A POSTERIORI ANALYSIS OF AN ITERATIVE ALGORITHM FOR NAVIER-STOKES PROBLEM

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A POSTERIORI ANALYSIS OF AN ITERATIVE ALGORITHM FOR NAVIER-STOKES PROBLEM

CHRISTINE BERNARDI †, JAD DAKROUB †‡, GIHANE MANSOUR ‡, TONI SAYAH ‡.

Abstract. This work deals with *a posteriori* error estimates for the Navier-Stokes equations. We propose a finite element discretization relying on the Galerkin method and we solve the discrete problem using an iterative method. Two sources of error appear, the discretization error and the linearization error. Balancing these two errors is very important to avoid performing an excessive number of iterations. Several numerical tests are provided to evaluate the efficiency of our indicators.

Keywords: *A posteriori* error estimation, Navier-Stokes problem, iterative method.

1. Introduction

The *a posteriori* analysis controls the overall discretization error of a problem by providing error indicators easy to compute. Once these error indicators are constructed, we prove their efficiency by bounding each indicator by the local error. This analysis was first introduced by I. Babuška [2], and developed by R. Verfürth [12]. The present work investigates *a posteriori* error estimates of the finite element discretization of the Navier-Stokes equations in polygonal domains. In fact, many works have been carried out in this field. In [3], C. Bernardi, F. Hecht and R. Verfürth considered a variational formulation of the three-dimensional Navier-Stokes equations with mixed boundary conditions and they proved that it admits a solution if the domain satisfies a suitable regularity assumption. In addition, they established the *a priori* and the *a posteriori* error estimates. As well, in [8], V. Ervin, W. Layton and J. Maubach present locally calculable *a posteriori* error estimators for the basic two-level discretization of the Navier-Stokes equations. In this work, we propose a finite element discretization of the Navier-Stokes equations relying on the Galerkin method. In order to solve the discrete problem we propose an iterative method. Therefore two sources of error appear, due to the discretization and the algorithm. Balancing these two errors leads to important computational savings. We apply this strategy on the following Navier-Stokes equations:

Let Ω be a connected open domain in $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary $\partial\Omega$. We consider, for a positive constant viscosity ν, the following system:

$$\begin{array}{rcll} -\nu\Delta\mathbf{u} + (\mathbf{u}.\nabla)\mathbf{u} + \nabla p & = & \mathbf{f} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} & = & 0 & \text{in } \Omega \\ \mathbf{u} & = & \mathbf{0} & \text{on } \partial\Omega, \end{array} \tag{1.1}$$

where the unknowns are the velocity $\mathbf{u}$ and the pressure p of the fluid. The right-hand side $\mathbf{f}$ belongs to $H^{-1}(\Omega)^d$, the dual of the Sobolev space $H^1_0(\Omega)^d$.

Using $\mathcal{P}_1$ Lagrange finite elements for the pressure and $\mathcal{P}_1$-bubble Lagrange finite elements for the velocity, the discrete variational problem amounts to a system of nonlinear equations. In order to solve it we propose an iterative algorithm which consists at each iteration to solve a linearized problem. We establish the corresponding *a posteriori* error estimates. Thus, two sources of error appear, namely linearization and discretization. The main goal of this work is to balance these two sources of error. In fact, if the

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† CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France
Sorbonne Universits, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France
‡ Unité de recherche EGFEM, Faculté des sciences, Université Saint-Joseph, Lebanon.
bernardi@ann.jussieu.fr, jad.dakroub@usj.edu.lb, gihane.mansour@usj.edu.lb, toni.sayah@usj.edu.lb.

discretization error dominates then the nonlinear solver iterations are reduced. Therefore, our objective is to calculate *a posteriori* error estimates distinguishing linearization and discretization errors in the context of an adaptive procedure. This type of analysis was introduced by A.-L. Chaillou and M. Suri [5, 6] for a general class of problems characterized by strongly monotone operators. It had been developed by L. El Alaoui, A. Ern and M. Vohralík [7] for a class of second-order monotone quasi-linear diffusion-type problems approximated by piecewise affine, continuous finite elements.

In this work we present a strategy for the linearization process. This strategy is iterative and can be outlined as follows:

(1) On the given mesh, perform an iterative linearization until a stopping criterion is satisfied.
(2) If the error is less than the desired precision, then stop, else refine the mesh adaptively and go to step (1).

An outline of the paper is as follows. In Section 2, we present the variational formulation of Navier-Stokes problem (1.1). We introduce in Section 3 the discrete variational problem with the *a priori* error estimate. The *a posteriori* analysis of the discretization of the iterative algorithm is performed in Section 4. Section 5 is devoted to the numerical experiments.

2. Analysis of Navier-Stokes equations

We describe in this section the Navier-Stokes problem (1.1) together with its variational formulation. First of all, we recall the main notion and results which we use later on. For the domain Ω, denote by $L^p(\Omega)$ the space of measurable functions v such that $|v|^p$ is integrable. For $v \in L^p(\Omega)$, the norm is defined by

$$\| v \|_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p d\mathbf{x} \right)^{1/p}.$$

Throughout this paper, we constantly use the classical Sobolev space

$$W^{m,r}(\Omega) = \left\{ v \in L^r(\Omega);\ \forall |k| \le m,\ \partial^k v \in L^r(\Omega) \right\},$$

where $k = (k_1, ..., k_d)$ is a d-tuple of positive integers such that $|k| = k_1 + ... + k_d$ and

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} ... \partial x_d^{k_d}}.$$

$W^{m,r}(\Omega)$ is equipped with the semi-norm

$$|v|_{m,r,\Omega} = \left(\sum_{|k|=m} \int_\Omega |\partial^k v|^r d\mathbf{x} \right)^{1/r},$$

and the norm

$$\| v \|_{m,r,\Omega} = \left(\sum_{\ell=0}^{m} |v|^r_{\ell,r,\Omega} d\mathbf{x} \right)^{1/r}.$$

For $r = 2$, we define the Hilbert space $H^m(\Omega) = W^{m,2}(\Omega)$. In particular, we consider the following space

$$X = H_0^1(\Omega)^d = \left\{ v \in H^1(\Omega)^d,\ v_{|\partial\Omega} = 0 \right\},$$

and its dual space $H^{-1}(\Omega)^d$.

We denote by $L_0^2(\Omega)$ the space of functions in $L^2(\Omega)$ with zero mean-value on Ω.

$$M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega);\quad \int_\Omega q \, d\mathbf{x} = 0 \right\}.$$

We recall the Sobolev imbeddings (see Adams [1], Chapter 3).

Lemma 2.1. *For all $j \leq 6$ and $d = 2, 3$, there exists a positive constant S_j such that*

$$\forall v \in H_0^1(\Omega), \quad \| v \|_{L^j(\Omega)} \leq S_j |v|_{1,\Omega}. \tag{2.1}$$

We now assume that the data $\mathbf{f}$ belongs to $H^{-1}(\Omega)^d$. Then system (1.1) is equivalent to the following variational problem:

Find $\mathbf{u} \in X,\ p \in M$ such that

$$\begin{array}{llcl} \forall \mathbf{v} \in X, & a(\mathbf{u},\mathbf{v}) + c(\mathbf{u};\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) & = & \langle \mathbf{f},\mathbf{v}\rangle, \\ \forall q \in M, & b(\mathbf{u},q) & = & 0, \end{array} \tag{2.2}$$

where the bilinear forms $a(.,.)$ and $b(.,.)$ and the trilinear form $c(.,.,.)$ are defined by

$$\begin{array}{rcl} a(\mathbf{u},\mathbf{v}) & = & \nu \displaystyle\int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}\ d\mathbf{x}, \\ b(\mathbf{v},q) & = & -\displaystyle\int_\Omega q \ \mathrm{div}\ \mathbf{v}\ d\mathbf{x}, \\ c(\mathbf{w};\mathbf{u},\mathbf{v}) & = & \displaystyle\int_\Omega (\mathbf{w}.\nabla)\mathbf{u}\mathbf{v}\ d\mathbf{x}. \end{array} \tag{2.3}$$

Furthermore, the bilinear form $b(.,.)$ satisfies the following inf-sup condition (see [9], Chapter I, Equation (5.14) for instance)

$$\inf_{q\in M, q\neq 0} \sup_{\mathbf{v}\in X} \frac{b(\mathbf{v},q)}{\| \mathbf{v} \|_X \| q \|_M} = \beta > 0. \tag{2.4}$$

Now we recall the following space

$$V = \Big\{ \mathbf{v} \in X; \quad \mathrm{div}\ \mathbf{v} = 0 \Big\}.$$

Then, problem (2.2) has the following form:

Find $\mathbf{u} \in V$ such that

$$\forall \mathbf{v} \in V, \quad \nu \int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}d\mathbf{x} + \int_\Omega (\mathbf{u}.\nabla)\mathbf{u}\mathbf{v}d\mathbf{x} \quad = \quad \langle \mathbf{f},\mathbf{v}\rangle. \tag{2.5}$$

The existence and the conditional uniqueness of the solution $(\mathbf{u}, p)$ of problem (2.2) is given in [9] (Chapter IV, Section 2).

