Analysis of a Simplified Coupled Fluid-Structure Model for Computational Hemodynamics

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Abstract

We analyze a simplified coupled model for aortic blood flow. The vessel wall reaction to the fluid is modeled by the Surface Pressure Model which assumes that the normal stress on the fluid is proportional to the displacement of the structure. This leads to a unique boundary problem for Navier-Stokes equations, where at the wall the velocity is normal to the wall, and proportional to the time derivative of the pressure. We prove that the problem is well posed and show that a semi-implicit time discretization converges. We present some numerical results and a comparison with a standard implementation of the Surface Pressure Model where the displacement is not eliminated.

1 Introduction

Computational hemodynamics is an important technique for the study of by-passes, stents and heart valves (see Thiriet [1] for instance). Modeling aortic flow can be done by a large variety of approximations ranging from nonlinear elasticity to fixed walls for the vessels and non-Newtonian Navier-Stokes to Stokes flow for the fluid [2, 3, 4, 5, 6].

Linear elasticity with small displacement coupled with Navier-Stokes equations has been shown to be well posed by Nobile and Vergana in [7]. However the numerical simulations are expensive and not unconditionally stable [8].

In the special case of aortic flow the geometry does not change much. Typically the aorta has a radius of 1 centimeter and for computational purposes a section of length of 5 to 10 centimeters; the thickness of the aortic wall is around 0.1cm; the heart pulse is about 1Hz and the pressure drop is roughly 6KPa.

It was shown in [7] that if lateral displacements are neglected, Koiter's shell linear elastic model reduces to a scalar equation for the normal displacement η on the mean position Σ of the vessel's wall,

$$\rho^s h\partial_{tt}\eta - \nabla\cdot(\mathbf{T}\nabla\eta) - \nabla\cdot(\mathbf{C}\nabla\partial_t\eta) + a\partial_t\eta + b\eta = f^s,\ \ \eta, \partial_t\eta \text{ given at } t=0. \quad (1)$$

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Here h denotes the average thickness of the vessel and ρ^s its volumetric mass; $\mathbf{T}$ is the pre-stress tensor, $\mathbf{C}$ is a damping term, a, b are viscoelastic terms and f^s the external normal force on the shell, i.e. $f^s = -\sigma^{\mathbf{s}}{}_{nn}$ the normal component of the normal stress at the surface of the solid.

Notice that in this context and due to the assumption of normal displacements the other components of the normal stress tensor cannot be matched with those of the fluid.

Finally assume that $[h, T, C, a] << b$; then the *Surface Pressure Model* is obtained:

$$-\sigma^{\mathbf{s}}{}_{nn} = b\eta, \text{ with } b = \frac{Eh\pi}{A(1-\xi^2)}, \tag{2}$$

where A is the vessel's cross section, E the Young modulus, ξ the Poisson coefficient. The following typical values (see [6]) will be used in the numerical tests:

$$E = 3M\text{Pa}, \quad \xi = 0.3, \quad A = \pi R^2, \; R = 0.01\text{m}, \quad h = 0.001\text{m} \quad \Rightarrow \; b = 3.310^7\text{ms}^{-2}. \tag{3}$$

The *Surface Pressure Model* is an interesting prototype to understand the complexity of fluid-structure interactions; it yields numerical models which do not require alternate resolutions between the fluid and the structure; hence, though less general, gives rise to a much more stable strategies.

From the mathematical standpoint it leads to unusual boundary conditions involving the pressure and the normal wall velocity. The object of this study is to investigate these new problems and show that they are well posed.

The paper is organized as follows. In Section 2 we introduce the simplified coupled model that we study in the paper. Section 3 is devoted to introduce a semi-discretization in time of the simplified model, that is proved to be well posed in Section 4. In Section 5 we prove the stability and convergence of the semi-discretization. Finally Section 6 presents some relevant numerical tests.

2 Modeling

Assuming blood flow to be Newtonian , the *Navier-Stokes equations* link the fluid velocity $\mathbf{u}$ and the pressure p by

$$\rho^f(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\cdot\sigma^{\mathbf{f}} = \mathbf{0}, \quad \nabla\cdot\mathbf{u} = 0, \tag{4}$$

where ρ^f is the volumetric mass density of the fluid, μ the viscosity and $\sigma^f = -p\mathbf{I}+\mu(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$ is the stress tensor.

With the *surface pressure model* (2) for the vessels Σ we have

$$-b\eta = (-p + 2\mu\partial_n\mathbf{u}\cdot\mathbf{n})|_\Sigma, \quad \partial_t\eta\;\mathbf{n} = \mathbf{u}|_\Sigma. \tag{5}$$

The second equation says that the derivative of the vessel's displacement is equal to the velocity of the fluid; note that it can also be written as

$$\mathbf{u}\times\mathbf{n} = \mathbf{0}, \quad \partial_t\eta = \mathbf{u}\cdot\mathbf{n} \text{ on } \Sigma. \tag{6}$$

In [9] it was shown that on a quasi-toroidal geometry of large radius R and small radius r the first condition in (5) is

$$-b\eta = -p + 2(1 + \frac{r}{R}\cos^2\theta)\frac{\mu}{r}\mathbf{u}\cdot\mathbf{n}$$

at all points on Σ of coordinates $[x = (R + r\cos\theta)\cos\phi,\ y = (R + r\cos\theta)\sin\phi,\ z = r\sin\theta]$. Therefore, on Σ:

$$\partial_t p = 2(1 + \frac{r}{R}\cos^2\theta)\frac{\mu}{r}\partial_t \mathbf{u}\cdot\mathbf{n} + b\mathbf{u}\cdot\mathbf{n}. \tag{7}$$

At the (artificial) input/output cross sections we shall assume that the flow is normal to the section, $\mathbf{u}\times\mathbf{n} = \mathbf{0}$, and the pressure is given by $p = p_\Gamma$. So a complete set of boundary and initial conditions for (4) is

$$\begin{aligned}
&\mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega, \quad p(0) = p_0 \text{ on } \Sigma,\\
&\mathbf{u}\times\mathbf{n} = \mathbf{0}, \quad \partial_t p = 2(1 + \frac{r}{R}\cos^2\theta)\frac{\mu}{r}\partial_t\mathbf{u}\cdot\mathbf{n} + b\mathbf{u}\cdot\mathbf{n} \text{ on } \Sigma\times(0,T),\\
&\mathbf{u}\times\mathbf{n} = \mathbf{0}, \quad p = p_\Gamma \text{ on } (\partial\Omega\backslash\Sigma)\times(0,T).
\end{aligned} \tag{8}$$

The vessel moves with the normal velocity η, so Ω is time dependent. As a first approximation (transpiration condition, see [10]) we may neglect the motion for the fluid computation. However in doing so (4)+(8) no longer preserves energy; therefore it is necessary to replace the nonlinear terms $\mathbf{u}\cdot\nabla\mathbf{u}$ in (4) by $-\mathbf{u}\times\nabla\times\mathbf{u}$ according to the identity,

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\mathbf{u}\times\nabla\times\mathbf{u} + \nabla\frac{|\mathbf{u}|^2}{2}. \tag{9}$$

This approximation has the same order as neglecting the time dependency of Ω because it amounts to replacing the dynamic pressure $p + \frac{1}{2}|u|^2$ by p on Σ and $\mathbf{u}|_\Sigma$ is small.

Finally recalling the identity:

$$-\Delta\mathbf{u} = \nabla\times\nabla\times\mathbf{u} - \nabla(\nabla\cdot\mathbf{u}), \tag{10}$$

the modified Navier-Stokes system for fluid-structure interactions in fixed domains such as those of aortic flows is

$$\begin{aligned}
&\partial_t\mathbf{u} - \mathbf{u}\times\nabla\times\mathbf{u} + \nu\nabla\times\nabla\times\mathbf{u} + \nabla\tilde{p} = \mathbf{0}, \quad \nabla\cdot\mathbf{u} = 0,\\
&\mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega,\\
&\mathbf{u}\times\mathbf{n} = \mathbf{0}, \quad \partial_t\tilde{p} = 2(1 + \frac{r}{R}\cos^2\theta)\frac{\nu}{r}\partial_t\mathbf{u}\cdot\mathbf{n} + \tilde{b}\mathbf{u}\cdot\mathbf{n} \text{ on } \Sigma\times(0,T),\\
&\mathbf{u}\times\mathbf{n} = \mathbf{0}, \quad p = p_\Gamma \text{ on } (\partial\Omega\backslash\Sigma)\times(0,T),
\end{aligned} \tag{11}$$

where $\nu := \frac{\mu}{\rho^f}$, $\tilde{p} = \frac{p}{\rho^f}$ and $\tilde{b} = \frac{b}{\rho^f}$. When $b >> \mu$ we may neglect the term containing $\partial_t\mathbf{u}\cdot\mathbf{n}$ and consider instead

$$\tilde{b}\,\mathbf{u} = \partial_t\tilde{p}\;\mathbf{n}, \tag{12}$$

but in any case, the following analysis can be extended to more general cases including the full equation (1).

In the sequel we drop the tilde over p and b. Also we take $\Gamma = \emptyset$, i.e. $\Sigma = \partial\Omega$, for clarity and leave to the reader the extension to the general case, assuming that p_0 and p_Γ have the right matching and regularity properties. Therefore we shall consider the system

$$\left\{\begin{array}{rll}
\partial_t\mathbf{u} - \mathbf{u}\times\nabla\times\mathbf{u} + \nu\,\nabla\times\nabla\times\mathbf{u} + \nabla p = \mathbf{f} & \text{in} & (0,T)\times\Omega,\\
\nabla\cdot\mathbf{u} = 0 & \text{in} & (0,T)\times\Omega,\\
b\,\mathbf{u} = \partial_t p\,\mathbf{n} & \text{on} & (0,T)\times\Sigma,\\
\mathbf{u}(0) = \mathbf{u}_0 & \text{in} & \Omega,\\
p(0) = p_0 & \text{in} & \Omega.
\end{array}\right. \tag{13}$$

3 Semi-discretization in time

In this section we propose a variational formulation for the Navier-Stokes boundary value problem (13). We use the Sobolev space $W^{k,p}(\Omega)$ and denote its norm by $\| \cdot \|_{k,p,\Omega}$. We denote $\mathbf{W}^k(\Omega) = [W^k(\Omega)]^3$, $H^k(\Omega) = W^{k,2}(\Omega)$, $\mathbf{H}^k(\Omega) = [H^k(\Omega)]^3$, $\mathbf{L}^k(\Omega) = [L^k(\Omega)]^3$; we assume that Ω is a Lipschitz domain. Our analysis is inspired by the early works on the Stokes and Navier-Stokes equations with boundary conditions on the pressure (Cf. [11, 12]).

Notice that a time discretization of (12) gives an expression for the pressure on Σ at each time step:

$$b\,\mathbf{u}^{n+1} \simeq \frac{p^{n+1} - p^n}{\delta t}\,\mathbf{n} \implies p^{n+1} \simeq p^n + b\,\delta t\,\mathbf{u}^{n+1} \cdot \mathbf{n}. \tag{14}$$

It can be used directly in the weak form of (13):

$$(\partial_t \mathbf{u}, \mathbf{w})_\Omega - (\mathbf{u} \times \nabla \times \mathbf{u}, \mathbf{w})_\Omega + \nu\,(\nabla \times \mathbf{u}, \nabla \times \mathbf{w})_\Omega - (p, \nabla \cdot \mathbf{w})_\Omega + (p, \mathbf{n} \cdot \mathbf{w})_\Sigma = (\mathbf{f}, \mathbf{w})_\Omega. \tag{15}$$

This weak form is obtained by integrating in Ω the first equation of (13) multiplied by a smooth test function $\mathbf{w}$ such that $\mathbf{n} \times \mathbf{w} = \mathbf{0}$ on Σ, and using the identity

$$(\nabla \times \nabla \times \mathbf{u}, \mathbf{w})_\Omega = (\nabla \times \mathbf{u}, \mathbf{w} \times \mathbf{n})_\Sigma + (\nabla \times \mathbf{u}, \nabla \times \mathbf{w})_\Omega = (\nabla \times \mathbf{u}, \nabla \times \mathbf{w})_\Omega.$$

As usual $(.,.)_\Omega$ denotes the $L^2(\Omega)$-scalar product.

