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To cite this version:
Christine Bernardi, Jad Dakroub, Gihane Mansour, Toni Sayah. A POSTERIORI ANALYSIS OF AN ITERATIVE ALGORITHM FOR NAVIER-STOKES PROBLEM. 2014. hal-01062832

HAL Id: hal-01062832
https://hal.sorbonne-universite.fr/hal-01062832
Submitted on 10 Sep 2014

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A POSTERIORI ANALYSIS OF AN ITERATIVE ALGORITHM FOR NAVIER-STOKES PROBLEM

CHRISTINE BERNARDI †, JAD DAKROUB ‡, GIHANE MANSOUR †, TONI SAYAH †.

ABSTRACT. This work deals with a posteriori error estimates for the Navier-Stokes equations. We propose a finite element discretization relying on the Galerkin method and we solve the discrete problem using an iterative method. Two sources of error appear, the discretization error and the linearization error. Balancing these two errors is very important to avoid performing an excessive number of iterations. Several numerical tests are provided to evaluate the efficiency of our indicators.

Keywords: A posteriori error estimation, Navier-Stokes problem, iterative method.

1. INTRODUCTION

The a posteriori analysis controls the overall discretization error of a problem by providing error indicators easy to compute. Once these error indicators are constructed, we prove their efficiency by bounding each indicator by the local error. This analysis was first introduced by I. Babuška [2], and developed by R. Verfürth [12]. The present work investigates a posteriori error estimates of the finite element discretization of the Navier-Stokes equations in polygonal domains. In fact, many works have been carried out in this field. In [3], C. Bernardi, F. Hecht and R. Verfürth considered a variational formulation of the three-dimensional Navier-Stokes equations with mixed boundary conditions and they proved that it admits a solution if the domain satisfies a suitable regularity assumption. In addition, they established the a priori and the a posteriori error estimates. As well, in [8], V. Ervin, W. Layton and J. Maubach present locally calculable a posteriori error estimators for the basic two-level discretization of the Navier-Stokes equations. In this work, we propose a finite element discretization of the Navier-Stokes equations relying on the Galerkin method. In order to solve the discrete problem we propose an iterative method. Therefore two sources of error appear, due to the discretization and the algorithm. Balancing these two errors leads to important computational savings. We apply this strategy on the following Navier-Stokes equations:

Let Ω be a connected open domain in \( \mathbb{R}^d \), \( d = 2, 3 \), with a Lipschitz continuous boundary \( \partial \Omega \). We consider, for a positive constant viscosity \( \nu \), the following system:

\[
-\nu \Delta u + (u, \nabla) u + \nabla p = f \quad \text{in } \Omega \\
\text{div } u = 0 \quad \text{in } \Omega \\
u u = 0 \quad \text{on } \partial \Omega, (1.1)
\]

where the unknowns are the velocity \( u \) and the pressure \( p \) of the fluid. The right-hand side \( f \) belongs to \( H^{-1}(\Omega)^d \), the dual of the Sobolev space \( H^1_0(\Omega)^d \).

Using \( P_1 \) Lagrange finite elements for the pressure and \( P_1 \)-bubble Lagrange finite elements for the velocity, the discrete variational problem amounts to a system of nonlinear equations. In order to solve it we propose an iterative algorithm which consists at each iteration to solve a linearized problem. We establish the corresponding a posteriori error estimates. Thus, two sources of error appear, namely linearization and discretization. The main goal of this work is to balance these two sources of error. In fact, if the

September 10, 2014.
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discretization error dominates then the nonlinear solver iterations are reduced. Therefore, our objective is to calculate \textit{a posteriori} error estimates distinguishing linearization and discretization errors in the context of an adaptive procedure. This type of analysis was introduced by A.-L. Chaillou and M. Suri [5, 6] for a general class of problems characterized by strongly monotone operators. It had been developed by L. El Alaoui, A. Ern and M. Vohralík [7] for a class of second-order monotone quasi-linear diffusion-type problems approximated by piecewise affine, continuous finite elements.

In this work we present a strategy for the linearization process. This strategy is iterative and can be outlined as follows:

1. On the given mesh, perform an iterative linearization until a stopping criterion is satisfied.
2. If the error is less than the desired precision, then stop, else refine the mesh adaptively and go to step (1).

An outline of the paper is as follows. In Section 2, we present the variational formulation of Navier-Stokes problem (1.1). We introduce in Section 3 the discrete variational problem with the \textit{a priori} error estimate. The \textit{a posteriori} analysis of the discretization of the iterative algorithm is performed in Section 4. Section 5 is devoted to the numerical experiments.

2. Analysis of Navier-Stokes equations

We describe in this section the Navier-Stokes problem (1.1) together with its variational formulation. First of all, we recall the main notion and results which we use later on. For the domain $\Omega$, denote by $L^p(\Omega)$ the space of measurable functions $v$ such that $|v|^p$ is integrable. For $v \in L^p(\Omega)$, the norm is defined by

$$\|v\|_{L^p(\Omega)} = \left( \int_{\Omega} |v(x)|^p dx \right)^{1/p}.$$ 

Throughout this paper, we constantly use the classical Sobolev space $W^{m,r}(\Omega) = \{ v \in L^r(\Omega); \forall |k| \leq m, \partial^k v \in L^r(\Omega) \}$, where $k = (k_1, ..., k_d)$ is a $d$-tuple of positive integers such that $|k| = k_1 + ... + k_d$ and

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} ... \partial x_d^{k_d}}.$$ 

$W^{m,r}(\Omega)$ is equipped with the semi-norm

$$|v|_{m,r,\Omega} = \left( \sum_{|k|=m} \int_{\Omega} |\partial^k v|^r d\mathbf{x} \right)^{1/r},$$

and the norm

$$\|v\|_{m,r,\Omega} = \left( \sum_{k=0}^m |v|_{r,\Omega}^r d\mathbf{x} \right)^{1/r}.$$ 

For $r = 2$, we define the Hilbert space $H^m(\Omega) = W^{m,2}(\Omega)$. In particular, we consider the following space

$$X = H^1_0(\Omega)^d = \{ v \in H^1(\Omega)^d, v_{|\partial\Omega} = 0 \},$$

and its dual space $H^{-1}(\Omega)^d$.