In order to calculate the *a posteriori* error estimate, we introduce the Stokes equations which are defined as follows:

$$\begin{array}{lcll} -\nu\Delta\mathbf{u} + \nabla p & = & \mathbf{f} & \text{in } \Omega \\ \mathrm{div}\ \mathbf{u} & = & 0 & \text{in } \Omega \\ \mathbf{u} & = & \mathbf{0} & \text{on } \partial\Omega. \end{array} \tag{2.6}$$

Using the previous notation, the Stokes problem amounts to the following variational form:

Find $\mathbf{u} \in X$, $p \in M$ such that

$$\begin{array}{llcl} \forall \mathbf{v} \in X, & a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) & = & \langle \mathbf{f},\mathbf{v}\rangle, \\ \forall q \in M, & b(\mathbf{u},q) & = & 0. \end{array} \tag{2.7}$$

The existence and the uniqueness of the solution $(\mathbf{u}, p) \in X \times M$ of problem (2.7) is given in [9], Chapter I, Section 5.1.

Remark 2.2. *In the sequel, we denote by C, a generic constant that can vary from line to line but is always independent of all discretization parameters.*

In what follows, for simplicity reasons, we suppose $d = 2$. In fact, the end of this work can easily be extended to $d = 3$ but requires some more technicalities that we prefer to avoid here.

3. Finite element discretization and the a priori estimate

This section collects some useful notation concerning the discrete setting and the *a priori* estimate.

Let $(\mathcal{T}_h)_h$ be a regular family of triangulations of the polygonal domain Ω, in the sense that, for each h:
- The union of all elements of $\mathcal{T}_h$ is equal to $\overline{\Omega}$.
- The intersection of two different elements of $\mathcal{T}_h$, if not empty, is a vertex or a whole edge of both triangles.
- The ratio of the diameter h_K of any element K of $\mathcal{T}_h$ to the diameter of its inscribed circle is smaller than a constant independent of h.

As usual, h stands for the maximum of the diameters h_K, $K \in \mathcal{T}_h$.

Let (X_h, M_h) be the couple of discrete spaces corresponding to (X, M) defined as follow :

$$M_h = \Big\{ q_h \in M,\ \forall K \in \mathcal{T}_h,\ \ q_{h_{|K}} \in \mathcal{P}_1(K) \Big\} \ \text{ and } \ X_h = \Big\{ \mathbf{v}_h \in X,\ \forall K \in \mathcal{T}_h,\ \mathbf{v}_{h_{|K}} \in (\mathcal{P}_1(K)\text{-bubble})^2 \Big\}$$

where $\mathcal{P}_1(K)$ stands for the space of restrictions to K of affine functions. $\mathcal{P}_1(K)$-bubble is defined by adding one extra degree of freedom to the barycenter of every simplex of the triangulation $\mathcal{T}_h$ of the domain Ω. We have the following inf-sup condition (see [9], Chapter II, Lemma 2.6) :

$$\inf_{q_h \in M_h, q_h \neq 0} \sup_{\mathbf{v}_h \in X_h} \frac{-\displaystyle\int_\Omega q_h \operatorname{div}\ \mathbf{v}_h \, d\mathbf{x}}{\| q_h \|_{L^2(\Omega)} \| \nabla \mathbf{v}_h \|_{L^2(\Omega)}} = \beta_* > 0. \tag{3.1}$$

We then consider the following finite element discretization of Navier-Stokes problem (2.2), obtained by the Galerkin method:

Find $\mathbf{u}_h \in X_h$, $p_h \in M_h$ such that

$$\begin{aligned} \forall \mathbf{v}_h \in X_h,\ \ \nu \int_\Omega \nabla \mathbf{u}_h \nabla \mathbf{v}_h d\mathbf{x} + \int_\Omega (\mathbf{u}_h.\nabla)\mathbf{u}_h \mathbf{v}_h d\mathbf{x} - \int_\Omega p_h \operatorname{div}\, \mathbf{v}_h \, d\mathbf{x} &= \langle \mathbf{f}, \mathbf{v}_h \rangle, \\ \forall q_h \in M_h,\ \ \int_\Omega q_h \operatorname{div}\, \mathbf{u}_h \, d\mathbf{x} &= 0. \end{aligned} \tag{3.2}$$

In order to solve the discrete problem (3.2), we introduce the following space

$$V_h = \Big\{ \mathbf{v}_h \in X_h; \forall q_h \in M_h,\ -\int_\Omega q_h \operatorname{div}\, \mathbf{v}_h d\mathbf{x} = 0 \Big\}.$$

Problem (3.2) is then equivalent to the problem:

Find $\mathbf{u}_h \in V_h$ such that

$$\forall \mathbf{v}_h \in V_h,\ \ \nu \int_\Omega \nabla \mathbf{u}_h \nabla \mathbf{v}_h d\mathbf{x} + \int_\Omega (\mathbf{u}_h.\nabla)\mathbf{u}_h \mathbf{v}_h d\mathbf{x} \ = \ \langle \mathbf{f}, \mathbf{v}_h \rangle, \tag{3.3}$$

and admits at least one solution $(\mathbf{u}_h, p_h) \in X_h \times M_h$ ([9], Chapter IV, Theorem 4.1) such that

$$|\mathbf{u}_h|_{1,\Omega} \leq \frac{c}{\nu} \| \mathbf{f} \|_{-1,\Omega} . \tag{3.4}$$

In addition, if $\mathbf{u} \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$, the *a priori* estimate can be proved by following the approach in [4]. Under some further assumptions, it reads ([9], Chapter IV, Theorem 4.1)

$$|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}+\parallel p-p_h\parallel_{0,\Omega}\leq Ch.$$

4. Iterative algorithm

In order to solve the Navier-Stokes discrete problem, we propose in this section a very simple iterative algorithm. In fact, we linearize the discrete problem and we set an initial guess $\mathbf{u}_h^0$. We will see later on that under suitable conditions, the solution of the iterative algorithm $(\mathbf{u}_h^{i+1}, p_h^{i+1})$ converges to the solution of the discrete problem $(\mathbf{u}_h, p_h)$.

Iterative algorithm. Let $\mathbf{u}_h^0$ be an initial guess. We introduce, for $i\geq 0$, the following algorithm:

Find $\mathbf{u}_h^{i+1}\in X_h$, $p_h^{i+1}\in M_h$ such that

$$\begin{array}{lll}\forall \mathbf{v}_h\in X_h,\ \ \nu\displaystyle\int_\Omega \nabla\mathbf{u}_h^{i+1}\nabla\mathbf{v}_h\, d\mathbf{x}+\int_\Omega(\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1}\mathbf{v}_h\, d\mathbf{x}-\int_\Omega p_h^{i+1}\,\mathrm{div}\,\mathbf{v}_h\, d\mathbf{x} &=& \langle\mathbf{f},\mathbf{v}_h\rangle,\\ \forall q_h\in M_h,\ \ \displaystyle\int_\Omega q_h\,\mathrm{div}\,\mathbf{u}_h^{i+1}\, d\mathbf{x} &=& 0.\end{array}\tag{4.1}$$

We clearly see that problem (4.1) has the following form:

Find $\mathbf{u}_h^{i+1}\in V_h$, such that

$$\forall\ \mathbf{v}_h\in V_h,\quad \nu\int_\Omega\nabla\mathbf{u}_h^{i+1}\nabla\mathbf{v}_h\, d\mathbf{x}+\int_\Omega(\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1}\mathbf{v}_h\, d\mathbf{x}\quad=\quad\langle\mathbf{f},\mathbf{v}_h\rangle.\tag{4.2}$$