Therefore using (14), equation (15) is discretized at times $t = t_n = n\,\delta t$ by

$$\begin{aligned}(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t}, \mathbf{w})_\Omega - (\mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n, \mathbf{w})_\Omega + \nu\,(\nabla \times \mathbf{u}^{n+1}, \nabla \times \mathbf{w})_\Omega - (p^{n+1}, \nabla \cdot \mathbf{w})_\Omega \\ +(p^n + b\,\delta t\,\mathbf{u}^{n+1} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma = (\mathbf{f}^{n+1}, \mathbf{w})_\Omega\end{aligned} \tag{16}$$

where $\mathbf{f}^{n+1}$ is some averaged value of f in $[t_n, t_{n+1}]$, with $t_n = n\,\delta t$. However this formulation only makes sense if $p^n_{|\Sigma}$ has some L^p regularity, in particular if $p^n_{|\Sigma} \in L^2(\Sigma)$. Hence we will first re-formulate (16) appropriately and then analyze its well-posedness, stability and convergence when $\delta t \to 0$ to the solution of a suitable continuous problem.

Consider the velocity and pressure spaces

$$\mathbf{W} = \{\mathbf{w} \in \mathbf{L}^2(\Omega) \,|\, \nabla \times \mathbf{w} \in \mathbf{L}^2(\Omega),\ \nabla \cdot \mathbf{w} \in L^2(\Omega),\ \mathbf{n} \times \mathbf{w}_{|_\Sigma} = \mathbf{0}\}, \tag{17}$$

$$M = L^2(\Omega). \tag{18}$$

Then it holds (Cf. Bernardi et al. [13])

Lemma 1 *The space* $\mathbf{W}$ *is well defined and is a Hilbert space endowed with the norm*

$$\|\mathbf{w}\|_{\mathbf{W}} = (\|\mathbf{w}\|^2_{0,2,\Omega} + \|\nabla \times \mathbf{w}\|^2_{0,2,\Omega} + \|\nabla \cdot \mathbf{w}\|^2_{0,2,\Omega})^{1/2}.$$

Moreover when Ω *is either convex or* $\mathcal{C}^{1,1}$*, then* $\mathbf{W}$ *is continuously embedded in* $\mathbf{H}^1(\Omega)$ *and there exists a constant* $C > 0$ *such that*

$$\|\mathbf{w}\|_{1,2,\Omega} \leq C\,\|\mathbf{w}\|_{\mathbf{W}} \ \ \forall \mathbf{w} \in \mathbf{W}.$$

Observe that the condition $\partial_t p\mathbf{n} = b\,\mathbf{u}$ may be re-written $p(t)\,\mathbf{n} = p(0)\,\mathbf{n} + b\displaystyle\int_0^t \mathbf{u}(s)\,ds$. This suggests the following discretization of problem (13): Assume $\mathbf{f} \in L^2((0,T) \times \Omega)$, $\mathbf{u}_0 \in \mathbf{W}$, $p_0 \in L^2(\Sigma)$, let $N \geq 1$ integer, $\delta t = T/N$. Set $\mathbf{u}^0 = \mathbf{u}_0$, $p^0 = p_0$.

For $n = 0, \cdots, N-1$, find $(\mathbf{u}^{n+1}, p^{n+1}) \in \mathbf{W} \times M$ such that for any $(\mathbf{w}, q) \in \mathbf{W} \times M$,

$$\left\{ \begin{array}{rcl} (\dfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t}, \mathbf{w})_\Omega - (\mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n, \mathbf{w})_\Omega + \nu\,(\nabla \times \mathbf{u}^{n+1}, \nabla \times \mathbf{w})_\Omega & & \\ -(p^{n+1}, \nabla \cdot \mathbf{w})_\Omega + (p^0 + b\,\mathbf{U}^{n+1} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma & = & (\mathbf{f}^{n+1}, \mathbf{w})_\Omega, \\ (\nabla \cdot \mathbf{u}^{n+1}, q) & = & 0, \end{array} \right. \tag{19}$$

where

$$\mathbf{U}^{n+1} = \delta t \sum_{k=1}^{n+1} \mathbf{u}^k, \quad \mathbf{f}^{n+1} = \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} \mathbf{f}(s)\,ds.$$

This discretization is equivalent to (16),

Set $\mathbf{u}_0 = \mathbf{u}^0$, $p_0 = p^0$. For $n = 0, \cdots, N-1$, find $(\mathbf{u}^{n+1}, p^{n+1}) \in \mathbf{W} \times M$ such that for any $(\mathbf{w}, q) \in \mathbf{W} \times M$,

$$\left\{ \begin{array}{rcl} (\dfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t}, \mathbf{w})_\Omega - (\mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n, \mathbf{w})_\Omega + \nu\,(\nabla \times \mathbf{u}^{n+1}, \nabla \times \mathbf{w})_\Omega & & \\ -(p^{n+1}, \nabla \cdot \mathbf{w})_\Omega + (p^n + b\,\delta t\,\mathbf{u}^{n+1} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma & = & (\mathbf{f}^{n+1}, \mathbf{w})_\Omega, \\ (\nabla \cdot \mathbf{u}^{n+1}, q) & = & 0. \end{array} \right. \tag{20}$$

Proposition 1 *Assume that problem (20) admits a solution such that* $p^n \in H^1(\Omega)$, $\mathbf{u}^n \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ *and* $\nabla \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ *for all* $n = 0, \cdots, N$*, then the sequence* $(\mathbf{u}^n, p^n)$, $n = 0, \cdots, N$ *is also a solution of (19).*

Conversely, if (19) admits a solution such that $p^n \in H^1(\Omega)$, $\mathbf{u}^n \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ *and* $\nabla \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ *for all* $n = 0, \cdots, N$*, then the sequence* $(\mathbf{u}^n, p^n)$, $n = 0, \cdots, N$ *is also a solution of (20)*

Proof. Assume for instance that problem (20) admits a solution such that $p^n \in H^1(\Omega)$, $\mathbf{u}^n \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ and $\nabla \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ for all $n = 0, \cdots, N$. Integrating by parts in (20) we obtain

$$\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t} - \mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n + \nu \nabla \times \nabla \times \mathbf{u}^{n+1} + \nabla p^{n+1} - \mathbf{f}^{n+1}, \mathbf{w} \right)_\Omega = 0,$$

for all $\mathbf{w} \in \mathbf{W}$ such that $\mathbf{w} \cdot \mathbf{n} = 0$ on Σ. Then this holds for all $\mathbf{w} \in \mathcal{D}(\Omega)^d$ and we deduce

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t} - \mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n + \nu \nabla \times \nabla \times \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} \text{ in } \mathbf{L}^2(\Omega). \tag{21}$$

Thus by (20),

$$(-p^{n+1} + p^n + b\,\delta t\,\mathbf{u}^{n+1} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma = 0 \text{ for all } \mathbf{w} \in \mathbf{W}.$$

Applying recursively this identity, we deduce

$$(-p^{n+1} + p^0 + b\,\delta t \sum_{k=1}^{n} \mathbf{u}^{k+1} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma = 0 \text{ for all } \mathbf{w} \in \mathbf{W}.$$

Then the sequence $(\mathbf{u}^n, p^n)$, $n = 0, 1, \cdots, N$ is a solution of (19). □

Remark. If $\mathbf{f}^n \in \mathbf{L}^{3/2}(\Omega)$, $n = 0, 1, \cdots, N$, the regularity hypotheses of this result simplify to $p^n \in W^{1,3/2}(\Omega)$, $n = 0, 1, \cdots, N$, and no additional regularity is necessary for the velocities. However this proof is quite involved technically and we prefer not to include it for simplicity.

4 Analysis of the semi-discrete problem

In this section we analyze the discrete problem (19) for fixed n. We prove that it admits a unique solution $(\mathbf{u}^{n+1}, p^{n+1})$. Estimates for $\mathbf{u}^{n+1}$ and for a primitive in time of the pressure (instead of the pressure itself) will be obtained in Theorem 5.1 below (See (40), (46)).

Problem (19) is an Oseen-like problem, however it is non-standard due to the structure of the convection term, and the presence of the boundary terms issued from the discretization of condition (12).

The well-posedness of problem (19) is based upon the following inf-sup condition:

Lemma 2 *Assume that the domain Ω is Lipschitz. Then for some $\beta > 0$,*

$$\beta \, \|q\|_{0,2,\Omega} \le \sup_{\mathbf{w}\in\mathbf{W}} \frac{(q, \nabla\cdot\mathbf{w})}{\|\mathbf{w}\|_{\mathbf{W}}} \quad \textit{for all } q \in M. \tag{22}$$

Proof. Let $q \in M$. Consider the solution $\Phi \in H_0^1(\Omega)$ of the problem

$$-\Delta\Phi = q \ \text{ in } \Omega, \ \ \Phi = 0 \ \text{ on } \Sigma.$$

Let $\mathbf{w} = -\nabla\Phi$. Then $\mathbf{w} \in \mathbf{L}^2(\Omega)$, $\nabla\cdot\mathbf{w} = -q$, $\nabla\times\mathbf{w} = 0$ in Ω and $\mathbf{n}\times\mathbf{w} = \mathbf{n}\times\nabla\Phi = \mathbf{0}$ on Σ because the components of $\mathbf{n}\times\nabla\Phi$ are tangential derivatives of Φ on Σ (Cf. Girault-Raviart [14]). Then $\mathbf{w} \in \mathbf{W}$, and (22) follows with

$$\beta = \frac{1}{\sqrt{1+\mathcal{P}^2}},$$

where $\mathcal{P}$ is the constant of Poincaré's inequality. □

Notations Let us introduce the following multilinear forms for $\mathbf{u}, \mathbf{w}, \mathbf{z} \in \mathbf{W}, r \in M$:

$$a(\mathbf{u},\mathbf{w}) = \nu\,(\nabla\times\mathbf{u}, \nabla\times\mathbf{w})_\Omega, \tag{23}$$

$$c(\mathbf{u};\mathbf{z},\mathbf{w}) = -(\mathbf{z}\times\nabla\times\mathbf{u}, \mathbf{w})_\Omega, \tag{24}$$

$$d(r,\mathbf{w}) = -(r, \nabla\cdot\mathbf{w})_\Omega, \tag{25}$$

$$A(\mathbf{v};\mathbf{u},\mathbf{w}) = \frac{1}{\delta t}(\mathbf{u},\mathbf{w})_\Omega + c(\mathbf{v};\mathbf{u},\mathbf{w}) + a(\mathbf{u},\mathbf{w}) + b\,\delta t\,(\mathbf{u}\cdot\mathbf{n}, \mathbf{w}\cdot\mathbf{n})_\Sigma, \tag{26}$$

$$l_n(\mathbf{w}) = \frac{1}{\delta t}(\mathbf{u}^n,\mathbf{w})_\Omega + (\mathbf{f}^{n+1},\mathbf{w})_\Omega - (p^0 + b\,\mathbf{U}^n\cdot\mathbf{n}, \mathbf{w}\cdot\mathbf{n})_\Sigma. \tag{27}$$

