11}$$

Then, problem (2.5) – (2.6) *for $\varepsilon = 0$ has a unique solution $(\boldsymbol{u}, p)$. Moreover, this solution satisfies*

$$\|\boldsymbol{u}\|_{\mathbb{X}} + \|p\|_{L^2(\Omega)} \leq c \left(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{H^{\frac{1}{2}}(\partial\Omega)^d} + \|p_2\|_{H^{\frac{1}{2}}_{00}(\Gamma_2)} \right). \tag{2.12}$$

Proof: Let $\boldsymbol{w}$ be a function in $H^1(\Omega)^d$ such that its trace on $\partial\Omega$ coincides with $\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)$ and which moreover satisfies

$$\|\boldsymbol{w}\|_{H^1(\Omega)^d} \leq c \, \|\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{H^{\frac{1}{2}}(\partial\Omega)^d}.$$

Then, the pair $(\boldsymbol{u}_0, p)$, with $\boldsymbol{u}_0 = \boldsymbol{u} - \boldsymbol{w}$, must be found in $\mathbb{X} \times L^2(\Omega)$ and satisfy

$$\begin{aligned}
&\forall \boldsymbol{v} \in \mathbb{X}, \quad a(\boldsymbol{u}_0, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \, d\boldsymbol{x} - \langle p_2, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\Gamma_2} - a(\boldsymbol{w}, \boldsymbol{v}), \\
&\forall q \in L^2(\Omega), \quad b(\boldsymbol{u}_0, q) = -b(\boldsymbol{w}, q).
\end{aligned} \tag{2.13}$$

The well-posedness of this last problem follows from Lemmas 2.3 and 2.4, see [20, chap. I, cor. 4.1]. This yields the existence and uniqueness of a solution to problem (2.5) – (2.6), together with estimate (2.12).

Remark 2.6. All this study makes use of data p_2 in $H^{\frac{1}{2}}_{00}(\Gamma_2)$ for generality. However, it follows from [16] that it can often be less regular, for instance in $L^2(\Gamma_2)$ when Ω is a polygon or a polyhedron.

2.3. The Navier–Stokes equations.

In the case $\varepsilon = 1$ of the Navier-Stokes equations, we decide to work with homogeneous boundary conditions on the velocity, namely

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_1 \qquad \text{and} \qquad \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma_2, \tag{2.14}$$

in order to avoid the technical difficulties due to the Hopf lemma, see [20, chap. IV, lemma 2.3] for instance. Proving the existence of a solution relies on Brouwer's fixed point theorem and requires the next lemma.

Lemma 2.7. *The spaces $\mathbb{X}$ and $\mathbb{V}$ are separable.*

Proof: The space $\mathscr{D}(\overline{\Omega})^d$ is dense in $H(\mathrm{div};\Omega) \cap H(\mathbf{curl};\Omega)$, see [2, prop. 2.3], so that this space is separable. Since it is a Banach space and $\mathbb{X}$ is a closed subspace of it (this is due to the continuity of the trace), $\mathbb{X}$ is also separable, see [11, prop. 3.22] for instance. Finally, since $\mathbb{V}$ is a closed subspace of $\mathbb{X}$, it is once more separable.

The main result of this section requires a further assumption.

Assumption 2.8. The space $\mathbb{X}$ is compactly imbedded in $L^4(\Omega)^d$.

It follows from Remark 2.1 that this assumption always holds when Γ_2 is of class $\mathscr{C}^{1,1}$ or convex and also from [16] that it holds when Ω is a two-dimensional polygon. However, it seems weaker.

Theorem 2.9. *Assume that the data $\boldsymbol{f}$ and p_2 satisfy*

$$\boldsymbol{f} \in L^2(\Omega)^d, \qquad p_2 \in H_{00}^{\frac{1}{2}}(\Gamma_2). \tag{2.15}$$

Then, if Assumption 2.8 holds, problem (2.6) – (2.14) for $\varepsilon = 1$ has at least a solution $(\boldsymbol{u}, p)$. Moreover, this solution satisfies

$$\begin{aligned}
&\|\boldsymbol{u}\|_{\mathbb{X}} \le \frac{c}{\nu}\big(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|p_2\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}\big),\\
&\|p\|_{L^2(\Omega)} \le c\,\big(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|p_2\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}\big) + \frac{c'}{\nu^2}\big(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|p_2\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}\big)^2,
\end{aligned} \tag{2.16}$$

where both constants c and c' are independent of ν.

Proof: We proceed in several steps.
1) We first note that, if $(\boldsymbol{u}, p)$ is a solution of problem (2.6) – (2.14), its part $\boldsymbol{u}$ belongs to $\mathbb{V}$ and satisfies

$$\forall \boldsymbol{v} \in \mathbb{V}, \quad a(\boldsymbol{u}, \boldsymbol{v}) + \tilde{N}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} - \langle p_2, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\Gamma_2}, \tag{2.17}$$

where the new trilinear form $\tilde{N}(\cdot,\cdot,\cdot)$ is defined by

$$\tilde{N}(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{w})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}.$$

We first investigate the existence of a solution for this problem.
2) Let us introduce the mapping Φ, defined from $\mathbb{V}$ into its dual space by

$$\langle \Phi(\boldsymbol{u}), \boldsymbol{v} \rangle = a(\boldsymbol{u}, \boldsymbol{v}) + \tilde{N}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \, d\boldsymbol{x} + \langle p_2, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\Gamma_2}.$$

By noting that $\tilde{N}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u})$ is zero, we derive by the same arguments as in Lemma 2.4

$$\langle \Phi(\boldsymbol{u}), \boldsymbol{u} \rangle \geq \alpha \, \|\boldsymbol{u}\|_{\mathbb{X}}^2 - c(\boldsymbol{f}, p_2) \, \|\boldsymbol{u}\|_{\mathbb{X}},$$

where the constant $c(\boldsymbol{f}, p_2) = \|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|p_2\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}$ only depends on the data. Thus, $\langle \Phi(\boldsymbol{u}), \boldsymbol{u} \rangle$ is nonnegative on the sphere with radius $\frac{c(\boldsymbol{f},p_2)}{\alpha}$ (note that α is equal to $c\,\nu$).
3) It follows from Lemma 2.7 that there exists an increasing sequence of finite-dimensional subspaces $\mathbb{V}_n$ of $\mathbb{V}$ such that $\cup_n \mathbb{V}_n$ is dense in $\mathbb{V}$. For any fixed n, the function Φ satisfies the same properties as previously on $\mathbb{V}_n$. So applying Brouwer's fixed point theorem (see [20, chap. IV, cor. 1.1] for instance) yields that there exists a $\boldsymbol{u}_n$ in $\mathbb{V}_n$ which satisfies :

$$\forall \boldsymbol{v}_n \in \mathbb{V}_n, \quad \langle \Phi(\boldsymbol{u}_n), \boldsymbol{v}_n \rangle = 0.$$

Moreover this $\boldsymbol{u}_n$ belongs to the ball with radius $\frac{c(\boldsymbol{f},p_2)}{\alpha}$.
4) Since the sequence $(\boldsymbol{u}_n)_n$ is bounded in $\mathbb{X}$, Assumption 2.8 implies that there exists a subsequence, still denoted by $(\boldsymbol{u}_n)_n$ for simplicity, which converges to a function $\boldsymbol{u}$ of $\mathbb{V}$ weakly in $\mathbb{X}$ and strongly in $L^4(\Omega)^d$. Moreover, due to the weak lower semi-continuity of the norm, the limit $\boldsymbol{u}$ still belongs to the ball with radius $\frac{c(\boldsymbol{f},p_2)}{\alpha}$, hence satisfies the first part of estimate (2.16).
5) For a fixed $m \leq n$, since the sequence $(\mathbb{V}_n)_n$ is increasing, each function $\boldsymbol{u}_n$ satisfies

$$\forall \boldsymbol{v}_m \in \mathbb{V}_m, \quad \langle \Phi(\boldsymbol{u}_n), \boldsymbol{v}_m \rangle = 0.$$

Then, passing to the limit on n follows from the previous convergence properties. Due to the density of $\cup_m \mathbb{V}_m$ into $\mathbb{V}$, it is thus readily checked that the function $\boldsymbol{u}$ satisfies

$$\forall \boldsymbol{v} \in \mathbb{V}, \quad \langle \Phi(\boldsymbol{u}), \boldsymbol{v} \rangle = 0,$$

hence is a solution of problem (2.17).
6) From the previous lines and thanks to the definition of $\mathbb{V}$, the quantity

$$\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \, d\boldsymbol{x} - \langle p_2, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\Gamma_2} - a(\boldsymbol{u}, \boldsymbol{v}) - N(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})$$

vanishes for all $\boldsymbol{v}$ in $\mathbb{V}$. So, it follows from Lemma 2.3, see [20, chap. I, lemma 4.1], that there exists a p in $L^2(\Omega)$ such that

$$\forall \boldsymbol{v} \in \mathbb{X}, \quad b(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \, d\boldsymbol{x} - \langle p_2, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\Gamma_2} - a(\boldsymbol{u}, \boldsymbol{v}) - N(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}).$$

Thus, the pair $(\boldsymbol{u}, p)$ is a solution of problem (2.6) – (2.14).
7) It also follows from Lemma 2.3 that

$$\|p\|_{L^2(\Omega)} \leq \beta^{-1} \sup_{\boldsymbol{v}\in\mathbb{X}} \frac{\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\,d\boldsymbol{x} - \langle p_2, \boldsymbol{v}\,\cdot\,\boldsymbol{n}\rangle_{\Gamma_2} - a(\boldsymbol{u}, \boldsymbol{v}) - N(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{\mathbb{X}}}.$$

Thanks to the estimate on $\boldsymbol{u}$, the quantity p satisfies the second part of (2.16).

It is readily checked that any solution $(\boldsymbol{u}, p)$ of problem (2.6) – (2.14) satisfies estimate (2.16). This yields the uniqueness of the solution, but unfortunately with a rather restrictive condition on the data.