We denote by $L^2_0(\Omega)$ the space of functions in $L^2(\Omega)$ with zero mean-value on $\Omega$.

$$M = L^2_0(\Omega) = \{ q \in L^2(\Omega); \int_{\Omega} q d\mathbf{x} = 0 \}.$$ 

We recall the Sobolev imbeddings (see Adams [1], Chapter 3).
Lemma 2.1. For all $j \leq 6$ and $d = 2, 3$, there exists a positive constant $S_j$ such that
\[ \forall v \in H_0^j(\Omega), \quad \| v \|_{L^j(\Omega)} \leq S_j \| v \|_{1, \Omega}. \] (2.1)

We now assume that the data $f$ belongs to $H^{-1}(\Omega)^d$. Then system (1.1) is equivalent to the following variational problem:

Find $u \in X$, $p \in M$ such that
\[ \forall v \in X, \quad a(u, v) + c(u; u, v) + b(v, p) = \langle f, v \rangle, \]
\[ \forall q \in M, \quad b(u, q) = 0, \] (2.2)

where the bilinear forms $a(., .)$ and $b(., .)$ and the trilinear form $c(., .; .)$ are defined by
\[ a(u, v) = \nu \int_\Omega \nabla u \nabla v \, dx, \]
\[ b(v, q) = -\int_\Omega q \, \text{div} \, v \, dx, \] (2.3)
\[ c(w; u, v) = \int_\Omega (w, \nabla)uv \, dx. \]

Furthermore, the bilinear form $b(., .)$ satisfies the following inf-sup condition (see [9], Chapter I, Equation (5.14) for instance)
\[ \inf_{q \in M, q \neq 0} \sup_{v \in X} \frac{b(v, q)}{\| v \|_X \| q \|_M} = \beta > 0. \] (2.4)

Now we recall the following space
\[ V = \left\{ v \in X; \quad \text{div} \, v = 0 \right\}. \]

Then, problem (2.2) has the following form:

Find $u \in V$ such that
\[ \forall v \in V, \quad \nu \int_\Omega \nabla u \nabla v \, dx + \int_\Omega (u, \nabla)uv \, dx = \langle f, v \rangle. \] (2.5)

The existence and the conditional uniqueness of the solution $(u, p)$ of problem (2.2) is given in [9], Chapter IV, Section 2).

In order to calculate the a posteriori error estimate, we introduce the Stokes equations which are defined as follows:
\[ -\nu \Delta u + \nabla p = f \quad \text{in } \Omega \]
\[ \text{div} \, u = 0 \quad \text{in } \Omega \]
\[ u = 0 \quad \text{on } \partial \Omega. \] (2.6)

Using the previous notation, the Stokes problem amounts to the following variational form:

Find $u \in X$, $p \in M$ such that
\[ \forall v \in X, \quad a(u, v) + b(v, p) = \langle f, v \rangle, \]
\[ \forall q \in M, \quad b(u, q) = 0. \] (2.7)

The existence and the uniqueness of the solution $(u, p) \in X \times M$ of problem (2.7) is given in [9], Chapter I, Section 5.1.
In the sequel, we denote by \( C \), a generic constant that can vary from line to line but is always independent of all discretization parameters.

In what follows, for simplicity reasons, we suppose \( d = 2 \). In fact, the end of this work can easily be extended to \( d = 3 \) but requires some more technicalities that we prefer to avoid here.

3. Finite element discretization and the a priori estimate

This section collects some useful notation concerning the discrete setting and the \( a \) priori estimate.

Let \( (T_h)_h \) be a regular family of triangulations of the polygonal domain \( \Omega \), in the sense that, for each \( h \):

- The union of all elements of \( T_h \) is equal to \( \Omega \).
- The intersection of two different elements of \( T_h \), if not empty, is a vertex or a whole edge of both triangles.
- The ratio of the diameter \( h_K \) of any element \( K \) of \( T_h \) to the diameter of its inscribed circle is smaller than a constant independent of \( h \).

As usual, \( h \) stands for the maximum of the diameters \( h_K, K \in T_h \).

Let \( (X_h, M_h) \) be the couple of discrete spaces corresponding to \( (X, M) \) defined as follow:

\[
M_h = \left\{ q_h \in M, \forall K \in T_h, \; q_{h, K} \in \mathcal{P}_1(K) \right\} \quad \text{and} \quad X_h = \left\{ v_h \in X, \forall K \in T_h, \; v_{h, K} \in (\mathcal{P}_1(K)\text{-bubble})^2 \right\}
\]

where \( \mathcal{P}_1(K) \) stands for the space of restrictions to \( K \) of affine functions. \( \mathcal{P}_1(K)\text{-bubble} \) is defined by adding one extra degree of freedom to the barycenter of every simplex of the triangulation \( T_h \) of the domain \( \Omega \). We have the following inf-sup condition (see [9], Chapter II, Lemma 2.6):

\[
\inf_{q_h \in M_h, q_h \neq 0} \sup_{v_h \in X_h} \frac{- \int \Omega q_h \text{ div } v_h \, dx}{\| q_h \|_{L^2(\Omega)} \| \nabla v_h \|_{L^2(\Omega)}} = \beta > 0. \tag{3.1}
\]

We then consider the following finite element discretization of Navier-Stokes problem (2.2), obtained by the Galerkin method:

Find \( u_h \in X_h, \; p_h \in M_h \) such that

\[
\forall v_h \in X_h, \; \nu \int \Omega \nabla u_h \nabla v_h \, dx + \int \Omega (u_h \cdot \nabla) u_h v_h \, dx - \int \Omega p_h \text{ div } v_h \, dx = \langle f, v_h \rangle, \tag{3.2}
\]

\[
\forall q_h \in M_h, \; \int \Omega q_h \text{ div } u_h \, dx = 0.
\]

In order to solve the discrete problem (3.2), we introduce the following space

\[ V_h = \left\{ v_h \in X_h; \forall q_h \in M_h, \; - \int \Omega q_h \text{ div } v_h \, dx = 0 \right\}. \]

Problem (3.2) is then equivalent to the problem:

Find \( u_h \in V_h \) such that

\[
\forall v_h \in V_h, \; \nu \int \Omega \nabla u_h \nabla v_h \, dx + \int \Omega (u_h \cdot \nabla) u_h v_h \, dx = \langle f, v_h \rangle, \tag{3.3}
\]

and admits at least one solution \( (u_h, p_h) \in X_h \times M_h \) ([9], Chapter IV, Theorem 4.1) such that

\[
|u_h|_{1, \Omega} \leq \frac{c}{\nu} \| f \|_{-1, \Omega}. \tag{3.4}
\]

In addition, if \( u \in H^2(\Omega)^2 \) and \( p \in H^1(\Omega) \), the \( a \) priori estimate can be proved by following the approach in [4]. Under some further assumptions, it reads ([9], Chapter IV, Theorem 4.1)
\begin{equation}
|u - u_h|_{1, \Omega} + \|p - p_h\|_{0, \Omega} \leq Ch.
\end{equation}

4. Iterative Algorithm

In order to solve the Navier-Stokes discrete problem, we propose in this section a very simple iterative algorithm. In fact, we linearize the discrete problem and we set an initial guess \( u_h^0 \). We will see later on that under suitable conditions, the solution of the iterative algorithm \((u_h^{i+1}, p_h^{i+1})\) converges to the solution of the discrete problem \((u_h, p_h)\).

Iterative algorithm. Let \( u_h^0 \) be an initial guess. We introduce, for \( i \geq 0 \), the following algorithm:

Find \( u_h^{i+1} \in X_h, p_h^{i+1} \in M_h \) such that

\begin{equation}
\begin{aligned}
\forall v_h \in X_h, & \quad \nu \int_{\Omega} \nabla u_h^{i+1} \nabla v_h \, dx + \int_{\Omega} (u_h^{i+1}) u_h^{i+1} v_h \, dx - \int_{\Omega} p_h^{i+1} \, \text{div} \, v_h \, dx = \langle f, v_h \rangle, \\
\forall q_h \in M_h, & \quad \int_{\Omega} q_h \, \text{div} \, u_h^{i+1} \, dx = 0.
\end{aligned}
\end{equation}

(4.1)

We clearly see that problem (4.1) has the following form:

Find \( u_h^{i+1} \in V_h \), such that

\begin{equation}
\begin{aligned}
\forall v_h \in V_h, & \quad \nu \int_{\Omega} \nabla u_h^{i+1} \nabla v_h \, dx + \int_{\Omega} (u_h^{i+1}) u_h^{i+1} v_h \, dx = \langle f, v_h \rangle.
\end{aligned}
\end{equation}

(4.2)

Theorem 4.1. (The convergence Theorem). Let \((u_h^{i+1}, p_h^{i+1}) \in X_h \times M_h\) and \((u_h, p_h) \in X_h \times M_h\) be the solutions of the iterative problem (4.1) and the discrete problem (3.2), respectively. Then, for \( \nu > S_1 \sqrt{\|f\|_{-1, \Omega}} \), we have

\begin{equation}
|u_h^{i+1} - u_h|_{1, \Omega} \leq C_1 C_2^{-1} |u_h^i - u_h|_{1, \Omega},
\end{equation}

(4.3)

\begin{equation}
\|p_h^{i+1} - p_h\|_{L^2(\Omega)} \leq C_3 |u_h^i - u_h|_{1, \Omega},
\end{equation}

(4.4)

with

\[ C_1 = \frac{S_2^2}{\nu} \|f\|_{-1, \Omega}, \]

\[ C_2 = \frac{\nu}{C_1}, \]

\[ C_3 = \alpha^{-1}(1 + C_1 C_2^{-1} + C_1). \]

Moreover, the sequence \((u_h^i)\) converges to the solution \(u_h\) of problem (3.2) if \(C_1^{-1} C_2 < 1\).

Proof. (i) We start by estimate (4.3). We have (see once more [9], Chapter IV, Theorem 4.1)

\begin{equation}
|u_h^{i+1}|_{1, \Omega} \leq \frac{C}{\nu} \|f\|_{-1, \Omega}.
\end{equation}

(4.5)

We now subtract (3.3) from (4.2) to obtain, for all \(v_h \in X_h\),

\begin{equation}
\nu \int_{\Omega} \nabla (u_h^{i+1} - u_h) \nabla v_h \, dx + \int_{\Omega} ((u_h^{i+1}) u_h^{i+1} - (u_h u_h)) v_h \, dx = 0.
\end{equation}

(4.6)

Intercalating \(\pm \int_{\Omega} (u_h^{i+1}, \nabla) u_h v_h \, dx\) and taking \(v_h = u_h^{i+1} - u_h\) in (4.6), we obtain by applying the Cauchy-Schwarz inequality the following estimate

\begin{equation}
\nu |u_h^{i+1} - u_h|_{1, \Omega} \leq \frac{S_2^2}{\nu} \|f\|_{-1, \Omega} |u_h^i - u_h|_{1, \Omega} + \frac{S_2^2}{\nu} \|f\|_{-1, \Omega} |u_h^{i+1} - u_h|_{1, \Omega}.
\end{equation}

(4.7)

