Oscillatory Behavior near Blow-up of the Solutions to Some Nonlinear Singular Second Order ODE’s

Mama Abdelli, Faouzia Aloui

To cite this version:

Mama Abdelli, Faouzia Aloui. Oscillatory Behavior near Blow-up of the Solutions to Some Nonlinear Singular Second Order ODE’s. 2014. hal-01078363

HAL Id: hal-01078363
https://hal.sorbonne-universite.fr/hal-01078363
Preprint submitted on 28 Oct 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Oscillatory Behavior near Blow-up of the Solutions to Some Nonlinear Singular Second Order ODE’s

Mama ABDELLI
Université Djillali Liabés, Laboratoire de Mathématique,
B.P.89. Sidi Bel Abbés 22000, Algeria
e-mail: abdelli_mama@yahoo.fr

Faouzia ALOUI
UPMC Univ Paris 6, UMR 7598, Laboratoire Jacques Louis Lions
4 place jussieu, 75252 Paris cedex 05, France
e-mail: aloui@ann.jussieu.fr

Abstract

In this paper, we study the oscillation properties of solutions for the scalar second order nonlinear ODE: \((|u'|u')' + d|u|^\beta u = c|u'|^\alpha u'\), where \(\alpha, \beta, l, c, d\) are positive constants.

AMS classification numbers: 34C10, 34C15, 34D05, 34G20, 35B44

Keywords: Second order scalar ODE, nonlinear ODE, Oscillatory and non-Oscillatory Blow-up,
1 Introduction

We consider the scalar second order nonlinear ODE

\[ (|u'|u')' + d|u|^\beta u = c|u'|^\alpha u', \]  

(1.1)

where \( l, d, c, \beta, \alpha \) are positive constants.

For dissipative ordinary differential equation of the type

\[ (|u'|u')' + d|u|^\beta u + c|u'|^\alpha u' = 0, \]  

(1.2)

In [1] Abdelli and Haraux proved the existence and uniqueness of a global solution \( u(t) \) of (1.2) with initial data \( (u_0, u_1) \in \mathbb{R}^2 \). They established the decay rate and used a method introduced by Haraux [4] to study the oscillatory or non-oscillatory properties of nontrivial solutions. This method is based on a polar coordinate system and the oscillation properties appear to depend on the relation between \( \alpha \) and \( \beta \) \((l+1)\)

The results of [1] can be summarized as follows:

Let \((A_1), (A_2), (A_3), (A_4)\) be the assumptions defined as follows:

\[(A_1) \quad \alpha > \frac{\beta(l+1)+l}{\beta+2} \]
\[(A_2) \quad \alpha = \frac{\beta(l+1)+l}{\beta+2} \text{ and } c < d(\beta + 2)((\beta+2)(l+1)) \frac{\beta+1}{\beta+2} \]
\[(A_3) \quad \alpha < \frac{\beta(l+1)+l}{\beta+2} \]
\[(A_4) \quad \alpha = \frac{\beta(l+1)+l}{\beta+2} \text{ and } c \geq d(\beta + 2)((\beta+2)(l+1)) \frac{\beta+1}{\beta+2} \]

i) If \((A_1)\) or \((A_2)\) is satisfied, then any non-null solution \( u(t) \) of (1.2) and its derivative \( u'(t) \) have non-constant sign on each interval \((T, \infty)\).

ii) If \((A_3)\) is satisfied, any non-null solution \( u(t) \) of (1.2) has a finite number of zeroes on \((0, \infty)\). Moreover, for \( t \) large enough, \( u(t) \) and \( u'(t) \) have opposite sign and \( u(t) \) and \( u''(t) \) have the same sign.

iii) If \((A_4)\) is satisfied, then any non-null solution \( u(t) \) of (1.2) has at most one zero on \((0, \infty)\).

We can also consider the equation

\[ u'' + |u|^\beta u = \tilde{g}(u'), \]  

(1.3)

where \( \tilde{g} \) is a locally Lipschitz continuous function satisfying the following hypotheses

\[ \exists c > 0, \quad \forall v, \quad |g(v)| \leq c|v|^\alpha+1 \]  

(1.4)

\[ \exists \eta > 0, \quad \forall v, \quad g(v)v \geq \eta|v|^\alpha+1, \]  

(1.5)
The equation (1.3) has been studied by Aloui [2]. By using a method different from the ones from Souplet [5] and Balabane, Jazar and Souplet [6], the author recovers the oscillation (or non-oscillation) properties of the solution of (1.3) near the blow-up time $T$ by the same method as [4] when $1 < \alpha < \beta$. Moreover, the author generalized the results to (1.3) with $g$ a general function satisfying (1.4)-(1.5).

