Analysis of a coupled fluid-structure model with applications to hemodynamics

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Analysis of a Simplified Coupled Fluid-Structure Model for Computational Hemodynamics

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Abstract

We analyze a simplified coupled model for arterial blood flow. The vessel wall reaction to the fluid is modeled by the Surface Pressure Model which assumes that the normal stress on the fluid is proportional to the displacement of the structure. This leads to a unique boundary problem for Navier-Stokes equations, where at the wall the velocity is normal to the wall, and proportional to the time derivative of the pressure. We prove that the problem is well posed and show that a semi-implicit time discretization converges. We present some numerical results and a comparison with a standard test case. implementation of the Surface Pressure Model where the displacement is not eliminated. The analysis extends to more complex elastoplastic shell models for the vessel walls.

1 Introduction

Computational hemodynamics is potentially an important technique to study by-passes, stents and heart valves (see [39],[16] or [38] and the references therein).
Modeling blood flow in vessels can be done by a large variety of approximations ranging from nonlinear elasticity to fixed walls for the vessels and non-Newtonian Navier-Stokes to Stokes flow for the fluid.
Modeling the blood vessel is difficult as it is a complex material for which the rehology is unclear because different in vitro from in vivo [39]. In the future, no doubt computers will be able to handle this complexity and one will use large displacement nonlinear models for the structure [11]. However in the mean time there is a need for fast, well understood and appropriate though less accurate models.
To handle the complexity of moving walls, the method of immersed boundaries has been used (if not invented) by Charles Peskin, the pionner of computational hemodynamics [13, 14, 15, 40]. However the mathematical analysis of this method is difficult [1, 12] and it is also difficult to imbed elaborated viscoelastic model for the vessels. Most authors prefer to follow the moving boundaries and use separate models for the fluid and the structure and couple them at each

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time step. It is not possible to cite all the contributors to this field, the literature is too large; let us just mention some, for reference [3, 2, 17, 41, 16].

Linear elasticity with small displacement for the vessels can be applied either in 3D (see [2] for instance) on a thick wall or in 2D via a shell model as in [18] (see also the excellent book by articles [16]). Still the fact that elasticity is written on a fixed domain while the fluid domain is moving creates a computational difficulty and causes instabilities which also engendered a large literature (see for instance [6, 20, 21]).

In this paper we propose to investigate a system which is derived from Nobile and Vergara's variational fluid / shell-structure model by using transpiration boundary conditions (see also [4],[24]). Let us state clearly that we have neither the intension nor the authority for modeling blood flow. Our claim here is to show that transpiration is mathematically consistent with the small displacement approximation made to use linear visco-elasticity and that the set of derived equations have nice suitable properties for a mathematical proof of existence with a coherent set of boundary and initial conditions.

Transpiration is an old idea in CFD [23] and it has been used in the nineties to analyze wing flutter [5] and for conditioning the fluid-structure coupling algorithm [20]. The idea is simple. If a boundary condition like $u = g$ has to be imposed on a part of the boundary, $\Sigma_t = \{\mathbf{x} + \eta\mathbf{n} \ : \ \mathbf{x} \in \Sigma_0\}$ where $\eta(\mathbf{x}, t)$ denotes the motion of Σ_t measured in the normal direction $\mathbf{n}(\mathbf{x})$ with respect to its position at rest Σ_0, then one may write

$$u(\mathbf{y}) = g(\mathbf{y}) \ \forall \mathbf{y} := \mathbf{x} + \eta\mathbf{n} \in \Sigma_t \ \Leftrightarrow \ u(\mathbf{x}) + \eta\nabla u^T(\mathbf{x})\mathbf{n} = g(\mathbf{x}) + \eta\nabla g^T(\mathbf{x})\mathbf{n} + o(\eta), \quad \forall \mathbf{x} \in \Sigma_0 \quad (1)$$

Such approximation is in line with the small displacement hypothesis made to use linear elasticity. Typically a large vessel like aorta has a radius of 1 centimeter and for computational purposes a section of length of 5 to 10 centimeters; the thickness of the vessel wall is around 0.1cm; the heart pulse is about 1Hz and the pressure amplitude change is roughly 6KPa [16]; these numbers induce small displacements indeed [16].

Nobile-Vergara [18] make a second approximation, that lateral displacements can be neglected in the shell model. Then they show that Koiter's model reduces to a scalar equation for the normal displacement η on the mean position Σ of the vessel's wall. With a visco-elastic pre-stress model [20, 2] [7]) the normal displacement is governed by

$$\rho^s h\partial_{tt}\eta - \nabla\cdot(\mathbf{T}\nabla\eta) - \nabla\cdot(\mathbf{C}\nabla\partial_t\eta) + a\partial_t\eta + b\eta = f^s, \ \eta, \partial_t\eta \text{ given at } t = 0. \quad (2)$$

Here h denotes the average thickness of the vessel and ρ^s its volumetric mass; $\mathbf{T}$ is the pre-stress tensor, $\mathbf{C}$ and a ere visco-elastic damping terms and b is also a *visco-elastic parameter*; f^s the external normal force on the shell, i.e. $f^s = -\sigma^{\mathbf{s}}{}_{nn}$ the normal component of the normal stress at the surface of the solid.

Notice that in this context and due to the assumption of normal displacements the other components of the normal stress tensor do not appear and hence cannot be matched with those of the fluid.

A particular case is the so-called *Surface Pressure Model*, when everything is neglected on the left of (2) except the last term. For a cylinder b is computable, giving

$$b\eta = -\sigma^{\mathbf{s}}{}_{nn}, \text{ with } b = \frac{Eh\pi}{A(1-\xi^2)}, \quad (3)$$

where A is the vessel's cross section, E the Young modulus, ξ the Poisson coefficient. For more complex shapes b depends on the coefficients of the map of the cross section to a reference circle, but to higher order terms in η it is not a function of η. A general formula is given in [18] (see (2.2)). However note that there is a hidden assumption here: the vessel is shaped like a pipe with smooth and slowly varying cross sections; a bifurcation is ruled out, for instance (but Koiter's model doesn't allow it either).
The following typical values (see [16]) will be used in the numerical tests:

$$E = 3M\text{Pa}, \quad \xi = 0.3, \quad A = \pi R^2, \; R = 0.01\text{m}, \quad h = 0.001\text{m} \quad \Rightarrow \; b = 3.310^7\text{ms}^{-2}. \tag{4}$$

The *Surface Pressure Model* is an interesting prototype to understand the complexity of fluid-structure interactions. For brievety we will use it to state the mathematical results, keeping in mind, however, that the results are not hard to extend to the full equation (2).

The paper is organized as follows. In Section 2 we introduce the simplified coupled model that we study in the paper. Section 3 is devoted to introduce a semi-discretization in time of the simplified model, that is proved to be well posed in Section 4. In Section 5 we prove the stability and convergence of the semi-discretization. Finally Section 6 presents some relevant numerical tests and comparison to other models.

2 Modeling

Assuming blood flow to be Newtonian in a time varying domain Ω_t; let Σ_t be the part of the boundary at the interface between the solid and the fluid at time t. We denote Σ (resp. Ω) a reference position of Σ_t (resp Ω_t); it could be Σ_0. Let $\mathbf{n}$ be the normal to Σ pointing outside Ω. The model for the blood vessel gives

$$\Sigma_t = \{x + \eta(x,t)\mathbf{n}(x) \;:\; x \in \Sigma\} \tag{5}$$

The *Navier-Stokes equations* link the fluid velocity $\mathbf{u}$ and the pressure p by

$$\rho^f(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\cdot\sigma^{\mathbf{f}} = \mathbf{0}, \quad \nabla\cdot\mathbf{u} = 0, \text{ in } \Omega_t, \tag{6}$$

where ρ^f is the volumetric mass density of the fluid, μ the viscosity and $\sigma^f = -p\mathbf{I}+\mu(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$ is the stress tensor.

With the *surface pressure model* (3) for the vessels Σ we have

$$b\eta|_x = (p - \mu\;\mathbf{n}^T(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\mathbf{n})|_{x+\eta\mathbf{n}}, \;\; \partial_t\eta\;\mathbf{n}|_x = \mathbf{u}|_{x+\eta\mathbf{n}}, \quad \forall x \in \Sigma. \tag{7}$$

Notice however that as Σ_t moves normally to itself $\mathbf{n}|_x = \mathbf{n}|_{x+\eta\mathbf{n}}$, so that the second conditions imply $\mathbf{u}\times\mathbf{n} = 0$ on Σ_t.

A variational formulation for (6,7) has been studied in [18] and also in [4] for an approximation which neglects the motion of the moving domain and to which we will compare our numerical results in section 7.2. However the following has not been realized by the previous authors:

Proposition 1 *When Σ_t is smooth in a neighborhood $\mathcal{V}$ of $\tilde{x}\in\Sigma_t$ and $\nabla\cdot\mathbf{u} = 0$ in $\Omega_t\cap\mathcal{V}$ and $\mathbf{u}\times\mathbf{n} = 0$ on $\mathcal{V}\cap\Sigma_t$, then,*

$$\frac{1}{2}\mathbf{n}^T(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\mathbf{n} = \frac{\partial\mathbf{u}}{\partial n}\cdot\mathbf{n} = -2\frac{\mathbf{u}\cdot\mathbf{n}}{R} + O(|\nabla\tilde{R}^{-1}|) + O(|\nabla r^{-1}|) \;\; \text{at } \tilde{x}\in\Sigma_t \tag{8}$$

where $\tilde{R}, r$ are the principal radius of curvature and $R = (R^{-1}+r^{-1})^{-1}$ is the mean radius of curvature of Σ at x.

Proof Let us work with simple toroidal coordinates $(r, \theta, \phi) \to (x = \tilde{R}\cos\phi,\ y = \tilde{R}\sin\phi,\ z = r\sin\theta)$ where $\tilde{R} = R_0 + r\cos\theta$, such that $\tilde{x}$ is at $\phi = \theta = 0$ and Σ_t is tangent to the torus, i.e. $r, R_0 + r$ are its two principal radius of curvature at $\tilde{x}$. Recall that (see for example [9] Appendix A.)

$$\nabla \cdot \mathbf{u} = h_r h_\theta h_\phi (\partial_r \frac{u_r}{h_\theta h_\phi} + \partial_\theta \frac{u_\theta}{h_\phi h_r} + \partial_\phi \frac{u_\phi}{h_r h_\theta}) \tag{9}$$

with $h_r = 1,\ h_\theta = \frac{1}{r},\ h_\phi = \frac{1}{R}$ because, by definition

$$\frac{1}{h_k^2} = (\partial_k x)^2 + (\partial_k y)^2 + (\partial_k z)^2, \quad k = r, \theta, \phi \tag{10}$$

So $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} \times \mathbf{n} = 0$ imply

$$\nabla \cdot \mathbf{u} = \partial_r u_r + u_r \frac{R_0 + 2r\cos\theta}{r(R_0 + r\cos\theta)} = 0 \Rightarrow \partial_r u_r = -u_r(\frac{1}{r} + \frac{\cos\theta}{\tilde{R}}) \Rightarrow \partial_r u_r = -2\frac{u_r}{R} \text{ at } \tilde{x}. \tag{11}$$

Similarly

$$\nabla \mathbf{u} = \sum_i e^i h_i \otimes \partial_i \sum_k e^k \mathbf{u}_k, \quad i, k \in (r, \theta, \phi) \tag{12}$$

with

$$e^r = \cos\theta(\cos\phi, \sin\phi, \tan\theta)^T, e^\theta = \sin\theta(-\cos\phi, -\sin\phi, \frac{1}{\tan\theta})^T, e^\phi = (-\sin\phi, \cos\phi, 0)^T \tag{13}$$

As $u_\theta = u_\phi = 0$, we have $\mathbf{n}^T(\nabla\mathbf{u})\mathbf{n} = e^{rT}\big(e^r \otimes e^r h_r \partial_r u_r\big)e^r = \partial_r u_r$ which is also $\frac{\partial \mathbf{u} \cdot \mathbf{n}}{\partial n}$.
□

2.1 Transpiration Approximation

Following (1) and using (8), (7) becomes

$$b\eta = p + \frac{4\mu}{R}\mathbf{u} \cdot \mathbf{n} + \eta(\frac{\partial p}{\partial n} + \frac{4\mu}{R}\frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{n}),\ \partial_t \eta = (\mathbf{u} + \eta\frac{\partial \mathbf{u}}{\partial n}) \cdot \mathbf{n}, \quad \text{on } \Sigma. \tag{14}$$

Thanks to (8), with $\alpha = \frac{4\mu}{R}(1 - \frac{2}{R})$, it is also

$$b\eta = p + \alpha\mathbf{u} \cdot \mathbf{n} + \eta\frac{\partial p}{\partial n},\ \partial_t \eta = (1 - \eta\frac{2}{R})\mathbf{u} \cdot \mathbf{n}, \quad \text{on } \Sigma. \tag{15}$$

The last term is second order and may be droped.