(ii) We now prove the second estimate (4.4). By subtracting (3.3) from (4.1) we obtain

\begin{equation}
\int_{\Omega} (p_h^{i+1} - p_h) \, \text{div} \, v_h \, dx = \nu \int_{\Omega} (u_h^{i+1} - u_h) \nabla v_h \, dx + \int_{\Omega} ((u_h^{i+1}, \nabla) u_h^{i+1} - (u_h, \nabla) u_h) v_h \, dx.
\end{equation}

(4.8)

Intercalating \(\pm \int_{\Omega} (u_h^{i+1}, \nabla) u_h v_h \, dx\), using (3.1), (3.4), (4.5) and the Cauchy-Schwarz inequality we obtain

\begin{equation}
\alpha \|p_h^{i+1} - p_h\|_{L^2(\Omega)} \leq (1 + C_1) |u_h^{i+1} - u_h|_{1, \Omega} + C_1 |u_h^i - u_h|_{1, \Omega}.
\end{equation}

(4.9)

Finally, combining (4.7) and (4.9) yields the desired estimates and convergence property.
We start this section by introducing some additional notation needed for constructing and analyzing the error indicators in the sequel.

For any element \( K \in \mathcal{T}_h \) we denote by \( \mathcal{E}(K) \) the set of its edges and we set
\[
\mathcal{E}_h = \bigcup_{K \in \mathcal{T}_h} \mathcal{E}(K).
\]

With any edge \( e \in \mathcal{E}_h \) we associate a unit vector \( \mathbf{n} \) such that \( \mathbf{n} \) is orthogonal to \( e \). We split \( \mathcal{E}(K) \) in the form
\[
\mathcal{E}(K) = \mathcal{E}_{K,\partial \Omega} \cup \mathcal{E}_{K,\Omega},
\]
where \( \mathcal{E}_{K,\partial \Omega} \) is the set of edges in \( \mathcal{E}(K) \) that lie on \( \partial \Omega \) and \( \mathcal{E}_{K,\Omega} = \mathcal{E}(K) \setminus \mathcal{E}_{K,\partial \Omega} \). Furthermore, for \( K \in \mathcal{T}_h \) and \( e \in \mathcal{E}_h \), let \( h_K \) and \( h_e \) be their diameter and length respectively. An important tool in the construction of bounds for the total error is Clément’s interpolation operator \( \mathcal{R}_h \) with values in \( X_h \). The operator \( \mathcal{R}_h \) satisfies, for all \( v \in H^1_0(\Omega) \), the following local approximation properties (see R. Verfürth, [12], Chapter 1):
\[
\| v - \mathcal{R}_h v \|_{L^2(K)} \leq C h_K |v|_{1,\Delta_K},
\]
\[
\| v - \mathcal{R}_h v \|_{L^2(e)} \leq C h_e^{1/2} |v|_{1,\Delta_e},
\]
where \( \Delta_K \) and \( \Delta_e \) are the following sets:
\[
\Delta_K = \bigcup \left\{ K' \in \mathcal{T}_h; \ K' \cap K \neq \emptyset \right\}
\]
and
\[
\Delta_e = \bigcup \left\{ K' \in \mathcal{T}_h; \ K' \cap e \neq \emptyset \right\}.
\]
We now recall the following properties (see R. Verfürth, [12], Chapter 1):

\[\text{Proposition 5.1. Let } r \text{ be a positive integer. For all } v \in P_r(K), \text{ the following properties hold}\]
\[
C \| v \|_{L^2(K)} \leq \| v \psi_K^{1/2} \|_{L^2(K)} \leq \| v \|_{L^2(K)}, \quad (5.1)
\]
\[
|v|_{1,K} \leq C h_K^{1/2} \| v \|_{L^2(K)}, \quad (5.2)
\]
where \( \psi_K \) is the triangle-bubble function (equal to the product of the barycentric coordinates associated with the nodes of \( K \)).

Finally, we denote by \( [v_h] \) the jump of \( v_h \) across the common edge \( e \) of two adjacent elements \( K, K' \in \mathcal{T}_h \). We have now provided all prerequisites to establish bounds for the total error.

We start the \textit{a posteriori} analysis of the iterative algorithm. In order to prove an upper bound of the error, we first introduce an approximation \( f_h \) of the data \( f \) which is constant on each element \( K \) of \( \mathcal{T}_h \). Then, we distinguish the discretization and linearization errors. We first write the residual equation

\[
\nu \int_\Omega \nabla u \nabla v \, dx + \int_\Omega (u \nabla) uv \, dx - \int_\Omega p \, \text{div} \ v \, dx
\]
\[
- \nu \int_\Omega \nabla u_h^{i+1} \nabla v \, dx - \int_\Omega (u_h^i \nabla) u_h^{i+1} v \, dx + \int_\Omega p_h^{i+1} \, \text{div} \ v \, dx
\]
\[
= \langle f, v - v_h \rangle - \nu \int_\Omega \nabla u_h^{i+1} \nabla (v - v_h) \, dx - \int_\Omega (u_h^i \nabla) u_h^{i+1} (v - v_h) \, dx + \int_\Omega p_h^{i+1} \, \text{div} \ (v - v_h) \, dx. \quad (5.3)
\]

Adding and subtracting \( \int_\Omega (u_h^{i+1} \nabla) u_h^{i+1} v \, dx \) and using the Green formula, give
\[ \nu \int_{\Omega} \nabla u \nabla v \, dx + \int_{\Omega} (u \nabla)v \, dx - \int_{\Omega} p \, \text{div} \, v \, dx \]

\[-\nu \int_{\Omega} \nabla u_h^{i+1} \nabla v \, dx - \int_{\Omega} (u_h^{i+1} \nabla)u_h^{i+1} v \, dx + \int_{\Omega} p_h^{i+1} \, \text{div} \, v \, dx \]

\[= \sum_{K \in T_h} \int_{K} (f - f_h)(v - v_h) \, dx + \sum_{K \in T_h} \left\{ \int_{K} (f_h + \nu \Delta u_h^{i+1} - (u_h^{i+1} \nabla)u_h^{i+1} - \nabla p_h^{i+1})(v - v_h) \, dx \right\} \]

\[-\frac{1}{2} \sum_{e \in E_{K,\Omega}} \int_{e} \frac{\partial u_h^{i+1}}{\partial n} - p_h^{i+1} n.(v - v_h) \, d\tau \right\} + \int_{\Omega} ((u_h^{i} - u_h^{i+1}) \nabla)u_h^{i+1} v \, dx, \quad (5.4) \]

where \( \tau \) denotes the tangential coordinate on \( \partial K \).