The results of [2] can be summarized as follows:

i) The energy defined by $E(t) = \frac{u'^2}{2} + \frac{|u|^\beta + 2}{\beta + 2}$ blows-up as soon as $u \not\equiv 0$ and we have, denoting by $T$ the blow-up time

\[ a) \quad \text{If } 0 < \alpha \leq \frac{\beta}{\beta + 2}, \quad C_0(T - t)^{-\frac{2}{\beta}} \leq E(t) \leq C_1(T - t)^{-\frac{2}{\beta}}, \]

\[ b) \quad \text{If } \frac{\beta}{\beta + 2} < \alpha < \beta, \quad E(t) \leq C'(T - t)^{-\frac{(\beta + 2)(\alpha + 1)}{\beta - \alpha}}. \]

as $t \to T$, for some $C_0, C_1, C' > 0$.

ii) If $0 < \alpha < \frac{\beta}{\beta + 2}$ or $\alpha = \frac{\beta}{\beta + 2}$, $c < (\beta + 2)(\frac{\beta + 2}{2\beta + 2})^{\frac{\beta + 1}{\beta + 2}}$, then all nontrivial solutions have an oscillatory finite-time blow-up $T$ and

\[ \liminf_{t \to T} u(t) = \liminf_{t \to T} u'(t) = -\infty, \quad \limsup_{t \to T} u(t) = \limsup_{t \to T} u'(t) = +\infty \]

iii) If $\frac{\beta}{\beta + 2} < \alpha < \beta$, $g \in C^1$ and $g' > 0$, then all nontrivial solutions have a non-oscillatory finite-time blow-up $T$ and $u, u'$ have the same sign as $t \to T$.

iv) If $\alpha = \frac{\beta}{\beta + 2}$, $c \geq c_0 = (\beta + 2)(\frac{\beta + 2}{2\beta + 2})^{\frac{\beta + 1}{\beta + 2}}$. Then any solution $u(t)$ of (1.3) blows-up in finite time $T$ and has a finite number of zeroes in $[0, T]$.

Note that (1.3) with $\tilde{g}(v) = c|v|^\alpha v$ is a special case of (1.1) when $l = 0$.

The objective of this paper is to recover the oscillatory (or non-oscillatory) properties of solutions of (1.1) when $t \in [0, T]$ by the same method as in [1] when $l < \alpha < \beta$. Moreover, we use the techniques from [2].

The plan of the paper is as follows. In section 2 we prove the local existence of the solution of (1.1). In section 3, we show that any solution has an unbounded energy for any nontrivial initial data. In section 4 we show that, under natural conditions, all nontrivial solutions are blowing up and we obtain precise energy estimates of solutions when $t \to T$, with $T$ the blow-up time. Finally, oscillatory and non-oscillatory behavior’s are delimited in section 5 and 6.

2 Local existence

In this section, we shall discuss the local existence for the initial value problem associated to equation (1.1)
Proposition 2.1. assume that \( l \leq \inf\{\alpha, \beta\} \). Then for any \((u_0, u_1) \in \mathbb{R}^2\), there exists \( T > 0 \) for which problem (1.1) has a solution on \([0, T]\) in the following sense:

\[
u \in C^1[0, T], \quad |u'|u' \in C^1[0, T] \quad \text{and} \quad u_0 = u(0), \quad u_1 = u'(0).
\] (2.6)

Proof. To show the existence of the solution for (2.6), we consider for \( \varepsilon \in (0, 1) \)

\[
\begin{cases}
(\varepsilon + (l + 1)|u'_\varepsilon|)|u'_\varepsilon| + d|u_\varepsilon|^\beta u_\varepsilon = c|u'_\varepsilon|^\alpha u'_\varepsilon \\
u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = u_1.
\end{cases}
\] (2.7)

The existence and uniqueness of \( u_\varepsilon \) in the class \( C^2[0, T] \) for some \( T > 0 \) is classical. Multiplying (2.7) by \( u'_\varepsilon \), we have the following energy identity

\[
dt \left[ \frac{\varepsilon}{2} |u'_\varepsilon(t)|^2 + \frac{l + 1}{l + 2} |u'_\varepsilon(t)|^{l+2} + \frac{d}{\beta + 2} |u_\varepsilon(t)|^{\beta+2} \right] = c|u'_\varepsilon|^\alpha + 2.
\] (2.8)