In laminar flows at Reynolds number of a few thousands or less, $\frac{\partial p}{\partial n}$ is somewhat smooth across the boundary layer near Σ so when the problem is adimensionalized it is $O(\rho^f)$. On the other hand b is large for arterial flows. In that case $\eta\frac{\partial p}{\partial n}$ is dominated by $b\eta$ and so we may drop it; $\mu\mathbf{u} \cdot \mathbf{n}$ is also small at high Reynold nummber, but it does not simplify the model to remove it. So we make the hypothesis

Hypothesis 1.

$$b >> \frac{\partial p}{\partial n}|_\Sigma \tag{H1}$$

Then the matching conditions on Σ reduce to

$$\mathbf{u}\times\mathbf{n}=0,\ p+\alpha\mathbf{u}\cdot\mathbf{n}=b\int_0^t \mathbf{u}\cdot\mathbf{n}\mathrm{d}\tau+c,\ \text{with } c:=p_0+\alpha\mathbf{u}_0\cdot\mathbf{n} \tag{16}$$

Remark 2.1 If the full visco-elastic model is used, under the same conditions, on Σ we will have,

$$\begin{aligned}&\mathbf{u}\times\mathbf{n}=0,\quad U_n:=\int_0^t \mathbf{u}\cdot\mathbf{n}\mathrm{d}\tau,\\ &\rho^s h\partial_{tt}U_n-\nabla\cdot\big(\mathbf{T}\nabla U_n\big)-\nabla\cdot\big(\mathbf{C}\nabla(\mathbf{u}\cdot\mathbf{n})\big)+(a-\alpha)\mathbf{u}\cdot\mathbf{n}+bU_n=p-c\end{aligned} \tag{17}$$

2.2 Other Boundary and Initial Conditions

For a portion of blood vessel, at the (artificial) input/output cross sections Γ, we shall assume that either the pressure or the flux is given; more generally

- On $\Gamma_p\subset\Gamma$ the flow is normal, i.e. $\mathbf{u}\times\mathbf{n}=0$, and that the pressure is given by $p=p_\Gamma$;
- on $\Gamma_f=\Gamma\backslash\Gamma_p$ the flux is given: $-p\mathbf{n}+\mu(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\mathbf{n}=\mathbf{g}$.

Initial conditions are needed for (6), for instance

$$\mathbf{u}(0)=\mathbf{u}_0 \text{ in } \Omega,\quad p(0)=p_0 \text{ on } \Sigma, \tag{18}$$

2.3 Energy considerations

Now there is a problem! (6)+(16)+(18) no longer preserves energy; therefore it is necessary to replace the nonlinear terms $\mathbf{u}\cdot\nabla\mathbf{u}$ in (6) by $-\mathbf{u}\times\nabla\times\mathbf{u}$ according to the identity,

$$\mathbf{u}\cdot\nabla\mathbf{u}=-\mathbf{u}\times\nabla\times\mathbf{u}+\nabla\frac{|\mathbf{u}|^2}{2}. \tag{19}$$

In effect this will replaces the pressure by the dynamic pressure $p+\frac{1}{2}|u|^2$. On Σ this change is compatible with the small displacement hypothesis because the change is $|\mathbf{u}|^2_{|\Sigma}$; it is not so inside Ω.

Finally recalling the identities

$$-\nabla\cdot(\nabla\mathbf{u}+\nabla\mathbf{u}^T)=-\Delta\mathbf{u}=\nabla\times\nabla\times\mathbf{u}-\nabla(\nabla\cdot\mathbf{u}), \tag{20}$$

the modified Navier-Stokes system for fluid-structure interactions is written in a fixed domain Ω as

$$\begin{aligned}&\partial_t\mathbf{u}-\mathbf{u}\times\nabla\times\mathbf{u}+\nu\nabla\times\nabla\times\mathbf{u}+\nabla\tilde{p}=\mathbf{0},\quad \nabla\cdot\mathbf{u}=0,\\ &\mathbf{u}\times\mathbf{n}_{|\Sigma}=0,\quad U_n:=\int_0^t\mathbf{u}\cdot\mathbf{n}\mathrm{d}\tau,\end{aligned}$$

$$\gamma\partial_{tt}U_n - \nabla\cdot\big(\tilde{\mathbf{T}}\nabla U_n\big) - \nabla\cdot\big(\tilde{\mathbf{C}}\nabla(\mathbf{u}\cdot\mathbf{n})\big) + (\tilde{a}-\tilde{\alpha})\mathbf{u}\cdot\mathbf{n} + \tilde{b}U_n = \tilde{p} - \tilde{c} \tag{21}$$

where $\gamma := \frac{\rho^s}{\rho^f}h$, $\nu := \frac{\mu}{\rho^f}$,$\tilde{p} := \frac{p}{\rho^f} + \frac{1}{2}|\mathbf{u}|^2$ and the $\tilde{.}$ indicates a division by ρ^f; in particular $\tilde{b} := \frac{b}{\rho^f}$.

In the sequel we drop the tilde over p and b and rename $a \leftarrow a - \alpha$. Also for clarity and without loss of generality we assume that $\Gamma = \emptyset$, i.e. $\Sigma = \partial\Omega$ and we replace $\mathbf{g}$ and p_Γ by a volumic force $\mathbf{f}$. A remark will be given later concerning the full variational formulation without this hypothesis.

Therefore we shall consider the system

$$\left\{\begin{array}{rll} \partial_t\mathbf{u} - \mathbf{u}\times\nabla\times\mathbf{u} + \nu\,\nabla\times\nabla\times\mathbf{u} + \nabla p = \mathbf{f} & \text{in} & (0,T)\times\Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in} & (0,T)\times\Omega, \\ \mathbf{u}\times\mathbf{n}_{|\Sigma} = 0, \ \ U_n := \int_0^t \mathbf{u}\cdot\mathbf{n}\mathrm{d}\tau, & \text{on} & (0,T)\times\Sigma, \\ \gamma\partial_{tt}U_n - \nabla\cdot\big(\mathbf{T}\nabla U_n\big) - \nabla\cdot\big(\mathbf{C}\nabla(\mathbf{u}\cdot\mathbf{n})\big) + a\mathbf{u}\cdot\mathbf{n} + bU_n + c = p & \text{on} & (0,T)\times\Sigma, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in} & \Omega, \\ p(0) = p_0 & \text{in} & \Omega. \end{array}\right. \tag{22}$$

2.4 Variational Formulation

When the momentum equation is hit with $\mathbf{v}$ and integrated in Ω a boundary term for the pressure appears. This trick is to replace p by the expression given by the 4^{th} equation. Then we must find $\mathbf{u}, p$ such that $\mathbf{u}\times\mathbf{n}_{|\Sigma} = 0$ and

$$\begin{aligned} &\int_\Omega \Big(\mathbf{v}\cdot(\partial_t\mathbf{u} - \mathbf{u}\times\nabla\times\mathbf{u}) + \nu\,\nabla\times\mathbf{v}\cdot\nabla\times\mathbf{u} - p\nabla\cdot\mathbf{v} - q\nabla\cdot\mathbf{u}\Big) \\ &+ \int_\Sigma \Big(\gamma\partial_{tt}U_n\mathbf{v}\cdot\mathbf{n} + (\mathbf{T}\nabla U_n + \mathbf{C}\nabla(\mathbf{u}\cdot\mathbf{n}))\cdot\nabla(\mathbf{v}\cdot\mathbf{n}) + a\mathbf{u}\cdot\mathbf{n}\ \mathbf{v}\cdot\mathbf{n} \\ &\qquad + bU_n\mathbf{v}\cdot\mathbf{n} + c\mathbf{v}\cdot\mathbf{n}\Big) = \int_\Omega \mathbf{f}\cdot\mathbf{v} \quad \forall\mathbf{v}, q \text{ with } \mathbf{v}\times\mathbf{n}_{|\Sigma} = 0 \end{aligned} \tag{23}$$

where $U_n = \int_0^t \mathbf{u}\cdot\mathbf{n}\mathrm{d}\tau$ on Σ.

Remark 2.2 When $\Gamma \neq 0$, then one must add on the right hand side $\int_{\Gamma_p} p_\Gamma\mathbf{v}\cdot\mathbf{n} + \int_{\Gamma_f}\mathbf{g}\cdot\mathbf{v}$ and constrain $\mathbf{u}, \mathbf{v}$ to satisfy also $\mathbf{u}\times\mathbf{n} = \mathbf{v}\times\mathbf{n} = 0$ on Γ_p.

2.5 Discussion

In [26] it was shown that $\nabla\times\nabla\times\mathbf{u}$ is the right form for the second order term for the variational formulation of the Navier-Stokes equations to handle the condition $\mathbf{u}\times\mathbf{n} = 0$. However it is a theoretical argument; in a numerical code according to (20) it makes no difference to use either of the 3 forms, except for its spatial discretization.

There has been objections to the numerical use of $\mathbf{u}\times\nabla\times\mathbf{u}$ in [10] as not fit for boundary layers. But here again by (19) the term is identical to $\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{2}|u|^2$; the two expressions are different only when discretized in space. Indeed in the fully discrete scheme it may be better to use $\mathbf{u}\cdot\nabla\mathbf{u}-\frac{1}{2}|u|^2$. However note that (23) computes the dynamic pressure. So ∂_Γ is now the prescribed dynamic pressure at the inflow/outflow cross-sections.

The conclusion of this section is also that, up to higher order terms, using $\nabla\times\nabla\times\mathbf{u}$ in place of $\nu\nabla\cdot(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$ does not affect the coupling at Σ. Poiseuille and Wormersley flows are also special solutions of (23) for cylindrical two dimensional geometries with rigid walls ($b=+\infty$). But the same argument also shows that the following variational formulation is also feasible, and differs from (23) by higher order terms near Σ and by the fact that in (23) it is the dynamic pressure while bellow it is the pressure.

3 Semi-discretization in time

In this section we propose a variational formulation for the Navier-Stokes boundary value problem (22). We begin with the Surface Pressure Model, i.e. $\gamma=T=C=a=0$; then $c=p_0$. We use the Sobolev space $W^{k,p}(\Omega)$ and denote its norm by $\|\cdot\|_{k,p,\Omega}$. We denote $\mathbf{W}^k(\Omega)=[W^k(\Omega)]^3$, $H^k(\Omega)=W^{k,2}(\Omega)$, $\mathbf{H}^k(\Omega)=[H^k(\Omega)]^3$, $\mathbf{L}^k(\Omega)=[L^k(\Omega)]^3$; we assume that Ω is a Lipschitz domain. Our analysis is inspired by the early works on the Stokes and Navier-Stokes equations with boundary conditions on the pressure (Cf. [25, 26]).

Notice that a time discretization of (**??**) gives an expression for the pressure on Σ at each time step:

$$b\,\mathbf{u}^{n+1}\simeq\frac{p^{n+1}-p^n}{\delta t}\mathbf{n}\implies p^{n+1}\simeq p^n+b\,\delta t\,\mathbf{u}^{n+1}\cdot\mathbf{n}. \tag{24}$$

It can be used directly in the weak form of (22):

$$(\partial_t\mathbf{u},\mathbf{w})_\Omega-(\mathbf{u}\times\nabla\times\mathbf{u},\mathbf{w})_\Omega+\nu\,(\nabla\times\mathbf{u},\nabla\times\mathbf{w})_\Omega-(p,\nabla\cdot\mathbf{w})_\Omega+(p,\mathbf{n}\cdot\mathbf{w})_\Sigma=(\mathbf{f},\mathbf{w})_\Omega. \tag{25}$$

This weak form is obtained by integrating in Ω the first equation of (22) multiplied by a smooth test function $\mathbf{w}$ such that $\mathbf{n}\times\mathbf{w}=\mathbf{0}$ on Σ, and using the identity

$$(\nabla\times\nabla\times\mathbf{u},\mathbf{w})_\Omega=(\nabla\times\mathbf{u},\mathbf{w}\times\mathbf{n})_\Sigma+(\nabla\times\mathbf{u},\nabla\times\mathbf{w})_\Omega=(\nabla\times\mathbf{u},\nabla\times\mathbf{w})_\Omega.$$

As usual $(.,.)_\Omega$ denotes the $L^2(\Omega)$-scalar product.

Therefore using (24), equation (25) is discretized at times $t=t_n=n\,\delta t$ by

$$\begin{aligned}(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t},\mathbf{w})_\Omega-(\mathbf{u}^{n+1}\times\nabla\times\mathbf{u}^n,\mathbf{w})_\Omega+\nu\,(\nabla\times\mathbf{u}^{n+1},\nabla\times\mathbf{w})_\Omega-(p^{n+1},\nabla\cdot\mathbf{w})_\Omega\\+(p^n+b\,\delta t\,\mathbf{u}^{n+1}\cdot\mathbf{n},\mathbf{w}\cdot\mathbf{n})_\Sigma=(\mathbf{f}^{n+1},\mathbf{w})_\Omega\end{aligned} \tag{26}$$

where $\mathbf{f}^{n+1}$ is some averaged value of f in $[t_n,t_{n+1}]$, with $t_n=n\,\delta t$. However this formulation only makes sense if $p^n_{|_\Sigma}$ has some L^p regularity, in particular if $p^n_{|_\Sigma}\in L^2(\Sigma)$. Hence we will first re-formulate (26) appropriately and then analyze its well-posedness, stability and convergence when $\delta t\to 0$ to the solution of a suitable continuous problem.