On the other hand, for all \( q \in L^2(\Omega) \)

\[ b(u - u_h^{i+1}, q) = \int_{\Omega} q \, \text{div} \, u_h^{i+1} \, dx. \quad (5.5) \]

We now define the local linearization indicator \( \eta^{(L)}_{K,i} \) and the local discretization indicator \( \eta^{(D)}_{K,i} \), corresponding to an element \( K \in T_h \), by:

\[ \eta^{(L)}_{K,i} = |u_h^{i+1} - u_h^{i}|_{1,K}, \quad (5.6) \]

\[ \eta^{(D)}_{K,i} = h_K \| f_h + \nu \Delta u_h^{i+1} - (u_h^{i} \nabla)u_h^{i+1} - \nabla p_h^{i+1} \|_{L^2(K)} \]

\[ + \frac{1}{2} \sum_{e \in E_{K,\Omega}} h_e^{1/2} \| \frac{\partial u_h^{i+1}}{\partial n} - p_h^{i+1} n \|_{L^2(e)} + \| \text{div} \, u_h^{i+1} \|_{L^2(K)}. \quad (5.7) \]

In order to calculate the a posteriori error estimates, we denote by \( S \) the operator which associates with any \( f \) in \( H^{-1}(\Omega)^d \) the part \( w = u \) of the solution \((u,p)\) of the Stokes problem (2.6),

\[ S : \ H^{-1}(\Omega)^d \rightarrow X \]

\[ f \quad \mapsto \quad Sf = w. \]

We consider now the following mapping

\[ G : \ X \rightarrow \ H^{-1}(\Omega)^d \]

\[ w \mapsto \quad G(w) = (w \nabla)w - f. \]

and we observe that problem (2.2) can equivalently be written as

\[ F(u) = u + SG(u) = 0. \quad (5.8) \]

**Lemma 5.2.** Let \((u, p)\) be the solution of problem (2.2). There exists a real number \( L > 0 \), such that the following Lipschitz property holds

\[ \forall w \in X, \quad \| S(DG(u) - DG(w)) \|_{L(1(\Omega))} \leq L |u - w|_{1,\Omega}. \quad (5.9) \]

**Proof.** We have, for all \( w, z \in X \)

\[ \| S(DG(u).z - DG(w).z) \|_{1,\Omega} \leq \frac{1}{\nu} \| DG(u).z - DG(w).z \|_{-1,\Omega}. \quad (5.10) \]

We observe that

\[ DG(u).z - DG(w).z = z \nabla(u - w) + (u - w) \nabla z, \quad (5.10) \]

whence

\[ \| (DG(u) - DG(w)).z \|_{-1,\Omega} \leq 2S_2^2 |u - w|_{1,\Omega} |z|_{1,\Omega}. \quad (5.11) \]
Thus, combining (5.9) with (5.10) and (5.11) yields the desired property.

**Assumption 5.3.** The solution \((u, p) \in X \times M\) of problem (2.2) is such that the operator \(\text{Id} + SDG(u)\) is an isomorphism of \(X\).

**Remark 5.4.** Assumption 5.3 implies that the solution \(u\) is locally unique, which is more weaker than the global uniqueness of the solution.

We can now state the first result of this section:

**Theorem 5.5.** Let \((u_h^{i+1}, p_h^{i+1}) \in X_h \times M_h\) and \((u_h, p_h) \in X_h \times M_h\) be the solutions of the iterative problem (4.1) and the discrete problem (3.2), respectively. Suppose that the solution \((u, p)\) satisfies Assumption 5.3. Then, there exists a neighborhood \(\mathcal{O}\) of \(u\) in \(X\) such that any solution \((u_h^{i+1}, p_h^{i+1})\) of problem (4.1) with \(u_h^{i+1}\) in \(\mathcal{O}\) satisfies the following a posteriori error estimate

\[
|u - u_h^{i+1}|_{1, \Omega} + |p - p_h^{i+1}|_{L^2(\Omega)} \leq C \left( \sum_{K \in T_h} \left( (\eta_D)^2 + \frac{h^2}{h} \| \mathbf{f} - f_h \|_{L^2(K)}^2 \right) \right)^{1/2} + C' \left( \sum_{K \in T_h} \left( \eta_L^{i+1} \right)^2 \right)^{1/2}.
\]

**Proof.** (i) Owing to Lemma 5.2 and Assumption 5.3, it follows from [11] that, for any \(u_h^{i+1}\) in an appropriate neighborhood \(\mathcal{O}\) of \(u\)

\[
|u - u_h^{i+1}|_{1, \Omega} \leq C |u_h^{i+1} + \text{SG}(u_h^{i+1})|_{-1, \Omega}.
\]