Introducing

\[
E_\varepsilon(t) = \frac{\varepsilon}{2} |u'_\varepsilon(t)|^2 + \frac{l + 1}{l + 2} |u'_\varepsilon(t)|^{l+2} + \frac{d}{\beta + 2} |u_\varepsilon(t)|^{\beta+2},
\]

we have as a consequence of (2.8)

\[
\frac{d}{dt} E_\varepsilon(t) \leq \frac{l + 2}{l + 1} E_\varepsilon(t)^{\frac{\alpha + 2}{l + 2}} - \alpha - l \frac{\alpha - l}{l + 1}.
\]

Then

\[
-\frac{l + 2}{\alpha - l} \frac{d}{dt} E_\varepsilon(t)^{-\frac{\alpha - l}{l + 2}} \leq -\frac{\alpha - l}{l + 1}.
\]

By integrating over \((0, t)\), we have

\[
E_\varepsilon(t)^{-\frac{\alpha - l}{l + 2}} \geq -\frac{\alpha - l}{l + 1} t + \frac{1}{E_\varepsilon(0)^{\frac{\alpha - l}{l + 2}}}.
\]

Hence, we can estimate an existence time for \( u_\varepsilon \) as a consequence of the inequality

\[
E_\varepsilon(t) \leq -\frac{\alpha - l}{l + 1} t + \frac{1}{E_\varepsilon(0)^{\frac{\alpha - l}{l + 2}}} + \frac{l + 1}{c(\alpha - l)E_\varepsilon(0)^{\frac{\alpha - l}{l + 2}}}, \quad \forall 0 \leq t \leq T_\varepsilon = \frac{l + 1}{c(\alpha - l)E_\varepsilon(0)^{\frac{\alpha - l}{l + 2}}}.
\]

Introducing \( T_0 = \frac{l + 1}{c(\alpha - l)E_\varepsilon(0)^{\frac{\alpha - l}{l + 2}}} \), it is clear that \( T_0 < T_\varepsilon \) and for \( \varepsilon \) small enough, we have

\[
\forall t \in [0, T_0], \quad |u_\varepsilon(t)| \leq M_1, \quad |u'_\varepsilon(t)| \leq M_2.
\] (2.9)
where $M_1, M_2$ are positive constants independent of $\varepsilon$. Then $u_\varepsilon$, $u'_\varepsilon$ are uniformly bounded. From (2.7), we obtain $\forall t \in [0, T_0]$, 
\[ \left| \left( |u'_\varepsilon(t)|^l u'_\varepsilon(t) \right)' \right| = (l + 1)|u'_\varepsilon(t)|^l |u''_\varepsilon(t)| \leq \left| (\varepsilon + (l + 1)|u'_\varepsilon(t)|^l)u''_\varepsilon(t) \right|, \]
by using (2.9), we deduce 
\[ \forall t \in [0, T_0], \quad \left| \left( |u'_\varepsilon(t)|^l u'_\varepsilon(t) \right)' \right| \leq M_3. \tag{2.10} \]
Therefore the function $w_\varepsilon(t) := |u'_\varepsilon(t)|^l u'_\varepsilon(t)$ is uniformly Lipshitz on $[0, T_0]$ independently of $\varepsilon$. Then the family of functions $u'_\varepsilon(t) = |w_\varepsilon(t)|^{\frac{1}{l+1}} \text{sgn}w_\varepsilon(t)$ is uniformly equicontinuous (actually Hölder continuous ) on $[0, T_0]$.
We can now pass to the limit as $\varepsilon \to 0$. As a consequence of Ascoli’s theorem and a priori estimate (2.9), we may extract a subsequence which is still denoted for simplicity by $(u_\varepsilon)$ for which 
\[ u_\varepsilon \to u \quad \text{in} \quad C^1[0, T_0] \]
as $\varepsilon$ tends to 0. Integrating (2.7) over $(0, t)$, we get 
\[ |u'_\varepsilon(t)|^l u'_\varepsilon(t) - |u'_\varepsilon(0)|^l u'_\varepsilon(0) = c \int_0^t |u'_\varepsilon(s)|^\alpha u'_\varepsilon(s) ds - d \int_0^t |u_\varepsilon(s)|^\beta u_\varepsilon(s) ds - \varepsilon \int_0^t u''_\varepsilon(s) ds \]
\[ = c \int_0^t |u'_\varepsilon(s)|^\alpha u'_\varepsilon(s) ds - d \int_0^t |u_\varepsilon(s)|^\beta u_\varepsilon(s) ds - \varepsilon (u'_\varepsilon(t) - u_1). \tag{2.11} \]
From (2.11), we then have, as $\varepsilon$ tends to 0 
\[ |u'_\varepsilon|^{l} u'_\varepsilon \to c \int_0^t |u'(s)|^\alpha u'(s) ds - d \int_0^t |u(s)|^\beta u(s) ds + |u'(0)|^{l} u'(0) \quad \text{in} \quad C^0[0, T_0]. \]
Hence 
\[ |u'|^l u' = c \int_0^t |u'(s)|^\alpha u'(s) ds - d \int_0^t |u(s)|^\beta u(s) ds + |u'(0)|^{l} u'(0), \tag{2.12} \]
and $|u'|^l u' \in C^1[0, T_0]$. Finally by differentiating (2.12) we conclude that $u$ is a solution of (1.1). Hence, the result with $T = T_0$.