Consider the velocity and pressure spaces

$$\mathbf{W}=\{\mathbf{w}\in\mathbf{L}^2(\Omega)\,|\,\nabla\times\mathbf{w}\in\mathbf{L}^2(\Omega),\,\nabla\cdot\mathbf{w}\in L^2(\Omega),\,\mathbf{n}\times\mathbf{w}_{|_\Sigma}=\mathbf{0}\}, \tag{27}$$

$$M=L^2(\Omega). \tag{28}$$

Then it holds (Cf. Bernardi et al. [27])

Lemma 1 *The space* $\mathbf{W}$ *is well defined and is a Hilbert space endowed with the norm*

$$\|\mathbf{w}\|_{\mathbf{W}} = (\|\mathbf{w}\|_{0,2,\Omega}^2 + \|\nabla \times \mathbf{w}\|_{0,2,\Omega}^2 + \|\nabla \cdot \mathbf{w}\|_{0,2,\Omega}^2)^{1/2}.$$

Moreover when Ω *is either convex or* $\mathcal{C}^{1,1}$*, then* $\mathbf{W}$ *is continuously embedded in* $\mathbf{H}^1(\Omega)$ *and there exists a constant* $C > 0$ *such that*

$$\|\mathbf{w}\|_{1,2,\Omega} \leq C\, \|\mathbf{w}\|_{\mathbf{W}} \;\; \forall \mathbf{w} \in \mathbf{W}.$$

Observe that the condition $\partial_t p\mathbf{n} = b\,\mathbf{u}$ may be re-written $p(t)\,\mathbf{n} = p(0)\,\mathbf{n} + b\displaystyle\int_0^t \mathbf{u}(s)\,ds$. This suggests the following discretization of problem (22): Assume $\mathbf{f} \in L^2((0,T)\times\Omega)$, $\mathbf{u}_0 \in \mathbf{W}$, $p_0 \in L^2(\Sigma)$, let $N \geq 1$ integer, $\delta t = T/N$. Set $\mathbf{u}^0 = \mathbf{u}_0$, $p^0 = p_0$.

For $n = 0, \cdots, N-1$, find $(\mathbf{u}^{n+1}, p^{n+1}) \in \mathbf{W} \times M$ such that for any $(\mathbf{w}, q) \in \mathbf{W} \times M$,

$$\left\{ \begin{array}{rcl} (\dfrac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t}, \mathbf{w})_\Omega - (\mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n, \mathbf{w})_\Omega + \nu\,(\nabla \times \mathbf{u}^{n+1}, \nabla \times \mathbf{w})_\Omega & & \\ -(p^{n+1}, \nabla \cdot \mathbf{w})_\Omega + (p^0 + b\,\mathbf{U}^{n+1} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma & = & (\mathbf{f}^{n+1}, \mathbf{w})_\Omega, \\ (\nabla \cdot \mathbf{u}^{n+1}, q) & = & 0, \end{array} \right. \tag{29}$$

where

$$\mathbf{U}^{n+1} = \delta t \sum_{k=1}^{n+1} \mathbf{u}^k, \quad \mathbf{f}^{n+1} = \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} \mathbf{f}(s)\,ds.$$

This discretization is equivalent to (26),

Set $\mathbf{u}_0 = \mathbf{u}^0$, $p_0 = p^0$. For $n = 0, \cdots, N-1$, find $(\mathbf{u}^{n+1}, p^{n+1}) \in \mathbf{W} \times M$ such that for any $(\mathbf{w}, q) \in \mathbf{W} \times M$,

$$\left\{ \begin{array}{rcl} (\dfrac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t}, \mathbf{w})_\Omega - (\mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n, \mathbf{w})_\Omega + \nu\,(\nabla \times \mathbf{u}^{n+1}, \nabla \times \mathbf{w})_\Omega & & \\ -(p^{n+1}, \nabla \cdot \mathbf{w})_\Omega + (p^n + b\,\delta t\,\mathbf{u}^{n+1} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma & = & (\mathbf{f}^{n+1}, \mathbf{w})_\Omega, \\ (\nabla \cdot \mathbf{u}^{n+1}, q) & = & 0. \end{array} \right. \tag{30}$$

Proposition 2 *Assume that problem (30) admits a solution such that* $p^n \in H^1(\Omega)$, $\mathbf{u}^n \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ *and* $\nabla \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ *for all* $n = 0, \cdots, N$*, then the sequence* $(\mathbf{u}^n, p^n)$, $n = 0, \cdots, N$ *is also a solution of (29).*

Conversely, if (29) admits a solution such that $p^n \in H^1(\Omega)$, $\mathbf{u}^n \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ *and* $\nabla \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ *for all* $n = 0, \cdots, N$*, then the sequence* $(\mathbf{u}^n, p^n)$, $n = 0, \cdots, N$ *is also a solution of (30)*

Proof. Assume for instance that problem (30) admits a solution such that $p^n \in H^1(\Omega)$, $\mathbf{u}^n \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ and $\nabla \times \nabla \times \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ for all $n = 0, \cdots, N$. Integrating by parts in (30) we obtain

$$\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} - \mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n + \nu \nabla \times \nabla \times \mathbf{u}^{n+1} + \nabla p^{n+1} - \mathbf{f}^{n+1}, \mathbf{w} \right)_\Omega = 0,$$

for all $\mathbf{w} \in \mathbf{W}$ such that $\mathbf{w} \cdot \mathbf{n} = 0$ on Σ. Then this holds for all $\mathbf{w} \in \mathcal{D}(\Omega)^d$ and we deduce

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} - \mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n + \nu \nabla \times \nabla \times \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} \;\text{ in }\; \mathbf{L}^2(\Omega). \tag{31}$$

Thus by (30),

$$(-p^{n+1} + p^n + b\,\delta t\,\mathbf{u}^{n+1} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma = 0 \text{ for all } \mathbf{w} \in \mathbf{W}.$$

Applying recursively this identity, we deduce

$$(-p^{n+1} + p^0 + b\,\delta t \sum_{k=1}^{n} \mathbf{u}^{k+1} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma = 0 \text{ for all } \mathbf{w} \in \mathbf{W}.$$

Then the sequence $(\mathbf{u}^n, p^n)$, $n = 0, 1, \cdots, N$ is a solution of (29). □

Remark. If $\mathbf{f}^n \in \mathbf{L}^{3/2}(\Omega)$, $n = 0, 1, \cdots, N$, the regularity hypotheses of this result simplify to $p^n \in W^{1,3/2}(\Omega)$, $n = 0, 1, \cdots, N$, and no additional regularity is necessary for the velocities. However this proof is quite involved technically and we prefer not to include it for simplicity.

4 Analysis of the semi-discrete problem

In this section we analyze the discrete problem (29) for fixed n. We prove that it admits a unique solution $(\mathbf{u}^{n+1}, p^{n+1})$. Estimates for $\mathbf{u}^{n+1}$ and for a primitive in time of the pressure (instead of the pressure itself) will be obtained in Theorem 5.1 below (See (50), (56)).

Problem (29) is an Oseen-like problem, however it is non-standard due to the structure of the convection term, and the presence of the boundary terms issued from the discretization of condition (**??**).

The well-posedness of problem (29) is based upon the following inf-sup condition:

Lemma 2 *Assume that the domain Ω is Lipschitz. Then for some $\beta > 0$,*

$$\beta \, \|q\|_{0,2,\Omega} \le \sup_{\mathbf{w} \in \mathbf{W}} \frac{(q, \nabla \cdot \mathbf{w})}{\|\mathbf{w}\|_{\mathbf{W}}} \quad \textit{for all } q \in M. \tag{32}$$

Proof. Let $q \in M$. Consider the solution $\Phi \in H^1_0(\Omega)$ of the problem

$$-\Delta \Phi = q \text{ in } \Omega, \ \Phi = 0 \text{ on } \Sigma.$$

Let $\mathbf{w} = -\nabla \Phi$. Then $\mathbf{w} \in \mathbf{L}^2(\Omega)$, $\nabla \cdot \mathbf{w} = -q$, $\nabla \times \mathbf{w} = 0$ in Ω and $\mathbf{n} \times \mathbf{w} = \mathbf{n} \times \nabla \Phi = \mathbf{0}$ on Σ because the components of $\mathbf{n} \times \nabla \Phi$ are tangential derivatives of Φ on Σ (Cf. Girault-Raviart [28]). Then $\mathbf{w} \in \mathbf{W}$, and (32) follows with

$$\beta = \frac{1}{\sqrt{1 + \mathcal{P}^2}},$$

where $\mathcal{P}$ is the constant of Poincaré's inequality. □

Notations Let us introduce the following multilinear forms for $\mathbf{u}, \mathbf{w}, \mathbf{z} \in \mathbf{W}, r \in M$:

$$a(\mathbf{u}, \mathbf{w}) = \nu \, (\nabla \times \mathbf{u}, \nabla \times \mathbf{w})_\Omega, \tag{33}$$

$$c(\mathbf{u}; \mathbf{z}, \mathbf{w}) = -(\mathbf{z} \times \nabla \times \mathbf{u}, \mathbf{w})_\Omega, \tag{34}$$

$$d(r, \mathbf{w}) = -(r, \nabla \cdot \mathbf{w})_\Omega, \tag{35}$$

$$A(\mathbf{v}; \mathbf{u}, \mathbf{w}) = \frac{1}{\delta t}(\mathbf{u}, \mathbf{w})_\Omega + c(\mathbf{v}; \mathbf{u}, \mathbf{w}) + a(\mathbf{u}, \mathbf{w}) + b\,\delta t\,(\mathbf{u} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma, \tag{36}$$

$$l_n(\mathbf{w}) = \frac{1}{\delta t}(\mathbf{u}^n, \mathbf{w})_\Omega + (\mathbf{f}^{n+1}, \mathbf{w})_\Omega - (p^0 + b\,\mathbf{U}^n \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma. \tag{37}$$

Problem (29) for $(\mathbf{u}^{n+1}, p^{n+1})$ may be re-written as

$$\left\{ \begin{array}{rcll} A(\mathbf{u}^n; \mathbf{u}^{n+1}, \mathbf{w}) + d(p^{n+1}, \mathbf{w}) & = & l_n(\mathbf{w}) & \text{for any } \mathbf{w} \in \mathbf{W}, \\ d(q, \mathbf{u}^{n+1}) & = & 0 & \text{for any } q \in M. \end{array} \right. \tag{38}$$

To analyse this problem note that the form l_n is linear on $\mathbf{W}$, the form a is bilinear continuous on $\mathbf{W} \times \mathbf{W}$, the form c is trilinear continuous on $\mathbf{W} \times \mathbf{W} \times \mathbf{W}$ and the form d is bilinear continuous on $M \times \mathbf{W}$. Moreover,

Lemma 3 *Assume that Ω is convex or $\mathcal{C}^{1,1}$. Then there exists a constant C such that*

$$|c(\mathbf{u}; \mathbf{z}, \mathbf{w})| \leq C \, \|\mathbf{u}\|_{\mathbf{W}} \, \|\mathbf{z}\|_{\mathbf{W}} \, \|\mathbf{w}\|_{\mathbf{W}} \;\; \textit{for all } \mathbf{u}, \, \mathbf{z}, \, \mathbf{w} \in \mathbf{W} \tag{39}$$

Proof. By Sobolev's imbeddings, $H^1(\Omega)$ is imbedded into $L^p(\Omega)$ for $1 \leq p \leq 6$. Therefore

$$|c(\mathbf{u}; \mathbf{z}, \mathbf{w})| \leq C \, \|\nabla \times \mathbf{u}\|_{0,2,\Omega} \, \|\mathbf{z}\|_{0,4,\Omega} \, \|\mathbf{w}\|_{0,4,\Omega} \leq C \, \|\mathbf{u}\|_{\mathbf{W}} \, \|\mathbf{z}\|_{1,2,\Omega} \, \|\mathbf{w}\|_{1,2,\Omega}$$

The conclusion follows from Lemma 1. □

We are now in a position to prove the

Proposition 3 *Assume that Ω is convex or $\mathcal{C}^{1,1}$. Then problem (29) admits a unique solution.*

Proof. The forms $A(\mathbf{v}; \cdot, \cdot)$ and d are bilinear and respectively continuous on $\mathbf{W} \times \mathbf{W}$ and $M \times \mathbf{W}$. Also, $A(\mathbf{v}; \cdot, \cdot)$ is coercive on the kernel $\mathbf{W}_{div}$ of d in $\mathbf{W}$,

$$\mathbf{W}_{div} = \{\mathbf{w} \in \mathbf{W} \,|\, \nabla \cdot \mathbf{w} = 0, \;\text{ a. e. in }\; \Omega\,\}.$$

Indeed, as $c(\mathbf{v}; \mathbf{w}, \mathbf{w}) = 0$, the form

$$\mathbf{w} \in \mathbf{W}_{div} \mapsto [\mathbf{w}] := A(\mathbf{v}; \mathbf{w}, \mathbf{w})^{1/2} = \left(\frac{1}{\delta t} \, \|\mathbf{w}\|_{0,2,\Omega}^2 + \nu \, \|\nabla \times \mathbf{w}\|_{0,2,\Omega}^2 + b \, \delta t \, \|\mathbf{w} \cdot \mathbf{n}\|_{0,2,\Sigma}^2 \right)^{1/2}$$

is a norm on $\mathbf{W}_{div}$ equivalent to the norm of $\mathbf{W}$. In addition, the inf-sup condition (32) holds. As the form l_n is linear on $\mathbf{W}$, we deduce that problem (38) admits a unique solution $(\mathbf{u}, p)$ (Cf. [28]). □

5 Stability and convergence analysis

In this section we establish the stability of (29) in natural norms and prove its convergence to a weak solution of the boundary value problem (22) for the Navier-Stokes equations. We start by giving a weak formulation to this problem. We shall look for the primitive of the pressure as an unknown instead of the pressure itself. This primitive is naturally bounded in $L^\infty((0,T); L^2(\Omega))$, while it is much harder to bound the pressure in a Banach space.