Introducing \(F(u)\) in (5.12) (see equation (5.8)), and from equation (5.4), we obtain for all \(v_h \in X_h\)

\[
|u - u_h^{i+1}|_{1, \Omega} \leq C \left( \sup_{v_h \in X_h, v_h \neq 0} \frac{\langle f - f_h, v - v_h \rangle}{|v|_{1, \Omega}} + \langle J, v - v_h \rangle \right)
\]

\[
+ \sup_{v_h \in X, v_h \neq 0} \left( \frac{\int \left( (u_h - u_h^{i+1}) \nabla v_h \right) u_h^{i+1} v \, dx}{|v|_{1, \Omega}} \right) + \sup_{q \in M, q \neq 0} \left( \frac{\int q \text{ div } u_h^{i+1} \, dx}{q \|_{L^2(\Omega)}} \right),
\]

where

\[
\langle J, v - v_h \rangle = \sum_{K \in T_h} \left\{ \int_K (f_h + \nu \Delta u_h^{i+1} - (u_h^{i+1}) \nabla u_h^{i+1} - \nabla p_h^{i+1})(v - v_h) \, dx \right\} - \frac{1}{2} \sum_{e \in E_{K, \Omega}} \int_e \left( \frac{\partial u_h^{i+1}}{\partial n} - p_h^{i+1} n \cdot (v - v_h) \, d\tau \right).
\]

Taking \(v_h\) equal to the image \(\mathcal{R}_h v\) of \(v\) by the Clément operator in (5.13), we obtain the desired estimate for \(|u - u_h^{i+1}|_{1, \Omega}\).

(ii) Computing \(b(v, p - p_h^{i+1})\) from (5.4) and adding and subtracting \(\int \Omega (u_h^{i+1}) \nabla u_h^{i+1} \, dx\) we obtain

\[
b(v, p - p_h^{i+1}) = \nu \int \Omega \nabla (u_h^{i+1} - u) \nabla v \, dx + \int \Omega \left( (u_h^{i+1}) \nabla (u_h^{i+1} - u) \right) \nabla v \, dx + \int \Omega (u_h^{i+1} - u) \nabla u_h^{i+1} \nabla v \, dx
\]

\[
+ \sum_{K \in T_h} \int_K (f - f_h)(v - v_h) \, dx + \sum_{K \in T_h} \left\{ \int_K (f_h + \nu \Delta u_h^{i+1} - (u_h^{i+1}) \nabla u_h^{i+1} - \nabla p_h^{i+1})(v - v_h) \, dx \right\}
\]

\[
- \frac{1}{2} \sum_{e \in E_{K, \Omega}} \int_e \left( \frac{\partial u_h^{i+1}}{\partial n} - p_h^{i+1} n \cdot (v - v_h) \, d\tau \right) + \int \Omega (u_h^{i+1} - u) \nabla u_h^{i+1} v \, dx.
\]
Using Cauchy-Schwarz inequality and the fact that \( u \) and \( u_h^{i+1} \) are bounded independently of \( h \), we derive the following estimate
\[
b(v, p - p_h^{i+1}) \leq \sum_{K \in T_h} \left( \| f_h - f_h \|_{L^2(K)} + \| f_h + \nu \Delta u_h^{i+1} - (u_h, \nabla)u_h^{i+1} - \nabla p_h^{i+1} \|_{L^2(K)} \right) \| v \|_{L^2(K)} + \frac{1}{2} \sum_{e \in E_{K, \partial}} \left( \| \partial u_h^{i+1} \|_{L^2(e)} \| v \|_{L^2(K)} + \frac{S_e^2}{\nu} \| f \|_{L^2(e)} \| u_h^{i+1} - u_h^i \|_{L^2(e)} \right) (5.15)
\]

Taking \( v_h = \mathcal{R}_h v \) in (5.15) and using the inf-sup condition (2.4), we obtain the desired estimate for \( \| p - p_h^{i+1} \|_{L^2(\Omega)} \).

We address now the efficiency of the previous indicators.

**Theorem 5.6.** For each \( K \in T_h \), the following estimates hold for the indicators \( \eta_{K, i}^{(L)} \) defined in (5.6)
\[
\eta_{K, i}^{(L)} \leq \left\| u - u_h^{i+1} \right\|_{1, K} + \left\| u - u_h^i \right\|_{1, K},
\]
and for the indicators \( \eta_{K, i}^{(D)} \) defined in (5.7)
\[
\eta_{K, i}^{(D)} \leq C \left( \left\| u - u_h^{i+1} \right\|_{1, \omega_K} + \left\| u - u_h^i \right\|_{1, \omega_K} + \left\| p_h^{i+1} - p \right\|_{L^2(\omega_K)} + \sum_{\kappa \subseteq \omega_K} h_{\kappa} \left\| f - f_h \right\|_{L^2(\kappa)} \right),
\]

where \( \omega_K \) is the union of the elements sharing at least one edge with \( K \).

**Proof.** The estimation of the linearization indicator follows easily from the triangle inequality by introducing \( u \) in \( \eta_{K, i}^{(L)} \). We now estimate the discretization indicator \( \eta_{K, i}^{(D)} \). We proceed in two steps:

(i) We start by adding and subtracting \( \int_{\Omega} (u_h^{i+1}, \nabla)uv \, dx \) in (5.3). Taking \( v_h = 0 \), we obtain
\[
\sum_{K \in T_h} \left( \int_{\Omega} (f_h + \nu \Delta u_h^{i+1} - (u_h, \nabla)u_h^{i+1} - \nabla p_h^{i+1})v \, dx \right)
\]
\[
= \nu \int_{\Omega} \nabla (u - u_h^{i+1}) \nabla v \, dx + \int_{\Omega} ((u - u_h^{i+1}), \nabla)uv \, dx - \sum_{K \in T_h} \int_{\Omega} (f - f_h) v \, dx
\]
\[
+ \frac{1}{2} \sum_{e \in E_{K, \partial}} \left( \int_{\Omega} \nabla (u - u_h^{i+1}) \nabla v \, dx \right) + \int_{\Omega} ((u_h^{i+1}, \nabla)u_h^{i+1})v \, dx + \int_{\Omega} (p_h^{i+1} - p) \text{div} v \, dx.
\]

We choose \( v = v_K \) such that
\[
v_K = \begin{cases} (f_h + \nu \Delta u_h^{i+1} - (u_h, \nabla)u_h^{i+1} - \nabla p_h^{i+1}) \psi_K & \text{on } K, \\
0 & \text{on } \Omega \setminus K,
\end{cases}
\]
where \( \psi_K \) is the triangle-bubble function of the element \( K \).