Problem (19) for $(\mathbf{u}^{n+1}, p^{n+1})$ may be re-written as

$$\left\{ \begin{array}{rcll} A(\mathbf{u}^n;\mathbf{u}^{n+1},\mathbf{w}) + d(p^{n+1},\mathbf{w}) &=& l_n(\mathbf{w}) & \text{for any } \mathbf{w}\in\mathbf{W}, \\ d(q,\mathbf{u}^{n+1}) &=& 0 & \text{for any } q \in M. \end{array} \right. \tag{28}$$

To analyse this problem note that the form l_n is linear on $\mathbf{W}$, the form a is bilinear continuous on $\mathbf{W}\times\mathbf{W}$, the form c is trilinear continuous on $\mathbf{W}\times\mathbf{W}\times\mathbf{W}$ and the form d is bilinear continuous on $M\times\mathbf{W}$. Moreover,

Lemma 3 *Assume that Ω is convex or $\mathcal{C}^{1,1}$. Then there exists a constant C such that*

$$|c(\mathbf{u};\mathbf{z},\mathbf{w})| \le C\,\|\mathbf{u}\|_{\mathbf{W}}\,\|\mathbf{z}\|_{\mathbf{W}}\,\|\mathbf{w}\|_{\mathbf{W}} \quad \textit{for all } \mathbf{u}, \mathbf{z}, \mathbf{w} \in \mathbf{W} \tag{29}$$

Proof. By Sobolev's imbeddings, $H^1(\Omega)$ is imbedded into $L^p(\Omega)$ for $1 \le p \le 6$. Therefore

$$|c(\mathbf{u};\mathbf{z},\mathbf{w})| \le C\, \|\nabla\times\mathbf{u}\|_{0,2,\Omega}\, \|\mathbf{z}\|_{0,4,\Omega}\, \|\mathbf{w}\|_{0,4,\Omega} \le C\, \|\mathbf{u}\|_{\mathbf{W}}\, \|\mathbf{z}\|_{1,2,\Omega}\, \|\mathbf{w}\|_{1,2,\Omega}$$

The conclusion follows from Lemma 1. □

We are now in a position to prove the

Proposition 2 *Assume that Ω is convex or $\mathcal{C}^{1,1}$. Then problem (19) admits a unique solution.*

Proof. The forms $A(\mathbf{v};\cdot,\cdot)$ and d are bilinear and respectively continuous on $\mathbf{W}\times\mathbf{W}$ and $M\times\mathbf{W}$. Also, $A(\mathbf{v};\cdot,\cdot)$ is coercive on the kernel $\mathbf{W}_{div}$ of d in $\mathbf{W}$,

$$\mathbf{W}_{div} = \{\mathbf{w}\in\mathbf{W}\,|\,\nabla\cdot\mathbf{w}=0,\ \text{ a. e. in }\ \Omega\,\}.$$

Indeed, as $c(\mathbf{v};\mathbf{w},\mathbf{w})=0$, the form

$$\mathbf{w}\in\mathbf{W}_{div}\mapsto[\mathbf{w}] := A(\mathbf{v};\mathbf{w},\mathbf{w})^{1/2} = \left(\frac{1}{\delta t}\,\|\mathbf{w}\|^2_{0,2,\Omega} + \nu\,\|\nabla\times\mathbf{w}\|^2_{0,2,\Omega} + b\,\delta t\,\|\mathbf{w}\cdot\mathbf{n}\|^2_{0,2,\Sigma}\right)^{1/2}$$

is a norm on $\mathbf{W}_{div}$ equivalent to the norm of $\mathbf{W}$. In addition, the inf-sup condition (22) holds. As the form l_n is linear on $\mathbf{W}$, we deduce that problem (28) admits a unique solution $(\mathbf{u},p)$ (Cf. [14]). □

5 Stability and convergence analysis

In this section we establish the stability of (19) in natural norms and prove its convergence to a weak solution of the boundary value problem (13) for the Navier-Stokes equations. We start by giving a weak formulation to this problem. We shall look for the primitive of the pressure as an unknown instead of the pressure itself. This primitive is naturally bounded in $L^\infty((0,T);L^2(\Omega))$, while it is much harder to bound the pressure in a Banach space.

5.1 Variational formulation

For brevity we shall denote $L^p((0,T);B)$ by $L^p(B)$, where B is a Banach space. When $B = W^{k,p}(\Omega)$ we denote $L^p(W^{k,p}(\Omega))$ by $L^p(W^{k,p})$. Let us define the mapping $\mathbf{U}: L^2(\mathbf{W})\mapsto H^1(\mathbf{W})$ by

$$\mathbf{U}(\mathbf{z})(t) = \int_0^t \mathbf{z}(s)ds.$$

We define the weak formulation of problem (19) as follows. Denote $Q_T = \Omega\times(0,T)$,

Definition 1 *Let $\mathbf{f}\in L^2(\mathbf{W}')$, $\mathbf{u}_0\in\mathbf{W}'_{div}$, $p_0\in L^2(\Sigma)$. A pair $(\mathbf{u},p)\in\mathcal{D}'(Q_T)^d\times\mathcal{D}'(Q_T)$ is a weak solution of the boundary value problem (13) if $\mathbf{u}\in L^2(\mathbf{W}_{div})\cap L^\infty(\mathbf{L}^2)$, there exists $P\in L^2(L^2)$ such that $p=\partial_t P$, and for all $\mathbf{w}\in\mathbf{W}$, $\varphi\in\mathcal{D}([0,T])$ such that $\varphi(T)=0$,*

$$\left\{\begin{array}{l} -\displaystyle\int_0^T (\mathbf{u}(t),\mathbf{w})_\Omega\,\varphi'(t)\,dt - \langle\mathbf{u}_0,\mathbf{w}\rangle\,\varphi(0) \\ +\displaystyle\int_0^T \left[\,c(\mathbf{u}(t);\mathbf{u}(t),\mathbf{w})\,dt + a(\mathbf{u}(t),\mathbf{w}) + b\,(\mathbf{U}(\mathbf{u})(t)\cdot\mathbf{n},\mathbf{w}\cdot\mathbf{n})_\Sigma\,\right]\varphi(t)\,dt \\ +\displaystyle\int_0^T (P(t),\nabla\cdot\mathbf{w})_\Omega\,\varphi'(t)\,dt = \int_0^T\langle\mathbf{f}(t),\mathbf{w}\rangle\,\varphi(t)\,dt - \int_0^T (p_0,\mathbf{w}\cdot\mathbf{n})_\Sigma\,\varphi(t)\,dt. \end{array}\right. \tag{30}$$

This definition makes sense because due to the regularity asked for $\mathbf{u}$ and P, all terms in (30) are integrable in $(0,T)$. The weak solutions given by this definition are solutions of the Navier-Stokes equations in the following sense.

Proposition 3 *Assume that Ω is convex or $\mathcal{C}^{1,1}$. Let $(\mathbf{u},p) \in \mathcal{D}'(Q_T)^d \times \mathcal{D}'(Q_T)$ be a weak solution of the boundary value problem (13). Then*

i) Equations

$$\partial_t \mathbf{u} - \mathbf{u} \times \nabla \times \mathbf{u} + \nu\, \nabla \times \nabla \times \mathbf{u} + \nabla p = \mathbf{f} \quad \textit{and} \quad \nabla \cdot \mathbf{u} = 0 \tag{31}$$

hold respectively in $\mathcal{D}'(Q_T)^d$ and in $L^2(Q_T)$.

ii)

$$\mathbf{u} \in C^0([0,T], \mathbf{W}'_{div}) \quad \textit{and} \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \textit{in} \ \mathbf{W}'_{div}.$$

iii)

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \textit{in} \ L^2(\mathbf{L}^4(\Sigma)).$$

iv) If Ω is $\mathcal{C}^{1,1}$ or polyhedric, $\mathbf{u} \in L^2(\mathbf{H}^2)$, $\partial_t \mathbf{u} \in L^2(\mathbf{L}^2)$, $p_0 \in H^{1/2}(\Sigma)$ and $p \in L^2(H^1)$, then

$$\partial_t p = b\, \mathbf{u} \cdot \mathbf{n} \quad \textit{in} \ L^2(H^{1/2}(\Sigma)), \quad p(0) = p_0 \ \textit{a. e. in} \ \Sigma \times (0,T).$$

Proof.

i) As $\mathbf{u} \in L^1(Q_T)$, then $\mathbf{u}$ satisfies in the sense of distributions

$$\langle \partial_t \mathbf{u}, \mathbf{w} \otimes \varphi \rangle_{\mathcal{D}(Q_T)} = -\int_{Q_T} \mathbf{u}(\mathbf{x},t)\, \partial_t(\mathbf{w}(\mathbf{x})\varphi(t))\, d\mathbf{x}\, dt = -\int_0^T (\mathbf{u}(t), \mathbf{w})_\Omega\, \varphi'(t)\, dt,$$

for all $\mathbf{w} \in \mathcal{D}(\Omega)^d$, $\varphi \in \mathcal{D}(0,T)$. Similarly, as $P \in L^1(Q_T)$,

$$\langle \nabla(\partial_t P), \mathbf{w} \otimes \varphi \rangle_{\mathcal{D}(Q_T)} = \int_0^T (P(t), \nabla \cdot \mathbf{w})_\Omega\, \varphi'(t)\, dt.$$

Then, integrating by parts and using $(\mathbf{U}(\mathbf{u})(t) \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma = 0$ and $\nabla \cdot \mathbf{u} = 0$ a. e. in Q_T, (30) implies

$$\langle \partial_t \mathbf{u} - \mathbf{u} \times \nabla \times \mathbf{u} + \nu\, \nabla \times \nabla \times \mathbf{u} + \nabla p - \mathbf{f}, \mathbf{w} \otimes \varphi \rangle_{\mathcal{D}(Q_T)} = 0$$

for all $\mathbf{w} \in \mathcal{D}(\Omega)^d$, $\varphi \in \mathcal{D}(0,T)$. As $\mathcal{D}(\Omega) \otimes \mathcal{D}(0,T)$ is sequentially dense in $\mathcal{D}(Q_T)$ (Cf. [15], Theorem 39.2) we deduce (31).