5.1 Variational formulation

For brevity we shall denote $L^p((0,T); B)$ by $L^p(B)$, where B is a Banach space. When $B = W^{k,p}(\Omega)$ we denote $L^p(W^{k,p}(\Omega))$ by $L^p(W^{k,p})$. Let us define the mapping $\mathbf{U} : L^2(\mathbf{W}) \mapsto H^1(\mathbf{W})$ by

$$\mathbf{U}(\mathbf{z})(t) = \int_0^t \mathbf{z}(s) ds.$$

We define the weak formulation of problem (29) as follows. Denote $Q_T = \Omega \times (0,T)$,

Definition 1 *Let* $\mathbf{f} \in L^2(\mathbf{W}')$, $\mathbf{u}_0 \in \mathbf{W}'_{div}$, $p_0 \in L^2(\Sigma)$. *A pair* $(\mathbf{u}, p) \in \mathcal{D}'(Q_T)^d \times \mathcal{D}'(Q_T)$ *is a weak solution of the boundary value problem (22) if* $\mathbf{u} \in L^2(\mathbf{W}_{div}) \cap L^\infty(\mathbf{L}^2)$, *there exists* $P \in L^2(L^2)$ *such that* $p = \partial_t P$, *and for all* $\mathbf{w} \in \mathbf{W}$, $\varphi \in \mathcal{D}([0,T])$ *such that* $\varphi(T) = 0$,

$$\begin{cases} -\displaystyle\int_0^T (\mathbf{u}(t), \mathbf{w})_\Omega \, \varphi'(t)\, dt - \langle \mathbf{u}_0, \mathbf{w}\rangle \, \varphi(0) \\ +\displaystyle\int_0^T [\, c(\mathbf{u}(t); \mathbf{u}(t), \mathbf{w})\, dt + a(\mathbf{u}(t), \mathbf{w}) + b\, (\mathbf{U}(\mathbf{u})(t)\cdot \mathbf{n}, \mathbf{w}\cdot\mathbf{n})_\Sigma \,]\, \varphi(t)\, dt \\ +\displaystyle\int_0^T (P(t), \nabla\cdot \mathbf{w})_\Omega \, \varphi'(t)\, dt = \int_0^T \langle \mathbf{f}(t), \mathbf{w}\rangle \, \varphi(t)\, dt - \int_0^T (p_0, \mathbf{w}\cdot\mathbf{n})_\Sigma \, \varphi(t)\, dt. \end{cases} \tag{40}$$

This definition makes sense because due to the regularity asked for $\mathbf{u}$ and P, all terms in (40) are integrable in $(0, T)$. The weak solutions given by this definition are solutions of the Navier-Stokes equations in the following sense.

Proposition 4 *Assume that* Ω *is convex or* $\mathcal{C}^{1,1}$. *Let* $(\mathbf{u}, p) \in \mathcal{D}'(Q_T)^d \times \mathcal{D}'(Q_T)$ *be a weak solution of the boundary value problem (22). Then*

i) Equations

$$\partial_t \mathbf{u} - \mathbf{u}\times\nabla\times\mathbf{u} + \nu\, \nabla\times\nabla\times\mathbf{u} + \nabla p = \mathbf{f} \;\; \text{and} \;\; \nabla\cdot\mathbf{u} = 0 \tag{41}$$

hold respectively in $\mathcal{D}'(Q_T)^d$ *and in* $L^2(Q_T)$.

ii)

$$\mathbf{u} \in C^0([0,T], \mathbf{W}'_{div}) \quad \text{and} \;\; \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in} \;\; \mathbf{W}'_{div}.$$

iii)

$$\mathbf{n}\times\mathbf{u} = \mathbf{0} \quad \text{in} \;\; L^2(\mathbf{L}^4(\Sigma)).$$

iv) If Ω *is* $\mathcal{C}^{1,1}$ *or polyhedric,* $\mathbf{u} \in L^2(\mathbf{H}^2)$, $\partial_t \mathbf{u} \in L^2(\mathbf{L}^2)$, $p_0 \in H^{1/2}(\Sigma)$ *and* $p \in L^2(H^1)$, *then*

$$\partial_t p = b\, \mathbf{u}\cdot\mathbf{n} \quad \text{in} \;\; L^2(H^{1/2}(\Sigma)), \quad p(0) = p_0 \;\; \text{a. e. in} \;\; \Sigma\times(0,T).$$

Proof.

i) As $\mathbf{u} \in L^1(Q_T)$, then $\mathbf{u}$ satisfies in the sense of distributions

$$\langle \partial_t \mathbf{u}, \mathbf{w}\otimes\varphi\rangle_{\mathcal{D}(Q_T)} = -\int_{Q_T} \mathbf{u}(\mathbf{x}, t)\, \partial_t(\mathbf{w}(\mathbf{x})\varphi(t))\, d\mathbf{x}\, dt = -\int_0^T (\mathbf{u}(t), \mathbf{w})_\Omega \, \varphi'(t)\, dt,$$

for all $\mathbf{w} \in \mathcal{D}(\Omega)^d$, $\varphi \in \mathcal{D}(0,T)$. Similarly, as $P \in L^1(Q_T)$,

$$\langle \nabla(\partial_t P), \mathbf{w}\otimes\varphi\rangle_{\mathcal{D}(Q_T)} = \int_0^T (P(t), \nabla\cdot\mathbf{w})_\Omega \, \varphi'(t)\, dt.$$

Then, integrating by parts and using $(\mathbf{U}(\mathbf{u})(t)\cdot\mathbf{n}, \mathbf{w}\cdot\mathbf{n})_\Sigma = 0$ and $\nabla\cdot\mathbf{u} = 0$ a. e. in Q_T, (40) implies

$$\langle \partial_t \mathbf{u} - \mathbf{u}\times\nabla\times\mathbf{u} + \nu\, \nabla\times\nabla\times\mathbf{u} + \nabla p - \mathbf{f}, \mathbf{w}\otimes\varphi\rangle_{\mathcal{D}(Q_T)} = 0$$

for all $\mathbf{w} \in \mathcal{D}(\Omega)^d$, $\varphi \in \mathcal{D}(0,T)$. As $\mathcal{D}(\Omega)\otimes\mathcal{D}(0,T)$ is sequentially dense in $\mathcal{D}(Q_T)$ (Cf. [29], Theorem 39.2) we deduce (41).
Also, as $\mathbf{u} \in L^2(\mathbf{W}_{div})$, then $\nabla\cdot\mathbf{u} = 0$ in $L^2(Q_T)$.

ii) Let $\Phi(t) \in \mathbf{W}'$ defined a. e. in $(0,T)$ by

$$\langle \Phi(t), \mathbf{z}\rangle = c(\mathbf{u}(t); \mathbf{u}(t), \mathbf{z}) + a(\mathbf{u}(t), \mathbf{z}) + b\,(\mathbf{U}(\mathbf{u})(t)\cdot \mathbf{n}, \mathbf{z}\cdot\mathbf{n})_\Sigma - \langle \mathbf{f}(t), \mathbf{z}\rangle - (p_0, \mathbf{z}\cdot\mathbf{n})_\Sigma.$$

By estimates (39) and the boundedness of forms a and c, there exists a constant $C > 0$ such that

$$\|\Phi(t)\|_{\mathbf{W}'} \le C\,(\|\mathbf{u}(t)\|^2_{\mathbf{W}} + \|\mathbf{u}(t)\|_{\mathbf{W}} + \|\mathbf{f}(t)\|_{\mathbf{W}} + \|p_0\|_{0,2,\Sigma} + \|\mathbf{u}\|_{0,1,\mathbf{W}}).$$

Then $\Phi \in L^1(\mathbf{W}')$. From (40) we deduce that for all $\mathbf{w} \in \mathbf{W}_{div}$, and for all $\varphi \in \mathcal{D}(0,T)$

$$\int_0^T (\mathbf{u}(t), \mathbf{w})_\Omega\, \varphi'(t)\, dt = \int_0^T \langle \Phi(t), \mathbf{w}\rangle_{\mathbf{W}_{div}}\, \varphi'(t)\, dt.$$

Then $\partial_t \mathbf{u} = -\Phi \in L^1(\mathbf{W}'_{div})$, and $\mathbf{u} \in C^0([0,T], \mathbf{W}'_{div})$. Moreover, if $\varphi \in \mathcal{D}([0,T])$ is such that $\varphi(T) = 0$, then (Cf. Temam [30], Chapter 3)

$$\int_0^T \langle \partial_t \mathbf{u}(t), \mathbf{w}\rangle_{\mathbf{W}_{div}}\, \varphi(t)\, dt = -\langle \mathbf{u}(0), \mathbf{w}\rangle_{\mathbf{W}_{div}}\, \varphi(0) - \int_0^T (\mathbf{u}(t), \mathbf{w})_\Omega\, \varphi'(t)\, dt.$$

As $\mathbf{u}_0 \in \mathbf{W}' \hookrightarrow \mathbf{W}'_{div}$, by (40) it follows

$$\int_0^T \langle \partial_t \mathbf{u}(t) + \Phi(t), \mathbf{w}\rangle_{\mathbf{W}_{div}}\, \varphi(t)\, dt + \langle \mathbf{u}_0 - \mathbf{u}(0), \mathbf{w}\rangle_{\mathbf{W}_{div}} \varphi(0) = 0,$$

and so $\langle \mathbf{u}_0 - \mathbf{u}(0), \mathbf{w}\rangle_{\mathbf{W}_{div}} = 0$ for all $\mathbf{w} \in \mathbf{W}_{div}$. We conclude that $\mathbf{u}(0) = \mathbf{u}_0$ in $\mathbf{W}'_{div}$.

iii) As $\mathbf{W}$ is imbedded into $\mathbf{H}^1(\Omega)$, trace theorems and Sobolev's imbeddings imply that $\mathbf{u}_{|\Sigma} \in L^2(\mathbf{L}^4(\Sigma))$. As $\mathbf{n} \in \mathbf{L}^\infty(\Sigma)$, then $\mathbf{u} \times \mathbf{n} \in L^2(\mathbf{L}^4(\Sigma))$.

iv) Assume $\mathbf{u} \in L^2(\mathbf{H}^2)$, $\partial_t \mathbf{u} \in L^2(\mathbf{L}^2)$ and $p \in L^2(H^1)$. Integrating by parts in (40), yields

$$\int_0^T (\partial_t \mathbf{u}(t) - \mathbf{u}(t) \times \nabla \times \mathbf{u}(t) + \nu \nabla \times \nabla \times \mathbf{u}(t) + \nabla p(t) - \mathbf{f}(t), \mathbf{w})_\Omega\; \varphi(t)\, dt$$
$$+ \int_0^T (b\,\mathbf{U}(\mathbf{u})(t)\cdot\mathbf{n} + p_0 - p(t), \mathbf{w}\cdot\mathbf{n})_\Sigma\, \varphi(t)\, dt = 0,$$

for all $\mathbf{w} \in \mathbf{W}$, $\varphi \in \mathcal{D}(0,T)$. Using that the first equation in (41) now holds in $L^2(\mathbf{L^2})$, we deduce

$$(b\,\mathbf{U}(\mathbf{u})(t)\cdot\mathbf{n} + p_0 - p(t), \mathbf{w}\cdot\mathbf{n})_\Sigma = 0 \;\text{ a. e. in }\; (0,T). \tag{42}$$

Assume that Ω is $\mathcal{C}^{1,1}$. Then $\mathbf{n} \in \mathcal{C}^{0,1}(\Sigma)$ and by Gagliardo [31] (see also Grisvard [32] Sect. 1.5) it admits a lifting $\mathbf{N} \in \mathbf{W}^{1,p}(\Omega)$ for some $p > d$. Then $(p(t) - p_0)\,\mathbf{N} \in \mathbf{H}^1(\Omega)$ and since $\mathbf{N} \times \mathbf{n} = \mathbf{0}$ on Σ, it follows that $(p_0 - p(t))\,\mathbf{N} \in \mathbf{W}$. Consequently $b\,\mathbf{U}(\mathbf{u})(t) + (p_0 - p(t))\,\mathbf{N} \in \mathbf{W}$ and as

$$(b\,\mathbf{U}(\mathbf{u})(t)\cdot\mathbf{n} + p_0 - p(t), \mathbf{w}\cdot\mathbf{n})_\Sigma = (b\,\mathbf{U}(\mathbf{u})(t)\cdot\mathbf{n} + (p_0 - p(t))\,\mathbf{N}\cdot\mathbf{n}, \mathbf{w}\cdot\mathbf{n})_\Sigma, \;\text{ a. e. in }\; (0,T)$$

we deduce $b\,\mathbf{U}(\mathbf{u})\cdot\mathbf{n} = p - p_0$ in $L^2(H^{1/2}(\Sigma))$.