Using Cauchy-Schwarz inequality, (5.1) and (5.2) we obtain
\[
h_K \| f_h + \nu \Delta u_h^{i+1} - (u_h, \nabla)u_h^{i+1} - \nabla p_h^{i+1} \|^2_{L^2(K)} \leq \left( \nu + \frac{2C}{\nu} \| f \|_{L^2(\Omega)} \right) \| u - u_h^{i+1} \|_{1, K} \| v_K \|_{L^2(K)} + h_K \| f - f_h \|_{L^2(K)} \| v_K \|_{L^2(K)}
\]
\[
+ \left( \| p_h^{i+1} - p \|_{L^2(\Omega)} \right) \| v_K \|_{L^2(K)} + \frac{C}{\nu} \| f \|_{L^2(\Omega)} \| u_h^{i+1} - u_h^i \|_{1, K} \| v_K \|_{L^2(K)}.
\]

Therefore, we obtain the following estimate of the first term of the local discretization estimator \( \eta_{K, i}^{(D)} \).
where \( \psi \) is defined by (5.13) and constructed from a fixed operator on the reference element.

Thus, we have estimated the second term of the local discretization indicator \( \eta_{K,i} \). Similarly, taking \( v_h = 0 \) in (5.4) we infer

\[
\frac{1}{2} \sum_{e \in E_{K,i}} \int_{e} \left( \frac{\partial u_{h}^{i+1}}{\partial n} - p_{h}^{i+1} n \right) v \, d\tau = \int \left( f_h + \nu \Delta u_{h}^{i+1} - \left( u_{h}^{i}, \nabla \right) u_{h}^{i+1} - \nabla p_{h}^{i+1} \right) v \, d\tau \]

\[
+ \nu \int_{\Omega} \nabla \left( u_{h}^{i+1} - u \right) \nabla v \, dx + \int_{\Omega} \left( \left( u_{h}^{i+1} - u \right) \nabla \right) uv \, dx + \sum_{K \in T_h} \int_{K} \left( f - f_h \right) v \, dx \]

\[
+ \int_{\Omega} \left( \nabla \left( u_{h}^{i+1} - u \right) \right) \nabla v \, dx + \int_{\Omega} \left( \left( u_{h}^{i} - u_{h}^{i+1} \right) \nabla \right) u_{h}^{i+1} v \, dx \]

\[
+ \int_{\Omega} \left( p - p_{h}^{i+1} \right) \nabla v \, dx.
\]

We choose \( v = v_e \) such that

\[
v_e = \begin{cases} 
L_{e,k} \left( \left[ \frac{\partial u_{h}^{i+1}}{\partial n} - p_{h}^{i+1} n \right] \psi_e \right) & \kappa \in \{K,K'\} \\
0 & \text{on } \Omega \ \setminus \ (K \cup K')
\end{cases}
\]

where \( \psi_e \) is the edge-bubble function, \( K' \) denotes the other element of \( T_h \) that share \( e \) with \( K \) and \( L_{e,k} \) is a lifting operator from \( e \) into \( \kappa \) mapping polynomials vanishing on \( \partial e \) into polynomials vanishing in \( \partial e \) and constructed from a fixed operator on the reference element.

Using Cauchy-Schwarz inequality, (5.1) and (5.2) we get

\[
h_{e}^{1/2} \left\| \left[ \frac{\partial u_{h}^{i+1}}{\partial n} - p_{h}^{i+1} n \right] \right\|_{L^2(e)} \]

\[
\leq (\nu + \frac{2C}{\nu} \left\| f \right\|_{0,\Omega}) \left\| u - u_{h}^{i+1} \right\|_{1,K \cup K'} \left\| v_e \right\|_{L^2(e)} + h_e \left\| f - f_h \right\|_{L^2(K \cup K')} \left\| v_e \right\|_{L^2(e)} \]

\[
+ h_e \left\| f_h + \nu \Delta u_{h}^{i+1} - \left( u_{h}^{i}, \nabla \right) u_{h}^{i+1} - \nabla p_{h}^{i+1} \right\|_{L^2(K \cup K')} \left\| v_e \right\|_{L^2(e)} \]

\[
+ \left\| p_{h}^{i+1} - p \right\|_{L^2(K \cup K')} \left\| v_e \right\|_{L^2(e)} + \frac{C}{\nu} \left\| f \right\|_{0,\Omega} \eta_{K,i}^{(L)} \left\| v_e \right\|_{L^2(e)},
\]

with

\[
\left\| v_e \right\|_{L^2(e)} \leq \epsilon \left\| \left[ \frac{\partial u_{h}^{i+1}}{\partial n} - p_{h}^{i+1} n \right] \right\|.
\]

Thus, we have estimated the second term of the local discretization indicator \( \eta_{K,i}^{(D)} \).

(iii) Finally, we take \( q = q_K \) in (5.5) such that

\[
q_K = \begin{cases} 
\nabla u_{h}^{i+1} & \text{on } K \\
0 & \text{on } \Omega \ \setminus \ K
\end{cases}
\]

We obtain

\[
\left\| \nabla u_{h}^{i+1} \right\|_{L^2(K)} \leq \left\| u - u_{h}^{i+1} \right\|_{1,K}.
\]

Collecting the bounds above leads to the final result

\[
\eta_{K,i}^{(D)} \leq C \left( \left\| u - u_{h}^{i+1} \right\|_{1,\omega_K} + \left\| u - u_{h}^{i+1} \right\|_{1,\omega_K} + \left\| p - p_{h}^{i+1} \right\|_{L^2(\Omega)} + \sum_{\kappa \in \omega_K} h_{\kappa} \left\| f - f_h \right\|_{L^2(\kappa)} \right).
\]

According to standard criteria, these estimates of the local linearization and discretization indicators are fully optimal [12].
6. Numerical results

In this section, we present numerical results for the Navier-Stokes iterative algorithm. These simulations have been performed using the code FreeFem++ due to F. Hecht and O. Pironneau, see [10].