Also, as $\mathbf{u} \in L^2(\mathbf{W}_{div})$, then $\nabla \cdot \mathbf{u} = 0$ in $L^2(Q_T)$.

ii) Let $\Phi(t) \in \mathbf{W}'$ defined a. e. in $(0,T)$ by

$$\langle \Phi(t), \mathbf{z} \rangle = c(\mathbf{u}(t); \mathbf{u}(t), \mathbf{z}) + a(\mathbf{u}(t), \mathbf{z}) + b\, (\mathbf{U}(\mathbf{u})(t) \cdot \mathbf{n}, \mathbf{z} \cdot \mathbf{n})_\Sigma - \langle \mathbf{f}(t), \mathbf{z} \rangle - (p_0, \mathbf{z} \cdot \mathbf{n})_\Sigma.$$

By estimates (29) and the boundedness of forms a and c, there exists a constant $C > 0$ such that

$$\|\Phi(t)\|_{\mathbf{W}'} \le C\, (\|\mathbf{u}(t)\|_{\mathbf{W}}^2 + \|\mathbf{u}(t)\|_{\mathbf{W}} + \|\mathbf{f}(t)\|_{\mathbf{W}} + \|p_0\|_{0,2,\Sigma} + \|\mathbf{u}\|_{0,1,\mathbf{W}}).$$

Then $\Phi \in L^1(\mathbf{W}')$. From (30) we deduce that for all $\mathbf{w} \in \mathbf{W}_{div}$, and for all $\varphi \in \mathcal{D}(0,T)$

$$\int_0^T (\mathbf{u}(t), \mathbf{w})_\Omega \, \varphi'(t) \, dt = \int_0^T \langle \Phi(t), \mathbf{w} \rangle_{\mathbf{W}_{div}} \, \varphi'(t) \, dt.$$

Then $\partial_t \mathbf{u} = -\Phi \in L^1(\mathbf{W}'_{div})$, and $\mathbf{u} \in C^0([0,T], \mathbf{W}'_{div})$. Moreover, if $\varphi \in \mathcal{D}([0,T])$ is such that $\varphi(T) = 0$, then (Cf. Temam [16], Chapter 3)

$$\int_0^T \langle \partial_t \mathbf{u}(t), \mathbf{w} \rangle_{\mathbf{W}_{div}} \, \varphi(t) \, dt = -\langle \mathbf{u}(0), \mathbf{w} \rangle_{\mathbf{W}_{div}} \, \varphi(0) - \int_0^T (\mathbf{u}(t), \mathbf{w})_\Omega \, \varphi'(t) \, dt.$$

As $\mathbf{u}_0 \in \mathbf{W}' \hookrightarrow \mathbf{W}'_{div}$, by (30) it follows

$$\int_0^T \langle \partial_t \mathbf{u}(t) + \Phi(t), \mathbf{w} \rangle_{\mathbf{W}_{div}} \, \varphi(t) \, dt + \langle \mathbf{u}_0 - \mathbf{u}(0), \mathbf{w} \rangle_{\mathbf{W}_{div}} \varphi(0) = 0,$$

and so $\langle \mathbf{u}_0 - \mathbf{u}(0), \mathbf{w} \rangle_{\mathbf{W}_{div}} = 0$ for all $\mathbf{w} \in \mathbf{W}_{div}$. We conclude that $\mathbf{u}(0) = \mathbf{u}_0$ in $\mathbf{W}'_{div}$.

iii) As $\mathbf{W}$ is imbedded into $\mathbf{H}^1(\Omega)$, trace theorems and Sobolev's imbeddings imply that $\mathbf{u}_{|\Sigma} \in L^2(\mathbf{L}^4(\Sigma))$. As $\mathbf{n} \in \mathbf{L}^\infty(\Sigma)$, then $\mathbf{u} \times \mathbf{n} \in L^2(\mathbf{L}^4(\Sigma))$.

iv) Assume $\mathbf{u} \in L^2(\mathbf{H}^2)$, $\partial_t \mathbf{u} \in L^2(\mathbf{L}^2)$ and $p \in L^2(H^1)$. Integrating by parts in (30), yields

$$\int_0^T (\partial_t \mathbf{u}(t) - \mathbf{u}(t) \times \nabla \times \mathbf{u}(t) + \nu \nabla \times \nabla \times \mathbf{u}(t) + \nabla p(t) - \mathbf{f}(t), \mathbf{w})_\Omega \, \varphi(t) \, dt$$
$$+ \int_0^T (b\, \mathbf{U}(\mathbf{u})(t) \cdot \mathbf{n} + p_0 - p(t), \mathbf{w} \cdot \mathbf{n})_\Sigma \, \varphi(t) \, dt = 0,$$

for all $\mathbf{w} \in \mathbf{W}$, $\varphi \in \mathcal{D}(0,T)$. Using that the first equation in (31) now holds in $L^2(\mathbf{L^2})$, we deduce

$$(b\, \mathbf{U}(\mathbf{u})(t) \cdot \mathbf{n} + p_0 - p(t), \mathbf{w} \cdot \mathbf{n})_\Sigma = 0 \ \text{ a. e. in } \ (0,T). \tag{32}$$

Assume that Ω is $\mathcal{C}^{1,1}$. Then $\mathbf{n} \in \mathcal{C}^{0,1}(\Sigma)$ and by Gagliardo [17] (see also Grisvard [18] Sect. 1.5) it admits a lifting $\mathbf{N} \in \mathbf{W}^{1,p}(\Omega)$ for some $p > d$. Then $(p(t) - p_0)\, \mathbf{N} \in \mathbf{H}^1(\Omega)$ and since $\mathbf{N} \times \mathbf{n} = \mathbf{0}$ on Σ, it follows that $(p_0 - p(t))\, \mathbf{N} \in \mathbf{W}$. Consequently $b\, \mathbf{U}(\mathbf{u})(t) + (p_0 - p(t))\, \mathbf{N} \in \mathbf{W}$ and as

$$(b\, \mathbf{U}(\mathbf{u})(t) \cdot \mathbf{n} + p_0 - p(t), \mathbf{w} \cdot \mathbf{n})_\Sigma = (b\, \mathbf{U}(\mathbf{u})(t) \cdot \mathbf{n} + (p_0 - p(t))\, \mathbf{N} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma, \ \text{ a. e. in } \ (0,T)$$

we deduce $b\, \mathbf{U}(\mathbf{u}) \cdot \mathbf{n} = p - p_0$ in $L^2(H^{1/2}(\Sigma))$.

Assume now that Ω is polyhedric. Let S be any face of Σ and φ any function of the space

$$\Psi = \left\{ \varphi \in H^1(\Omega) \ \text{ such that } \ \gamma_0 \varphi \ \text{ is supported by } \ \Sigma \right\},$$

where γ_0 is the trace operator on Σ. Since $\mathbf{N} = \mathbf{n}_{|S}$ is a constant vector, the function $\varphi \mathbf{N}$ belongs to $H^1(\Omega)$ and $\gamma_0(\varphi \mathbf{N})$ is supported by S. Again $(\varphi \mathbf{N}) \times \mathbf{n} = \mathbf{0}$ and hence $\varphi \mathbf{N} \in \mathbf{W}$. Let $\mathbf{w} = \varphi \mathbf{N}$ for any $\varphi \in \Psi$. Then (33) reduces to

$$(b\, \mathbf{U}(\mathbf{u})(t) \cdot \mathbf{n} + p_0 - p(t), \varphi)_S = 0 \ \text{ a. e. in } \ (0,T). \tag{33}$$

Since by definition any function in $H^{1/2}_{00}(S)$ is the trace of a function $\varphi \in \Psi$, this implies that

$$b\, \mathbf{U}(\mathbf{u}) \cdot \mathbf{n} + p_0 - p = 0 \ \text{ a. e. in } \ S \times (0,T).$$

As this is valid for any face S of Σ, we derive

$$b\,\mathbf{U}(\mathbf{u})\cdot\mathbf{n}+p_0-p=0 \ \text{ a. e. in } \ \Sigma\times(0,T)$$

As $\mathbf{U}(\mathbf{u})\in H^1(H^{1/2}(\Sigma))$, then $\partial_t p=b\,\mathbf{u}\cdot\mathbf{n}$ in $L^2(H^{1/2}(\Sigma))$. Thus $p\in C^0([0,T];H^{1/2}(\Sigma))$ and $b\,\mathbf{U}(\mathbf{u})(t)=p(t)-p(0)$ in $[0,T]$. So $p(0)=p_0$ in $H^{1/2}(\Sigma)$.

□

Remark For general domains, the condition $\partial_t p=b\,\mathbf{u}\cdot\mathbf{n}$ a. e. on $\Sigma\times(0,T)$ will hold (for smooth enough $\mathbf{u}$, p_0 and p) if the set $\{\mathbf{w}\cdot\mathbf{n}\,|\,\mathbf{w}\in\mathbf{W}\}$ is dense in some $L^p(\Sigma)$.

5.2 Stability

We analyze in this section the stability of discretization (19). Let us consider the following functions:

- $\mathbf{u}_\delta\,:\,[0,T]\mapsto\mathbf{W}$ is the piecewise linear in time function that takes the value $\mathbf{u}^n$ at $t=t_n=n\delta t$,
$$\mathbf{u}_\delta(t):=\frac{t_{n+1}-t}{\delta t}\mathbf{u}^n+\frac{t-t_n}{\delta t}\mathbf{u}^{n+1}.$$
- $\tilde{p}_\delta\,:\,(0,T)\mapsto M$ is the piecewise constant in time function that takes the value p^n in the time interval (t_n,t_{n+1}). This function is defined a. e. in $(0,T)$.
- $P_\delta\,:\,[0,T]\mapsto M$ is the primitive of the discrete pressure function $\tilde{p}_\delta$.
$$P_\delta(t):=\int_0^t\tilde{p}_\delta(s)\,ds,$$
- $\tilde{\mathbf{u}}_\delta\,:\,(-\delta t,T)\mapsto\mathbf{W}$ is the piecewise constant function that takes the value $\mathbf{u}^{n+1}$ in (t_n,t_{n+1}), and $\tilde{\mathbf{u}}_\delta(t)=\mathbf{u}_\delta^0$ in $(-\delta t,0)$. This function is defined a. e. in $(-\delta t,T)$.
- $\tilde{\mathbf{u}}_\delta^-\,:\,(0,T)\mapsto\mathbf{W}$ is the piecewise constant function that takes the value $\mathbf{u}^n$ in (t_n,t_{n+1}). This function is defined a. e. in $(0,T)$.
- $\tilde{\mathbf{U}}_\delta\,:\,(0,T)\mapsto\mathbf{W}$ is the piecewise constant function that takes the value $\mathbf{U}^{n+1}$ in (t_n,t_{n+1}).
- $\tilde{\mathbf{f}}_\delta\,:\,(0,T)\mapsto\mathbf{W}'$ is the piecewise constant function that takes the value $\mathbf{f}^{n+1}$ in (t_n,t_{n+1}).

We estimate a fractional time derivative of the velocity in the Nikolskii spaces $N^{s,p}(0,T;B)$, which are sub-spaces of $L^p(0,T;B)$, where B is a Banach space. The Nikolskii space of order $r\in[0,1]$ and exponent $p\in[0,+\infty]$ is defined as

$$N^{r,p}(0,T;B)=\{f\in L^p(0,T;B)\ \text{ such that }\ \|f\|_{\tilde{N}^{r,p}}<+\infty\,\},$$

where

$$\|f\|_{\tilde{N}^{r,p}}=\sup_{\delta>0}\frac{1}{\delta^r}\|\tau_\delta f\|_{L^p(0,T-\delta;B)},$$

and $\tau_\delta f(t)=f(t+\delta)-f(t)$, $0\le t\le T-\delta$. The space $N^{r,p}(0,T;B)$, endowed with the norm

$$\|f\|_{N^{r,p}(0,T;B)}=\|f\|_{L^p(0,T;B)}+\|f\|_{\tilde{N}^{r,p}}$$

is a Banach space. We may think of $N^{r,p}(0,T;B)$ as being formed by functions whose fractional derivative in time of order r belongs to $L^p(0,T;B)$. Whenever there is no source of confusion, we shall denote $N^{s,p}(0,T;B)$ by $N^{s,p}(B)$. We also use the following (Corollary 3.19 of [13])

Lemma 4 *Assume that Ω is simply-connected and Σ is connected. Then the semi norm*

$$\|\nabla \times \mathbf{w}\|_{0,2,\Omega}$$

is a norm equivalent to the $\|\cdot\|_{\mathbf{W}}$ norm on the space $\mathbf{W}_{div}$.

We may now state the following stability result:

Theorem 5.1 *Assume that Ω is convex or is simply-connected with a $\mathcal{C}^{1,1}$ connected boundary Σ. Assume that $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$, $\mathbf{f} \in L^2(\mathbf{W}')$, $p_0 \in L^2(\Sigma)$. Then the solution of problem (19) satisfies the following estimates:*

$$\|\mathbf{u}_\delta\|_{L^\infty(\mathbf{L}^2)} + \sqrt{\nu}\,\|\mathbf{u}_\delta\|_{L^2(\mathbf{H}^1)} + \quad b\,\|\tilde{\mathbf{U}}_\delta \cdot \mathbf{n}\|_{L^\infty(\mathbf{L}^2(\Sigma))} \tag{34}$$
$$\le C_1 \left(\|\mathbf{u}_0\|_{0,2,\Omega} + \frac{1}{\sqrt{\nu}}\|\mathbf{f}\|_{L^2(\mathbf{W}')} + \frac{1}{\sqrt{\nu}}\|p_0\|_{L^2(\Sigma)} \right),$$

$$\|\mathbf{u}_\delta\|_{N^{1/4,2}(\mathbf{L}^2)} \le C_2, \tag{35}$$

and

$$\|P_\delta\|_{L^\infty(L^2)} \le C_2, \tag{36}$$

for some constant $C_1 > 0$ independent of h, δt and ν, and some constant $C_2 > 0$ independent of δt.