Theorem 2.10. *Assume that the data $\boldsymbol{f}$ and p_2 satisfy* (2.15) *and moreover*

$$\frac{\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|p_2\|_{H^{\frac{1}{2}}_{00}(\Gamma_2)}}{\nu^2} \leq c, \tag{2.18}$$

for an appropriate constant c. Then, if Assumption 2.8 *holds, problem* (2.6) – (2.14) *for $\varepsilon = 1$ has at most a solution $(\boldsymbol{u}, p)$.*

Proof: Let $(\boldsymbol{u}_1, p_1)$ and $(\boldsymbol{u}_2, p_2)$ be two solutions of (2.6) – (2.14). Then, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ belong to $\mathbb{V}$ and their difference satisfies

$$\forall \boldsymbol{v} \in \mathbb{V}, \quad a(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{v}) = \tilde{N}(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{v}) - \tilde{N}(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{v}).$$

Next, taking $\boldsymbol{v}$ equal to $\boldsymbol{u}_1 - \boldsymbol{u}_2$ and noting that $\tilde{N}(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{v})$ vanishes for all $\boldsymbol{v}$, we obtain

$$\nu\,\|\mathbf{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} = \tilde{N}(\boldsymbol{u}_2 - \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2).$$

We recall that

$$\forall \boldsymbol{w} \in \mathbb{V}, \quad \|\boldsymbol{w}\|_{\mathbb{X}} \leq c\,\|\mathbf{curl}\,\boldsymbol{w}\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}},$$

so that using estimate (2.16) for $\boldsymbol{u}_2$ yields

$$\nu\,\|\mathbf{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} \leq \frac{c}{\nu}\left(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|p_2\|_{H^{\frac{1}{2}}_{00}(\Gamma_2)}\right)\|\mathbf{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}}.$$

Thus, when (2.18) is satisfied with a small enough constant c, $\mathbf{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)$ vanishes. It thus follows from [2, cor. 3.19] that, since both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are divergence-free, they coincide.
In this case, the functions p_1 and p_2 satisfy

$$\forall \boldsymbol{v} \in \mathbb{X}, \quad b(\boldsymbol{v}, p_1 - p_2) = 0,$$

so that, owing to Lemma 2.3, they coincide. This concludes the proof.

2.4. A final remark.

We consider once more problem (2.5) – (2.6) or (2.6) – (2.14) but now with the form $a(\cdot,\cdot)$ replaced by

$$a_\lambda(\boldsymbol{u},\boldsymbol{v}) = \nu \int_\Omega \big((\mathbf{curl}\,\boldsymbol{u})(\boldsymbol{x}) \,\cdot\, \mathbf{curl}\,\boldsymbol{v}(\boldsymbol{x}) + \lambda\,\mathrm{div}\,\boldsymbol{u}(\boldsymbol{x})\mathrm{div}\,\boldsymbol{v}(\boldsymbol{x})\big)\,d\boldsymbol{x}.$$

It is easy to check that, for a positive parameter λ, this modification does not change at all the problems and that all the previous results are still valid with the modified problems.

The main difference between the forms $a(\cdot,\cdot)$ and $a_\lambda(\cdot,\cdot)$ is that this new form satisfies the next stronger ellipticity property. The interest of this new property for the discretization is obvious: It leads to the stabilization of the divergence term.

Lemma 2.11. *For any positive parameter λ, there exists a constant $\alpha > 0$ such that the following ellipticity property holds*

$$\forall \boldsymbol{v} \in \mathbb{X}, \quad a_\lambda(\boldsymbol{v},\boldsymbol{v}) \geq \alpha\,\min\{1,\lambda\}\,\|\boldsymbol{v}\|^2_{\mathbb{X}}. \tag{2.19}$$

3. Discretization of the Stokes problem.

From now on, we assume that Ω is a polygon or a polyhedron. We introduce a regular family of triangulations of Ω (by triangles or tetrahedra), in the usual sense that, for each h,

- $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_h$;
- The intersection of two different elements of $\mathcal{T}_h$, if not empty, is a vertex or a whole edge or a whole face of both of them;
- The ratio of the diameter h_K of any element K of $\mathcal{T}_h$ to the diameter of its inscribed circle or sphere is smaller than a constant independent of h.

As usual, h stands for the maximum of the diameters h_K. We make the further and non restrictive assumption that $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are the union of whole edges ($d = 2$) or faces ($d = 3$) of elements of $\mathcal{T}_h$. From now on, c, c', ... stand for generic constants that can vary from line to line but are always independent of h.

3.1. The discrete problem and its well-posedness.

Setting

$$\mathbb{Y}_h = \Big\{ v_h \in H^1(\Omega);\ \forall K \in \mathcal{T}_h,\ v_h|_K \in \mathcal{P}_2(K) \Big\},$$

we define the space of discrete velocities

$$\mathbb{X}_h = \mathbb{Y}_h^d \cap \mathbb{X},$$

and the space of discrete pressures

$$\mathbb{M}_h = \Big\{ q_h \in H^1(\Omega);\ \forall K \in \mathcal{T}_h,\ q_h|_K \in \mathcal{P}_1(K) \Big\}.$$

Even if the following analysis is valid for general mixed finite elements, we have chosen this one, called Taylor–Hood element, see [22], which is highly used in the case of standard boundary conditions, we refer to [20, chap. II, section 4.2] for its main properties. We denote by $\mathcal{I}_h$ the standard Lagrange interpolation operator with values in $\mathbb{Y}_h$.

In view of Lemma 2.11, we have decided to work with $\lambda = 1$, i.e. with the form $a_1(\cdot, \cdot)$. The discrete problem is then constructed by the Galerkin method, it reads:

Find $(\boldsymbol{u}_h, p_h)$ *in* $\mathbb{Y}_h^d \times \mathbb{M}_h$ *such that*

$$\boldsymbol{u}_h = \mathcal{I}_h \boldsymbol{u}_1 \text{ on } \Gamma_1 \qquad \text{and} \qquad \boldsymbol{u}_h \times \boldsymbol{n} = \mathcal{I}_h \boldsymbol{u}_2 \times \boldsymbol{n} \text{ on } \Gamma_2, \tag{3.1}$$

and

$$\begin{aligned} &\forall \boldsymbol{v}_h \in \mathbb{X}_h, \quad a_1(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, d\boldsymbol{x} - \langle p_2, \boldsymbol{v}_h \cdot \boldsymbol{n} \rangle_{\Gamma_2}, \\ &\forall q_h \in \mathbb{M}_h, \quad b(\boldsymbol{u}_h, q_h) = 0. \end{aligned} \tag{3.2}$$

Proving its well-posedness relies on the same arguments as for the continuous problem, however a further assumption is required for the first inf-sup condition.

Assumption 3.1. At least an edge ($d=2$) or a face ($d=3$) of an element of $\mathcal{T}_h$ is contained in Γ_2.

This assumption is not restrictive at all since it is always true for h small enough and leads to the following lemma.

Lemma 3.2. *If Assumption* 3.1 *holds, there exists a constant* $\beta_* > 0$ *such that the following inf-sup condition holds*

$$\forall q_h \in \mathbb{M}_h, \quad \sup_{\boldsymbol{v}_h \in \mathbb{X}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\mathbb{X}}} \geq \beta_* \|q_h\|_{L^2(\Omega)}. \tag{3.3}$$

Proof: For any q_h in $\mathbb{M}_h$, we use the expansion

$$q_h = \tilde{q} + \overline{q}, \qquad \text{with} \quad \overline{q} = \frac{1}{\text{meas}(\Omega)} \int_\Omega q_h(\boldsymbol{x})\, d\boldsymbol{x}.$$

Next, we proceed in three steps.
1) Since $\tilde{q}$ has a null integral on Ω, the standard inf-sup condition, see [20, chap. II, thm 4.2] for instance, implies that there exists a function $\tilde{\boldsymbol{v}}$ in $\mathbb{Y}_h^d \cap H_0^1(\Omega)^d$, hence in $\mathbb{X}_h$, such that

$$\operatorname{div} \tilde{\boldsymbol{v}} = -\tilde{q} \quad \text{and} \quad \|\tilde{\boldsymbol{v}}\|_{\mathbb{X}} \leq c\, \|\tilde{q}\|_{L^2(\Omega)}. \tag{3.4}$$

2) Since $\overline{q}$ is a constant, we observe that, for any $\boldsymbol{v}$ in $\mathbb{X}$,

$$b(\boldsymbol{v}, \overline{q}) = -\overline{q} \int_{\Gamma_2} (\boldsymbol{v} \cdot \boldsymbol{n})(s)\, ds.$$

We introduce a function φ in $\mathscr{D}(\Omega \cup \Gamma_2)$ such that $\int_{\Gamma_2} \varphi(s)\, ds$ is a positive constant c_0. And we note that

$$\int_{\Gamma_2} \mathcal{I}_h \varphi(s)\, ds \geq \int_{\Gamma_2} \varphi(s)\, ds - \|\varphi - \mathcal{I}_h \varphi\|_{L^1(\Gamma_2)} \geq c_0 - c\, h^2,$$

so that it is larger than $\frac{c_0}{2}$ for h small enough (this requires Assumption 3.1). Now, we consider a regular extension $\boldsymbol{n}^*$ of $\boldsymbol{n}$ to Ω and we take $\overline{\boldsymbol{v}}$ equal to $-\overline{q}\, \mathcal{I}_h(\varphi\, \boldsymbol{n}^*)$, which gives

$$b(\overline{\boldsymbol{v}}, \overline{q}) \geq \frac{c_0}{2} \overline{q}^2 = \frac{c_0}{2\text{meas}\,(\Omega)} \|\overline{q}\|^2_{L^2(\Omega)} \quad \text{and} \quad \|\overline{\boldsymbol{v}}\|_{\mathbb{X}} \leq c\, \|\overline{q}\|_{L^2(\Omega)}. \tag{3.5}$$

3) We conclude by using the argument due to Boland and Nicolaides [10]. We take $\boldsymbol{v}_h$ equal to $\tilde{\boldsymbol{v}} + \mu\, \overline{\boldsymbol{v}}$ for a positive constant μ and, noting that $b(\tilde{\boldsymbol{v}}, \overline{q})$ is zero, we derive from (3.4) and (3.5) that

$$\begin{aligned} b(\boldsymbol{v}_h, q_h) &= b(\tilde{\boldsymbol{v}}, \tilde{q}) + \mu b(\overline{\boldsymbol{v}}, \overline{q}) + \mu b(\overline{\boldsymbol{v}}, \tilde{q}) \\ &\geq \|\tilde{q}\|^2_{L^2(\Omega)} + \frac{\mu c_0}{2\text{meas}\,(\Omega)} \|\overline{q}\|^2_{L^2(\Omega)} - c\mu\, \|\tilde{q}\|_{L^2(\Omega)} \|\overline{q}\|_{L^2(\Omega)}. \end{aligned}$$

Using a Young's inequality thus yields

$$b(\boldsymbol{v}_h, q_h) \geq \frac{1}{2}\|\tilde{q}\|^2_{L^2(\Omega)} + \mu\big(\frac{c_0}{2\text{meas}\,(\Omega)} - \frac{c^2\mu}{2}\big)\|\overline{q}\|^2_{L^2(\Omega)},$$

whence, by taking μ equal to $\frac{c_0}{2c^2\text{meas}\,(\Omega)}$ and using the orthogonality of $\tilde{q}$ and $\overline{q}$ in $L^2(\Omega)$,

$$b(\boldsymbol{v}_h, q_h) \geq c'\|q_h\|^2_{L^2(\Omega)}.$$

On the other hand, we have

$$\|\boldsymbol{v}_h\|_{\mathbb{X}} \leq \|\tilde{\boldsymbol{v}}\|_{\mathbb{X}} + \mu\,\|\overline{\boldsymbol{v}}\|_{\mathbb{X}} \leq c''\|q_h\|_{L^2(\Omega)}.$$

This yields the desired inf-sup condition.