3 The maximal solution

In this section, we still assume $0 \leq l \leq \inf\{\alpha, \beta\}$. Then as a consequence of [1] the solution $u$ of (1.1) with $u(0) = u_0$ and $u'(0) = u_1$ is unique on $[0, T_0]$. Moreover, if $v$ is another solution of the same problem on $[0, T_1]$ with $T_1 > T_0$, then $u = v$ on $[0, T_0]$. This allows us to obtain a maximal solution on $[0, T^*)$ with $0 < T^* \leq +\infty$. 

5
Remark 3.1. Integrating (2.8) over $(0,t)$, we then have, by passing to the limit as $\varepsilon$ tends to 0
\[
E(t) - E(0) = c \int_0^t |u'(s)|^{\alpha+2} ds,
\]
where
\[
E(t) = \frac{l+1}{l+2} |u'(t)|^{l+2} + \frac{d}{\beta+2} |u(t)|^{\beta+2}.
\]
It follows that $E$ is differentiable at any point $t \in [0,T^*)$ and
\[
\frac{d}{dt} E(t) = c |u'(t)|^{\alpha+2}.
\]

Proposition 3.2. Let $(u_0,u_1) \neq (0,0)$ be such that the unique solution of (2.6) is global. Then, $u$ is unbounded and $E(t) \to \infty$ as $t$ tends to $\infty$.

Proof. Assuming $u$ to be global and bounded, we can introduce the compact metric space
\[
Z = \bigcup_{t \geq 0} \{(u(t),u'(t))\} \quad \text{endowed with the distance associated to the euclidian norm in } \mathbb{R}^2.
\]
Let \{S(t)\}_{t \geq 0} be the dynamical system such that
\[
S(t) : Z \to Z
\]
\[
(v_0,v_1) \mapsto (v(t),v'(t))
\]
where $v$ is the solution of problem (1.1) with $v_0 = v(0)$ and $v'(0) = v_1$.

For $(\varphi,\psi) \in Z$, we set
\[
\Phi(\varphi,\psi) = -E(\varphi,\psi) = -\left(\frac{l+1}{l+2} |\psi|^{l+2} + \frac{d}{\beta+2} |\varphi|^{\beta+2}\right).
\]
Then if $(\varphi(t),\psi(t)) = S(t)(\varphi_0,\psi_0)$, we have as previously shown in remark 3.1:
\[
\frac{d}{dt} \Phi(\varphi(t),\psi(t)) = -c |\psi(t)|^{\alpha+2} \leq 0.
\]
In particular
\[
\Phi(S(t)(\varphi,\psi)) \leq \Phi(\varphi,\psi), \quad \forall (\varphi,\psi) \in Z, \quad \forall t \geq 0.
\]
Let $\omega(u_0,u_1)$ be the $\omega$-limit set of the $(u(t),u'(t))$ as $t \to +\infty$. It is clear that
$\omega(u_0,u_1) \subseteq \{(v_0,v_1) \in Z, \quad (v(t),v'(t)) \text{ is global and bounded where } (v(t),v'(t)) = S(t)(v_0,v_1)\}$.

Since $\Phi(u(t),u'(t))$ is non-increasing and bounded, it has a limit $L$ as $t \to \infty$.

Hence,
\[
\forall (v_0,v_1) \in \omega(u_0,u_1), \quad \Phi(S(t)(v_0,v_1)) = L, \quad \forall t \geq 0.
\]

Because
\[
\frac{d}{dt} \Phi(v(t),v'(t)) = -c |v'(t)|^{\alpha+2} = 0, \quad \forall t \geq 0,
\]
this implies $v' \equiv 0$ on $\mathbb{R}^+$ and by the equation (1.1) we derive $v \equiv 0$.