Assume now that Ω is polyhedric. Let S be any face of Σ and φ any function of the space

$$\Psi = \left\{\varphi \in H^1(\Omega) \;\text{ such that }\; \gamma_0 \varphi \;\text{ is supported by }\; \Sigma\right\},$$

where γ_0 is the trace operator on Σ. Since $\mathbf{N} = \mathbf{n}_{|S}$ is a constant vector, the function $\varphi\,\mathbf{N}$ belongs to $H^1(\Omega)$ and $\gamma_0(\varphi\mathbf{N})$ is supported by S. Again $(\varphi\mathbf{N}) \times \mathbf{n} = \mathbf{0}$ and hence $\varphi\mathbf{N} \in \mathbf{W}$. Let $\mathbf{w} = \varphi\mathbf{N}$ for any $\varphi \in \Psi$. Then (43) reduces to

$$(b\,\mathbf{U}(\mathbf{u})(t) \cdot \mathbf{n} + p_0 - p(t), \varphi)_S = 0 \ \text{ a. e. in } \ (0, T). \tag{43}$$

Since by definition any function in $H^{1/2}_{00}(S)$ is the trace of a function $\varphi \in \Psi$, this implies that

$$b\,\mathbf{U}(\mathbf{u}) \cdot \mathbf{n} + p_0 - p = 0 \ \text{ a. e. in } \ S \times (0, T).$$

As this is valid for any face S of Σ, we derive

$$b\,\mathbf{U}(\mathbf{u}) \cdot \mathbf{n} + p_0 - p = 0 \ \text{ a. e. in } \ \Sigma \times (0, T)$$

As $\mathbf{U}(\mathbf{u}) \in H^1(H^{1/2}(\Sigma))$, then $\partial_t p = b\,\mathbf{u} \cdot \mathbf{n}$ in $L^2(H^{1/2}(\Sigma))$. Thus $p \in C^0([0, T]; H^{1/2}(\Sigma))$ and $b\,\mathbf{U}(\mathbf{u})(t) = p(t) - p(0)$ in $[0, T]$. So $p(0) = p_0$ in $H^{1/2}(\Sigma)$.

□

Remark For general domains, the condition $\partial_t p = b\,\mathbf{u} \cdot \mathbf{n}$ a. e. on $\Sigma \times (0, T)$ will hold (for smooth enough $\mathbf{u}$, p_0 and p) if the set $\{\mathbf{w} \cdot \mathbf{n} \,|\, \mathbf{w} \in \mathbf{W}\}$ is dense in some $L^p(\Sigma)$.

5.2 Stability

We analyze in this section the stability of discretization (29). Let us consider the following functions:

- $\mathbf{u}_\delta : [0, T] \mapsto \mathbf{W}$ is the piecewise linear in time function that takes the value $\mathbf{u}^n$ at $t = t_n = n\delta t$,
$$\mathbf{u}_\delta(t) := \frac{t_{n+1} - t}{\delta t}\mathbf{u}^n + \frac{t - t_n}{\delta t}\mathbf{u}^{n+1}.$$
- $\tilde{p}_\delta : (0, T) \mapsto M$ is the piecewise constant in time function that takes the value p^n in the time interval (t_n, t_{n+1}). This function is defined a. e. in $(0, T)$.
- $P_\delta : [0, T] \mapsto M$ is the primitive of the discrete pressure function $\tilde{p}_\delta$.
$$P_\delta(t) := \int_0^t \tilde{p}_\delta(s)\, ds,$$
- $\tilde{\mathbf{u}}_\delta : (-\delta t, T) \mapsto \mathbf{W}$ is the piecewise constant function that takes the value $\mathbf{u}^{n+1}$ in (t_n, t_{n+1}), and $\tilde{\mathbf{u}}_\delta(t) = \mathbf{u}^0_\delta$ in $(-\delta t, 0)$. This function is defined a. e. in $(-\delta t, T)$.
- $\tilde{\mathbf{u}}^-_\delta : (0, T) \mapsto \mathbf{W}$ is the piecewise constant function that takes the value $\mathbf{u}^n$ in (t_n, t_{n+1}). This function is defined a. e. in $(0, T)$.
- $\tilde{\mathbf{U}}_\delta : (0, T) \mapsto \mathbf{W}$ is the piecewise constant function that takes the value $\mathbf{U}^{n+1}$ in (t_n, t_{n+1}).
- $\tilde{\mathbf{f}}_\delta : (0, T) \mapsto \mathbf{W}'$ is the piecewise constant function that takes the value $\mathbf{f}^{n+1}$ in (t_n, t_{n+1}).

We estimate a fractional time derivative of the velocity in the Nikolskii spaces $N^{s,p}(0, T; B)$, which are sub-spaces of $L^p(0, T; B)$, where B is a Banach space. The Nikolskii space of order $r \in [0, 1]$ and exponent $p \in [0, +\infty]$ is defined as

$$N^{r,p}(0, T; B) = \{f \in L^p(0, T; B) \ \text{ such that } \ \|f\|_{\tilde{N}^{r,p}} < +\infty\},$$

where

$$\|f\|_{\tilde{N}^{r,p}} = \sup_{\delta>0} \frac{1}{\delta^r} \|\tau_\delta f\|_{L^p(0,T-\delta;B)},$$

and $\tau_\delta f(t) = f(t+\delta) - f(t)$, $0 \le t \le T-\delta$. The space $N^{r,p}(0,T;B)$, endowed with the norm

$$\|f\|_{N^{r,p}(0,T;B)} = \|f\|_{L^p(0,T;B)} + \|f\|_{\tilde{N}^{r,p}}$$

is a Banach space. We may think of $N^{r,p}(0,T;B)$ as being formed by functions whose fractional derivative in time of order r belongs to $L^p(0,T;B)$. Whenever there is no source of confusion, we shall denote $N^{s,p}(0,T;B)$ by $N^{s,p}(B)$. We also use the following (Corollary 3.19 of [27])

Lemma 4 *Assume that Ω is simply-connected and Σ is connected. Then the semi norm*

$$\|\nabla \times \mathbf{w}\|_{0,2,\Omega}$$

is a norm equivalent to the $\|\cdot\|_{\mathbf{W}}$ norm on the space $\mathbf{W}_{div}$.

We may now state the following stability result:

Theorem 5.1 *Assume that Ω is convex or is simply-connected with a $\mathcal{C}^{1,1}$ connected boundary Σ. Assume that $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$, $\mathbf{f} \in L^2(\mathbf{W}')$, $p_0 \in L^2(\Sigma)$. Then the solution of problem (29) satisfies the following estimates:*

$$\begin{aligned} \|\mathbf{u}_\delta\|_{L^\infty(\mathbf{L}^2)} + \sqrt{\nu}\,\|\mathbf{u}_\delta\|_{L^2(\mathbf{H}^1)} + \quad & b\,\|\tilde{\mathbf{U}}_\delta \cdot \mathbf{n}\|_{L^\infty(\mathbf{L}^2(\Sigma))} \\ & \le C_1 \left(\|\mathbf{u}_0\|_{0,2,\Omega} + \frac{1}{\sqrt{\nu}}\|\mathbf{f}\|_{L^2(\mathbf{W}')} + \frac{1}{\sqrt{\nu}}\|p_0\|_{L^2(\Sigma)} \right), \end{aligned} \tag{44}$$

$$\|\mathbf{u}_\delta\|_{N^{1/4,2}(\mathbf{L}^2)} \le C_2, \tag{45}$$

and

$$\|P_\delta\|_{L^\infty(L^2)} \le C_2, \tag{46}$$

for some constant $C_1 > 0$ independent of δt and ν, and some constant $C_2 > 0$ independent of δt.

Proof. We proceed by steps.

Step 1. **Velocity estimates.** To obtain estimate (44) we use

$$(\mathbf{u}^{n+1} - \mathbf{u}^n)\cdot \mathbf{u}^{n+1} = \frac{1}{2}(\mathbf{u}^{n+1} - \mathbf{u}^n)\cdot(\mathbf{u}^{n+1} + \mathbf{u}^n) + \frac{1}{2}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2, \tag{47}$$

and

$$(\mathbf{U}^{n+1}\cdot\mathbf{n}, \mathbf{u}^{n+1}\cdot\mathbf{n})_\Sigma = \left(\mathbf{U}^{n+1}\cdot\mathbf{n}, \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\delta t}\cdot\mathbf{n}\right)_\Sigma,$$

where we assume $\mathbf{U}^0 = \mathbf{0}$. Then, setting $\mathbf{w} = \mathbf{u}^{n+1}$ and $q = p^{n+1}$ in (29) yields

$$\begin{aligned} &\frac{1}{2}\|\mathbf{u}^{n+1}\|^2_{0,2,\Omega} + \frac{1}{2}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega} + \delta t\,\nu\|\nabla\times\mathbf{u}^{n+1}\|^2_{0,2,\Omega} + \frac{b}{2}\|\mathbf{U}^{n+1}\cdot\mathbf{n}\|^2_{0,2,\Sigma} \\ &+\frac{b}{2}\|\mathbf{U}^{n+1}\cdot\mathbf{n} - \mathbf{U}^n\cdot\mathbf{n}\|^2_{0,2,\Sigma} \\ &= \frac{1}{2}\|\mathbf{u}^n\|^2_{0,2,\Omega} + \frac{b}{2}\|\mathbf{U}^n\cdot\mathbf{n}\|^2_{0,2,\Sigma} + \delta t\, < \mathbf{f}^{n+1}, \mathbf{u}^{n+1} > + \delta t\,(p_0, \mathbf{u}^{n+1}\cdot\mathbf{n})_\Sigma. \end{aligned} \tag{48}$$

Using Lemma 4 and Young's inequality,

$$\|\mathbf{u}^{n+1}\|^2_{0,2,\Omega} + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega} + \delta t\,\nu \|\nabla \times \mathbf{u}^{n+1}\|^2_{0,2,\Omega} + b\,\|\mathbf{U}^{n+1}\cdot \mathbf{n}\|^2_{0,2,\Sigma} \\ \leq \|\mathbf{u}^n\|^2_{0,2,\Omega} + b\,\|\mathbf{U}^n \cdot \mathbf{n}\|^2_{0,2,\Sigma} + C\,\delta t \nu^{-1}\|\mathbf{f}^{n+1}\|^2_{\mathbf{W}'} + C\,\delta t\nu^{-1}\|p_0\|^2_{0,2,\Sigma}, \tag{49}$$

for some constant $C > 0$. Summing estimates (49) for $n = 0, 1, \cdots, k$ for some $k \leq N-1$,

$$\|\mathbf{u}^{k+1}\|^2_{0,2,\Omega} + \sum_{n=0}^{k} \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega} + \nu\,\delta t \sum_{n=0}^{k} \|\nabla \times \mathbf{u}^{n+1}\|^2_{0,2,\Omega} + b\,\|\mathbf{U}^{k+1}\cdot \mathbf{n}\|^2_{0,2,\Sigma} \tag{50}$$

$$\leq \|\mathbf{u}_0\|^2_{0,2,\Omega} + C\,\delta t\,\nu^{-1} \sum_{n=0}^{k} \|\mathbf{f}^{n+1}\|^2_{\mathbf{W}'} + C\,T\,\nu^{-1}\,\|p_0\|^2_{0,2,\Sigma}.$$

This yields estimate (44), as

$$\sum_{n=0}^{N-1} \delta t\,\|\mathbf{f}^{n+1}\|^2_{\mathbf{W}'} \leq \|\mathbf{f}\|^2_{L^2(\mathbf{W}')}, \quad \|\mathbf{u}_\delta\|_{L^\infty(\mathbf{L}^2)} = \max_{n=0,1,\cdots,N} \|\mathbf{u}^n\|_{0,2,\Omega},$$

$$\|\tilde{\mathbf{U}}_\delta \cdot \mathbf{n}\|_{L^\infty(\mathbf{L}^2(\Sigma))} = \max_{n=0,1,\cdots,N} \|\mathbf{U}^n\cdot\mathbf{n}\|_{0,2,\Sigma}, \text{ and } \|\mathbf{u}_\delta\|^2_{L^2(\mathbf{W})} \leq C\,\delta t \sum_{n=0}^{N} \|\nabla \times \mathbf{u}^n\|^2_{0,2,\Omega},$$

for some constant $C > 0$ independent of δt.