Theorem 4.1. ***(The convergence Theorem).*** *Let $(\boldsymbol{u}_h^{i+1}, p_h^{i+1})\in X_h\times M_h$ and $(\boldsymbol{u}_h, p_h)\in X_h\times M_h$ be the solutions of the iterative problem* (4.1) *and the discrete problem* (3.2)*, respectively. Then, for $\nu > S_4\sqrt{\parallel \boldsymbol{f}\parallel_{-1,\Omega}}$, we have*

$$|\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_h|_{1,\Omega}\leq C_1C_2^{-1}|\boldsymbol{u}_h^i-\boldsymbol{u}_h|_{1,\Omega},\tag{4.3}$$

$$\parallel p_h^{i+1}-p_h\parallel_{L^2(\Omega)}\leq C_3|\boldsymbol{u}_h^i-\boldsymbol{u}_h|_{1,\Omega},\tag{4.4}$$

with

$$\begin{array}{lll} C_1 &=& \dfrac{S_4^2}{\nu}\parallel\boldsymbol{f}\parallel_{-1,\Omega},\\ C_2 &=& \nu-C_1,\\ C_3 &=& \alpha^{-1}\big((1+C_1)C_1C_2^{-1}+C_1\big).\end{array}$$

Moreover, the sequence $(\boldsymbol{u}_h^i)_i$ converges to the solution $\boldsymbol{u}_h$ of problem (3.2) *if $C_1^{-1}C_2<1$.*

Proof. (i) We start by estimate (4.3). We have (see once more [9], Chapter IV, Theorem 4.1)

$$|\mathbf{u}_h^{i+1}|_{1,\Omega}\leq\frac{c}{\nu}\parallel\mathbf{f}\parallel_{-1,\Omega}.\tag{4.5}$$

We now subtract (3.3) from (4.2) to obtain, for all $\mathbf{v}_h\in X_h$,

$$\nu\int_\Omega\nabla(\mathbf{u}_h^{i+1}-\mathbf{u}_h)\nabla\mathbf{v}_h\, d\mathbf{x}+\int_\Omega\big((\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1}-(\mathbf{u}_h.\nabla)\mathbf{u}_h\big)\mathbf{v}_h\, d\mathbf{x}=0.\tag{4.6}$$

Intercalating $\pm\displaystyle\int_\Omega(\mathbf{u}_h^i.\nabla)\mathbf{u}_h\mathbf{v}_h\, d\mathbf{x}$ and taking $\mathbf{v}_h=\mathbf{u}_h^{i+1}-\mathbf{u}_h$ in (4.6), we obtain by applying the Cauchy-Schwarz inequality the following estimate

$$\nu|\mathbf{u}_h^{i+1}-\mathbf{u}_h|_{1,\Omega}\leq\frac{S_4^2}{\nu}\parallel\mathbf{f}\parallel_{-1,\Omega}|\mathbf{u}_h^i-\mathbf{u}_h|_{1,\Omega}+\frac{S_4^2}{\nu}\parallel\mathbf{f}\parallel_{-1,\Omega}|\mathbf{u}_h^{i+1}-\mathbf{u}_h|_{1,\Omega}.\tag{4.7}$$

(ii) We now prove the second estimate (4.4). By subtracting (3.3) from (4.1) we obtain

$$\int_\Omega\big(p_h^{i+1}-p_h\big)\mathrm{div}\,\mathbf{v}_h\, d\mathbf{x}=\int_\Omega\nabla(\mathbf{u}_h^{i+1}-\mathbf{u}_h)\nabla\mathbf{v}_h\, d\mathbf{x}+\int_\Omega\big((\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1}-(\mathbf{u}_h.\nabla)\mathbf{u}_h\big)\mathbf{v}_h\, d\mathbf{x}.\tag{4.8}$$

Intercalating $\pm\displaystyle\int_\Omega(\mathbf{u}_h^i.\nabla)\mathbf{u}_h\mathbf{v}_h d\mathbf{x}$, using (3.1), (3.4), (4.5) and the Cauchy-Schwarz inequality we obtain

$$\alpha\parallel p_h^{i+1}-p_h\parallel_{L^2(\Omega)}\leq\big(1+C_1\big)|\mathbf{u}_h^{i+1}-\mathbf{u}_h|_{1,\Omega}+C_1|\mathbf{u}_h^i-\mathbf{u}_h|_{1,\Omega}.\tag{4.9}$$

Finally, combining (4.7) and (4.9) yields the desired estimates and convergence property.

5. A posteriori error analysis

We start this section by introducing some additional notation needed for constructing and analyzing the error indicators in the sequel.

For any element $K \in \mathcal{T}_h$ we denote by $\mathcal{E}(K)$ the set of its edges and we set

$$\mathcal{E}_h = \bigcup_{K \in \mathcal{T}_h} \mathcal{E}(K).$$

With any edge $e \in \mathcal{E}_h$ we associate a unit vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to e. We split $\mathcal{E}(K)$ in the form

$$\mathcal{E}(K) = \mathcal{E}_{K,\partial\Omega} \cup \mathcal{E}_{K,\Omega},$$

where $\mathcal{E}_{K,\partial\Omega}$ is the set of edges in $\mathcal{E}(K)$ that lie on $\partial\Omega$ and $\mathcal{E}_{K,\Omega} = \mathcal{E}(K) \setminus \mathcal{E}_{K,\partial\Omega}$. Furthermore, for $K \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$, let h_K and h_e be their diameter and length respectively. An important tool in the construction of bounds for the total error is Clément's interpolation operator $\mathcal{R}_h$ with values in X_h. The operator $\mathcal{R}_h$ satisfies, for all $v \in H_0^1(\Omega)$, the following local approximation properties (see R. Verfürth, [12], Chapter 1):

$$\begin{aligned} \| v - R_h v \|_{L^2(K)} &\leq C h_K |v|_{1,\Delta_K}, \\ \| v - R_h v \|_{L^2(e)} &\leq C h_e^{1/2} |v|_{1,\Delta_e}, \end{aligned}$$

where Δ_K and Δ_e are the following sets:

$$\Delta_K = \bigcup \Big\{ K' \in \mathcal{T}_h;\ \ K' \cap K \neq \emptyset \Big\} \quad \text{and} \quad \Delta_e = \bigcup \Big\{ K' \in \mathcal{T}_h;\ \ K' \cap e \neq \emptyset \Big\}.$$

We now recall the following properties (see R. Verfürth, [12], Chapter 1):

Proposition 5.1. *Let r be a positive integer. For all $v \in P_r(K)$, the following properties hold*

$$C \| v \|_{L^2(K)} \leq \| v \psi_K^{1/2} \|_{L^2(K)} \leq \| v \|_{L^2(K)} , \tag{5.1}$$

$$|v|_{1,K} \leq C h_K^{-1} \| v \|_{L^2(K)} . \tag{5.2}$$

where ψ_K is the triangle-bubble function (equal to the product of the barycentric coordinates associated with the nodes of K).

Finally, we denote by $[v_h]$ the jump of v_h across the common edge e of two adjacent elements $K, K' \in \mathcal{T}_h$. We have now provided all prerequisites to establish bounds for the total error.