Proof. We proceed by steps.

STEP 1. **Velocity estimates.** To obtain estimate (34) we use

$$(\mathbf{u}^{n+1} - \mathbf{u}^n) \cdot \mathbf{u}^{n+1} = \frac{1}{2}(\mathbf{u}^{n+1} - \mathbf{u}^n) \cdot (\mathbf{u}^{n+1} + \mathbf{u}^n) + \frac{1}{2}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2, \tag{37}$$

and

$$(\mathbf{U}^{n+1} \cdot \mathbf{n}, \mathbf{u}^{n+1} \cdot \mathbf{n})_\Sigma = \left(\mathbf{U}^{n+1} \cdot \mathbf{n}, \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\delta t} \cdot \mathbf{n} \right)_\Sigma,$$

where we assume $\mathbf{U}^0 = \mathbf{0}$. Then, setting $\mathbf{w} = \mathbf{u}^{n+1}$ and $q = p^{n+1}$ in (19) yields

$$\frac{1}{2}\|\mathbf{u}^{n+1}\|^2_{0,2,\Omega} + \frac{1}{2}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega} + \delta t\,\nu\|\nabla \times \mathbf{u}^{n+1}\|^2_{0,2,\Omega} + \frac{b}{2}\|\mathbf{U}^{n+1} \cdot \mathbf{n}\|^2_{0,2,\Sigma}$$
$$+\frac{b}{2}\|\mathbf{U}^{n+1} \cdot \mathbf{n} - \mathbf{U}^n \cdot \mathbf{n}\|^2_{0,2,\Sigma}$$
$$= \frac{1}{2}\|\mathbf{u}^n\|^2_{0,2,\Omega} + \frac{b}{2}\|\mathbf{U}^n \cdot \mathbf{n}\|^2_{0,2,\Sigma} + \delta t\ < \mathbf{f}^{n+1}, \mathbf{u}^{n+1} > + \delta t\,(p_0, \mathbf{u}^{n+1} \cdot \mathbf{n})_\Sigma. \tag{38}$$

Using Lemma 4 and Young's inequality,

$$\|\mathbf{u}^{n+1}\|^2_{0,2,\Omega} + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega} + \delta t\,\nu\|\nabla \times \mathbf{u}^{n+1}\|^2_{0,2,\Omega} + b\,\|\mathbf{U}^{n+1} \cdot \mathbf{n}\|^2_{0,2,\Sigma}$$
$$\le \|\mathbf{u}^n\|^2_{0,2,\Omega} + b\,\|\mathbf{U}^n \cdot \mathbf{n}\|^2_{0,2,\Sigma} + C\,\delta t \nu^{-1}\|\mathbf{f}^{n+1}\|^2_{\mathbf{W}'} + C\,\delta t \nu^{-1}\|p_0\|^2_{0,2,\Sigma}, \tag{39}$$

for some constant $C > 0$. Summing estimates (39) for $n = 0, 1, \cdots, k$ for some $k \le N - 1$,

$$\|\mathbf{u}^{k+1}\|^2_{0,2,\Omega} + \sum_{n=0}^{k} \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega} + \nu\,\delta t \sum_{n=0}^{k} \|\nabla \times \mathbf{u}^{n+1}\|^2_{0,2,\Omega} + b\,\|\mathbf{U}^{k+1} \cdot \mathbf{n}\|^2_{0,2,\Sigma} \tag{40}$$
$$\le \|\mathbf{u}_0\|^2_{0,2,\Omega} + C\,\delta t\,\nu^{-1} \sum_{n=0}^{k} \|\mathbf{f}^{n+1}\|^2_{\mathbf{W}'} + C\,T\,\nu^{-1}\,\|p_0\|^2_{0,2,\Sigma}.$$

This yields estimate (34), as

$$\sum_{n=0}^{N-1} \delta t \, \|\mathbf{f}^{n+1}\|^2_{\mathbf{W}'} \le \|\mathbf{f}\|^2_{L^2(\mathbf{W}')}, \quad \|\mathbf{u}_\delta\|_{L^\infty(\mathbf{L}^2)} = \max_{n=0,1,\cdots,N} \|\mathbf{u}^n\|_{0,2,\Omega},$$

$$\|\tilde{\mathbf{U}}_\delta \cdot \mathbf{n}\|_{L^\infty(\mathbf{L}^2(\Sigma))} = \max_{n=0,1,\cdots,N} \|\mathbf{U}^n \cdot \mathbf{n}\|_{0,2,\Sigma}, \text{ and } \|\mathbf{u}_\delta\|^2_{L^2(\mathbf{W})} \le C\, \delta t \sum_{n=0}^{N} \|\nabla \times \mathbf{u}^n\|^2_{0,2,\Omega},$$

for some constant $C > 0$ independent of δt.

Step 2. **Velocity time increment estimates.** Let us re-state problem (19) as

$$\left\{ \begin{array}{r} (\partial_t \mathbf{u}_\delta(t), \mathbf{w}) + c(\tilde{\mathbf{u}}_\delta(t-\delta t); \tilde{\mathbf{u}}_\delta(t), \mathbf{w}) + a(\tilde{\mathbf{u}}_\delta(t), \mathbf{w}) + b\,(\tilde{\mathbf{U}}_\delta(t) \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma \\ -(\tilde{p}_\delta(t), \nabla \cdot \mathbf{w})_\Omega = \langle \tilde{\mathbf{f}}_\delta(t), \mathbf{w} \rangle - (p_0, \mathbf{w} \cdot \mathbf{n})_\Sigma \\ (\nabla \cdot \tilde{\mathbf{u}}_\delta(t), q)_\Omega = 0, \end{array} \right. \tag{41}$$

a.e. in $(0, T)$, for all $\mathbf{w} \in \mathbf{W}$. Let us integrate (41) in $(t, t+\delta)$ for $t \in [0, T-\delta]$,

$$(\tau_\delta \mathbf{u}_\delta(t), \mathbf{w})_\Omega = \int_t^{t+\delta} \langle \mathcal{F}_\delta(s), \mathbf{w} \rangle \, ds + \int_t^{t+\delta} (\tilde{p}_\delta(s), \nabla \cdot \mathbf{w})_\Omega \, ds, \tag{42}$$

where (we recall) $\tau_\delta \mathbf{u}_\delta(t) = \mathbf{u}_\delta(t+\delta t) - \mathbf{u}_\delta(t)$, and $\mathcal{F}_\delta(s) \in \mathbf{W}'$ is defined a. e. in $(0, T)$ by

$$\begin{array}{rcl} \langle \mathcal{F}_\delta(s), \mathbf{w} \rangle &=& -c(\tilde{\mathbf{u}}_\delta(s-\delta t); \tilde{\mathbf{u}}_\delta(s), \mathbf{w}) - a(\tilde{\mathbf{u}}_\delta(s), \mathbf{w}) - b\,(\tilde{\mathbf{U}}_\delta(s) \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma \\ &+& \langle \tilde{\mathbf{f}}_\delta(s), \mathbf{w} \rangle - (p_0, \mathbf{w} \cdot \mathbf{n})_\Sigma, \text{ for all } \mathbf{w} \in \mathbf{W}. \end{array}$$

Setting $\mathbf{w} = \tau_\delta \mathbf{u}_\delta(t)$ and integrating from 0 to $T - \delta$,

$$\int_0^{T-\delta} \|\tau_\delta \mathbf{u}_\delta(t)\|^2_{0,2,\Omega} \, dt = \int_0^{T-\delta} \int_t^{t+\delta} \langle \mathcal{F}_\delta(s), \tau_\delta \mathbf{u}_\delta(t) \rangle \, ds \, dt, \tag{43}$$

were we have used that $(\nabla \cdot \tau_\delta \mathbf{u}_\delta(t), \tilde{p}_\delta(s)) = 0$, a. e. for t, $s \in (0, T)$. Using the imbedding of $\mathbf{W}$ in $\mathbf{H}^1(\Omega)$,

$$\|\mathcal{F}_\delta(s)\|_{\mathbf{W}'} \le C \Big[\|\tilde{\mathbf{u}}_\delta(s-\delta t)\|^2_{\mathbf{W}} + \|\tilde{\mathbf{u}}_\delta(s)\|^2_{\mathbf{W}} + \|\nabla \times \tilde{\mathbf{u}}_\delta(s)\|^2_{0,2,\Omega} + \|\tilde{\mathbf{f}}_\delta(s)\|_{\mathbf{W}'} + \|\tilde{\mathbf{U}}_\delta \cdot \mathbf{n}\|_{0,2,\Sigma} + \|p_0\|_{0,2,\Sigma} \Big].$$

Due to estimate (34), this implies that $\mathcal{F}_\delta \in L^1(\mathbf{W}')$, and

$$\|\mathcal{F}_\delta\|_{L^1(\mathbf{W}')} \le C \tag{44}$$

for some constant $C > 0$ independent of h and δt. Now, we use Fubini's theorem to estimate the r.h.s. of (43), as follows

$$\begin{aligned} \int_0^{T-\delta} \|\tau_\delta \mathbf{u}_\delta(t)\|^2_{0,2,\Omega} \, dt &= \left| \int_0^T \int_{s-\delta}^s \langle \mathcal{F}_\delta(s), \widetilde{\tau_\delta \mathbf{u}_\delta}(t) \rangle \, dt \, ds \right| \\ &\le \int_0^T \|\mathcal{F}_\delta(s)\|_{\mathbf{W}'} \left(\int_{s-\delta}^s \|\widetilde{\tau_\delta \mathbf{u}_\delta}(t)\|_{\mathbf{W}} \, dt \right) ds \\ &\le \int_0^T \|\mathcal{F}_\delta(s)\|_{\mathbf{W}'} \, \delta^{1/2} \left(\int_{s-\delta}^s \|\widetilde{\tau_\delta \mathbf{u}_\delta}(t)\|^2_{\mathbf{W}} \, dt \right)^{1/2} ds \\ &\le C \delta^{1/2} \|\mathbf{u}_\delta\|_{L^2(\mathbf{W})} \le C\, \delta^{1/2}, \end{aligned} \tag{45}$$

for some constant C independent of h, where $\widetilde{v}$ denotes the extension by zero outside $[0, T-\delta]$ of a function v. The last line of estimates follows from (34) and (44). Estimate (45) yields (35).