STEP 2. **Velocity time increment estimates.** Let us re-state problem (29) as

$$\left\{ \begin{array}{r} (\partial_t \mathbf{u}_\delta(t), \mathbf{w}) + c(\tilde{\mathbf{u}}_\delta(t-\delta t); \tilde{\mathbf{u}}_\delta(t), \mathbf{w}) + a(\tilde{\mathbf{u}}_\delta(t), \mathbf{w}) + b\,(\tilde{\mathbf{U}}_\delta(t)\cdot\mathbf{n}, \mathbf{w}\cdot\mathbf{n})_\Sigma \\ -(\tilde{p}_\delta(t), \nabla\cdot\mathbf{w})_\Omega = \langle \tilde{\mathbf{f}}_\delta(t), \mathbf{w}\rangle - (p_0, \mathbf{w}\cdot\mathbf{n})_\Sigma \\ (\nabla\cdot\tilde{\mathbf{u}}_\delta(t), q)_\Omega = 0, \end{array} \right. \tag{51}$$

a.e. in $(0, T)$, for all $\mathbf{w} \in \mathbf{W}$. Let us integrate (51) in $(t, t+\delta)$ for $t \in [0, T-\delta]$,

$$(\tau_\delta \mathbf{u}_\delta(t), \mathbf{w})_\Omega = \int_t^{t+\delta} \langle \mathcal{F}_\delta(s), \mathbf{w}\rangle\, ds + \int_t^{t+\delta} (\tilde{p}_\delta(s), \nabla\cdot\mathbf{w})_\Omega\, ds, \tag{52}$$

where (we recall) $\tau_\delta \mathbf{u}_\delta(t) = \mathbf{u}_\delta(t+\delta t) - \mathbf{u}_\delta(t)$, and $\mathcal{F}_\delta(s) \in \mathbf{W}'$ is defined a. e. in $(0,T)$ by

$$\begin{array}{rcl} \langle \mathcal{F}_\delta(s), \mathbf{w}\rangle & = & -c(\tilde{\mathbf{u}}_\delta(s-\delta t); \tilde{\mathbf{u}}_\delta(s), \mathbf{w}) - a(\tilde{\mathbf{u}}_\delta(s), \mathbf{w}) - b\,(\tilde{\mathbf{U}}_\delta(s)\cdot\mathbf{n}, \mathbf{w}\cdot\mathbf{n})_\Sigma \\ & + & \langle \tilde{\mathbf{f}}_\delta(s), \mathbf{w}\rangle - (p_0, \mathbf{w}\cdot\mathbf{n})_\Sigma, \text{ for all } \mathbf{w} \in \mathbf{W}. \end{array}$$

Setting $\mathbf{w} = \tau_\delta \mathbf{u}_\delta(t)$ and integrating from 0 to $T-\delta$,

$$\int_0^{T-\delta} \|\tau_\delta \mathbf{u}_\delta(t)\|^2_{0,2,\Omega}\, dt = \int_0^{T-\delta}\int_t^{t+\delta} \langle \mathcal{F}_\delta(s), \tau_\delta \mathbf{u}_\delta(t)\rangle\, ds\, dt, \tag{53}$$

were we have used that $(\nabla\cdot\tau_\delta \mathbf{u}_\delta(t), \tilde{p}_\delta(s)) = 0$, a. e. for t, $s \in (0,T)$. Using the imbedding of $\mathbf{W}$ in $\mathbf{H}^1(\Omega)$,

$$\|\mathcal{F}_\delta(s)\|_{\mathbf{W}'} \leq C\left[\|\tilde{\mathbf{u}}_\delta(s-\delta t)\|^2_{\mathbf{W}} + \|\tilde{\mathbf{u}}_\delta(s)\|^2_{\mathbf{W}} + \|\nabla\times\tilde{\mathbf{u}}_\delta(s)\|^2_{0,2,\Omega} + \|\tilde{\mathbf{f}}_\delta(s)\|_{\mathbf{W}'} + \|\tilde{\mathbf{U}}_\delta\cdot\mathbf{n}\|_{0,2,\Sigma} \right. \\ \left. + \|p_0\|_{0,2,\Sigma}\right].$$

Due to estimate (44), this implies that $\mathcal{F}_\delta \in L^1(\mathbf{W}')$, and

$$\|\mathcal{F}_\delta\|_{L^1(\mathbf{W}')} \le C \tag{54}$$

for some constant $C > 0$ independent of h and δt. Now, we use Fubini's theorem to estimate the r.h.s. of (53), as follows

$$\begin{aligned}\int_0^{T-\delta} \|\tau_\delta \mathbf{u}_\delta(t)\|^2_{0,2,\Omega}\, dt &= \left| \int_0^T \int_{s-\delta}^s \langle \mathcal{F}_\delta(s), \widetilde{\tau_\delta \mathbf{u}_\delta}(t)\rangle \, dt\, ds \right| \\ &\le \int_0^T \|\mathcal{F}_\delta(s)\|_{\mathbf{W}'} \left(\int_{s-\delta}^s \|\widetilde{\tau_\delta \mathbf{u}_\delta}(t)\|_{\mathbf{W}}\, dt \right) ds \\ &\le \int_0^T \|\mathcal{F}_\delta(s)\|_{\mathbf{W}'}\, \delta^{1/2} \left(\int_{s-\delta}^s \|\widetilde{\tau_\delta \mathbf{u}_\delta}(t)\|^2_{\mathbf{W}}\, dt \right)^{1/2} ds \\ &\le C\delta^{1/2} \|\mathbf{u}_\delta\|_{L^2(\mathbf{W})} \le C\, \delta^{1/2},\end{aligned} \tag{55}$$

for some constant C independent of h, where $\widetilde{v}$ denotes the extension by zero outside $[0, T-\delta]$ of a function v. The last line of estimates follows from (44) and (54). Estimate (55) yields (45).

Step 3. **Estimate of the primitive of the pressure.** Let $\mathbf{w} \in \mathbf{W}$. Equation (51) yields

$$\begin{aligned}(P_\delta(t), \nabla\cdot \mathbf{w})_\Omega &= (\mathbf{u}_\delta(t) - \mathbf{u}_0, \mathbf{w})_\Omega - \int_0^t \langle \mathcal{F}_\delta(s), \mathbf{w}\rangle\, ds \\ &\le C \left(\|\mathbf{u}_\delta\|_{L^\infty(\mathbf{L}^2)} + \|\mathbf{u}_0\|_{0,2,\Omega} + \|\mathcal{F}_\delta\|_{L^1(\mathbf{W}')} \right) \|\mathbf{w}\|_{\mathbf{W}} \\ &\le C\, \|\mathbf{w}\|_{\mathbf{W}},\end{aligned} \tag{56}$$

where the last estimate follows from estimates (44) and (54). Then, by the inf-sup condition (32), estimate (46) follows. □

We next prove the convergence, we need some preliminary results:

Lemma 5 *Let $\mathbf{z} \in L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{L}^4)$. Then $\mathbf{z} \in \mathbf{L}^3(Q_T)$ and*

$$\|\mathbf{z}\|_{0,3,Q_T} \le \|\mathbf{z}\|^{1/3}_{L^\infty(\mathbf{L}^2)}\, \|\mathbf{z}\|^{2/3}_{L^2(\mathbf{L}^4)}. \tag{57}$$

Proof. Let $r \in [2, 4]$. By Hölder's inequality,

$$\|\mathbf{z}(t)\|^r_{0,r,\Omega} \le \|\mathbf{z}(t)\|^{2\theta}_{0,2,\Omega}\, \|\mathbf{z}(t)\|^{4(1-\theta)}_{0,4,\Omega} \le \|\mathbf{z}\|^{2\theta}_{L^\infty(\mathbf{L}^2)}\, \|\mathbf{z}(t)\|^{4(1-\theta)}_{0,4,\Omega}, \text{ a. e. in } (0,T),$$

where $r = 2\theta + 4(1-\theta)$. Setting $r = 3$ we obtain $\theta = 1/2$. Integrating in time the above inequality yields (57). □

Lemma 6 *Assume that the domain Ω is convex or $\mathcal{C}^{1,1}$. Assume that the sequence $\{\mathbf{v}_\delta\}_{\delta>0} \subset L^3(Q_T)$ strongly converges to $\mathbf{v}$ in $L^3(Q_T)$. Let $\varphi \in \mathcal{D}([0,T])$, $\mathbf{w} \in \mathbf{W}$. Then $\mathbf{v}_\delta(\mathbf{x},t) \otimes \mathbf{w}(\mathbf{x})\, \varphi(t)$ strongly converges to $\mathbf{v}(\mathbf{x},t) \otimes \mathbf{w}(\mathbf{x})\, \varphi(t)$ in $L^2(Q_T)^{3\times 3}$.*

Proof. By Hölder's inequality,

$$\|\mathbf{v}_\delta \otimes \mathbf{w}\, \varphi - \mathbf{v} \otimes \mathbf{w}\ \varphi\|_{0,2,Q_T} \le C\, \|\mathbf{v}_\delta - \mathbf{v}\|_{0,3,Q_T}\, \|\mathbf{w}\|_{0,6,\Omega}\, \|\varphi\|_{0,\infty,(0,T)}$$

for some constant that does not depend on δ. The conclusion follows. □

We also need the following compactness result for space-time functions (Cf. [33])

Lemma 7 *Let X, E, Y be Banach spaces such that $X \hookrightarrow E \hookrightarrow Y$ where the imbedding $X \hookrightarrow E$ is compact. Then the imbedding*

$$L^p(0,T;X) \cap N^{r,p}(0,T;Y) \hookrightarrow L^p(0,T;E) \text{ with } 0 < r < 1,\ 1 \le p \le +\infty$$

is compact.

We are now in a position to state the convergence result:

Theorem 5.2 *Assume that Ω is convex or is simply-connected with a $\mathcal{C}^{1,1}$ connected boundary Σ. Assume that $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$, $\mathbf{f} \in L^2(\mathbf{W}')$ and $p_0 \in L^2(\Sigma)$. Then the sequence $((\mathbf{u}_\delta, p_\delta))_{\delta>0}$ contains a sub-sequence $((\mathbf{u}_{\delta'}, p_{\delta'}))_{\delta'>0}$that is weakly convergent in $L^2(\mathbf{W}) \times H^{-1}(\mathbf{L}^2)$ to a weak solution $(\mathbf{v}, p)$ of the boundary value problem (22). Moreover $(\mathbf{u}_{\delta'})_{\delta'>0}$ is weakly-* convergent in $L^\infty(\mathbf{L}^2)$ to $\mathbf{u}$, strongly in $L^2(\mathbf{L}^r)$ for $1 \le r < 6$, and the primitives in time of the pressures $(p_{\delta'})_{\delta'>0}$ are weakly-* convergent in $L^\infty(L^2)$ to a primitive in time of the pressure p.*

If the solution of the problem (40) is unique, then the whole sequence converges to it.

Proof. We proceed by steps.

Step 1. **Extraction of convergent sub-sequences.** By estimates (44) and (45), $\mathbf{u}_\delta$ is uniformly bounded in $L^2(\mathbf{H}^1)$, in $L^\infty(\mathbf{L}^2)$ and in $N^{1/4,2}(\mathbf{L}^2)$. The imbedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ is compact for $1 \le r < 6$ (Cf. Brézis [34], Chapter 9), and then the imbedding $\mathbf{W} \hookrightarrow \mathbf{L}^r(\Omega)$ also is compact. Applying Lemma 7 with $X = \mathbf{H}^1(\Omega)$, $E = \mathbf{L}^r(\Omega)$ and $Y = \mathbf{L}^2(\Omega)$, it follows that the sequence $(\mathbf{u}_\delta)_{\delta>0}$ is compact in $L^2(\mathbf{L}^r)$ for $1 \le r < 6$.

By estimate (46), the sequence $(P_\delta)_{\delta>0}$ is uniformly bounded in $L^\infty(L^2)$. Then the sequence $((\mathbf{u}_\delta, P_\delta))_{\delta>0}$ contains a sub-sequence (that we still denote in the same way) such that $(\mathbf{u}_\delta)_{\delta>0}$ is strongly convergent in $L^2(\mathbf{L}^r)$ to some $\mathbf{u}$, for any $1 \le r < 6$, weakly in $L^2(\mathbf{H}^1)$ and weakly-* in $L^\infty(\mathbf{L}^2)$, and $(P_\delta)_{\delta>0}$ is weakly-* convergent in $L^\infty(L^2)$ to some P. We prove in the sequel that the pair $(\mathbf{u}, \partial_t P)$ is a weak solution of Navier-Stokes equations (40) in the sense of Definition 1.

Also, by (44) the sequence $\tilde{\mathbf{u}}_\delta$ is uniformly bounded in $L^2(\mathbf{H}^1)$ and in $L^\infty(\mathbf{L}^2)$. Then, it contains a subsequence (that we may assume to be a sub-sequence of the preceding one) weakly convergent in $L^2(\mathbf{H}^1)$, weakly-* convergent in $L^\infty(\mathbf{L}^2)$ and strongly convergent in $L^2(\mathbf{L}^r)$, for any $1 \le r < 6$, to some $\tilde{\mathbf{u}}$. Both limit functions $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are equal. Indeed,

$$\begin{aligned}
\|\mathbf{u}_\delta - \tilde{\mathbf{u}}_\delta\|^2_{L^2(\mathbf{L}^2)} &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \Big\| \frac{t_{n+1}-t}{\delta t}\mathbf{u}^n + \frac{t-t_n}{\delta t}\mathbf{u}^{n+1} - \mathbf{u}^{n+1}\Big\|^2_{0,2,\Omega}\, dt \\
&\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \frac{t_{n+1}-t}{\delta t} \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega}\, dt \\
&\le \delta t \sum_{n=0}^{N-1} \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega} \le C(\nu, \mathbf{f}, \mathbf{u}_0, p_0)\, \delta t.
\end{aligned}$$

Similarly, $\tilde{\mathbf{u}}_\delta^-$ contains a subsequence (again assumed to be a sub-sequence of the preceding one) weakly convergent in $L^2(\mathbf{H}^1)$, weakly-* convergent in $L^\infty(\mathbf{L}^2)$ and strongly convergent in $L^2(\mathbf{L}^r)$, for any $1 \le r < 6$, to the same limit $\mathbf{u}$. Indeed,

$$\begin{aligned}
\|\mathbf{u}_\delta - \tilde{\mathbf{u}}_\delta^-\|^2_{L^2(\mathbf{L}^2)} &\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \Big\| \frac{t_{n+1}-t}{\delta t}\mathbf{u}^n + \frac{t-t_n}{\delta t}\mathbf{u}^{n+1} - \mathbf{u}^n\Big\|^2_{0,2,\Omega}\, dt \\
&\le C(\nu, \mathbf{f}, \mathbf{u}_0, p_0)\, \delta t.
\end{aligned}$$

Step 2. **Limit of the momentum conservation equation.** To pass to the limit in the momentum conservation equation in (51) we re-formulate it as

$$\begin{aligned}
- & \int_0^T (\mathbf{u}_\delta(t), \mathbf{w})_\Omega \varphi'(t)\, dt - (\mathbf{u}_0, \mathbf{w})_\Omega\, \varphi(0) + \int_0^T c(\tilde{\mathbf{u}}_\delta^-(t); \tilde{\mathbf{u}}_\delta(t), \mathbf{w})\, \varphi(t)\, dt \\
+ & \int_0^T a(\tilde{\mathbf{u}}_\delta(t), \mathbf{w})\, \varphi(t)\, dt + b \int_0^T (\tilde{\mathbf{U}}_\delta(t) \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma\, \varphi(t)\, dt \qquad (58) \\
+ & \int_0^T (P_\delta(t), \nabla \cdot \mathbf{w})_\Omega \varphi'(t)\, dt = \int_0^T \langle \tilde{\mathbf{f}}_\delta(t), \mathbf{w} \rangle\, \varphi(t)\, dt - \int_0^T (p_0, \mathbf{w} \cdot \mathbf{n})_\Sigma\, \varphi(t)\, dt, \text{ for all } \mathbf{w} \in \mathbf{W},
\end{aligned}$$

for any function $\varphi \in \mathcal{D}([0,T])$ such that $\varphi(T) = 0$.