We start the *a posteriori* analysis of the iterative algorithm. In order to prove an upper bound of the error, we first introduce an approximation f_h of the data f which is constant on each element K of $\mathcal{T}_h$. Then, we distinguish the discretization and linearization errors. We first write the residual equation

$$\begin{aligned} &\nu \int_\Omega \nabla \mathbf{u} \nabla \mathbf{v}\, d\mathbf{x} + \int_\Omega (\mathbf{u}.\nabla)\mathbf{u}\mathbf{v}\, d\mathbf{x} - \int_\Omega p \,\mathrm{div}\, \mathbf{v}\, d\mathbf{x} \\ &\quad -\nu \int_\Omega \nabla \mathbf{u}_h^{i+1} \nabla \mathbf{v}\, d\mathbf{x} - \int_\Omega (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1}\mathbf{v}\, d\mathbf{x} + \int_\Omega p_h^{i+1} \,\mathrm{div}\, \mathbf{v}\, d\mathbf{x} \\ &\quad = \langle \mathbf{f}, \mathbf{v} - \mathbf{v}_h \rangle - \nu \int_\Omega \nabla \mathbf{u}_h^{i+1} \nabla(\mathbf{v} - \mathbf{v}_h)\, d\mathbf{x} - \int_\Omega (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1}(\mathbf{v} - \mathbf{v}_h)\, d\mathbf{x} + \int_\Omega p_h^{i+1} \,\mathrm{div}\, (\mathbf{v} - \mathbf{v}_h)\, d\mathbf{x}. \end{aligned} \tag{5.3}$$

Adding and subtracting $\int_\Omega (\mathbf{u}_h^{i+1}.\nabla)\mathbf{u}_h^{i+1}\mathbf{v}\, d\mathbf{x}$ and using the Green formula, give

$$\nu \int_{\Omega} \nabla \mathbf{u} \nabla \mathbf{v} \, d\mathbf{x} + \int_{\Omega} (\mathbf{u}.\nabla)\mathbf{u}\mathbf{v} \, d\mathbf{x} - \int_{\Omega} p \, \mathrm{div} \, \mathbf{v} \, d\mathbf{x}$$

$$-\nu \int_{\Omega} \nabla \mathbf{u}_h^{i+1} \nabla \mathbf{v} \, d\mathbf{x} - \int_{\Omega} (\mathbf{u}_h^{i+1}.\nabla)\mathbf{u}_h^{i+1}\mathbf{v} \, d\mathbf{x} + \int_{\Omega} p_h^{i+1} \, \mathrm{div} \, \mathbf{v} \, d\mathbf{x}$$

$$= \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{f} - \mathbf{f}_h)(\mathbf{v} - \mathbf{v}_h) \, d\mathbf{x} + \sum_{K \in \mathcal{T}_h} \Big\{ \int_K (\mathbf{f}_h + \nu \Delta \mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1})(\mathbf{v} - \mathbf{v}_h) \, d\mathbf{x}$$

$$-\frac{1}{2} \sum_{e \in \mathcal{E}_{K,\Omega}} \int_e [\frac{\partial \mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1}\mathbf{n}].(\mathbf{v} - \mathbf{v}_h) \, d\tau \Big\} + \int_{\Omega} \big((\mathbf{u}_h^i - \mathbf{u}_h^{i+1}).\nabla\big)\mathbf{u}_h^{i+1}\mathbf{v} \, d\mathbf{x}, \tag{5.4}$$

where τ denotes the tangential coordinate on ∂K.

On the other hand, for all $q \in L^2(\Omega)$

$$b(\mathbf{u} - \mathbf{u}_h^{i+1}, q) = \int_{\Omega} q \, \mathrm{div} \, \mathbf{u}_h^{i+1} \, d\mathbf{x}. \tag{5.5}$$

We now define the local linearization indicator $\eta_{K,i}^{(L)}$ and the local discretization indicator $\eta_{K,i}^{(D)}$, corresponding to an element $K \in \mathcal{T}_h$, by:

$$\eta_{K,i}^{(L)} = |\mathbf{u}_h^{i+1} - \mathbf{u}_h^i|_{1,K}, \tag{5.6}$$

$$\begin{aligned} \eta_{K,i}^{(D)} \quad = \quad & h_K \parallel \mathbf{f}_h + \nu \Delta \mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1} \parallel_{L^2(K)} \\ & + \frac{1}{2} \sum_{e \in \mathcal{E}_{K,\Omega}} h_e^{1/2} \parallel [\frac{\partial \mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1}\mathbf{n}] \parallel_{L^2(e)} + \parallel \mathrm{div} \, \mathbf{u}_h^{i+1} \parallel_{L^2(K)} . \end{aligned} \tag{5.7}$$

In order to calculate the *a posteriori* error estimates, we denote by $\mathcal{S}$ the operator which associates with any $\mathbf{f}$ in $H^{-1}(\Omega)^d$ the part $\mathbf{w} = \mathbf{u}$ of the solution $(\mathbf{u}, p)$ of the Stokes problem (2.6),

$$\begin{array}{rccc} \mathcal{S}: & H^{-1}(\Omega)^d & \rightarrow & X \\ & \mathbf{f} & \mapsto & \mathcal{S}\mathbf{f} = \mathbf{w}. \end{array}$$

We consider now the following mapping

$$\begin{array}{rccc} G: & X & \rightarrow & H^{-1}(\Omega)^d \\ & \mathbf{w} & \mapsto & G(\mathbf{w}) = (\mathbf{w}.\nabla)\mathbf{w} - \mathbf{f}. \end{array}$$

and we observe that problem (2.2) can equivalently be written as

$$F(\mathbf{u}) = \mathbf{u} + SG(\mathbf{u}) = 0. \tag{5.8}$$

Lemma 5.2. *Let $(\boldsymbol{u}, p)$ be the solution of problem* (2.2)*. There exists a real number $L > 0$, such that the following Lipschitz property holds*

$$\forall \boldsymbol{w} \in X, \ \parallel \mathcal{S}\big(DG(\boldsymbol{u}) - DG(\boldsymbol{w})\big) \parallel_{\mathcal{L}(H^1(\Omega))} \leq L|\boldsymbol{u} - \boldsymbol{w}|_{1,\Omega}.$$

Proof. We have, for all $\mathbf{w}, \ \mathbf{z} \in X$

$$\parallel \mathcal{S}\big(DG(\mathbf{u}).\mathbf{z} - DG(\mathbf{w}).\mathbf{z}\big) \parallel_{1,\Omega} \leq \frac{1}{\nu} \parallel DG(\mathbf{u}).\mathbf{z} - DG(\mathbf{w}).\mathbf{z} \parallel_{-1,\Omega} . \tag{5.9}$$

We observe that

$$DG(\mathbf{u}).\mathbf{z} - DG(\mathbf{w}).\mathbf{z} = \mathbf{z}.\nabla(\mathbf{u} - \mathbf{w}) + (\mathbf{u} - \mathbf{w}).\nabla\mathbf{z}, \tag{5.10}$$

whence

$$\parallel \big(DG(\mathbf{u}) - DG(\mathbf{w})\big).\mathbf{z} \parallel_{-1,\Omega} \leq 2S_4^2|\mathbf{u} - \mathbf{w}|_{1,\Omega}|\mathbf{z}|_{1,\Omega}. \tag{5.11}$$

Thus, combining (5.9) with (5.10) and (5.11) yields the desired property.

Assumption 5.3. *The solution $(\boldsymbol{u},p) \in X \times M$ of problem* (2.2) *is such that the operator $Id + SDG(\boldsymbol{u})$ is an isomorphism of X.*

Remark 5.4. *Assumption* 5.3 *implies that the solution* $\mathbf{u}$ *is locally unique, which is more weaker than the global uniqueness of the solution.*

We can now state the first result of this section:

Theorem 5.5. *Let $(\boldsymbol{u}_h^{i+1}, p_h^{i+1}) \in X_h \times M_h$ and $(\boldsymbol{u}_h, p_h) \in X_h \times M_h$ be the solutions of the iterative problem* (4.1) *and the discrete problem* (3.2)*, respectively. Suppose that the solution $(\boldsymbol{u},p)$ satisfies Assumption* 5.3*. Then, there exists a neighborhood $\mathcal{O}$ of $\boldsymbol{u}$ in X such that any solution $(\boldsymbol{u}_h^{i+1}, p_h^{i+1})$ of problem* (4.1) *with $\boldsymbol{u}_h^{i+1}$ in $\mathcal{O}$ satisfies the following a posteriori error estimate*

$$|\boldsymbol{u} - \boldsymbol{u}_h^{i+1}|_{1,\Omega} + \| p - p_h^{i+1} \|_{L^2(\Omega)} \leq C \left(\sum_{K\in\mathcal{T}_h} ((\eta_{K,i}^{(D)})^2 + h_K^2 \| \boldsymbol{f} - \boldsymbol{f}_h \|^2_{L^2(K)}) \right)^{1/2} + C' \left(\sum_{K\in\mathcal{T}_h} (\eta_{K,i}^{(L)})^2 \right)^{1/2}.$$