STEP 3. **Estimate of the primitive of the pressure.** Let $\mathbf{w} \in \mathbf{W}$. Equation (41) yields

$$\begin{aligned} (P_\delta(t), \nabla \cdot \mathbf{w})_\Omega &= (\mathbf{u}_\delta(t) - \mathbf{u}_0, \mathbf{w})_\Omega - \int_0^t \langle \mathcal{F}_\delta(s), \mathbf{w} \rangle \, ds \\ &\leq C \left(\|\mathbf{u}_\delta\|_{L^\infty(\mathbf{L}^2)} + \|\mathbf{u}_0\|_{0,2,\Omega} + \|\mathcal{F}_\delta\|_{L^1(\mathbf{W}')} \right) \|\mathbf{w}\|_{\mathbf{W}} \\ &\leq C \, \|\mathbf{w}\|_{\mathbf{W}}, \end{aligned} \tag{46}$$

where the last estimate follows from estimates (34) and (44). Then, by the inf-sup condition (22), estimate (36) follows. □

We next prove the convergence, we need some preliminary results:

Lemma 5 *Let $\mathbf{z} \in L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{L}^4)$. Then $\mathbf{z} \in \mathbf{L}^3(Q_T)$ and*

$$\|\mathbf{z}\|_{0,3,Q_T} \leq \|\mathbf{z}\|_{L^\infty(\mathbf{L}^2)}^{1/3} \, \|\mathbf{z}\|_{L^2(\mathbf{L}^4)}^{2/3}. \tag{47}$$

Proof. Let $r \in [2,4]$. By Hölder's inequality,

$$\|\mathbf{z}(t)\|_{0,r,\Omega}^r \leq \|\mathbf{z}(t)\|_{0,2,\Omega}^{2\theta} \, \|\mathbf{z}(t)\|_{0,4,\Omega}^{4(1-\theta)} \leq \|\mathbf{z}\|_{L^\infty(\mathbf{L}^2)}^{2\theta} \, \|\mathbf{z}(t)\|_{0,4,\Omega}^{4(1-\theta)}, \text{ a. e. in } (0,T),$$

where $r = 2\theta + 4(1-\theta)$. Setting $r = 3$ we obtain $\theta = 1/2$. Integrating in time the above inequality yields (47). □

Lemma 6 *Assume that the domain Ω is convex or $\mathcal{C}^{1,1}$. Assume that the sequence $\{\mathbf{v}_\delta\}_{\delta>0} \subset L^3(Q_T)$ strongly converges to $\mathbf{v}$ in $L^3(Q_T)$. Let $\varphi \in \mathcal{D}([0,T])$, $\mathbf{w} \in \mathbf{W}$. Then $\mathbf{v}_\delta(\mathbf{x},t) \otimes \mathbf{w}(\mathbf{x}) \, \varphi(t)$ strongly converges to $\mathbf{v}(\mathbf{x},t) \otimes \mathbf{w}(\mathbf{x}) \, \varphi(t)$ in $L^2(Q_T)^{3\times 3}$.*

Proof. By Hölder's inequality,

$$\|\mathbf{v}_\delta \otimes \mathbf{w} \, \varphi - \mathbf{v} \otimes \mathbf{w} \ \varphi\|_{0,2,Q_T} \leq C \, \|\mathbf{v}_\delta - \mathbf{v}\|_{0,3,Q_T} \, \|\mathbf{w}\|_{0,6,\Omega} \, \|\varphi\|_{0,\infty,(0,T)}$$

for some constant that does not depend on δ. The conclusion follows. □

We also need the following compactness result for space-time functions (Cf. [19])

Lemma 7 *Let X, E, Y be Banach spaces such that $X \hookrightarrow E \hookrightarrow Y$ where the imbedding $X \hookrightarrow E$ is compact. Then the imbedding*

$$L^p(0,T;X) \cap N^{r,p}(0,T;Y) \hookrightarrow L^p(0,T;E) \ \text{with} \ 0 < r < 1, \ 1 \leq p \leq +\infty$$

is compact.

We are now in a position to state the convergence result:

Theorem 5.2 *Assume that Ω is convex or is simply-connected with a $\mathcal{C}^{1,1}$ connected boundary Σ. Assume that $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$, $\mathbf{f} \in L^2(\mathbf{W}')$ and $p_0 \in L^2(\Sigma)$. Then the sequence $((\mathbf{u}_\delta, p_\delta))_{\delta>0}$ contains a sub-sequence $((\mathbf{u}_{\delta'}, p_{\delta'}))_{\delta'>0}$that is weakly convergent in $L^2(\mathbf{W}) \times H^{-1}(\mathbf{L}^2)$ to a weak solution $(\mathbf{v}, p)$ of the boundary value problem (13). Moreover $(\mathbf{u}_{\delta'})_{\delta'>0}$ is weakly-* convergent in $L^\infty(\mathbf{L}^2)$ to $\mathbf{u}$, strongly in $L^2(\mathbf{L}^r)$ for $1 \leq r < 6$, and the primitives in time of the pressures $(p_{\delta'})_{\delta'>0}$ are weakly-* convergent in $L^\infty(L^2)$ to a primitive in time of the pressure p.*

If the solution of the problem (30) is unique, then the whole sequence converges to it.

Proof. We proceed by steps.

STEP 1. **Extraction of convergent sub-sequences.** By estimates (34) and (35), $\mathbf{u}_\delta$ is uniformly bounded in $L^2(\mathbf{H}^1)$, in $L^\infty(\mathbf{L}^2)$ and in $N^{1/4,2}(\mathbf{L}^2)$. The imbedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ is compact for $1 \le r < 6$ (Cf. Brézis [20], Chapter 9), and then the imbedding $\mathbf{W} \hookrightarrow \mathbf{L}^r(\Omega)$ also is compact. Applying Lemma 7 with $X = \mathbf{H}^1(\Omega)$, $E = \mathbf{L}^r(\Omega)$ and $Y = \mathbf{L}^2(\Omega)$, it follows that the sequence $(\mathbf{u}_\delta)_{\delta>0}$ is compact in $L^2(\mathbf{L}^r)$ for $1 \le r < 6$.

By estimate (36), the sequence $(P_\delta)_{\delta>0}$ is uniformly bounded in $L^\infty(L^2)$. Then the sequence $((\mathbf{u}_\delta, P_\delta))_{\delta>0}$ contains a sub-sequence (that we still denote in the same way) such that $(\mathbf{u}_\delta)_{\delta>0}$ is strongly convergent in $L^2(\mathbf{L}^r)$ to some $\mathbf{u}$, for any $1 \le r < 6$, weakly in $L^2(\mathbf{H}^1)$ and weakly-* in $L^\infty(\mathbf{L}^2)$, and $(P_\delta)_{\delta>0}$ is weakly-* convergent in $L^\infty(L^2)$ to some P. We prove in the sequel that the pair $(\mathbf{u}, \partial_t P)$ is a weak solution of Navier-Stokes equations (30) in the sense of Definition 1.

Also, by (34) the sequence $\tilde{\mathbf{u}}_\delta$ is uniformly bounded in $L^2(\mathbf{H}^1)$ and in $L^\infty(\mathbf{L}^2)$. Then, it contains a subsequence (that we may assume to be a sub-sequence of the preceding one) weakly convergent in $L^2(\mathbf{H}^1)$, weakly-* convergent in $L^\infty(\mathbf{L}^2)$ and strongly convergent in $L^2(\mathbf{L}^r)$, for any $1 \le r < 6$, to some $\tilde{\mathbf{u}}$. Both limit functions $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are equal. Indeed,

$$
\begin{aligned}
\|\mathbf{u}_\delta - \tilde{\mathbf{u}}_\delta\|^2_{L^2(\mathbf{L}^2)} &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\frac{t_{n+1}-t}{\delta t}\mathbf{u}^n + \frac{t-t_n}{\delta t}\mathbf{u}^{n+1} - \mathbf{u}^{n+1}\|^2_{0,2,\Omega}\, dt \\
&\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \frac{t_{n+1}-t}{\delta t} \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega}\, dt \\
&\le \delta t \sum_{n=0}^{N-1} \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega} \le C(\nu, \mathbf{f}, \mathbf{u}_0, p_0)\, \delta t.
\end{aligned}
$$

Similarly, $\tilde{\mathbf{u}}_\delta^-$ contains a subsequence (again assumed to be a sub-sequence of the preceding one) weakly convergent in $L^2(\mathbf{H}^1)$, weakly-* convergent in $L^\infty(\mathbf{L}^2)$ and strongly convergent in $L^2(\mathbf{L}^r)$, for any $1 \le r < 6$, to the same limit $\mathbf{u}$. Indeed,

$$
\begin{aligned}
\|\mathbf{u}_\delta - \tilde{\mathbf{u}}_\delta^-\|^2_{L^2(\mathbf{L}^2)} &\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\frac{t_{n+1}-t}{\delta t}\mathbf{u}^n + \frac{t-t_n}{\delta t}\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega}\, dt \\
&\le C(\nu, \mathbf{f}, \mathbf{u}_0, p_0)\, \delta t.
\end{aligned}
$$

STEP 2. **Limit of the momentum conservation equation.** To pass to the limit in the momentum conservation equation in (41) we re-formulate it as

$$
\begin{aligned}
&- \int_0^T (\mathbf{u}_\delta(t), \mathbf{w})_\Omega \varphi'(t)\, dt - (\mathbf{u}_0, \mathbf{w})_\Omega\, \varphi(0) + \int_0^T c(\tilde{\mathbf{u}}_\delta^-(t); \tilde{\mathbf{u}}_\delta(t), \mathbf{w})\, \varphi(t)\, dt \\
&+ \int_0^T a(\tilde{\mathbf{u}}_\delta(t), \mathbf{w})\, \varphi(t)\, dt + b \int_0^T (\tilde{\mathbf{U}}_\delta(t)\cdot \mathbf{n}, \mathbf{w}\cdot\mathbf{n})_\Sigma\, \varphi(t)\, dt \qquad (48) \\
&+ \int_0^T (P_\delta(t), \nabla\cdot\mathbf{w})_\Omega \varphi'(t)\, dt = \int_0^T \langle \tilde{\mathbf{f}}_\delta(t), \mathbf{w}\rangle\, \varphi(t)\, dt - \int_0^T (p_0, \mathbf{w}\cdot\mathbf{n})_\Sigma\, \varphi(t)\, dt, \quad \text{for all } \mathbf{w} \in \mathbf{W},
\end{aligned}
$$

for any function $\varphi \in \mathcal{D}([0,T])$ such that $\varphi(T) = 0$.

Let $\mathbf{w} \in \mathbf{W}$. The sequences $\tilde{\mathbf{u}}_\delta^-$ and $\tilde{\mathbf{u}}_\delta$ are bounded in $L^\infty(\mathbf{L}^2)$ and convergent in $L^2(\mathbf{L}^4)$, so by Lemma 5, both sequences strongly converge to $\mathbf{u}$ in $L^3(Q_T)^3$. Then

$$\lim_{\delta t\to 0} \int_0^T (\mathbf{u}_\delta(t), \mathbf{w})_\Omega \varphi'(t)\, dt = \int_0^T (\mathbf{u}(t), \mathbf{w})_\Omega \varphi'(t)\, dt.$$

To pass to the limit in the convection term, observe that by Lemma 6, $\tilde{\mathbf{u}}_\delta^-(\mathbf{x}, t) \otimes \mathbf{w}(\mathbf{x})\, \varphi(t)$ strongly converges to $\mathbf{u}(\mathbf{x}, t) \otimes \mathbf{w}\, \varphi(t)$ in $L^2(Q_T)^{3\times 3}$. Then, as $\nabla \times \tilde{\mathbf{u}}_\delta(t)$ weakly converges to $\nabla \times \mathbf{u}$ in $\mathbf{L}^2(Q_T)$,

$$\lim_{\delta t\to 0} \int_0^T (\tilde{\mathbf{u}}_\delta^-(t) \times \nabla \times \tilde{\mathbf{u}}_\delta(t), \mathbf{w})_\Omega \varphi(t)\, dt = \int_0^T (\mathbf{u}(t) \times \nabla \times \mathbf{u}(t), \mathbf{w})_\Omega \varphi(t)\, dt.$$

As $\tilde{\mathbf{u}}_\delta(t)$ is weakly convergent to $\mathbf{u}$ in $L^2(\mathbf{H}^1)$,

$$\lim_{\delta t\to 0} \int_0^T a(\tilde{\mathbf{u}}_\delta(t), \mathbf{w})\, \varphi(t)\, dt = \int_0^T a(\mathbf{u}(t), \mathbf{w})\, \varphi(t)\, dt.$$