From now on, we suppose that Assumption 3.1 holds. On the other hand, since $\mathbb{X}_h$ is imbedded in $\mathbb{X}$, the ellipticity of the form $a_1(\cdot,\cdot)$ is a direct consequence of Lemma 2.11. So, we now state the well-posedness result.

Theorem 3.3. *Assume that the data $\boldsymbol{f}$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and p_2 satisfy, for a real number $\sigma > \frac{d-1}{2}$,*

$$f \in L^2(\Omega)^d, \qquad \mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2) \in H^\sigma(\partial\Omega)^d, \qquad p_2 \in H^{\frac{1}{2}}_{00}(\Gamma_2). \tag{3.6}$$

Then, problem (3.1) – (3.2) *has a unique solution* $(\boldsymbol{u}_h, p_h)$*. Moreover, this solution satisfies*

$$\|\boldsymbol{u}_h\|_{\mathbb{X}} + \|p_h\|_{L^2(\Omega)} \leq c\,\big(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{H^\sigma(\partial\Omega)^d} + \|p_2\|_{H^{\frac{1}{2}}_{00}(\Gamma_2)}\big). \tag{3.7}$$

Proof: The lifting $\boldsymbol{w}$ of the trace $\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)$ introduced in the proof of Theorem 2.5 can now be chosen in $H^{\sigma+\frac{1}{2}}(\Omega)$, at least for σ small enough, hence is continuous on $\overline{\Omega}$. Thus standard arguments yield

$$\|\mathcal{I}_h\boldsymbol{w}\|_{H^1(\Omega)^d} \leq c\,\|\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{H^\sigma(\partial\Omega)^d}.$$

Writing the problem satisfied by $(\boldsymbol{u}_h - \mathcal{I}_h\boldsymbol{w}, p_h)$ and combining [20, chap. I, cor. 4.1] with Lemmas 2.11 and 3.2 imply that problem (3.1) – (3.2) has a unique solution. Then estimate (3.7) obviously follows.

3.2. A priori analysis.

Using the same lifting $\boldsymbol{w}$ as previously, we observe that the pair $(\boldsymbol{u}_{0h}, p_h)$, with $\boldsymbol{u}_{0h} = \boldsymbol{u}_h - \mathcal{I}_h\boldsymbol{w}$, is a solution in $\mathbb{X}_h \times \mathbb{M}_h$ of

$$\begin{aligned}
&\forall \boldsymbol{v}_h \in \mathbb{X}_h, \quad a_1(\boldsymbol{u}_{0h}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) \\
&\qquad\qquad = \int_\Omega \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{v}_h(\boldsymbol{x})\,d\boldsymbol{x} - \langle p_2, \boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_{\Gamma_2} - a_1(\mathcal{I}_h\boldsymbol{w}, \boldsymbol{v}_h), \\
&\forall q_h \in \mathbb{M}_h, \quad b(\boldsymbol{u}_{0h}, q_h) = -b(\mathcal{I}_h\boldsymbol{w}, q_h).
\end{aligned} \tag{3.8}$$

On the other hand, the pair $(\boldsymbol{u}_0, p_0)$, with $\boldsymbol{u}_0 = \boldsymbol{u} - \boldsymbol{w}$, is a solution of the analogous continuous problem (2.13) with $a(\cdot,\cdot)$ replaced by $a_1(\cdot,\cdot)$. So standard arguments, see [20, chap. II, thm 1.1], relying once more on Lemmas 2.11 and 3.2, yield the following version of the Strang lemma.

Lemma 3.4. *The following error estimate holds between the pairs* $(\boldsymbol{u}_0, p)$ *and* $(\boldsymbol{u}_{0h}, p_h)$

$$\begin{aligned}\|\boldsymbol{u}_0 - \boldsymbol{u}_{0h}\|_{\mathbb{X}} &+ \|p - p_h\|_{L^2(\Omega)} \\ &\le c\,\big(\inf_{\boldsymbol{v}_h \in \mathbb{X}_h} \|\boldsymbol{u}_0 - \boldsymbol{v}_h\|_{\mathbb{X}} + \inf_{q_h \in \mathbb{M}_h} \|p - q_h\|_{L^2(\Omega)}\big) + c'\,\|\boldsymbol{w} - \mathcal{I}_h \boldsymbol{w}\|_{\mathbb{X}}.\end{aligned} \tag{3.9}$$

By using the triangle inequality

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbb{X}} \le \|\boldsymbol{u}_0 - \boldsymbol{u}_{0h}\|_{\mathbb{X}} + \|\boldsymbol{w} - \mathcal{I}_h \boldsymbol{w}\|_{\mathbb{X}},$$

and the approximation properties of the spaces $\mathbb{X}_h$ and $\mathbb{M}_h$ together with that of $\mathcal{I}_h$ (see [9, chap. IX] for instance), we can now state the a priori estimate.

Theorem 3.5. *Assume that the data* $\boldsymbol{f}$*,* $\boldsymbol{u}_1$*,* $\boldsymbol{u}_2$ *and* p_2 *satisfy* (3.6) *for a real number* σ*,* $\frac{d-1}{2} < \sigma \le \frac{5}{2}$*, and that the solution* $(\boldsymbol{u}, p)$ *of problem* (2.5) – (2.6) *for* $\varepsilon = 0$ *belongs to* $H^{s+1}(\Omega)^d \times H^s(\Omega)$ *for a real number* s*,* $0 \le s \le 2$*. Then, the following a priori error estimate holds between this solution and the solution* $(\boldsymbol{u}_h, p_h)$ *of problem* (3.1) – (3.2)

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbb{X}} &+ \|p - p_h\|_{L^2(\Omega)} \\ &\le c\, h^s \big(\|\boldsymbol{u}\|_{H^{s+1}(\Omega)^d} + \|p\|_{H^s(\Omega)}\big) + c'\, h^{\sigma - \frac{1}{2}}\, \|\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{H^\sigma(\partial\Omega)^d}.\end{aligned} \tag{3.10}$$

Clearly, this estimate is fully optimal and, when combined with (3.7), proves the convergence of the discretization for all solutions $(\boldsymbol{u}, p)$. On the other hand, for a smooth solution $(\boldsymbol{u}, p)$, the error behaves like h^2, so that the method is of order 2.

3.3. A posteriori analysis.

This analysis requires some further notation: For each element K of $\mathcal{T}_h$,

- $\mathcal{E}_K$ stands for the set of edges ($d = 2$) or faces ($d = 3$) of K which are not contained in $\partial\Omega$;
- $\mathcal{E}_K^2$ stands for the set of edges ($d = 2$) or faces ($d = 3$) of K which are contained in $\overline{\Gamma}_2$;
- ω_K denotes the union of elements of $\mathcal{T}_h$ that share at least an edge ($d = 2$) or a face ($d = 3$) with K;
- for each e in $\mathcal{E}_K$, $[\cdot]_e$ denotes the jump through e (making its sign precise is not necessary in what follows);
- for each e in $\mathcal{E}_K$ or $\mathcal{E}_K^2$, h_e stands for the length ($d = 2$) or diameter ($d = 3$) of e.

We now intend to prove an a posteriori error estimate between the pairs $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ solutions of problems (2.5) – (2.6) and (3.1) – (3.2), respectively. The first residual equation now reads, for all $\boldsymbol{v}$ in $\mathbb{X}$ and $\boldsymbol{v}_h$ in $\mathbb{X}_h$,

$$\begin{aligned}a_1(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p - p_h) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot (\boldsymbol{v} - \boldsymbol{v}_h)(\boldsymbol{x})\, d\boldsymbol{x} - \langle p_2, (\boldsymbol{v} - \boldsymbol{v}_h) \cdot \boldsymbol{n} \rangle_{\Gamma_2} \\ - a_1(\boldsymbol{u}_h, \boldsymbol{v} - \boldsymbol{v}_h) - b(\boldsymbol{v} - \boldsymbol{v}_h, p_h).\end{aligned}$$

When integrating by parts on each element K of $\mathcal{T}_h$, this gives

$$
\begin{aligned}
&a_1(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})+b(\boldsymbol{v},p-p_h)\\
&=\sum_{K\in\mathcal{T}_h}\Big(\int_K\big(\boldsymbol{f}-\nu\,\mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}_h)+\nu\,\mathbf{grad}(\operatorname{div}\boldsymbol{u}_h)-\mathbf{grad}\,p_h\big)(\boldsymbol{x})\cdot(\boldsymbol{v}-\boldsymbol{v}_h)(\boldsymbol{x})\,d\boldsymbol{x}\\
&\qquad+\frac{1}{2}\sum_{e\in\mathcal{E}_K}\int_e\nu\big([\mathbf{curl}\,\boldsymbol{u}_h]_e(\tau)\cdot(\boldsymbol{v}-\boldsymbol{v}_h)\times\boldsymbol{n}(\tau)-[\operatorname{div}\boldsymbol{u}_h]_e(\tau)(\boldsymbol{v}-\boldsymbol{v}_h)\cdot\boldsymbol{n}(\tau)\big)\,d\tau\\
&\qquad\qquad+\sum_{e\in\mathcal{E}_K^2}\int_e(p_2-\nu\operatorname{div}\boldsymbol{u}_h-p_h)(\tau)(\boldsymbol{v}-\boldsymbol{v}_h)\cdot\boldsymbol{n}(\tau)\,d\tau\Big).
\end{aligned}
\tag{3.11}
$$

Fortunately, the second residual equation is much simpler. It reads, for any q in $L^2(\Omega)$,

$$b(\boldsymbol{u}-\boldsymbol{u}_h,q)=-b(\boldsymbol{u}_h,q). \tag{3.12}$$

To go further, we introduce an approximation $\boldsymbol{f}_h$ of $\boldsymbol{f}$ in $\mathbb{M}_h^d$ for instance and an approximation p_{2h} of p_2 which is continuous and affine on each edge ($d=2$) or face ($d=3$) contained in Γ_2. Thanks to equations (3.11) and (3.12), we are now in a position to define the error indicators. They read, for each K in $\mathcal{T}_h$,

$$
\begin{aligned}
\eta_K&=h_K\,\|\boldsymbol{f}_h-\nu\,\mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}_h)-\mathbf{grad}\,p_h\|_{L^2(K)^d}+\|\operatorname{div}\boldsymbol{u}_h\|_{L^2(K)}\\
&\qquad+\sum_{e\in\mathcal{E}_K}h_e^{\frac{1}{2}}\,\|[\mathbf{curl}\,\boldsymbol{u}_h]_e\|_{L^2(e)^{\frac{d(d-1)}{2}}}+\sum_{e\in\mathcal{E}_K^2}h_e^{\frac{1}{2}}\,\|p_{2h}-p_h\|_{L^2(e)}.
\end{aligned}
\tag{3.13}
$$

These indicators are very easy to compute since they only involve polynomials of low degree.