Proof. (i) Owing to Lemma 5.2 and Assumption 5.3, it follows from [11] that, for any $\mathbf{u}_h^{i+1}$ in a appropriate neighborhood $\mathcal{O}$ of $\mathbf{u}$

$$|\mathbf{u} - \mathbf{u}_h^{i+1}|_{1,\Omega} \leq C \| \mathbf{u}_h^{i+1} + \mathcal{S}G(\mathbf{u}_h^{i+1}) \|_{-1,\Omega} . \tag{5.12}$$

Introducing $F(\mathbf{u})$ in (5.12) (see equation (5.8)), and from equation (5.4), we obtain for all $\mathbf{v}_h \in X_h$

$$|\mathbf{u} - \mathbf{u}_h^{i+1}|_{1,\Omega} \leq C \Bigg(\sup_{\substack{\mathbf{v}\in X\\ \mathbf{v}\neq 0}} \frac{\langle \mathbf{f} - \mathbf{f}_h, \mathbf{v} - \mathbf{v}_h\rangle + \langle \mathcal{J}, \mathbf{v} - \mathbf{v}_h\rangle}{|\mathbf{v}|_{1,\Omega}} + \sup_{\substack{\mathbf{v}\in X\\ \mathbf{v}\neq 0}} \frac{\int_\Omega ((\mathbf{u}_h^i - \mathbf{u}_h^{i+1}).\nabla)\mathbf{u}_h^{i+1}\mathbf{v}\, d\mathbf{x}}{|\mathbf{v}|_{1,\Omega}} + \sup_{\substack{q\in M\\ q\neq 0}} \frac{\int_\Omega q \,\mathrm{div}\, \mathbf{u}_h^{i+1}\, d\mathbf{x}}{\| q \|_{L^2(\Omega)}} \Bigg), \tag{5.13}$$

where

$$\langle \mathcal{J}, \mathbf{v} - \mathbf{v}_h\rangle = \sum_{K\in\mathcal{T}_h} \Bigg\{ \int_K (\mathbf{f}_h + \nu\Delta\mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1})(\mathbf{v} - \mathbf{v}_h)\, d\mathbf{x} - \frac{1}{2} \sum_{e\in\mathcal{E}_{K,\Omega}} \int_e [\frac{\partial \mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1}\mathbf{n}].(\mathbf{v} - \mathbf{v}_h)\, d\tau \Bigg\}.$$

Taking $\mathbf{v}_h$ equal to the image $\mathcal{R}_h\mathbf{v}$ of $\mathbf{v}$ by the Clément operator in (5.13), we obtain the desired estimate for $|\mathbf{u} - \mathbf{u}_h^{i+1}|_{1,\Omega}$.

(ii) Computing $b(\mathbf{v}, p - p_h^{i+1})$ from (5.4) and adding and subtracting $\int_\Omega (\mathbf{u}_h^{i+1}.\nabla)\mathbf{u}\mathbf{v}\, d\mathbf{x}$ we obtain

$$\begin{aligned} b(\mathbf{v}, p - p_h^{i+1}) &= \nu \int_\Omega \nabla(\mathbf{u}_h^{i+1} - \mathbf{u})\nabla\mathbf{v}\, d\mathbf{x} + \int_\Omega (\mathbf{u}_h^{i+1}.\nabla)(\mathbf{u}_h^{i+1} - \mathbf{u})\mathbf{v}\, d\mathbf{x} + \int_\Omega (\mathbf{u}_h^{i+1} - \mathbf{u})\nabla\mathbf{u}\mathbf{v}\, d\mathbf{x} \\ &+ \sum_{K\in\mathcal{T}_h} \int_K (\mathbf{f} - \mathbf{f}_h)(\mathbf{v} - \mathbf{v}_h)\, d\mathbf{x} + \sum_{K\in\mathcal{T}_h} \Bigg\{ \int_K (\mathbf{f}_h + \nu\Delta\mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1})(\mathbf{v} - \mathbf{v}_h)\, d\mathbf{x} \\ &- \frac{1}{2} \sum_{e\in\mathcal{E}_{K,\Omega}} \int_e [\frac{\partial \mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1}\mathbf{n}].(\mathbf{v} - \mathbf{v}_h)\, d\tau \Bigg\} + \int_\Omega ((\mathbf{u}_h^i - \mathbf{u}_h^{i+1}).\nabla)\mathbf{u}_h^{i+1}\mathbf{v}\, d\mathbf{x}. \end{aligned} \tag{5.14}$$

Using Cauchy-Shwarz inequality and the fact that $\mathbf{u}$ and $\mathbf{u}_h^{i+1}$ are bounded independently of h, we derive the following estimate

$$\begin{aligned}
b(\mathbf{v}, p-p_h^{i+1}) \le \sum_{K\in\mathcal{T}_h} \Big(\| \mathbf{f}-\mathbf{f}_h \|_{L^2(K)} + \| \mathbf{f}_h + \nu\Delta\mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1} \|_{L^2(K)} \Big) \| \mathbf{v}-\mathbf{v}_h \|_{L^2(K)} \\
+\frac{1}{2}\sum_{e\in\mathcal{E}_{K,\Omega}} \| [\frac{\partial \mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1}\mathbf{n}] \|_{L^2(e)} \| \mathbf{v}-\mathbf{v}_h \|_{L^2(K)} + \frac{S_4^2}{\nu} \| \mathbf{f} \|_{0,\Omega} |\mathbf{u}_h^{i+1} - \mathbf{u}_h^i|_{1,\Omega} |\mathbf{v}|_{1,\Omega} \\
+\Big(2\frac{S_4^2}{\nu} \| \mathbf{f} \|_{0,\Omega} + \nu\Big) |\mathbf{u}_h^{i+1} - \mathbf{u}|_{1,\Omega} |\mathbf{v}|_{1,\Omega}.
\end{aligned} \tag{5.15}$$

Taking $\mathbf{v}_h$ equal $\mathcal{R}_h\mathbf{v}$ in (5.15) and using the inf-sup condition (2.4), we obtain the desired estimate for $\| p-p_h^{i+1} \|_{L^2(\Omega)}$.

We address now the efficiency of the previous indicators.

Theorem 5.6. *For each $K \in \mathcal{T}_h$, the following estimates hold for the indicators $\eta_{K,i}^{(L)}$ defined in* (5.6)

$$\eta_{K,i}^{(L)} \le \| \boldsymbol{u}-\boldsymbol{u}_h^{i+1} \|_{1,K} + \| \boldsymbol{u}-\boldsymbol{u}_h^i \|_{1,K}, \tag{5.16}$$

and for the indicators $\eta_{K,i}^{(D)}$ defined in (5.7)

$$\begin{aligned}
\eta_{K,i}^{(D)} \le C\Big(\| \boldsymbol{u}-\boldsymbol{u}_h^{i+1} \|_{1,\omega_K} + \| \boldsymbol{u}-\boldsymbol{u}_h^i \|_{1,\omega_K} + \| p_h^{i+1} - p \|_{L^2(\omega_K)} \\
+ \sum_{\kappa\subset\omega_K} h_\kappa \| \boldsymbol{f}-\boldsymbol{f}_h \|_{L^2(\kappa)} \Big),
\end{aligned} \tag{5.17}$$

where ω_K is the union of the elements sharing at least one edge with K.