To treat the boundary term, observe that there exists a sub-sequence of $\tilde{\mathbf{U}}_\delta \cdot \mathbf{n}$ (that we assume to be a sub-sequence of the previous one) which is weakly-* convergent in $L^\infty(L^2(\Sigma))$ to some l. Let $\mathbf{w} \in \mathbf{W}$, $\varphi \in L^1(0, T)$. Then $\mathbf{w}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, \varphi(t) \in L^1(L^2(\Sigma))$, and so

$$\lim_{\delta \to 0} \int_0^T (\tilde{\mathbf{U}}_\delta(t) \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma\, \varphi(t)\, dt = \int_0^T (l(t), \mathbf{w} \cdot \mathbf{n})_\Sigma\, \varphi(t)\, dt.$$

To identify the limit l we use the Green formula (Cf. [13, 14])

$$(\nabla \cdot \mathbf{w}, \sigma)_\Omega = (\mathbf{w} \cdot \mathbf{n}, \sigma)_\Sigma - (\mathbf{w}, \nabla \sigma)_\Omega, \quad \forall \mathbf{w} \in \mathbf{W},\ \sigma \in H^1(\Omega). \tag{49}$$

Then, as $\nabla \cdot \tilde{\mathbf{u}}_\delta = 0$,

$$(\tilde{\mathbf{u}}_\delta(t) \cdot \mathbf{n}, \sigma)_\Sigma = (\tilde{\mathbf{u}}_\delta(t), \nabla \sigma)_\Omega, \text{ for all } \sigma \in H^1(\Omega), \text{ a. e. in } (0, T).$$

Hence

$$\int_0^T (\tilde{\mathbf{U}}_\delta(t) \cdot \mathbf{n}, \sigma)_\Sigma\, \varphi(t)\, dt = \left(\int_0^T \int_0^t \tilde{\mathbf{u}}_\delta(s)\, \varphi(t)\, ds\, dt, \nabla \sigma \right)_\Omega, \text{ for all } \varphi \in L^1(0, T).$$

Thus, taking the limit $\delta \to 0$,

$$\int_0^T (l(t), \sigma)_\Sigma\, \varphi(t)\, dt = \left(\int_0^T \int_0^t \mathbf{u}(s)\, \varphi(t)\, ds\, dt, \nabla \sigma \right)_\Omega = \int_0^T (\mathbf{U}(\mathbf{u})(t) \cdot \mathbf{n}, \sigma)_\Sigma\, \varphi(t)\, dt.$$

Then

$$(l(t), \sigma)_\Sigma = (\mathbf{U}(\mathbf{u})(t) \cdot \mathbf{n}, \sigma)_\Sigma \text{ for all } \sigma \in \mathcal{D}(\overline{\Omega}) \text{ a. e. in } (0, T),$$

and we conclude that $l = \mathbf{U}(\mathbf{u}) \cdot \mathbf{n}$ in $L^\infty(L^2(\Sigma))$.

To pass to the limit in the pressure term, observe that $(P_\delta)_{\delta>0}$ is weakly-* convergent in $L^\infty(L^2)$ to P,

$$\lim_{\delta t\to 0} \int_0^T (P_\delta, \nabla \cdot \mathbf{w}(\mathbf{x}))_\Omega\, \varphi'(t)\, dt = \int_0^T (P, \nabla \cdot \mathbf{w}(\mathbf{x}))_\Omega\, \varphi'(t)\, dt.$$

Also, as $\tilde{\mathbf{f}}_\delta$ strongly converges to $\mathbf{f}$ in $L^2(\mathbf{W}')$,

$$\lim_{\delta t\to 0}\int_0^T \langle \tilde{\mathbf{f}}_\delta(t),\mathbf{w}\rangle\varphi(t)\,dt = \int_0^T \langle \mathbf{f}(t),\mathbf{w}\rangle\varphi(t)\,dt.$$

STEP 3. **Limit of the continuity equation.** Let us consider some function $q\in L^2(\Omega)$. As $\nabla\cdot\mathbf{u}_\delta$ weakly converges to $\nabla\cdot\mathbf{u}$ in $L^2(L^2)$,

$$\int_0^T (\nabla\cdot\mathbf{u}(t),q)_\Omega\,\varphi(t)\,dt = \lim_{\delta t\to 0}\int_0^T (\nabla\cdot\mathbf{u}_\delta(t),q)_\Omega\,\varphi(t)\,dt.$$

Consequently,

$$\int_0^T (\nabla\cdot\mathbf{u}(t),q)_\Omega\,\varphi(t)\,dt = 0,\quad \forall q\in L^2(\Omega),\ \forall\varphi\in\mathcal{D}(0,T). \tag{50}$$

As $\mathcal{D}(\Omega)\otimes\mathcal{D}(0,T)$ is sequentially dense in $\mathcal{D}(Q_T)$, we deduce that

$$\nabla\cdot\mathbf{u}=0\quad \text{a. e. in}\quad \Omega\times(0,T).$$

STEP 4. **Conclusion.** As a consequence of the preceding analysis, $\mathbf{u}$ belongs to $L^2(\mathbf{W}_{div})\cap L^\infty(\mathbf{L}^2)$, P belongs to $L^\infty(L^2)$, and the pair $(\mathbf{u},P)$ satisfies (30). Thus, the pair $(\mathbf{u},\partial_t P)$ is a weak solution of the Navier-Stokes problem (13) in the sense of Definition 1. As P_δ weakly converges to P in $L^2(L^2)$, then $p_\delta=\partial_t P_\delta$ weakly converges to $p=\partial_t P$ in $H^{-1}(L^2)$.

If the solution of Navier-Stokes equations (30) is unique, then the whole sequence converges to it, as this proof is based upon a standard compactness argument. □

6 Other Time Discretizations

The above scheme may be extended to second order by means of the θ-scheme,

Find $\mathbf{u}_\delta^{n+1}\in\mathbf{W}$, $p_\delta^{n+1}\in M$ such that for all $\mathbf{w}\in\mathbf{W}$, $q\in M$,

$$\left\{\begin{array}{rcl} \left(\dfrac{\mathbf{u}_\delta^{n+1}-\mathbf{u}_\delta^n}{\delta t},\mathbf{w}\right)_\Omega + c(\mathbf{u}_\delta^{n+\varepsilon\theta};\mathbf{u}_\delta^{n+\theta},\mathbf{w}) + a(\mathbf{u}_\delta^{n+\theta},\mathbf{w}) & & \\ +(\mathbf{U}_\delta^{n+\theta}\cdot\mathbf{n},\mathbf{w}\cdot\mathbf{n}) - (p_\delta^{n+1},\nabla\cdot\mathbf{w})_\Omega & = & \langle\mathbf{f}^{n+\theta},\mathbf{w}\rangle, -(p_0,\mathbf{w}\cdot\mathbf{n})_\Sigma, \\ (\nabla\cdot\mathbf{u}_\delta^{n+\theta},q)_\Omega & = & 0, \end{array}\right. \tag{51}$$

where $0\le\theta\le 1$, $\varepsilon=0$ or 1, and

$$\mathbf{u}_\delta^{n+\theta}=\theta\mathbf{u}_\delta^{n+1}+(1-\theta)\mathbf{u}_\delta^n,\ \mathbf{U}_\delta^{n+\theta}=\mathbf{U}_\delta^n+\delta t\,\mathbf{u}_\delta^{n+\theta},$$
$$\mathbf{f}^{n+\theta}=\theta\mathbf{f}^n+(1-\theta)\mathbf{f}^{n+1}.$$

The choice $\varepsilon=1$, $\theta=1/2$ corresponds to the Crank-Nicolson scheme, which is second-order accurate in time. When $\varepsilon=1$, for any θ this is a fully implicit scheme, in particular $\theta=1$ corresponds to the fully implicit Euler scheme.

The stability for $\theta\ge 1/2$ follows from the identity

$$(\mathbf{u}_\delta^{n+1}-\mathbf{u}_\delta^n,\mathbf{u}_\delta^{n+\theta})_\Omega = \frac{1}{2}\,\|\mathbf{u}_\delta^{n+1}\|_{0,2,\Omega}^2 - \frac{1}{2}\,\|\mathbf{u}_\delta^{n+1}\|_{0,2,\Omega}^2 + \left(\theta-\frac{1}{2}\right)\|\mathbf{u}_\delta^{n+1}-\mathbf{u}_\delta^n\|_{0,2,\Omega}^2.$$

while the convergence is proved in a similar way.

7 Full Discretization and Numerical tests

7.1 Discretization with a Finite Element Method

Assume that Ω is polyhedric. Let T_h be a triangulation made of K tetraedra $\{T_k\}_1^K$ with the usual conformity hypotheses; let $\Omega := \cup_k T_k \subset \mathbb{R}^3$.

Consider the Taylor-Hood ($P^2 - P^1$) element , see for instance [21] or [14], built from

$$\begin{aligned}\mathbf{V}_h &= \{\mathbf{u} \in C^0(\overline{\Omega})^3 \ : \ v_i|_{T_k} \in P^2, \forall k \in T_h, \ i = 1,2,3\},\\ Q_h &= \{q \in C^0(\overline{\Omega}) \ : \ q|_{T_k} \in P^1, \forall k \in T_h\}.\end{aligned} \tag{52}$$

In practice the boundary Σ is decomposed into the inflow region Σ^-, the outflow region Σ^+, and the vessel walls Σ^w. The boundary conditions are then

$$\begin{aligned}&p = p^- \ \text{ on } \ \Sigma^- \times (0,T), \, p = p^+ \ \text{ on } \ \Sigma^+ \times (0,T),\\ &\partial_t p = b\, \mathbf{u} \cdot \mathbf{n} \ \text{ on } \ \Sigma^w \times (0,T), \, p(0) = p^0 \ \text{ on } \ \Sigma^w,\\ &\mathbf{u} \times \mathbf{n} = \mathbf{0} \ \text{ on } \ \Sigma \times (0,T).\end{aligned}$$

We denote for simplicity $\Gamma = \Sigma^- \cup \Sigma^+$, $p_\Gamma = \begin{cases} p^- & \text{on} \quad \Sigma^-, \\ p^+ & \text{on} \quad \Sigma^+ \end{cases}$.

A feasible discretization of (20) is to find $\mathbf{u}^{n+1} \in \mathbf{V}_h, \ p^{n+1} \in Q_h$ such that for all $\mathbf{w} \in \mathbf{V}_h, q \in Q_h$,

$$\begin{aligned}&\int_\Omega [\mathbf{w} \cdot (\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} - \mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n) - p^{n+1} \nabla \cdot \mathbf{w} - q \nabla \cdot \mathbf{u}^{n+1}] + \nu \int_\Omega \nabla \times \mathbf{u}^{n+1} \cdot \nabla \times \mathbf{w}\\ &+ \frac{1}{\epsilon} \int_\Sigma (\mathbf{u}^{n+1} \times \mathbf{n}) \cdot (\mathbf{w} \times \mathbf{n}) + \int_\Sigma b \mathbf{w} \cdot \mathbf{n} (\mathbf{u}^{n+1} \delta t + \mathbf{U}^n) \cdot \mathbf{n} = \int_\Omega \mathbf{f}^{n+1}\, \mathbf{w} \ - \int_\Gamma p_\Gamma\, \mathbf{w} \cdot \mathbf{n},\\ &\mathbf{U}^{n+1} = \mathbf{U}^n + \mathbf{u}^{n+1} \delta t.\end{aligned} \tag{53}$$

Notice that $\mathbf{u}^{n+1} \times \mathbf{n}|_\Sigma = \mathbf{0}$ is implemented by penalty. Indeed, as shown by V. Girault in [22] it would be vain to require $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ in strong form unless Nedelec elements of degree 2 at least be used.