Let $\mathbf{w} \in \mathbf{W}$. The sequences $\tilde{\mathbf{u}}_\delta^-$ and $\tilde{\mathbf{u}}_\delta$ are bounded in $L^\infty(\mathbf{L}^2)$ and convergent in $L^2(\mathbf{L}^4)$, so by Lemma 5, both sequences strongly converge to $\mathbf{u}$ in $L^3(Q_T)^3$. Then

$$\lim_{\delta t \to 0} \int_0^T (\mathbf{u}_\delta(t), \mathbf{w})_\Omega \varphi'(t)\, dt = \int_0^T (\mathbf{u}(t), \mathbf{w})_\Omega \varphi'(t)\, dt.$$

To pass to the limit in the convection term, observe that by Lemma 6, $\tilde{\mathbf{u}}_\delta^-(\mathbf{x}, t) \otimes \mathbf{w}(\mathbf{x})\, \varphi(t)$ strongly converges to $\mathbf{u}(\mathbf{x}, t) \otimes \mathbf{w}\, \varphi(t)$ in $L^2(Q_T)^{3\times 3}$. Then, as $\nabla \times \tilde{\mathbf{u}}_\delta(t)$ weakly converges to $\nabla \times \mathbf{u}$ in $\mathbf{L}^2(Q_T)$,

$$\lim_{\delta t \to 0} \int_0^T (\tilde{\mathbf{u}}_\delta^-(t) \times \nabla \times \tilde{\mathbf{u}}_\delta(t), \mathbf{w})_\Omega \varphi(t)\, dt = \int_0^T (\mathbf{u}(t) \times \nabla \times \mathbf{u}(t), \mathbf{w})_\Omega \varphi(t)\, dt.$$

As $\tilde{\mathbf{u}}_\delta(t)$ is weakly convergent to $\mathbf{u}$ in $L^2(\mathbf{H}^1)$,

$$\lim_{\delta t \to 0} \int_0^T a(\tilde{\mathbf{u}}_\delta(t), \mathbf{w})\, \varphi(t)\, dt = \int_0^T a(\mathbf{u}(t), \mathbf{w})\, \varphi(t)\, dt.$$

To treat the boundary term, observe that there exists a sub-sequence of $\tilde{\mathbf{U}}_\delta \cdot \mathbf{n}$ (that we assume to be a sub-sequence of the previous one) which is weakly-* convergent in $L^\infty(L^2(\Sigma))$ to some l. Let $\mathbf{w} \in \mathbf{W}$, $\varphi \in L^1(0,T)$. Then $\mathbf{w}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, \varphi(t) \in L^1(L^2(\Sigma))$, and so

$$\lim_{\delta \to 0} \int_0^T (\tilde{\mathbf{U}}_\delta(t) \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n})_\Sigma\, \varphi(t)\, dt = \int_0^T (l(t), \mathbf{w} \cdot \mathbf{n})_\Sigma\, \varphi(t)\, dt.$$

To identify the limit l we use the Green formula (Cf. [27, 28])

$$(\nabla \cdot \mathbf{w}, \sigma)_\Omega = (\mathbf{w} \cdot \mathbf{n}, \sigma)_\Sigma - (\mathbf{w}, \nabla \sigma)_\Omega, \quad \forall \mathbf{w} \in \mathbf{W},\ \sigma \in H^1(\Omega). \qquad (59)$$

Then, as $\nabla \cdot \tilde{\mathbf{u}}_\delta = 0$,

$$(\tilde{\mathbf{u}}_\delta(t) \cdot \mathbf{n}, \sigma)_\Sigma = (\tilde{\mathbf{u}}_\delta(t), \nabla \sigma)_\Omega, \text{ for all } \sigma \in H^1(\Omega), \text{ a. e. in } (0,T).$$

Hence

$$\int_0^T (\tilde{\mathbf{U}}_\delta(t) \cdot \mathbf{n}, \sigma)_\Sigma\, \varphi(t)\, dt = \left(\int_0^T \int_0^t \tilde{\mathbf{u}}_\delta(s)\, \varphi(t)\, ds\, dt, \nabla \sigma \right)_\Omega, \text{ for all } \varphi \in L^1(0,T).$$

Thus, taking the limit $\delta \to 0$,

$$\int_0^T (l(t),\sigma)_\Sigma\, \varphi(t)\, dt = \left(\int_0^T \int_0^t \mathbf{u}(s)\, \varphi(t)\, ds\, dt, \nabla\sigma \right)_\Omega = \int_0^T (\mathbf{U}(\mathbf{u})(t)\cdot \mathbf{n}, \sigma)_\Sigma\, \varphi(t)\, dt.$$

Then

$$(l(t),\sigma)_\Sigma = (\mathbf{U}(\mathbf{u})(t)\cdot\mathbf{n},\sigma)_\Sigma \text{ for all } \sigma \in \mathcal{D}(\overline{\Omega}) \text{ a. e. in } (0,T),$$

and we conclude that $l = \mathbf{U}(\mathbf{u})\cdot\mathbf{n}$ in $L^\infty(L^2(\Sigma))$.

To pass to the limit in the pressure term, observe that $(P_\delta)_{\delta>0}$ is weakly-* convergent in $L^\infty(L^2)$ to P,

$$\lim_{\delta t\to 0} \int_0^T (P_\delta, \nabla\cdot \mathbf{w}(\mathbf{x}))_\Omega\, \varphi'(t)\, dt = \int_0^T (P, \nabla\cdot\mathbf{w}(\mathbf{x}))_\Omega\, \varphi'(t)\, dt.$$

Also, as $\tilde{\mathbf{f}}_\delta$ strongly converges to $\mathbf{f}$ in $L^2(\mathbf{W}')$,

$$\lim_{\delta t\to 0} \int_0^T \langle \tilde{\mathbf{f}}_\delta(t), \mathbf{w}\rangle \varphi(t)\, dt = \int_0^T \langle \mathbf{f}(t), \mathbf{w}\rangle \varphi(t)\, dt.$$

Step 3. **Limit of the continuity equation.** Let us consider some function $q \in L^2(\Omega)$. As $\nabla\cdot\mathbf{u}_\delta$ weakly converges to $\nabla\cdot\mathbf{u}$ in $L^2(L^2)$,

$$\int_0^T (\nabla\cdot\mathbf{u}(t), q)_\Omega\, \varphi(t)\, dt = \lim_{\delta t\to 0}\int_0^T (\nabla\cdot\mathbf{u}_\delta(t), q)_\Omega\, \varphi(t)\, dt.$$

Consequently,

$$\int_0^T (\nabla\cdot\mathbf{u}(t), q)_\Omega\, \varphi(t)\, dt = 0, \quad \forall q \in L^2(\Omega),\ \forall \varphi \in \mathcal{D}(0,T). \tag{60}$$

As $\mathcal{D}(\Omega)\otimes\mathcal{D}(0,T)$ is sequentially dense in $\mathcal{D}(Q_T)$, we deduce that

$$\nabla\cdot\mathbf{u} = 0 \quad \text{a. e. in} \quad \Omega\times(0,T).$$

Step 4. **Conclusion.** As a consequence of the preceding analysis, $\mathbf{u}$ belongs to $L^2(\mathbf{W}_{div})\cap L^\infty(\mathbf{L}^2)$, P belongs to $L^\infty(L^2)$, and the pair $(\mathbf{u},P)$ satisfies (40). Thus, the pair $(\mathbf{u},\partial_t P)$ is a weak solution of the Navier-Stokes problem (22) in the sense of Definition 1. As P_δ weakly converges to P in $L^2(L^2)$, then $p_\delta = \partial_t P_\delta$ weakly converges to $p = \partial_t P$ in $H^{-1}(L^2)$.

If the solution of Navier-Stokes equations (40) is unique, then the whole sequence converges to it, as this proof is based upon a standard compactness argument. □

6 Other Time Discretizations

The above scheme may be extended to second order by means of the θ-scheme, Find $\mathbf{u}_\delta^{n+1} \in \mathbf{W}$, $p_\delta^{n+1} \in M$ such that for all $\mathbf{w} \in \mathbf{W}$, $q \in M$,

$$\left\{ \begin{array}{rcl} \left(\dfrac{\mathbf{u}_\delta^{n+1} - \mathbf{u}_\delta^n}{\delta t}, \mathbf{w}\right)_\Omega + c(\mathbf{u}_\delta^{n+\varepsilon\theta}; \mathbf{u}_\delta^{n+\theta}, \mathbf{w}) + a(\mathbf{u}_\delta^{n+\theta}, \mathbf{w}) & & \\ +(\mathbf{U}_\delta^{n+\theta} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n}) - (p_\delta^{n+1}, \nabla \cdot \mathbf{w})_\Omega & = & \langle \mathbf{f}^{n+\theta}, \mathbf{w} \rangle, -(p_0, \mathbf{w} \cdot \mathbf{n})_\Sigma, \\ (\nabla \cdot \mathbf{u}_\delta^{n+\theta}, q)_\Omega & = & 0, \end{array} \right. \tag{61}$$

where $0 \leq \theta \leq 1$, $\varepsilon = 0$ or 1, and

$$\mathbf{u}_\delta^{n+\theta} = \theta \mathbf{u}_\delta^{n+1} + (1-\theta)\mathbf{u}_\delta^n, \;\; \mathbf{U}_\delta^{n+\theta} = \mathbf{U}_\delta^n + \delta t \, \mathbf{u}_\delta^{n+\theta},$$

$$\mathbf{f}^{n+\theta} = \theta \mathbf{f}^n + (1-\theta)\mathbf{f}^{n+1}.$$

The choice $\varepsilon = 1$, $\theta = 1/2$ corresponds to the Crank-Nicolson scheme, which is second-order accurate in time. When $\varepsilon = 1$, for any θ this is a fully implicit scheme, in particular $\theta = 1$ corresponds to the fully implicit Euler scheme.

The stability for $\theta \geq 1/2$ follows from the identity

$$(\mathbf{u}_\delta^{n+1} - \mathbf{u}_\delta^n, \mathbf{u}_\delta^{n+\theta})_\Omega = \frac{1}{2}\|\mathbf{u}_\delta^{n+1}\|_{0,2,\Omega}^2 - \frac{1}{2}\|\mathbf{u}_\delta^{n+1}\|_{0,2,\Omega}^2 + \left(\theta - \frac{1}{2}\right)\|\mathbf{u}_\delta^{n+1} - \mathbf{u}_\delta^n\|_{0,2,\Omega}^2.$$

while the convergence is proved in a similar way.

7 Full Discretization and Numerical tests

7.1 Discretization with a Finite Element Method

Assume that Ω is polyhedric. Let T_h be a triangulation made of K tetraedra $\{T_k\}_1^K$ with the usual conformity hypotheses; let $\Omega := \cup_k T_k \subset \mathbb{R}^3$.

Consider the Taylor-Hood ($P^2 - P^1$) element , see for instance [35] or [28], built from

$$\begin{array}{l} \mathbf{V}_h = \{\mathbf{u} \in C^0(\overline{\Omega})^3 \;:\; v_i|_{T_k} \in P^2, \, \forall k \in T_h, \; i = 1,2,3\}, \\ Q_h = \{q \in C^0(\overline{\Omega}) \;:\; q|_{T_k} \in P^1, \, \forall k \in T_h\}. \end{array} \tag{62}$$

In practice the boundary Σ is decomposed into the inflow region Σ^-, the outflow region Σ^+, and the vessel walls Σ^w. The boundary conditions are then

$$\begin{array}{l} p = p^- \text{ on } \Sigma^- \times (0,T), \, p = p^+ \text{ on } \Sigma^+ \times (0,T), \\ \partial_t p = b \, \mathbf{u} \cdot \mathbf{n} \text{ on } \Sigma^w \times (0,T), \, p(0) = p^0 \text{ on } \Sigma^w, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ on } \Sigma \times (0,T). \end{array}$$

We denote for simplicity $\Gamma = \Sigma^- \cup \Sigma^+$, $p_\Gamma = \left\{ \begin{array}{lll} p^- & \text{on} & \Sigma^-, \\ p^+ & \text{on} & \Sigma^+ \end{array} \right.$.