Remark 3.6. The term due to the jump of $\mathbf{curl}\,\boldsymbol{u}_h$ in the indicator η_K defined by (3.13) may be simplified to

$$\sum_{e\in\mathcal{E}_K}h_e^{1/2}\,\|[\partial_n\boldsymbol{u}_{ht}\|_{L^2(e)^{\frac{d(d-1)}{2}}},$$

where ∂_n denotes the normal derivative and $\boldsymbol{u}_{ht}$ are the tangential components of the velocity $\boldsymbol{u}_h$ on e. This occurs because in (3.11) we have

$$[\mathbf{curl}\,\boldsymbol{u}_h\times\boldsymbol{n}]_e=[(\mathbf{curl}\,\boldsymbol{u}_h\times\boldsymbol{n})_t]_e=[\partial_n\boldsymbol{u}_{h\tau}]_e\quad\text{on } e,$$

where the second equality holds because the tangential derivatives do not jump across e. Similarly the term $\boldsymbol{f}_h-\nu\,\mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}_h)-\mathbf{grad}\,p_h$ can be replaced by $\boldsymbol{f}_h+\nu\Delta\boldsymbol{u}_h-\mathbf{grad}\,p_h$. With these modifications, it is may be easier to see that these indicators are of residual type (which means that, when suppressing the indices h, they vanish). However the expression for the **curl** term in (3.13) leads to an easier computation in practice.

We are now in a position to state the a posteriori error estimate. For this, we introduce a neighbourhood $\mathcal{V}$ of the reentrant comers and edges in Γ_2 and set

$$s_K=\begin{cases}\frac{1}{2} & \text{if } K\subset\mathcal{V},\\ 0 & \text{otherwise.}\end{cases} \tag{3.14}$$

Theorem 3.7. *The following a posteriori error estimate holds between the solution* $(\boldsymbol{u}, p)$ *of problem* (2.5) – (2.6) *for* $\varepsilon = 0$ *and the solution* $(\boldsymbol{u}_h, p_h)$ *of problem* (3.1) – (3.2)

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbb{X}} + \|p - p_h\|_{L^2(\Omega)} \leq c\Big(\sum_{K \in \mathcal{T}_h} h_K^{-2s_K} \eta_K^2\Big)^{\frac{1}{2}} + \varepsilon_h, \tag{3.15}$$

where the quantity ε_h *is defined by*

$$\varepsilon_h = \Big(\sum_{K \in \mathcal{T}_h} \big(h_K^{2(1-s_K)} \|\boldsymbol{f} - \boldsymbol{f}_h\|^2_{L^2(K)^d} + \sum_{e \in \mathcal{E}_K^2} h_e^{1-2s_K} \|p_2 - p_{2h}\|^2_{L^2(e)}\big)\Big)^{\frac{1}{2}} \\ + \|\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2) - \mathcal{I}_h \mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)\|_{H^{\frac{1}{2}}(\partial\Omega)^d}. \tag{3.16}$$

Proof: We observe from (3.11) and (3.12) that the pair $(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)$ is a solution of problem (2.5) – (2.6) with data equal to the right-hand side $\mathcal{R}$ of (3.11), the quantity $\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2) - \mathcal{I}_h \mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)$ and the right-hand side of (3.12). Thus, estimate (3.15) will follow by applying estimate (2.12) to this new problem . The quantity $\mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2) - \mathcal{I}_h \mathcal{C}(\boldsymbol{u}_1, \boldsymbol{u}_2)$ and the right-hand side of (3.12) are obviously bounded. To evaluate $\mathcal{R}$, we apply a Cauchy–Schwartz inequality, take $\boldsymbol{v}_h$ equal to the image of $\boldsymbol{v}$ by a Clément type regularization operator $\mathcal{R}_h$ with values in $\mathbb{X}_h$ and recall from [9, section IX.3] or [34, prop. 3.33] that, for any $s \geq \frac{1}{2}$ and for any e in $\mathcal{E}_K$ or in $\mathcal{E}_K^2$

$$\|\boldsymbol{v} - \mathcal{R}_h \boldsymbol{v}\|_{L^2(K)^d} \leq c\, h_K^s \|\boldsymbol{v}\|_{H^s(\omega_K)}, \qquad \|\boldsymbol{v} - \mathcal{R}_h \boldsymbol{v}\|_{L^2(e)^d} \leq c\, h_e^{s-\frac{1}{2}} \|\boldsymbol{v}\|_{H^s(\omega_K)}.$$

To conclude, we note from Remark 2.1 that functions $\boldsymbol{v}$ in $\mathbb{X}$ belongs to $H^1(\Omega \setminus \mathcal{V})$ but only to $H^{\frac{1}{2}}(\mathcal{V})$ and we get rid of the further terms involving $\operatorname{div} \boldsymbol{u}_h$ by using the inverse inequalities [9, chap. VII, prop. 4.1] [34, prop. 3.37]

$$h_K \|\mathbf{grad}(\operatorname{div} \boldsymbol{u}_h)\|_{L^2(K)^d} \leq c\, \|\operatorname{div} \boldsymbol{u}_h\|_{L^2(K)}, \quad h_e^{\frac{1}{2}} \|\operatorname{div} \boldsymbol{u}_h\|_{L^2(e)} \leq c\, \|\operatorname{div} \boldsymbol{u}_h\|_{L^2(K)}. \tag{3.17}$$

All this yields the desired estimate.

Estimate (3.15) is optimal when the domain Ω is convex in a neighbourhood of Γ_2. Moreover the lack of optimality in the general case is local, limited to $\mathcal{V}$, and exactly the same was noted in [8, prop. 5.3] for another type of mixed boundary conditions. We now prove a local upper bound for the indicators. For each K in $\mathcal{T}_h$, we denote by $\|\cdot\|_{\mathbb{X}(K)}$ the restriction of the norm $\|\cdot\|_{\mathbb{X}}$ to K, with obvious extension to ω_K.

Proposition 3.8. *Each indicator* η_K, $K \in \mathcal{T}_h$, *defined in* (3.13) *satisfies*

$$\eta_K \leq c\,(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbb{X}(\omega_K)} + \|p - p_h\|_{L^2(\omega_K)} + \varepsilon_K), \tag{3.18}$$

where the quantity ε_K *is defined by*

$$\varepsilon_K = h_K \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(\omega_K)^d} + \sum_{e \in \mathcal{E}_K^2} h_e^{\frac{1}{2}} \|p_2 - p_{2h}\|_{L^2(e)}. \tag{3.19}$$

Proof: Since the arguments are fully standard, we only give an abridged version of the proof. We bound successively the four terms in η_K.
1) We set:

$$\boldsymbol{v}_K = \begin{cases} \big(\boldsymbol{f}_h - \nu\,\mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}_h) + \nu\,\mathbf{grad}(\operatorname{div}\boldsymbol{u}_h) - \mathbf{grad}\,p_h\big)\psi_K & \text{on } K, \\ \mathbf{0} & \text{elsewhere,} \end{cases}$$

where ψ_K is the bubble function on K (equal to the product of the barycentric coordinates associated with the vertices of K). Next, we take $\boldsymbol{v}$ equal to $\boldsymbol{v}_K$ and $\boldsymbol{v}_h$ equal to zero in (3.11). Standard inverse inequalities (see [34, prop. 3.37]) lead to

$$\begin{aligned} &h_K\,\|\boldsymbol{f}_h - \nu\,\mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}_h) + \nu\,\mathbf{grad}(\operatorname{div}\boldsymbol{u}_h) - \mathbf{grad}\,p_h\|_{L^2(K)^d} \\ &\qquad\qquad \le c\,(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbb{X}(K)} + \|p-p_h\|_{L^2(K)} + h_K\|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(K)^d}), \end{aligned}$$

or, equivalently, by using (3.17),

$$\begin{aligned} &h_K\,\|\boldsymbol{f}_h - \nu\,\mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}_h) - \mathbf{grad}\,p_h\|_{L^2(K)^d} \\ &\quad \le c\,(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbb{X}(K)} + \|p-p_h\|_{L^2(K)} + h_K\|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(K)^d}) + c'\,\|\operatorname{div}\boldsymbol{u}_h\|_{L^2(K)}. \end{aligned} \tag{3.20}$$

2) We set:

$$q_K = \begin{cases} (\operatorname{div}\boldsymbol{u}_h)\chi_K & \text{on } K, \\ 0 & \text{elsewhere,} \end{cases}$$

where χ_K is the characteristic function of K. Taking q equal to q_K in (3.12) gives

$$\|\operatorname{div}\boldsymbol{u}_h\|_{L^2(K)} \le \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbb{X}(K)}. \tag{3.21}$$

Combining (3.20) and (3.21) gives the estimate for the first two terms in η_K.
3) For each edge ($d=2$) or face ($d=3$) e of K, we consider a lifting operator $\mathcal{L}_{e,K}$ that maps polynomials of fixed degree on e vanishing on ∂e into polynomials vanishing on $\partial K \setminus e$ and is constructed from a fixed lifting operator on the reference triangle or tetrahedron. If an element e of $\mathcal{E}_K$ is shared by two elements K and K', we set:

$$\boldsymbol{v}_e = \begin{cases} \mathcal{L}_{e,\kappa}\big([\mathbf{curl}\,\boldsymbol{u}_h]_e\psi_e\big) & \text{on } \kappa \in \{K, K'\}, \\ \mathbf{0} & \text{elsewhere,} \end{cases}$$

where ψ_e is now the bubble function on e. We take $\boldsymbol{v}$ equal to $\tilde{\boldsymbol{v}}_e$ and $\boldsymbol{v}_h$ equal to zero in (3.11), where $\tilde{\boldsymbol{v}}_e$ is such that

$$\tilde{\boldsymbol{v}}_e \times \boldsymbol{n} = \boldsymbol{v}_e \times \boldsymbol{n} \quad \text{and} \quad \tilde{\boldsymbol{v}}_e \cdot \boldsymbol{n} = 0 \quad \text{on } e.$$

Standard arguments [34, prop. 3.37], combined with (3.20) and (3.21), yield

$$\begin{aligned} &h_e^{\frac{1}{2}}\,\|[\mathbf{curl}\,\boldsymbol{u}_h]_e\|_{L^2(e)^{\frac{d(d-1)}{2}}} \\ &\qquad \le c\,(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbb{X}(K\cup K')} + \|p-p_h\|_{L^2(K\cup K')} + h_K\|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(K\cup K')^d}). \end{aligned} \tag{3.22}$$

4) For each e in $\mathcal{E}_K^2$, we set:

$$\boldsymbol{v}_e = \begin{cases} \mathcal{L}_{e,K}\big((p_{2h} - p_h)\boldsymbol{n}\psi_e\big) & \text{on } K, \\ \boldsymbol{0} & \text{elsewhere.} \end{cases}$$

We finally take $\boldsymbol{v}$ equal to $\boldsymbol{v}_e$ and $\boldsymbol{v}_h$ equal to zero in (3.11), which gives

$$\begin{aligned} & h_e^{\frac{1}{2}} \|p_{2h} - p_h\|_{L^2(e)} \\ & \leq c\,\big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbb{X}(K)} + \|p - p_h\|_{L^2(K)} + h_K \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(K)^d} + h_e^{\frac{1}{2}}\,\|p_2 - p_{2h}\|_{L^2(e)}\big). \end{aligned} \tag{3.23}$$

Owing to the definition (3.19) of ε_K, estimate (3.18) follows from (3.20) to (3.23).