Proof. The estimation of the linearization indicator follows easily from the triangle inequality by introducing $\mathbf{u}$ in $\eta_{K,i}^{(L)}$. We now estimate the discretization indicator $\eta_{K,i}^{(D)}$. We proceed in two steps:

(i) We start by adding and subtracting $\int_\Omega (\mathbf{u}_h^{i+1}.\nabla)\mathbf{u}\mathbf{v}\, d\mathbf{x}$ in (5.3). Taking $\mathbf{v}_h = 0$, we obtain

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} \Big(\int_K (\mathbf{f}_h + \nu\Delta\mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1})\mathbf{v} \Big)\, d\mathbf{x} \\
= \nu\int_\Omega \nabla(\mathbf{u}-\mathbf{u}_h^{i+1})\nabla\mathbf{v}\, d\mathbf{x} + \int_\Omega \big((\mathbf{u}-\mathbf{u}_h^{i+1}).\nabla\big)\mathbf{u}\mathbf{v}\, d\mathbf{x} - \sum_{K\in\mathcal{T}_h}\int_K (\mathbf{f}-\mathbf{f}_h)\mathbf{v}\, d\mathbf{x} \\
+\frac{1}{2}\sum_{e\in\mathcal{E}_{K,\Omega}} \int_e [\frac{\partial\mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1}\mathbf{n}].\mathbf{v}\, d\tau \Big\} + \int_\Omega \big(\mathbf{u}_h^{i+1}.\nabla\big)(\mathbf{u}-\mathbf{u}_h^{i+1})\mathbf{v}\, d\mathbf{x} \\
+ \int_\Omega \big((\mathbf{u}_h^{i+1}-\mathbf{u}_h^i).\nabla\big)\mathbf{u}_h^{i+1}\mathbf{v}\, d\mathbf{x} + \int_\Omega (p_h^{i+1}-p)\, \mathrm{div}\, \mathbf{v}\, d\mathbf{x}.
\end{aligned} \tag{5.18}$$

We choose $\mathbf{v} = \mathbf{v}_K$ such that

$$\mathbf{v}_K = \begin{cases} (\mathbf{f}_h + \nu\Delta\mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1})\psi_K & \text{on } K \\ 0 & \text{on } \Omega\setminus K, \end{cases}$$

where ψ_K is the triangle-bubble function of the element K.

Using Cauchy-Schwarz inequality, (5.1) and (5.2) we obtain

$$\begin{aligned}
h_K \| \mathbf{f}_h + \nu\Delta\mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1} \|_{L^2(K)}^2 \\
\le (\nu + \frac{2C}{\nu} \| \mathbf{f} \|_{0,\Omega}) \| \mathbf{u}-\mathbf{u}_h^{i+1} \|_{1,K} \| \mathbf{v}_K \|_{L^2(K)} + h_K \| \mathbf{f}-\mathbf{f}_h \|_{L^2(K)} \| \mathbf{v}_K \|_{L^2(K)} \\
+ \| p_h^{i+1} - p \|_{L^2(\Omega)} \| \mathbf{v}_K \|_{L^2(K)} + \frac{C}{\nu} \| \mathbf{f} \|_{0,\Omega} \| \mathbf{u}_h^i - \mathbf{u}_h^{i+1} \|_{1,K} \| \mathbf{v}_K \|_{L^2(K)}\, .
\end{aligned} \tag{5.19}$$

Therefore, we obtain the following estimate of the first term of the local discretization estimator $\eta_{K,i}^{(D)}$

$$h_K \parallel \mathbf{f}_h + \nu\Delta\mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1} \parallel_{L^2(K)}$$
$$\le C\Big(\parallel \mathbf{u} - \mathbf{u}_h^{i+1} \parallel_{1,K} + \parallel \mathbf{u} - \mathbf{u}_h^i \parallel_{1,K} + h_K \parallel \mathbf{f} - \mathbf{f}_h \parallel_{L^2(K)} + \parallel p_h^{i+1} - p \parallel_{L^2(\Omega)} + \eta_{K,i}^{(L)}\Big). \quad (5.20)$$

(ii) We now estimate the second term of $\eta_{K,i}^{(D)}$. Similarly, taking $\mathbf{v}_h = 0$ in (5.4) we infer

$$\frac{1}{2}\sum_{e\in\mathcal{E}_{K,\Omega}} \int_e [\frac{\partial \mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1}\mathbf{n}].\mathbf{v}\, d\tau = \int_K (\mathbf{f}_h + \nu\Delta\mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1})\mathbf{v}\, d\mathbf{x}$$
$$+\nu \int_\Omega \nabla(\mathbf{u}_h^{i+1} - \mathbf{u})\nabla\mathbf{v}\, d\mathbf{x} + \int_\Omega \big((\mathbf{u}_h^{i+1} - \mathbf{u}).\nabla\big)\mathbf{u}\mathbf{v}\, d\mathbf{x} + \sum_{K\in\mathcal{T}_h}\int_K (\mathbf{f} - \mathbf{f}_h)\mathbf{v}\, d\mathbf{x} \quad (5.21)$$
$$+ \int_\Omega (\mathbf{u}_h^{i+1}.\nabla)(\mathbf{u}_h^{i+1} - \mathbf{u})\mathbf{v}\, d\mathbf{x} + \int_\Omega \big((\mathbf{u}_h^i - \mathbf{u}_h^{i+1}).\nabla\big)\mathbf{u}_h^{i+1}\mathbf{v}\, d\mathbf{x}$$
$$+ \int_\Omega (p - p_h^{i+1})\ \mathrm{div}\ \mathbf{v}\, d\mathbf{x}.$$

We choose $\mathbf{v} = \mathbf{v}_e$ such that

$$\mathbf{v}_e = \begin{cases} L_{e,\kappa}\left([\frac{\partial \mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1}\mathbf{n}]\psi_e \right) & \kappa \in \{K, K'\} \\ 0 & \text{on } \Omega\setminus(K\cup K') \end{cases}$$

where ψ_e is the edge-bubble function, K' denotes the other element of $\mathcal{T}_h$ that share e with K and $L_{e,\kappa}$ is a lifting operator from e into κ mapping polynomials vanishing on ∂e into polynomials vanishing in $\partial\backslash e$ and constructed from a fixed operator on the reference element.

Using Cauchy-Schwarz inequality, (5.1) and (5.2) we get

$$h_e^{1/2} \parallel [\frac{\partial \mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1}\mathbf{n}] \parallel_{L^2(e)}^2$$
$$\le (\nu + \frac{2C}{\nu} \parallel \mathbf{f} \parallel_{0,\Omega}) \parallel \mathbf{u} - \mathbf{u}_h^{i+1} \parallel_{1,K\cup K'} \parallel \mathbf{v}_e \parallel_{L^2(e)} + h_e \parallel \mathbf{f} - \mathbf{f}_h \parallel_{L^2(K\cup K')} \parallel \mathbf{v}_e \parallel_{L^2(e)}$$
$$+ h_e \parallel \mathbf{f}_h + \nu\Delta\mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1} \parallel_{L^2(K\cup K')} \parallel \mathbf{v}_e \parallel_{L^2(e)} \quad (5.22)$$
$$+ \parallel p_h^{i+1} - p \parallel_{L^2(K\cup K')} \parallel \mathbf{v}_e \parallel_{L^2(e)} + \frac{C}{\nu} \parallel \mathbf{f} \parallel_{0,\Omega} \eta_{K,i}^{(L)} \parallel \mathbf{v}_e \parallel_{L^2(e)},$$

with

$$\parallel \mathbf{v}_e \parallel_{L^2(e)}, \le c \parallel [\frac{\partial \mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1}\mathbf{n}] \parallel .$$

Thus, we have estimated the second term of the local discretization indicator $\eta_{K,i}^{(D)}$.

(iii) Finally, we take $q = q_K$ in (5.5) such that

$$q_K = \begin{cases} \mathrm{div}\ \mathbf{u}_h^{i+1} & \text{on } K \\ 0 & \text{on } \Omega\setminus K \end{cases}$$

We obtain

$$\parallel \mathrm{div}\ \mathbf{u}_h^{i+1} \parallel_{L^2(K)} \le \parallel \mathbf{u} - \mathbf{u}_h^{i+1} \parallel_{1,K} . \quad (5.23)$$

Collecting the bounds above leads to the final result

$$\eta_{K,i}^{(D)} \le C\big(\parallel \mathbf{u} - \mathbf{u}_h^i \parallel_{1,\omega_K} + \parallel \mathbf{u} - \mathbf{u}_h^{i+1} \parallel_{1,\omega_K} + \parallel p - p_h^{i+1} \parallel_{L^2(\kappa)} + \sum_{\kappa\subset\omega_K} h_\kappa \parallel \mathbf{f} - \mathbf{f}_h \parallel_{L^2(\kappa)} \big).$$

According to standard criteria, these estimates of the local linearization and discretization indicators are fully optimal [12].

6. Numerical results

In this section, we present numerical results for the Navier-Stokes iterative algorithm. These simulations have been performed using the code FreeFem++ due to F. Hecht and O. Pironneau, see [10].