A feasible discretization of (30) is to find $\mathbf{u}^{n+1} \in \mathbf{V}_h$, $p^{n+1} \in Q_h$ such that for all $\mathbf{w} \in \mathbf{V}_h, q \in Q_h$,

$$\int_\Omega [\mathbf{w} \cdot (\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} - \mathbf{u}^{n+1} \times \nabla \times \mathbf{u}^n) - p^{n+1} \nabla \cdot \mathbf{w} - q \nabla \cdot \mathbf{u}^{n+1}] + \nu \int_\Omega \nabla \times \mathbf{u}^{n+1} \cdot \nabla \times \mathbf{w}$$
$$+ \frac{1}{\epsilon} \int_\Sigma (\mathbf{u}^{n+1} \times \mathbf{n}) \cdot (\mathbf{w} \times \mathbf{n}) + \int_\Sigma b \mathbf{w} \cdot \mathbf{n} (\mathbf{u}^{n+1} \delta t + \mathbf{U}^n) \cdot \mathbf{n} = \int_\Omega \mathbf{f}^{n+1} \mathbf{w} - \int_\Gamma p_\Gamma \, \mathbf{w} \cdot \mathbf{n},$$

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \mathbf{u}^{n+1}\delta t. \tag{63}$$

Notice that $\mathbf{u}^{n+1} \times \mathbf{n}|_\Sigma = \mathbf{0}$ is implemented by penalty. Indeed, as shown by V. Girault in [36] it would be vain to require $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ in strong form unless Nedelec elements of degree 2 at least be used.

Notice also that it is more convenient for the implementation to define $\mathbf{U}$ everywhere, not just on Σ.

Letting $\mathbf{w} = \mathbf{u}^{n+1},\ q = -p^{n+1}$ gives the following energy estimate:

$$\frac{1}{2\delta t}(\|\mathbf{u}^{n+1}\|^2_{0,2,\Omega} - \|\mathbf{u}^n\|^2_{0,2,\Omega}) + \frac{1}{2\delta t}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{0,2,\Omega} + \nu\|\nabla\times\mathbf{u}^{n+1}\|^2_{0,2,\Omega} + \frac{1}{\epsilon}\|\mathbf{u}^{n+1}\times\mathbf{n}\|^2_{0,2,\Sigma}$$
$$+\frac{b\delta t}{2}\|\mathbf{u}^{n+1}\cdot\mathbf{n}\|^2_{0,2,\Sigma} + \frac{1}{\delta t}(\|\mathbf{U}^{n+1}\cdot\mathbf{n}\|^2_{0,2,\Sigma} - \|\mathbf{U}^n\cdot\mathbf{n}\|^2_{0,2,\Sigma}) = \int_\Omega \mathbf{f}^{n+1}\,\mathbf{u}^{n+1} - \int_\Gamma p_\Gamma\,\mathbf{u}^{n+1}\cdot\mathbf{n}. \tag{64}$$

This implies the stability of the scheme, similarly to the analysis performed in the preceding section. Moreover, we deduce

$$\|\mathbf{u}^{n+1}\times\mathbf{n}\|_{0,2,\Sigma} \le C\,(\|\mathbf{f}^{n+1}\|_{\mathbf{W}'} + \|p_\Gamma\|_{0,2,\Sigma})\,\sqrt{\epsilon}.$$

In practice if the domains has curved boundaries it should be approximated by polyhedric domains. It is well known that this generates an error of order $\sqrt{h}$ in the approximation of $\mathbf{u}^{n+1}\times\mathbf{n} = \mathbf{0}$. Then the optimal choice is $\epsilon = h$.

7.2 Comparison with Another Scheme

Now we consider the boundary conditions (7) directly as studied in [24, 37, 35] with the following scheme: $\forall\ [\mathbf{w}, q, \zeta] \in \mathbf{V}_h \times Q_h \times Q_h$,

$$\begin{aligned}&\int_\Omega[\mathbf{w}\cdot(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t} - \mathbf{u}^{n+1}\times\nabla\times\mathbf{u}^n) - p^{n+1}\nabla\cdot\mathbf{w} - q\nabla\cdot\mathbf{u}^{n+1}]\\&+\int_\Omega[\nu\nabla\mathbf{u}^{n+1}:\nabla\mathbf{w} + \epsilon\nabla\eta^{n+1}\cdot\nabla\zeta]\\&+\int_\Sigma b[\eta^{n+1}\mathbf{w}\cdot\mathbf{n} - \zeta(\mathbf{u}\cdot\mathbf{n}^{n+1} - \frac{1}{\delta t}(\eta^{n+1}-\eta^n)) + \frac{1}{\epsilon}(\mathbf{u}^{n+1}\times\mathbf{n})\cdot(\mathbf{w}\times\mathbf{n})]\\&=\int_\Omega \mathbf{f}^{n+1}\,\mathbf{w} - \int_\Gamma p_\Gamma\mathbf{w}\cdot\mathbf{n},\end{aligned} \tag{65}$$

where ϵ is any small positive parameter.

An energy conservation identity is derived by choosing $\mathbf{w} = \mathbf{u}^{n+1}$, $q = -p^{n+1}$, $\zeta = \eta^{n+1}$:

$$\frac{1}{2\delta t}(\|\mathbf{u}^{n+1}\|^2_{0,2,\Omega} - \|\mathbf{u}^n\|^2_{0,2,\Omega}) + \frac{1}{2\delta t}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2_{0,2,\Omega} + \nu\|\nabla\mathbf{u}^{n+1}\|^2_{0,2,\Omega} + \frac{1}{\epsilon}\|\mathbf{u}^{n+1}\times\mathbf{n}\|^2_{0,2,\Sigma}$$
$$+\epsilon\|\eta^{n+1}\|^2_{0,2,\Omega} + \frac{1}{2\delta t}(\|\eta^{n+1}\|^2_{0,2,\Omega} - \|\eta^n\|^2_{0,2,\Omega}) + \frac{1}{2\delta t}\|\eta^{n+1}-\eta^n\|^2_{0,2,\Omega}$$
$$= \int_\Omega \mathbf{f}^{n+1}\,\mathbf{u}^{n+1} - \int_\Gamma p_\Gamma\mathbf{u}^{n+1}\cdot\mathbf{n}$$

Again this implies the stability of the scheme.

7.3 Numerical Tests

The full model requires that at every time step Σ^w be moved along its normal by a quantity $\delta t\mathbf{u}\cdot\mathbf{n}$. To preserve the triangulation we follow the literature [20] and solve an additional problem

$$-\Delta d^{n+1}=0 \text{ in } \Omega,\quad d^{n+1}|_{\Sigma^w}=d^n+\delta t\mathbf{u}^n\cdot\mathbf{n},\quad d^{n+1}|_{\Gamma^-\cup\Gamma^+}=0, \tag{66}$$

and then move every vertex q^j of the triangulation: $q^j\mapsto q^j+\kappa d$. In theory $\kappa=1$ but for graphic enhancement it can be adjusted. Note however that (66) is expensive.

The geometry is a quarter of a torus with $R=4$, $r=2$. The parameters of the problem are

$$p^-=0,\quad p^+=1\quad \delta t=0.05,\quad \nu=0.001,\quad b=200,\quad \epsilon=0.001.$$

The geometry is updated for visualization purposes with a multiplicative factor 100. The surfaces of constant pressure are shown for both methods at $T=0.8$.

Two time schemes have been tested for both problems: Euler's scheme as written in (63) and (65); and Crank-Nicolson's scheme which would be second order if we had symmetrized the nonlinear terms, which we did not do because it jeopardizes the stability of the method. The scheme is obtained by changing δt into $\delta t/2$ and setting $\mathbf{u}^{n+1}=2\mathbf{u}^{n+\frac{1}{2}}-\mathbf{u}^n$ where $\mathbf{u}^{n+\frac{1}{2}}=\tilde{\mathbf{u}}^{n+1}$ computed by solving (63) or (65).

The surfaces of equal pressures are shown on figure 1. Notice that there are more differences between the results obtained by Euler and Crank-Nicolson schemes than by (63) and (65). This comforts us in trusting the small modifications done to the setting of the model to pass from (65) to (63).

The computations have been made with the software *freefem++*[8].

8 Conclusion

By a few minor modifications to the Surface Pressure model for blood flow we have obtained a model which gives similar numerical results on our preliminary tests and which is fully analyzed mathematically in the continuous case. It remains to show that the finite element discretization is stable. The penalty of the condition $\mathbf{u}\times\mathbf{n}$ probably weakens the error estimates unless $\epsilon\sim h^2$, the size of the tetrahedra. But convergence might be difficult to establish on a polygonal surface with non-parametric elements. Assuming that it converges with the mesh size and the time step decreasing to zero, the scheme is a truly implicit fluid-structure method and, being on a fixed mesh, it is much more stable than those on moving meshes which require iterations between the solid part and the fluid part and preconditioning by things like added mass.

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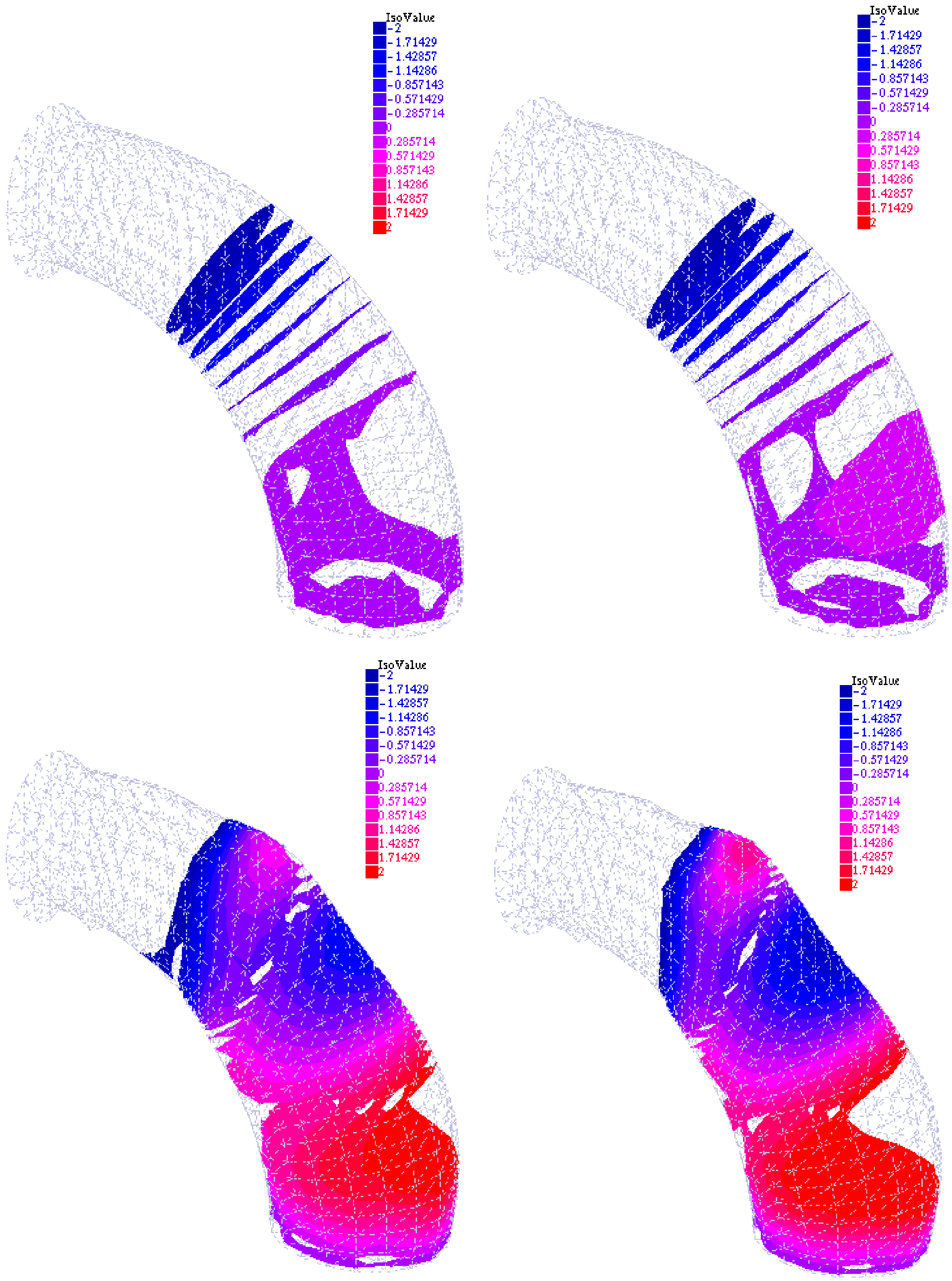

Figure 1: Surface of equal pressure at $t = 0.8$. Top left: computed by solving (63) with Euler's scheme. Top right: computed by solving (65) with Euler's scheme. Bottom left: computed by solving (63) with Crank-Nicolson's scheme. Bottom right: computed by solving (65) with Crank-Nicolson's scheme.

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