Estimate (3.18) is fully optimal. Moreover it is local, which proves the efficiency of our indicators for mesh adaptivity.

4. Discretization of the Navier–Stokes equations.

We use here all the notation of Secion 3. We write the nonlinear discrete problem. Next, we prove simultaneously the existence of a solution and the a priori error estimate by following the approach due to Brezzi, Rappaz and Raviart [12]. We conclude by extending the results of a posteriori analysis to the nonlinear case.

4.1. The discrete problem.

As previously, the discrete problem associated with problem (2.6) – (2.14) (for $\varepsilon = 1$) is constructed by the Galerkin method. It reads

Find $(\boldsymbol{u}_h, p_h)$ *in* $\mathbb{X}_h \times \mathbb{M}_h$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v}_h \in \mathbb{X}_h, \quad a_1(\boldsymbol{u}_h, \boldsymbol{v}_h) + N(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) \\ &\qquad\qquad = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, d\boldsymbol{x} - \langle p_2, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\Gamma_2}, \\ &\forall q_h \in \mathbb{M}_h, \quad b(\boldsymbol{u}_h, q_h) = 0. \end{aligned} \tag{4.1}$$

The existence of a solution for this problem can be proved by the same arguments as in Section 2.3. However, we prefer to perform directly its numerical analysis.

4.2. A priori analysis.

We now introduce a different notation. Let $\mathcal{S}$ denote the operator which associates with $(\boldsymbol{f}, p_2)$ in $L^2(\Omega)^d \times H_{00}^{\frac{1}{2}}(\Gamma_2)$, the solution $(\boldsymbol{u}, p)$ of problem (2.6) – (2.14) with $\varepsilon = 0$, namely of the Stokes problem with zero boundary conditions on the velocity. Then, problem (2.6) – (2.14) with $\varepsilon = 1$, can equivalently be written

$$\mathcal{F}(\boldsymbol{u}, p) = (\boldsymbol{u}, p) - \mathcal{S}(\boldsymbol{g}(\boldsymbol{u}), p_2) = 0, \tag{4.2}$$

where the function $\boldsymbol{g}$ is defined by duality

$$\langle \boldsymbol{g}(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} - N(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}). \tag{4.3}$$

Similarly, let $\mathcal{S}_h$ denote the operator which associates with $(\boldsymbol{f}, p_2)$ in $L^2(\Omega)^d \times H_{00}^{\frac{1}{2}}(\Gamma_2)$, the solution $(\boldsymbol{u}_h, p_h)$ of problem (3.1) – (3.2) with zero boundary conditions $\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{0}$ on the velocity, more precisely of

Find $(\boldsymbol{u}_h, p_h)$ *in* $\mathbb{X}_h \times \mathbb{M}_h$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v}_h \in \mathbb{X}_h, \quad a_1(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, d\boldsymbol{x} - \langle p_2, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\Gamma_2}, \\ &\forall q_h \in \mathbb{M}_h, \quad b(\boldsymbol{u}_h, q_h) = 0. \end{aligned} \tag{4.4}$$

Then, problem (4.1) can equivalently be written

$$\mathcal{F}_h(\boldsymbol{u}_h, p_h) = (\boldsymbol{u}_h, p_h) - \mathcal{S}_h(\boldsymbol{g}(\boldsymbol{u}_h), p_2) = 0. \tag{4.5}$$

Denoting by $\mathcal{Z}$ the space $\mathbb{X} \times L^2(\Omega)$, we recall from Theorems 3.3 and 3.5 the main properties of the operators $\mathcal{S}_h$: its stability

$$\|\mathcal{S}_h(\boldsymbol{f}, p_2)\|_{\mathcal{Z}} \le c \left(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|p_2\|_{H^{\frac{1}{2}}_{00}(\Gamma_2)} \right), \tag{4.6}$$

and the error estimate, for a smooth enough solution $\mathcal{S}(\boldsymbol{f}, p_2)$ and $s \le 2$,

$$\|(\mathcal{S} - \mathcal{S}_h)(\boldsymbol{f}, p_2)\|_{\mathcal{Z}} \le c h^s \, \|\mathcal{S}(\boldsymbol{f}, p_2)\|_{H^{s+1}(\Omega)^d \times H^s(\Omega)}. \tag{4.7}$$

All this gives the convergence property, for any $(\boldsymbol{f}, p_2)$ in $L^2(\Omega)^d \times H^{\frac{1}{2}}_{00}(\Gamma_2)$,

$$\lim_{h \to 0} \|(\mathcal{S} - \mathcal{S}_h)(\boldsymbol{f}, p_2)\|_{\mathcal{Z}} = 0. \tag{4.8}$$

Due to Lemma 3.4, this convergence easily extends to data $(\boldsymbol{f}, p_2)$ in $\mathbb{X}' \times H^{\frac{1}{2}}_{00}(\Gamma_2)$, where $\mathbb{X}'$ stands for the dual space of $\mathbb{X}$.

We are thus in a position to prove some preliminary lemmas. As usual, they require a further assumption.

Assumption 4.1. We consider a solution $(\boldsymbol{u}, p)$ of problem (2.6) – (2.14) with $\varepsilon = 1$
(i) which belongs to $H^{s+1}(\Omega)^d \times H^s(\Omega)$ for a real number s, $0 < s \le 2$,
(ii) is such that $D\mathcal{F}(\boldsymbol{u}, p)$ is an isomorphism of $\mathcal{Z}$ (where D denotes the differential operator).

This assumption is much weaker than the uniqueness of the solution established in Theorem 2.10, since part (ii) of it only implies the local uniqueness of the solution. We denote by $\mathcal{L}(\mathcal{Z})$ the space of endomorphisms of $\mathcal{Z}$.

Lemma 4.2. *If Assumptions* 2.8 *and* 4.1 *hold, there exists a* $h_0 > 0$ *such that, for* $h \le h_0$, $D\mathcal{F}_h(\boldsymbol{u}, p)$ *is an isomorphism of* $\mathcal{Z}$ *and the norm of its inverse is bounded independently of* h.

Proof: We use the expansion

$$D\mathcal{F}_h(\boldsymbol{u}, p) = D\mathcal{F}(\boldsymbol{u}, p) + (\mathcal{S} - \mathcal{S}_h)(D\boldsymbol{g}(\boldsymbol{u}), 0).$$

Indeed, thanks to part (ii) of Assumption 4.1, it suffices to check that the quantity $\|(\mathcal{S} - \mathcal{S}_h)(D\boldsymbol{g}(\boldsymbol{u}), 0)\|_{\mathcal{L}(\mathcal{Z})}$ tends to zero when h tends to zero. Next, we observe that, for any $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{X}$,

$$\langle D\boldsymbol{g}(\boldsymbol{u})\boldsymbol{w}, \boldsymbol{v} \rangle = -N(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) - N(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}).$$

So, owing to Assumption 2.8, this convergence is an obvious consequence of (4.8).

Lemma 4.3. *If Assumption* 2.8 *holds, there exist a neighbourhood* $\mathcal{V}$ *of* $(\boldsymbol{u}, p)$ *in* $\mathcal{Z}$ *and a constant* $\Lambda > 0$ *such that the mapping* $D\mathcal{F}_h$ *satisfies the following Lipschitz property*

$$\forall(\boldsymbol{v}, q) \in \mathcal{V}, \quad \|D\mathcal{F}_h(\boldsymbol{u}, p) - D\mathcal{F}_h(\boldsymbol{v}, q)\|_{\mathcal{L}(\mathcal{Z})} \leq \Lambda \, \|(\boldsymbol{u}, p) - (\boldsymbol{v}, q)\|_{\mathcal{Z}}. \tag{4.9}$$

Proof: Since the nonlinearity that we consider is quadratic, choosing $\mathcal{V}$ bounded and using (4.6) give the desired result.

We now set:

$$E_h = \|\mathcal{F}_h(\boldsymbol{u}, p)\|_{\mathcal{Z}}.$$

Due to equation (4.2), bounding E_h is a direct consequence of (4.7).

Lemma 4.4. *If Assumption* 4.1 *holds, the quantity* E_h *satisfies the following bound*

$$E_h \leq c(\boldsymbol{u}, p)\, h^s, \tag{4.10}$$

for a constant $c(\boldsymbol{u}, p)$ *only depending on the regularity of* $(\boldsymbol{u}, p)$.

Owing to Lemmas 4.2 to 4.4, all the assumptions needed for [12, thm 1] (see also [20, chap. IV, thm 3.1]) are satisfied. So applying this theorem leads to the main result of this section.

Theorem 4.5. *If Assumptions* 2.8 *and* 4.1 *hold, there exist a* $h_* > 0$ *and a neighbourhood* $\mathcal{V}_*$ *of* $(\boldsymbol{u}, p)$ *in* $\mathcal{Z}$ *such that, for* $h \leq h_*$, *problem* (4.1) *has a unique solution* $(\boldsymbol{u}_h, p_h)$ *in* $\mathcal{V}_*$. *Moreover, the following a priori error estimate is satisfied*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbb{X}} + \|p - p_h\|_{L^2(\Omega)} \leq c(\boldsymbol{u}, p)\, h^s, \tag{4.11}$$

for a constant $c(\boldsymbol{u}, p)$ *only depending on* $(\boldsymbol{u}, p)$.