Notice also that it is more convenient for the implementation to define $\mathbf{U}$ everywhere, not just on Σ.

Letting $\mathbf{w} = \mathbf{u}^{n+1}, \ q = -p^{n+1}$ gives the following energy estimate:

$$\begin{aligned}&\frac{1}{2\delta t}(\|\mathbf{u}^{n+1}\|^2_{0,2,\Omega} - \|\mathbf{u}^n\|^2_{0,2,\Omega}) + \frac{1}{2\delta t}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega} + \nu \|\nabla \times \mathbf{u}^{n+1}\|^2_{0,2,\Omega} + \frac{1}{\epsilon}\|\mathbf{u}^{n+1} \times \mathbf{n}\|^2_{0,2,\Sigma}\\ &+ \frac{b\delta t}{2}\|\mathbf{u}^{n+1} \cdot \mathbf{n}\|^2_{0,2,\Sigma} + \frac{1}{\delta t}(\|\mathbf{U}^{n+1} \cdot \mathbf{n}\|^2_{0,2,\Sigma} - \|\mathbf{U}^n \cdot \mathbf{n}\|^2_{0,2,\Sigma}) = \int_\Omega \mathbf{f}^{n+1}\, \mathbf{u}^{n+1} - \int_\Gamma p_\Gamma\, \mathbf{u}^{n+1} \cdot \mathbf{n}.\end{aligned} \tag{54}$$

This implies the stability of the scheme, similarly to the analysis performed in the preceding section. Moreover, we deduce

$$\|\mathbf{u}^{n+1} \times \mathbf{n}\|_{0,2,\Sigma} \leq C\,(\|\mathbf{f}^{n+1}\|_{\mathbf{W}'} + \|p_\Gamma\|_{0,2,\Sigma})\,\sqrt{\epsilon}.$$

In practice if the domains has curved boundaries it should be approximated by polyhedric domains. It is well known that this generates an error of order $\sqrt{h}$ in the approximation of $\mathbf{u}^{n+1} \times \mathbf{n} = \mathbf{0}$. Then the optimal choice is $\epsilon = h$.

7.2 Comparison with Another Scheme

Now we consider the boundary conditions (5) directly as studied in [10, 23, 21] with the following scheme: $\forall\ [\mathbf{w}, q, \zeta] \in \mathbf{V}_h \times Q_h \times Q_h$,

$$
\begin{aligned}
&\int_\Omega [\mathbf{w}\cdot(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t} - \mathbf{u}^{n+1}\times\nabla\times\mathbf{u}^n) - p^{n+1}\nabla\cdot\mathbf{w} - q\nabla\cdot\mathbf{u}^{n+1}] \\
&+\int_\Omega [\nu\nabla\mathbf{u}^{n+1} : \nabla\mathbf{w} + \epsilon\nabla\eta^{n+1}\cdot\nabla\zeta] \\
&+\int_\Sigma b[\eta^{n+1}\mathbf{w}\cdot\mathbf{n} - \zeta(\mathbf{u}\cdot\mathbf{n}^{n+1} - \frac{1}{\delta t}(\eta^{n+1}-\eta^n)) + \frac{1}{\epsilon}(\mathbf{u}^{n+1}\times\mathbf{n})\cdot(\mathbf{w}\times\mathbf{n})] \\
&= \int_\Omega \mathbf{f}^{n+1}\,\mathbf{w} - \int_\Gamma p_\Gamma \mathbf{w}\cdot\mathbf{n},
\end{aligned}
\tag{55}
$$

where ϵ is any small positive parameter.

An energy conservation identity is derived by choosing $\mathbf{w} = \mathbf{u}^{n+1}$, $q = -p^{n+1}$, $\zeta = \eta^{n+1}$:

$$
\begin{aligned}
&\frac{1}{2\delta t}(\|\mathbf{u}^{n+1}\|^2_{0,2,\Omega} - \|\mathbf{u}^n\|^2_{0,2,\Omega}) + \frac{1}{2\delta t}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2_{0,2,\Omega} + \nu\|\nabla\mathbf{u}^{n+1}\|^2_{0,2,\Omega} + \frac{1}{\epsilon}\|\mathbf{u}^{n+1}\times\mathbf{n}\|^2_{0,2,\Sigma} \\
&+\epsilon\|\eta^{n+1}\|^2_{0,2,\Omega} + \frac{1}{2\delta t}(\|\eta^{n+1}\|^2_{0,2,\Omega} - \|\eta^n\|^2_{0,2,\Omega}) + \frac{1}{2\delta t}\|\eta^{n+1}-\eta^n\|^2_{0,2,\Omega} \\
&= \int_\Omega \mathbf{f}^{n+1}\,\mathbf{u}^{n+1} - \int_\Gamma p_\Gamma \mathbf{u}^{n+1}\cdot\mathbf{n}
\end{aligned}
$$

Again this implies the stability of the scheme.

7.3 Numerical Tests

The full model requires that at every time step Σ^w be moved along its normal by a quantity $\delta t\mathbf{u}\cdot\mathbf{n}$. To preserve the triangulation we follow the literature [8] and solve an additional problem

$$
-\Delta d^{n+1} = 0 \text{ in } \Omega, \quad d^{n+1}|_{\Sigma^w} = d^n + \delta t\mathbf{u}^n\cdot\mathbf{n}, \quad d^{n+1}|_{\Gamma^-\cup\Gamma^+} = 0, \tag{56}
$$

and then move every vertex q^j of the triangulation: $q^j \mapsto q^j + \kappa d$. In theory $\kappa = 1$ but for graphic enhancement it can be adjusted. Note however that (56) is expensive.

The geometry is a quarter of a torus with $R = 4$, $r = 2$. The parameters of the problem are

$$
p^- = 0, \quad p^+ = 1 \quad \delta t = 0.05, \quad \nu = 0.001, \quad b = 200, \quad \epsilon = 0.001.
$$

The geometry is updated for visualization purposes with a multiplicative factor 100. The surfaces of constant pressure are shown for both methods at $T = 0.8$.

Two time schemes have been tested for both problems: Euler's scheme as written in (53) and (55); and Crank-Nicolson's scheme which would be second order if we had symmetrized the nonlinear terms, which we did not do because it jeopardizes the stability of the method. The scheme is obtained by changing δt into $\delta t/2$ and setting $\mathbf{u}^{n+1} = 2\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}^n$ where $\mathbf{u}^{n+\frac{1}{2}} = \tilde{\mathbf{u}}^{n+1}$ computed by solving (53) or (55).

The surfaces of equal pressures are shown on figure 1. Notice that there are more differences between the results obtained by Euler and Crank-Nicolson schemes than by (53) and (55). This comforts us in trusting the small modifications done to the setting of the model to pass from (55) to (53).

The computations have been made with the software *freefem++*[24].

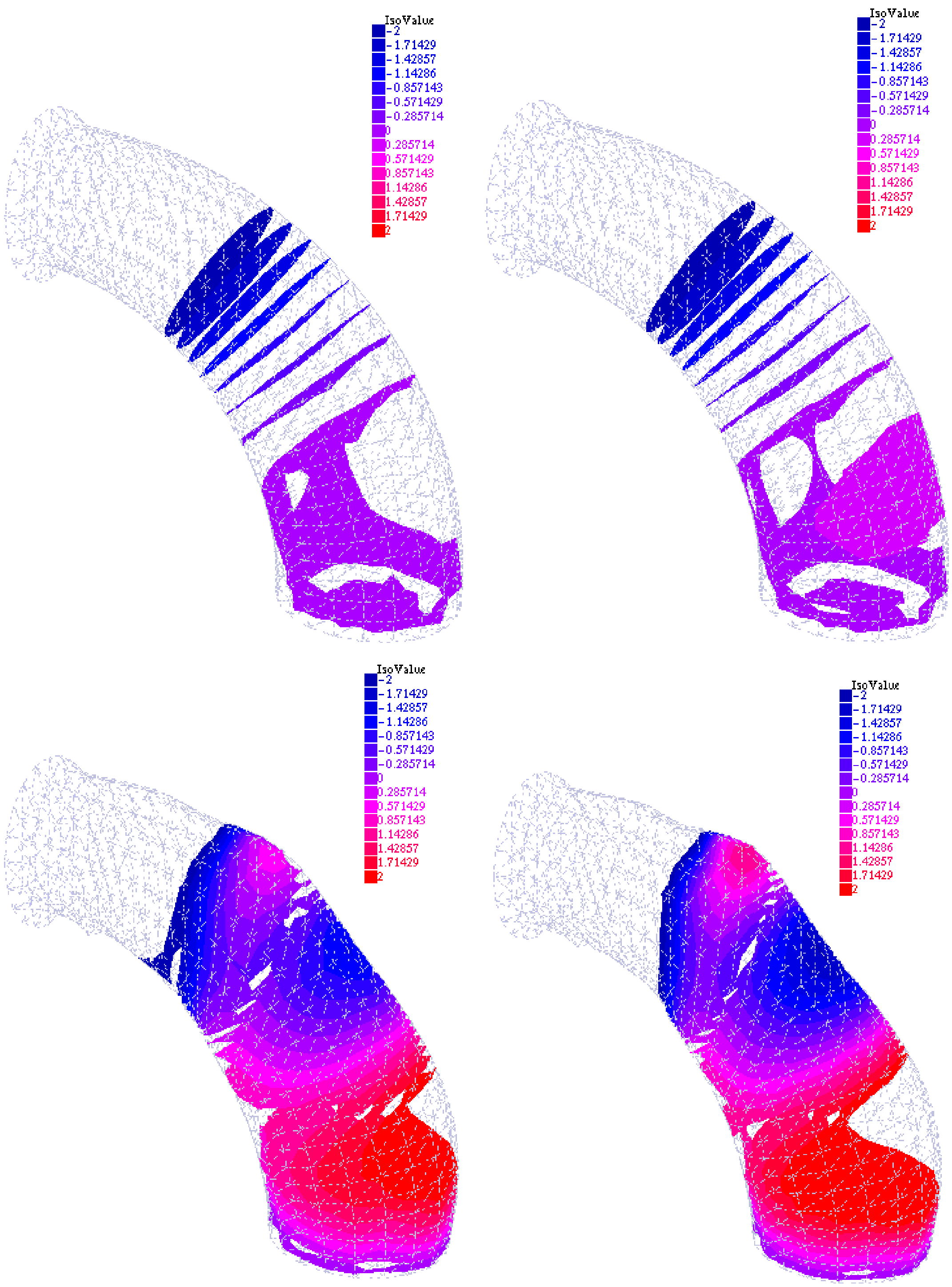

Figure 1: Surface of equal pressure at $t = 0.8$. Top left: computed by solving (53) with Euler's scheme. Top right: computed by solving (55) with Euler's scheme. Bottom left: computed by solving (53) with Crank-Nicolson's scheme. Bottom right: computed by solving (55) with Crank-Nicolson's scheme.

8 Conclusion

By a few minor modifications to the Surface Pressure model for blood flow we have obtained a model which gives similar numerical results on our preliminary tests and which is fully analyzed mathematically in the continuous case. It remains to show that the finite element discretization is stable. The penalty of the condition $\mathbf{u}\times\mathbf{n}$ probably weakens the error estimates unless $\epsilon \sim h^2$, the size of the tetrahedra. But convergence might be difficult to establish on a polygonal surface with non-parametric elements. Assuming that it converges with the mesh size and the time step decreasing to zero, the scheme is a truly implicit fluid-structure method and, being on a fixed mesh, it is much more stable than those on moving meshes which require iterations between the solid part and the fluid part and preconditioning by things like added mass.

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