6.1. A priori estimation. We consider the square $\Omega = [0, 3]^2$. Each edge is divided into $N$ equal segments so that $\Omega$ is divided into $2N^2$ triangles. We consider the iterative Navier-Stokes algorithm and the theoretical solution $(u, p) = (\text{rot } \psi, p)$ where $\psi$ and $p$ are defined as follows

$$\psi(x, y) = e^{-30((x-1)^2+(y-1)^2)},$$

$$p(x, y) = \cos(2\pi x)\cos(2\pi y).$$

Figures 1 and 2 compare the exact and the numerical solution of the pressure $p$ for $N = 100$. We can clearly see that the two solutions are coherent.

As well, Figures 3, 4, 5 and 6 compare the different components of the numerical and exact solutions of the velocity $u$ for $N = 100$. 

---

**Figure 1.** Numerical pressure  
**Figure 2.** Exact pressure

**Figure 3.** First component of the numerical velocity  
**Figure 4.** First component of the exact velocity
Figures 7 and 8 present the error curve of the velocity and the pressure as a function of $h$ in logarithmic scales. We test the algorithm for the number of segments $N$ going from 60 to 100. The slope of the velocity error curve is equal to 0.92 while it is 1.08 for the pressure error curve.

Remark 6.1. Note that the error curves of the pressure and the velocity are coherent with the theoretical results in Section 3.

6.2. A posteriori analysis. In this section, we test our a posteriori error estimates on the iterative Navier-Stokes problem. On the same domain as previously, we consider the exact solution $(u, p) = (\text{rot } \psi, p)$ where $\psi$ and $p$ are defined as follows:

\[
\psi(x, y) = e^{-30 \left( (x-1)^2 + (y-1)^2 \right)},
\]

\[
p(x, y) = \cos(2\pi \frac{x}{3}) \cos(2\pi \frac{y}{3}).
\]
We define two different stopping criteria: the classical one $\eta_i^{(L)} \leq 10^{-5}$ and the new one $\eta_i^{(L)} \leq \gamma \eta_i^{(D)}$, where $\gamma$ is a positive parameter. The linearization error indicator $\eta_i^{(L)}$ and the discretization error indicator $\eta_i^{(D)}$ are defined by

$$\eta_i^{(L)} = \left( \sum_{K \in \mathcal{T}_h} (\eta_{K,i}^{(L)})^2 \right)^{1/2},$$

$$\eta_i^{(D)} = \left( \sum_{K \in \mathcal{T}_h} (\eta_{K,i}^{(D)})^2 \right)^{1/2},$$

with

$$\eta_{K,i}^{(L)} = |u_h^{i+1} - u_h^i|_{1,K},$$

$$\eta_{K,i}^{(D)} = h_K \| f_h + \nu \Delta u_h^{i+1} - (u_h^i, \nabla) u_h^{i+1} - \nabla p_h^{i+1} \|_{L^2(K)} + \frac{1}{2} \sum_{e \in \mathcal{T}_K} h_e \| \frac{\partial u^{i+1}_h}{\partial n} - p_h^{i+1} n \|_{L^2(e)} + \| \text{div} u_h^{i+1} \|_{L^2(K)}.$$

Figures 9 to 12 show the evolution of the mesh (see [12], Introduction) using the iterative Navier-Stokes algorithm. An adaptive mesh refinement can be outlined as follows:

For $i \geq 0$,

1. Construct an initial mesh $\mathcal{T}_i$.
2. Solve the discrete problem on $\mathcal{T}_i$.
3. For each element $K$ in $\mathcal{T}_i$ compute the a posteriori error estimate.
4. If the estimated global error is sufficiently small then STOP. Otherwise refine locally the mesh (see [10] for details), recall $\mathcal{T}_{i+1}$ the new mesh; take $i = i + 1$ and return to step (2).
Figures 13 and 14 present the numerical and the exact first component of the velocity for the mesh refinement of Figure 12.

We observe that the numerical velocity and the exact velocity are perfectly coherent.

Figure 15 presents the error curve for uniform (red) and adaptive (blue) mesh refinement using the new stopping criterion. We note that the error using an adaptive mesh is much smaller than the error using a uniform mesh.

Figure 16 illustrates the performance of our new stopping criterion with $\gamma = 0.01$ by comparing it to the classical stopping criterion $\eta_i^{(L)} \leq 10^{-5}$. We can clearly observe that our new stopping criterion reduces the number of iterations.
Finally, the following table presents the CPU time of each level of refinement for both criteria, the classical one and the new one. We can see clearly the efficiency of the new stopping criterion with $\gamma = 0.01$.

<table>
<thead>
<tr>
<th>Method</th>
<th>Level of refinement</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>New criterion</td>
<td></td>
<td>6.466s</td>
<td>8.331s</td>
<td>14.439s</td>
<td>11.591s</td>
<td>15.351s</td>
</tr>
<tr>
<td>Classical criterion</td>
<td></td>
<td>30.609s</td>
<td>13.104s</td>
<td>21.279s</td>
<td>29.25s</td>
<td>49.483s</td>
</tr>
</tbody>
</table>

6.3. Conclusion. In this work we have derived a postiori error estimates for the finite element discretization of the Navier-Stokes equations. These estimates yield a fully computable upper bound which allow to distinguish the discretization and the linearization errors. We have shown in this work that balancing these two errors leads to important computational savings; in fact, it avoid performing an excessive number of iterations.

REFERENCES