4.3. A posteriori analysis.

The second residual equation (3.12) is the same as in the linear case but unfortunately the first residual equation is a little more complex. After integration by parts on each K, it reads for all $\boldsymbol{v}$ in $\mathbb{X}$ and $\boldsymbol{v}_h$ in $\mathbb{X}_h$,

$$a_1(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) + N(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - N(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p - p_h) = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3, \tag{4.12}$$

where

$$\begin{aligned}
\mathcal{R}_1 &= \sum_{K \in \mathcal{T}_h} \Big(\int_K \big(\boldsymbol{f} - \nu \mathbf{curl}(\mathbf{curl}\, \boldsymbol{u}_h) + \nu \mathbf{grad}(\mathrm{div}\, \boldsymbol{u}_h) \\
&\qquad\qquad - (\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h - \mathbf{grad}\, p_h\big)(\boldsymbol{x}) \cdot (\boldsymbol{v} - \boldsymbol{v}_h)(\boldsymbol{x})\, d\boldsymbol{x} \\
\mathcal{R}_2 &= \frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} \int_e \nu \big([\mathbf{curl}\, \boldsymbol{u}_h]_e(\tau) \cdot (\boldsymbol{v} - \boldsymbol{v}_h) \times \boldsymbol{n}(\tau) - [\mathrm{div}\, \boldsymbol{u}_h]_e(\tau)(\boldsymbol{v} - \boldsymbol{v}_h) \cdot \boldsymbol{n}(\tau)\big)\, d\tau \\
\mathcal{R}_3 &= \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K^2} \int_e (p_2 - \nu \mathrm{div}\, \boldsymbol{u}_h - p_h - \frac{1}{2}|\boldsymbol{u}_h|^2)(\tau)(\boldsymbol{v} - \boldsymbol{v}_h) \cdot \boldsymbol{n}(\tau)\, d\tau.
\end{aligned} \tag{4.13}$$

This leads to the definition of the error indicators: For each K in $\mathcal{T}_h$, with the same notation as previously,

$$\eta_K = h_K \,\|\boldsymbol{f}_h - \nu\,\mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}_h) - (\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h - \mathbf{grad}\,p_h\|_{L^2(K)^d} + \|\mathrm{div}\,\boldsymbol{u}_h\|_{L^2(K)} + \sum_{e \in \mathcal{E}_K} h_e^{\frac{1}{2}} \|[\mathbf{curl}\,\boldsymbol{u}_h]_e\|_{L^2(e)^{\frac{d(d-1)}{2}}} + \sum_{e \in \mathcal{E}_K^2} h_e^{\frac{1}{2}} \|p_{2h} - p_h - \frac{1}{2}|\boldsymbol{u}_h|^2\|_{L^2(e)}. \tag{4.14}$$

Even if the nonlinear terms add polynomials of higher degree, these indicators are still easy to compute.

In order to apply the theorem due to Pousin and Rappaz [30], we need a further notation: Let $\mathcal{S}^*$ denote the operator which associates with $(\boldsymbol{f}, \chi, p_2)$ in $L^2(\Omega)^d \times L^2(\Omega) \times H_{00}^{\frac{1}{2}}(\Gamma_2)$, the solution $(\boldsymbol{u}, p)$ in $\mathbb{X} \times L^2(\Omega)$ of the problem

$$\begin{aligned} &\forall \boldsymbol{v} \in \mathbb{X}, \quad a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} - \langle p_2, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\Gamma_2}, \\ &\forall q \in L^2(\Omega), \quad b(\boldsymbol{u}, q) = \int_\Omega \chi(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned} \tag{4.15}$$

(the introduction of this more complex operator is due to the fact that the right-hand side of (3.12) is not zero). Then, problem (2.6) – (2.14) with $\varepsilon = 1$, can equivalently be written

$$\mathcal{F}^*(\boldsymbol{u}, p) = (\boldsymbol{u}, p) - \mathcal{S}^*(\boldsymbol{g}(\boldsymbol{u}), 0, p_2) = 0, \tag{4.16}$$

We are now in a position to prove the a posteriori error estimate.

Theorem 4.6. *For any solution $(\boldsymbol{u}, p)$ of problem (2.6) – (2.14) with $\varepsilon = 1$ such that $D\mathcal{F}^*(\boldsymbol{u}, p)$ is an isomorphism of $\mathcal{Z}$, there exists a neighbourhood $\mathcal{V}_{**}$ of $(\boldsymbol{u}, p)$ in $\mathcal{Z}$ such that the following a posteriori error estimate is satisfied for any solution $(\boldsymbol{u}_h, p_h)$ of problem (4.1) in $\mathcal{V}_{**}$*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbb{X}} + \|p - p_h\|_{L^2(\Omega)} \le c \Big(\sum_{K \in \mathcal{T}_h} h_K^{-2s_K} \eta_K^2 \Big)^{\frac{1}{2}} + \varepsilon_h, \tag{4.17}$$

where the parameter s_K is defined in (3.14) and the quantity ε_h in (3.16).

Proof: The same arguments as for Lemma 4.3 imply that $D\mathcal{F}^*$ is Lipschitz-continuous in a neighbourhood of $(\boldsymbol{u}, p)$. So we apply the theorem due to Pousin and Rappaz [30] (see also [34, prop. 5.1]): Any solution of problem (4.1) in this neighbourhood satisfies

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbb{X}} + \|p - p_h\|_{L^2(\Omega)} \le \|\mathcal{F}^*(\boldsymbol{u}_h, p_h)\|_{\mathcal{Z}}.$$

whence, due to (4.16),

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbb{X}} + \|p - p_h\|_{L^2(\Omega)} \le \|\mathcal{F}^*(\boldsymbol{u}, p) - \mathcal{F}^*(\boldsymbol{u}_h, p_h)\|_{\mathcal{Z}}.$$

Due to the stability property of $\mathcal{S}^*$, estimating the right-hand side of this equation relies on equations (4.12) and (3.12) and is performed by the same arguments as for Theorem 3.7.

To prove the converse estimate, we observe that

$$N(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - N(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}) = N(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{v}) + N(\boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}).$$

So, when working with bounded $\boldsymbol{u}$ and $\boldsymbol{u}_h$, proving the next proposition relies on exactly the same arguments as for Proposition 3.8, now applied to equations (4.12) and (3.12).

Proposition 4.7. *For any solution* $(\boldsymbol{u}_h, p_h)$ *of problem* (4.1) *in a neighbourhood of* $(\boldsymbol{u}, p)$, *each indicator* η_K, $K \in \mathcal{T}_h$, *defined in* (4.14) *satisfies*

$$\eta_K \leq c\,(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbb{X}(\omega_K)} + \|p - p_h\|_{L^2(\omega_K)} + \varepsilon_K), \tag{4.18}$$

where the quantity ε_K *is defined in* (3.19).

There also, this estimate is fully optimal.

5. Numerical experiments.

The next computations are performed on the code FreeFem++ due to F. Hecht and O.Pironneau, see [21]. We start from a coarse initial mech and perform adaptivity following the next criterion: The diameter of each new triangle containing an element K or contained in an element K of the old triangulation is proportional to $h_K \frac{\overline{\eta}}{\eta_K}$, where $\overline{\eta}$ is the mean value of the η_K.

We work with the Navier-Stokes equations for the data $\boldsymbol{f} = \boldsymbol{0}$. So we use the following iterative algorithm to treat the nonlinear term: Assuming that the solution of the time-dependent problem with time-independent data converges to the solution $(\boldsymbol{u}_h, p_h)$ of our problem, we solve the time-dependent problem via an implicit Euler's scheme where the nonlinear term is treated in a semi-explicit way. On each mesh, we iterate this algorithm till its convergence, i.e. till the difference between two consecutive solutions becomes smaller than a fixed tolerance.

First, we consider the two-dimensional domain made of two pipes, see Figure 1. Let P_1 be the horizontal pipe and P_2 the vertical one. The boundary Γ_2 is made of the vertical edge of P_1 (on the left) and of the two horizontal edges of P_2, while Γ_1 is equal to $\partial\Omega \setminus \overline{\Gamma}_2$.

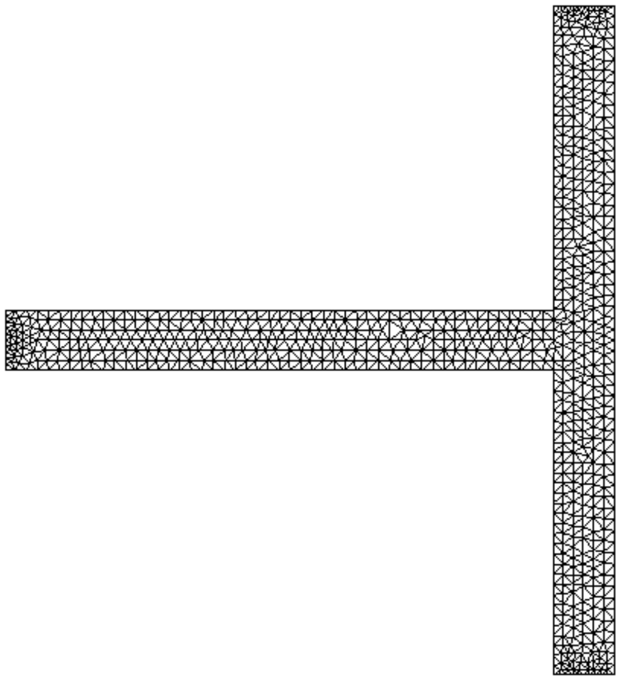

Figure 1. The domain Ω and its initial mesh.

We take the viscosity ν equal to 0.025. The geometry and the data are similar to those suggested in [4, Section 3.4.1], in particular the data on the velocity are zero as in (2.14) and the data on the pressure are a constant on each connected component of Γ_2 (see Remark 2.6 for the justification of that).

In the first test case, the constants on the two edges of P_2 are equal, so that, since the viscosity ν is large enough, the flow remains symmetric. More precisely and with obvious notation, these constants are given by

$$c_1 = 0, \quad c_{2-} = c_{2+} = -2.$$

Figure 2 presents a zoom of the final adapted mesh near the re-entrant corners. Figure 3 illistrates the velocity $\boldsymbol{u}_h$ and the pressure p_h on this last mesh.

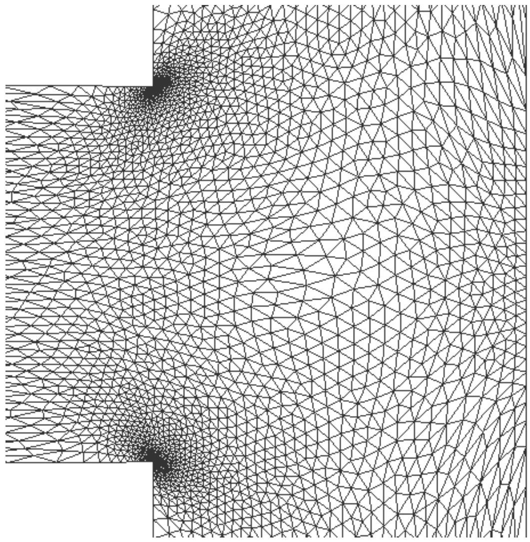

Figure 2. Zoom of the adapted mesh.