6.1. **A priori estimation.** We consider the square $\Omega =]0,3[^2$. Each edge is divided into N equal segments so that Ω is divided into $2N^2$ triangles. We consider the iterative Navier-Stokes algorithm and the theoretical solution $(\mathbf{u}, p) = (\text{rot}\ \psi, p)$ where ψ and p are defined as follows

$$\psi(x,y) = \mathrm{e}^{-30\big((x-1)^2+(y-1)^2\big)},$$

$$p(x,y) = cos(2\pi x)cos(2\pi y).$$

Figures 1 and 2 compare the exact and the numerical solution of the pressure p for $N = 100$. We can clearly see that the two solutions are coherent.

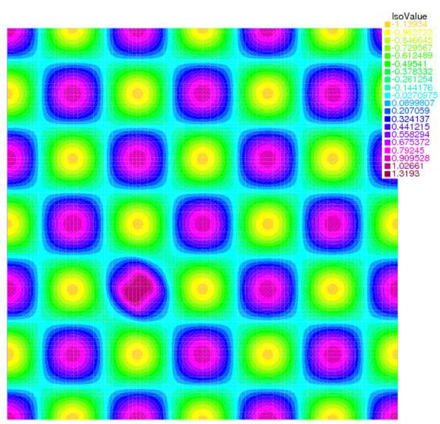

Figure 1. Numerical pressure

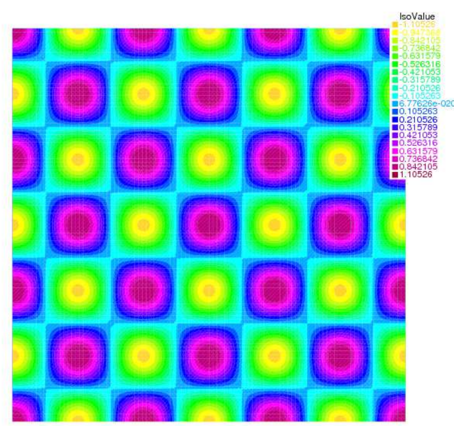

Figure 2. Exact pressure

As well, Figures 3, 4, 5 and 6 compare the different components of the numerical and exact solutions of the velocity $\mathbf{u}$ for $N = 100$.

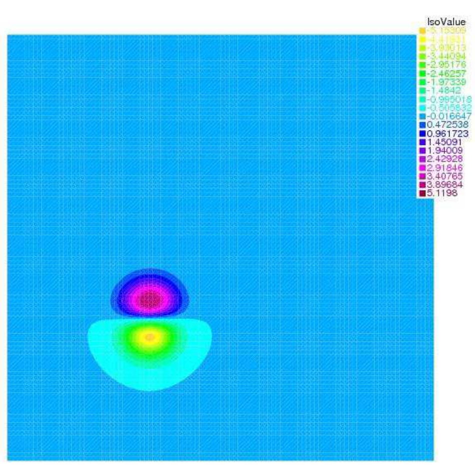

Figure 3. First component of the numerical velocity

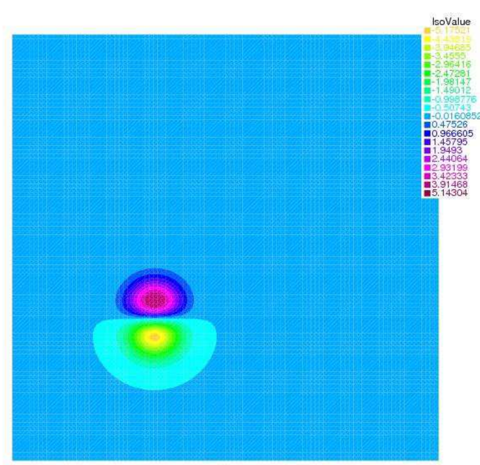

Figure 4. First component of the exact velocity

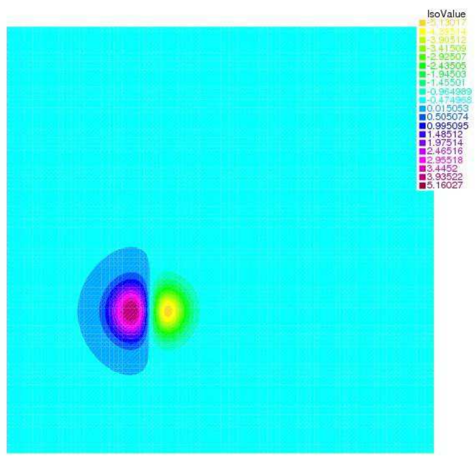

FIGURE 5. Second component of the numerical velocity

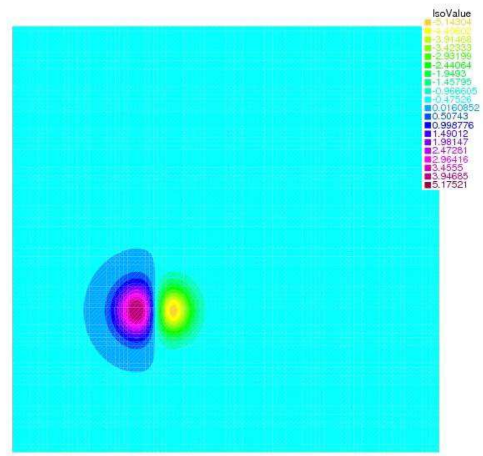

FIGURE 6. Second component of the exact velocity

Figures 7 and 8 present the error curve of the velocity and the pressure as a function of h in logarithmic scales. We test the algorithm for the number of segments N going from 60 to 100. The slope of the velocity error curve is equal to 0.92 while it is 1.08 for the pressure error curve

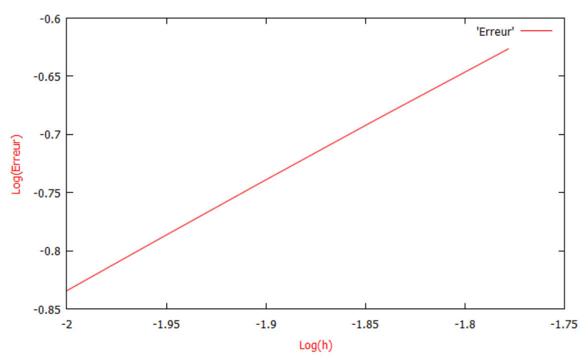

FIGURE 7. Error curve of the velocity for N going from 60 to 100

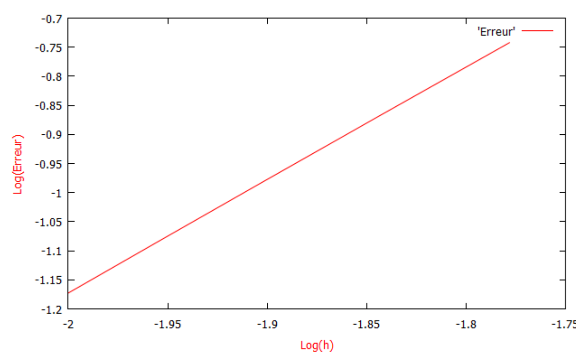

FIGURE 8. Error curve of the pressure for N going from 60 to 100

Remark 6.1. *Note that the error curves of the pressure and the velocity are coherent with the theoretical results in Section* 3.

6.2. **A posteriori analysis.** In this section, we test our *a posteriori* error estimates on the iterative Navier-Stokes problem. On the same domain as previously, we consider the exact solution $(\mathbf{u}, p) = (\text{rot}\ \psi, p)$ where ψ and p are defined as follows

$$\psi(x,y) = \mathrm{e}^{-30\left((x-1)^2+(y-1)^2\right)},$$

$$p(x,y) = cos(2\pi\frac{x}{3})cos(2\pi\frac{y}{3}).$$

We define two different stopping criteria: the classical one $\eta_i^{(L)} \leq 10^{-5}$ and the new one $\eta_i^{(L)} \leq \gamma \eta_i^{(D)}$, where γ is a positive parameter. The linearization error indicator $\eta_i^{(L)}$ and the discretization error indicator $\eta_i^{(D)}$ are defined by

$$\eta_i^{(L)} = \left(\sum_{K \in \mathcal{T}_h} \left(\eta_{K,i}^{(L)}\right)^2 \right)^{1/2},$$

$$\eta_i^{(D)} = \left(\sum_{K \in \mathcal{T}_h} \left(\eta_{K,i}^{(D)}\right)^2 \right)^{1/2},$$

with

$$\begin{aligned} \eta_{K,i}^{(L)} &= |u_h^{i+1} - u_h^i|_{1,K}, \\ \eta_{K,i}^{(D)} &= h_K \parallel \mathbf{f}_h + \nu \Delta \mathbf{u}_h^{i+1} - (\mathbf{u}_h^i.\nabla)\mathbf{u}_h^{i+1} - \nabla p_h^{i+1} \parallel_{L^2(K)} \\ &\qquad + \frac{1}{2} \sum_{e \in \Upsilon_K} h_e \parallel [\frac{\partial \mathbf{u}_h^{i+1}}{\partial n} - p_h^{i+1} \mathbf{n}] \parallel_{L^2(e)} + \parallel \operatorname{div} \mathbf{u}_h^{i+1} \parallel_{L^2(K)} . \end{aligned}$$