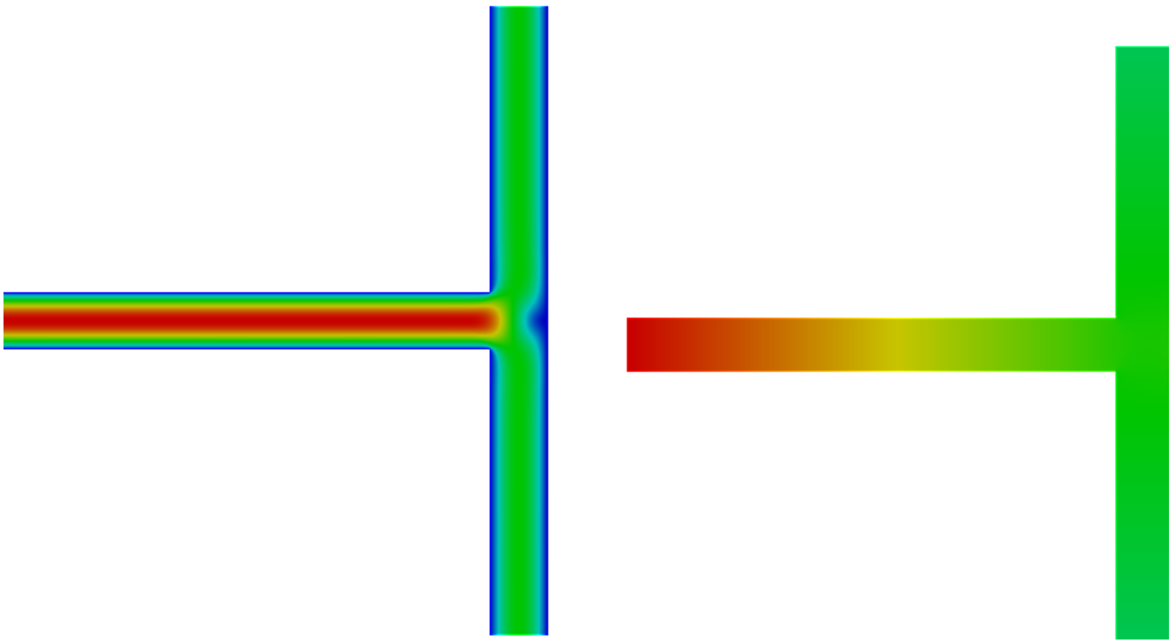

Figure 3. The discrete velocity $\boldsymbol{u}_h$ (left part) and pressure p_h (right part).

In the second test case, the data are the same but the constants on the two edges of P_2 are rather different, given by

$$c_1 = 0, \quad c_{2-} = -4, \quad c_{2+} = -2.$$

Figure 4 presents a zoom of the final adapted mesh. Figure 5 illistrates the velocity $\boldsymbol{u}_h$ and the pressure p_h for these new values. All these results are in good coherence with [4, Figures 3.2 & 3.3].

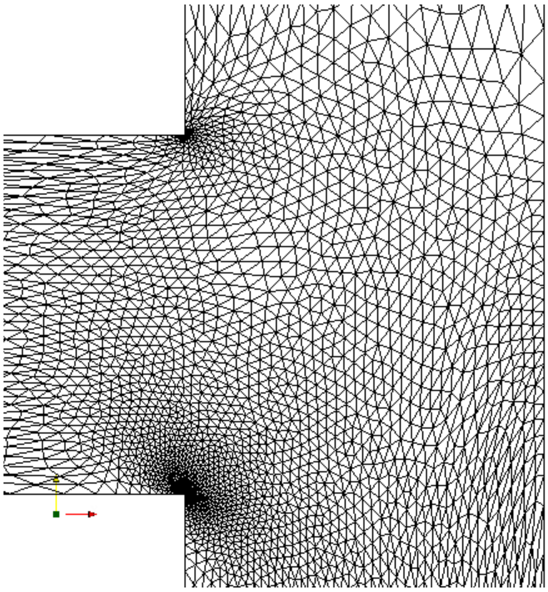

Figure 4. Zoom of the adapted mesh.

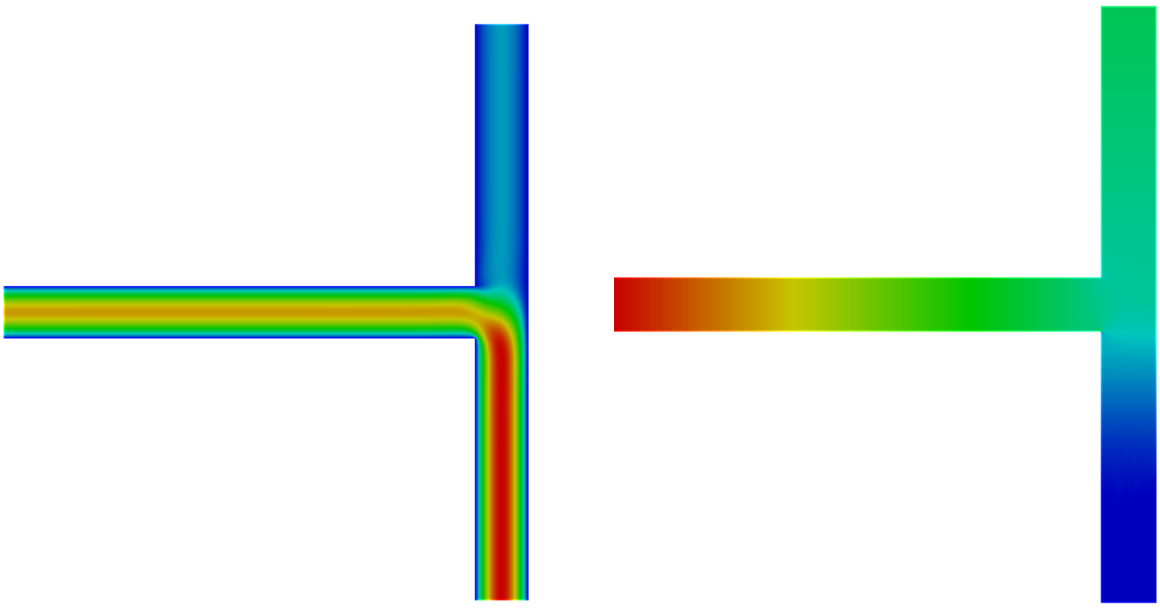

Figure 5. The discrete velocity $\boldsymbol{u}_h$ (left part) and pressure p_h (right part).

Next, we study the case of a flow behind a spherical obstacle, as illustrated in the left part of Figure 6. The viscosity is taken equal to $\frac{1}{55}$, and, with Γ_2 equal to the union of the two vertical edges of $\partial\Omega$, the pressure is given equal to 5 in the left edge, to 3 on the right edge. The final adapted mesh is presented in the right part of Figure 6 and the corresponding velocity in Figure 7. The existence of the Von Karman vortex street is undeniable. There also, these results are very similar to those in [4, Figure 3.4].

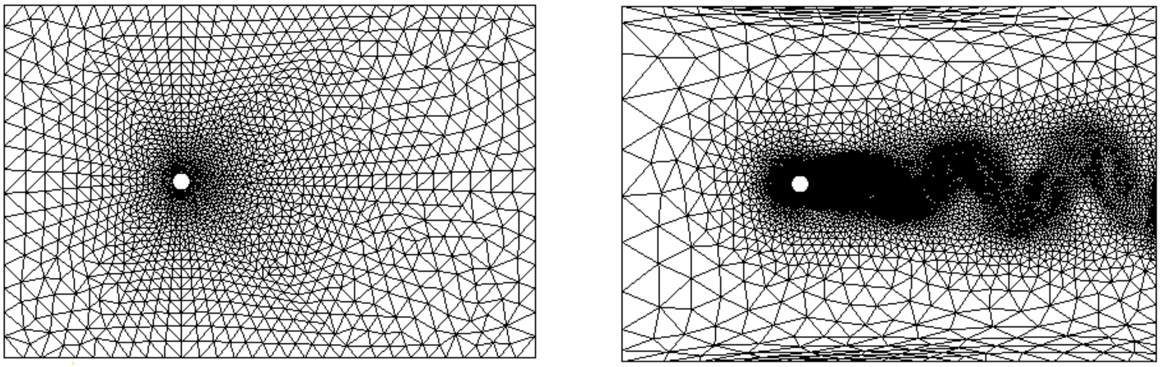

Figure 6. The initial and adapted meshes.

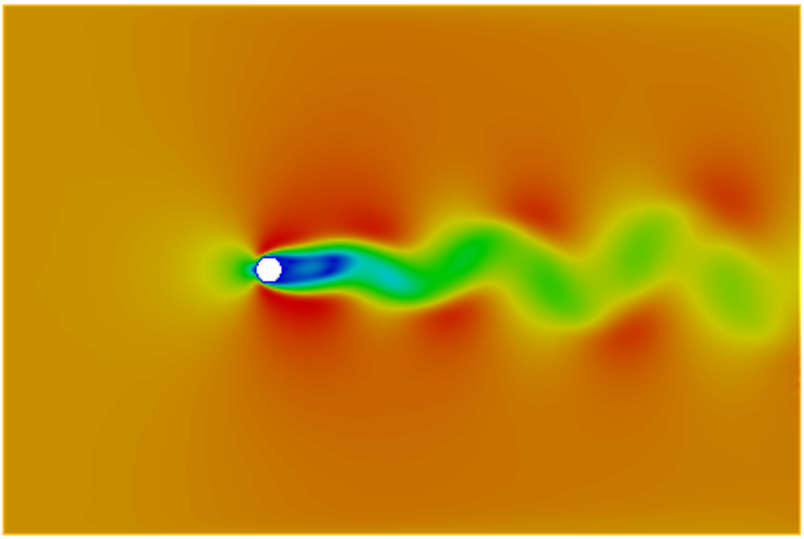

Figure 7. The discrete velocity $\boldsymbol{u}_h$.

We conclude with the case of a three-dimensional channel flow which is one of the most popular test problems for the investigation of wall bounded turbulent flows. This is a well fitted flow to test our pressure boundary conditions for the Navier-Stokes equations, as it is driven by a pressure jump between the inflow and outflow boundaries. In the usual formulation of Navier-Stokes equations this pressure jump is modeled by means of a forcing term.

The characteristic parameter of the turbulent channel flow is the friction Reynolds number

$$Re_\tau = \frac{\boldsymbol{u}_\tau\,\delta}{\nu},$$

where $u_\tau = \sqrt{\nu\,|\partial_n \mathbf{u}_t|}$ is the turbulent wall-shear velocity ($\mathbf{u}_t$ denotes the tangential velocity at the wall), and δ is the channel half-width. We consider the computational domain

$$\Omega = (0, L_1) \times (-\delta, \delta) \times (0, L_3),$$

with $\delta = 1$ (wall-normal direction), $L_1 = 2\pi$ (stream-wise direction), and $L_3 = (4/3)\pi$ (span-wise direction). The boundary conditions are periodic in both the stream-wise and span-wise directions. The viscosity is $\nu = 1/180$. The Reynolds number based on a unit friction velocity reachable at a steady state is then $Re_\tau = 180$.

In the standard formulation of Navier-Stokes equations, the flow is driven by a constant forcing $\boldsymbol{f} = (f_p, 0, 0) = (1, 0, 0)$, that models an imposed pressure gradient in the stream-wise direction. The specific choice of a unit value for f_p aims at obtaining a unit value for u_τ in the statistically steady state, subject to the relation $u_\tau = \sqrt{f_p h}$ (cf. [19]). This corresponds to a pressure jump $p_{out} - p_{in} = L_1$.

We use the projection-based VMS (Variational Multi-Scale) turbulence model described in [14, Chapter 11], that we not detail here for brevity. In this model the sub-grid flow is modeled by means of Smagorkinsky-like eddy diffusion term with projection structure. To impose the boundary conditions on the pressure we just reformulate the Navier-Stokes equations as in (2.6) and keep the same sub-grid modeling terms as in the VMS model. We impose no-slip boundary conditions on the upper and lower walls.

We compare second-order statistics as measure of turbulence intensities, for three models: The original VMS method (Method 1) with forcing term, the present method with Dirichlet pressure boundary conditions (Method 2), both with $32 \times 32 \times 32$ degrees of freedom, and a Direct Numerical Simulation (DNS) of Moser, Kim and Mansour [27] with forcing term, obtained with $128 \times 128 \times 128$ degrees of freedom. Figures 8, 9 and 10 display the normalized (by u_τ) root-mean-square (r.m.s.) values of velocity fluctuations in wall coordinates

$$y^+ = \frac{u_\tau}{\nu}\, y$$

at the upper half-width of the channel. The errors with respect to the DNS simulation of Methods 1 and 2 are comparable for all three fluctuations. The errors for the stream wise velocity fluctuations are smaller for Method 1, while those for the cross-wise velocities fluctuations are smaller for Method 2. All these results are in good coherence with the computations performed by Rubino, see [13]. We thus obtain similar results with our formulation imposing pressure jump conditions, as we could expect.

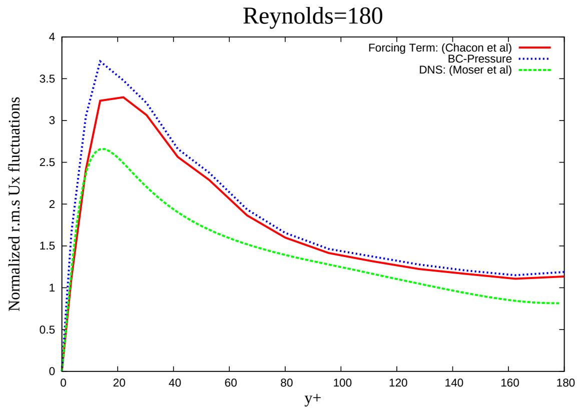

Figure 8. Normalized r.m.s. U_x velocity fluctuations profiles in wall coordinates y^+.

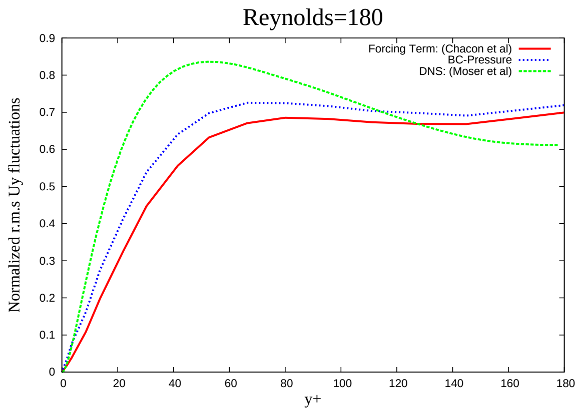

Figure 9. Normalized r.m.s. U_y velocity fluctuations profiles in wall coordinates y^+.

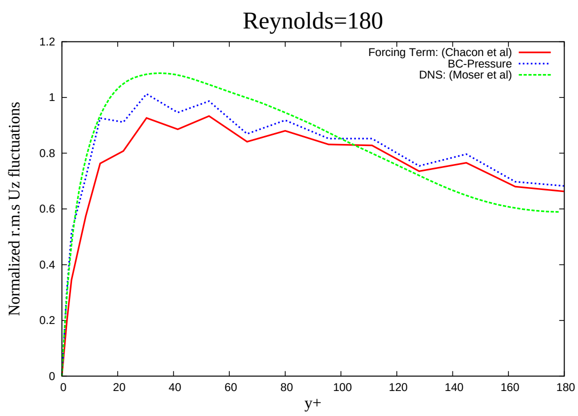

Figure 10/ Normalized r.m.s. U_z velocity fluctuations profiles in wall coordinates y^+.

References

[1] R.A. Adams, J. Fournier — *Sobolev Spaces*, Academic Press (2003).

[2] C. Amrouche, C. Bernardi, M. Dauge, V. Girault — Vector potentials in three-dimensional nonsmooth domains, *Math. Meth. in the Applied Sciences* **21** (1998), 823–864.

[3] C. Amrouche, N.E.H. Seloula — L^p-theory for the Navier–Stokes equations with pressure boundary conditions, *Discrete Contin. Dyn. Syst. Ser. S* **6** (2013), 1113–1137.

[4] C. Bègue, C. Conca, F. Murat, O. Pironneau — Les équations de Stokes et de Navier–Stokes avec des conditions aux limites sur la pression, in *Nonlinear Partial Differential Equations and their Applications*, Collège de France Seminar **IX**, H. Brezis and J.-L.Lions, eds., Pitman (1988), 179–264.

[5] C. Bernardi, C. Canuto, Y. Maday — Spectral approximations of the Stokes equations with boundary conditions on the pressure, *SIAM J. Numer. Anal.* **28** (1991), 333–362.

[6] C. Bernardi, T. Chacón Rebollo, R. Lewandowski, F. Murat — A model for two coupled turbulent fluids. Part I: analysis of the system, *Collège de France Seminar* **XIV**, D. Cioranescu & J.-L. Lions eds., North-Holland (2002), 69–102.

[7] C. Bernardi, N. Chorfi — Spectral discretization of the vorticity, velocity and pressure formulation of the Stokes problem, *SIAM J. Numer. Anal.* **44** (2006), 826–850.

[8] C. Bernardi, F. Hecht, R. Verfürth — A finite element discretization of the three-dimensional Navier-Stokes equations with mixed boundary conditions, *Math. Model. Numer. Anal.* **43** (2009), 1185–1201.

[9] C. Bernardi, Y. Maday, F. Rapetti — *Discrétisations variationnelles de problèmes aux limites elliptiques*, Collection "Mathématiques et Applications" **45**, Springer-Verlag (2004).

[10] J. Boland, R. Nicolaides — Stability of finite elements under divergence constraints, *SIAM J. Numer. Anal.* **20** (1983), 722–731.

[11] H. Brezis — *Analyse fonctionnelle, Théorie et applications*, Masson (1987).

[12] F. Brezzi, J. Rappaz, P.-A. Raviart — Finite dimensional approximation of nonlinear problems, Part I: Branches of nonsingular solutions, *Numer. Math.* **36** (1980), 1–25.

[13] T. Chacón Rebollo, M. Gómez Mármol, S. Rubino — *TurboMathS: a FreeFem++ numerical code for VMS turbulence models*, Research Group "Modelado Matemtico y Simulacin de Sistemas Medioambientales", University of Seville (2014).

[14] T. Chacón Rebollo, R. Lewandowski — *Mathematical and Numerical Foundations of Turbulence Models and Applications*, Birkhauser (2014).

[15] C. Conca, F. Murat, O. Pironneau — The Stokes and Navier–Stokes equations with boundary conditions involving the pressure, *Japan. J. Math.* **20** (1994), 279–318.

[16] M. Costabel — A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains, *Math. Meth. in Appl. Sc.* **12** (1990), 365–368.

[17] F. Dubois — Vorticity–velocity–pressure formulation for the Stokes problem, *Math. Meth.*

Applied Sciences **25** (2002), 1091–1119.

[18] F. Dubois, M. Salaün, S. Salmon — Vorticity–velocity–pressure and stream function–vorticity formulations for the Stokes problem, *J. Math. Pures Appl.* **82** (2003), 1395–1451.

[19] T. Dubois, F. Jauberteau, R. Temam — *Dynamic Multilevel Methods and the Numerical Simulation of Turbulence*, Cambridge University Press (1999).

[20] V. Girault, P.-A. Raviart — *Finite Element Methods for Navier–Stokes Equations, Theory and Algorithms*, Springer–Verlag (1986).

[21] F. Hecht — New development in FreeFem++, *Journal of Numerical Mathematics* **20** (2012), 251–266.

[22] P. Hood, C. Taylor — A numerical solution of the Navier–Stokes equations using the finite element technique, *Comp. and Fluids* **1** (1973), 73–100.

[23] S.M. Hosseini, J.J. Feng — Pressure boundary conditions for computing incompressible flows with SPH, *J. Comput. Phys.* **230** (2011), 7473–7487.

[24] H. Johnston, J.-G. Liu —Finite difference schemes for incompressible flow based on local pressure boundary conditions, *J. Comput. Phys.* **180** (2002), 12–154.

[25] J.-L. Lions, E. Magenes — *Problèmes aux limites non homogènes et applications*, Vol. I, Dunod, Paris (1968).

[26] S. Marušić — On the Navier–Stokes system with pressure boundary condition, *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **53** (2007), 31–331.

[27] R. Moser, J. Kim, N.N. Mansour — Direct numerical simulation of turbulent channel flow up to $Re_\tau = 590$, *Phys. Fluids* **11** (1999), 943–945.

[28] N.A. Petersson — Stability of pressure boundary conditions for Stokes and Navier–Stokes equations, *J. Comput. Phys.* **172** (2001), 4–70.

[29] O. Pironneau — Conditions aux limites sur la pression pour les équations de Stokes et de Navier–Stokes, *C.R. Acad. Sc. Paris, Série I* **303** (1986), 403–406.

[30] J. Pousin, J. Rappaz — Consistency, stability, a priori and a posteriori errors for Petrov–Galerkin methods applied to nonlinear problems, *Numer. Math.* **69** (1994), 213–231.

[31] S. Salmon — Développement numérique de la formulation tourbillon-vitesse-pression pour le problème de Stokes, Ph.D. thesis, Université Pierre et Marie Curie (1999).

[32] R.L. Sani, J. Shen, O. Pironneau, P.M. Gresho — Pressure boundary condition for the time-dependent incompressible Navier–Stokes equations, *Internat. J. Numer. Methods Fluids* **50** (2006), 673–682.

[33] R. Temam — *Navier-Stokes Equations. Theory and Numerical Analysis*, Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford (1977).

[34] R. Verfürth — *A Posteriori Error Estimation Techniques for Finite Element Methods*, Oxford University Press (2013).