Figures 9 to 12 show the evolution of the mesh (see [12], Introduction) using the iterative Navier-Stokes algorithm. An adaptive mesh refinement can be outlined as follows:

For $i \geq 0$,

(1) Construct an initial mesh $\mathcal{T}_i$
(2) Solve the discrete problem on $\mathcal{T}_i$
(3) For each element K in $\mathcal{T}_i$ compute the *a posteriori* error estimate.
(4) If the estimated global error is sufficiently small then **STOP**. Otherwise refine locally the mesh (see [10] for details), recall $\mathcal{T}_{i+1}$ the new mesh; take $i = i + 1$ and return to step (2).

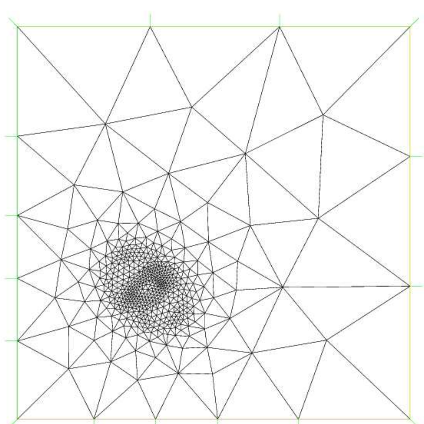

FIGURE 9. 273 vertices

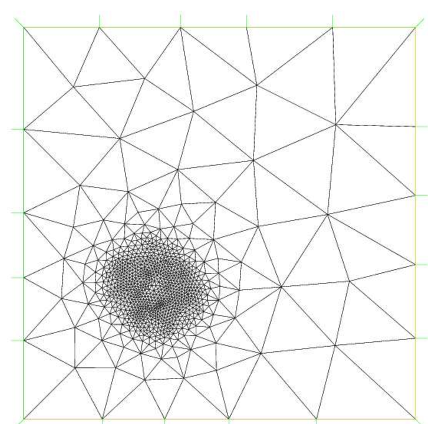

FIGURE 10. 507 vertices

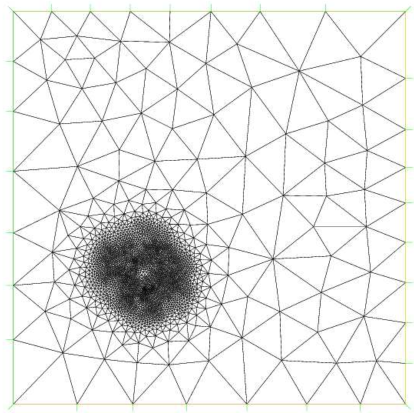

FIGURE 11. 891 vertices

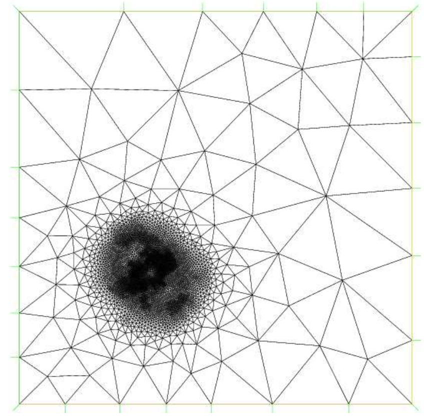

FIGURE 12. 1615 vertices

Figures 13 and 14 present the numerical and the exact first component of the velocity for the mesh refinement of Figure 12.

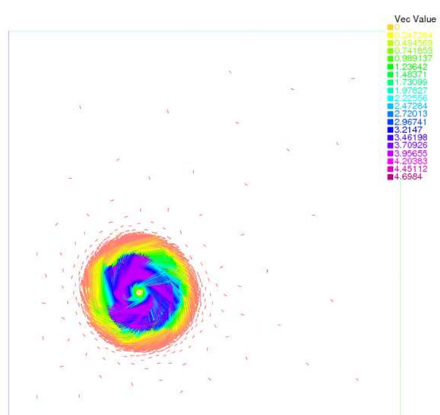

FIGURE 13. Numerical velocity

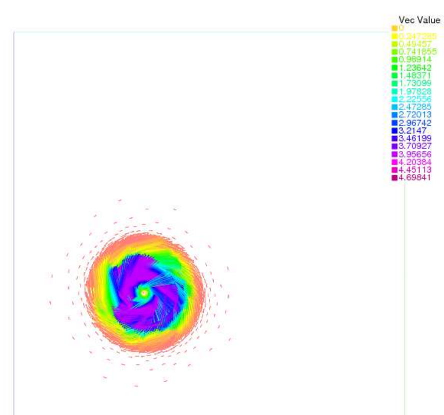

FIGURE 14. Exact velocity

We observe that the numerical velocity and the exact velocity are perfectly coherent.

Figure 15 presents the error curve for uniform (red) and adaptive (blue) mesh refinement using the new stopping criterion. We note that the error using an adaptive mesh is much smaller than the error using a uniform mesh.

Figure 16 illustrates the performance of our new stopping criterion with $\gamma = 0.01$ by comparing it to the classical stopping criterion $\eta_i^{(L)} \leq 10^{-5}$. We can clearly observe that our new stopping criterion reduces the number of iterations.

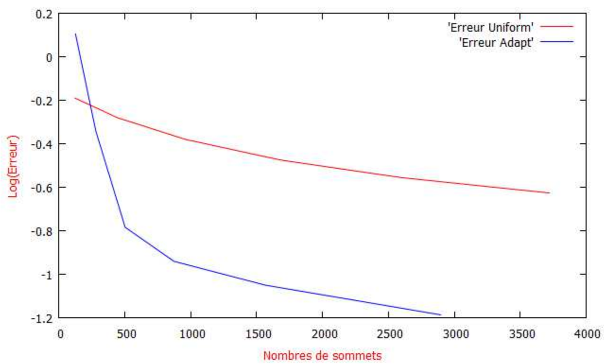

FIGURE 15. Error curve as a function of the global vertices number. Uniform mesh (top), adaptive mesh (bottom).

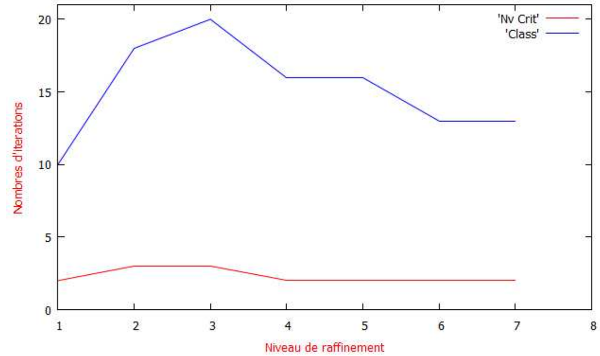

FIGURE 16. Iterations number as a function of the refinement level. Classical criterion (top), new criterion (bottom).

Finally, the following table presents the CPU time of each level of refinement for both criteria, the classical one and the new one. We can see clearly the efficiency of the new stopping criterion with $\gamma = 0.01$.

Level of refinement / Method	3	4	5	6	7
New criterion	6.466s	8.331s	14.439s	11.591s	15.351s
Classical criterion	30.609s	13.104s	21.279s	29.25s	49.483s

6.3. **Conclusion.** In this work we have derived *a posteriori* error estimates for the finite element discretization of the Navier-Stokes equations. These estimates yield a fully computable upper bound which allow to distinguish the discretization and the linearization errors. We have shown in this work that balancing these two errors leads to important computational savings; in fact, it avoid performing an excessive number of iterations.

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