We now know that $\omega(u_0,u_1) = \{0,0\}$. In particular, as $t \to \infty$, $\Phi(u(t),u'(t)) \to \Phi(0,0) = 0$.

But by hypothesis $\Phi(u(t),u'(t))$ is non-increasing and $\Phi(u(0),u'(0)) < 0$. This is contradictory hence $E(t)$ cannot be bounded. \qed
4 Blow-up of nontrivial solutions and energy estimates near blow-up

Theorem 4.1. Let \( \beta > \alpha > l \) and let \( u \neq 0 \) be a solution of (1.1), then \( u \) blows-up in a finite time. Moreover, if \( T > 0 \) denotes the blow-up time,

i) If \( l < \alpha \leq \frac{\beta(l+1)+l}{\beta+2} \), then there exist \( C_0, C_1 > 0 \) such that
\[
C_0(T - t)^{-\frac{l+2}{\alpha-1}} \leq E(t) \leq C_1(T - t)^{-\frac{l+2}{\alpha-1}}, \quad \text{as} \ t \to T, \tag{4.13}
\]

ii) If \( \frac{\beta(l+1)+l}{\beta+2} < \alpha < \beta \), then there exists \( C' > 0 \) such that
\[
E(t) \leq C'(T - t)^{-\frac{(\beta+2)(\alpha+1)}{\alpha-\beta}}, \quad \text{as} \ t \to T. \tag{4.14}
\]

Proof. We consider the functional:
\[
F(t) = E(t) - \epsilon|u|\gamma u|u'|^lu',
\]
where \( l > 0, \gamma > 0 \) and \( \epsilon > 0 \).

By using Young’s inequality with exponents \( l + 2 \) and \( \frac{l+2}{l+1} \), we obtain
\[
|u|\gamma u|u'|^lu' \leq c_1|u|^{(\gamma+1)(l+2)} + c_2|u'|^{l+2},
\]
we assume that
\[
(\gamma + 1)(l + 2) \leq \beta + 2,
\]
which reduces to the condition
\[
\gamma \leq \frac{\beta - l}{l + 2}, \tag{4.15}
\]
therefore
\[
\forall u \in \mathbb{R}, \quad |u|^{(\gamma+1)(l+2)} \leq \max\{|u|^{\beta+2}, 1\} \leq |u|^{eta+2} + 1.
\]

Then, we obtain the existence of \( K > 0 \) such that
\[
-C_1 + E(t)(1 - K\epsilon) \leq F(t) \leq E(t)(1 + K\epsilon) + C_2,
\]
for \( \epsilon \) small enough, we have
\[
\frac{1}{2}E(t) - C_1 \leq F(t) \leq 2E(t) + C_2, \quad \forall t \in [0, T]. \tag{4.16}
\]

On the other hand
\[
F'(t) = \frac{d}{dt}E(t) - \epsilon(|u|\gamma u)'|u'|^lu' - \epsilon|u|\gamma u(|u'|^lu')'
\]
\[
= c|u'|^\alpha + 2 + d\epsilon|u|^{\gamma + \beta + 2} - \epsilon(\gamma + 1)|u|^\gamma |u'|^{l+2} - c\epsilon|u|^\gamma u|u'|^\alpha u'. \tag{4.17}
\]

By using Young’s inequality in the third term with exponents $\frac{\alpha+2}{\alpha-l}$ and $\frac{\alpha+2}{l+2}$, we obtain

$$|u|^\gamma |u'|^{l+2} \leq \delta |u|^\gamma \left(\frac{\alpha+2}{\alpha-l}\right) + c(\delta) |u'|^{\alpha+2},$$

we assume that

$$\gamma \left(\frac{\alpha+2}{\alpha-l}\right) \leq \gamma + \beta + 2,$$

this is equivalent to the condition

$$\gamma \leq (\beta+2) \left(\frac{\alpha-l}{l+2}\right),$$

in order that

$$\forall u \in \mathbb{R}, |u|^\gamma \left(\frac{\alpha+2}{\alpha-l}\right) \leq |u|^\beta + \gamma + 2 + 1.$$

Taking $\delta$ small enough, we have for some $P > 0$ and $\rho_1 > 0$

$$-\epsilon(\gamma + 1)|u|^\gamma |u'|^{l+2} \geq -\frac{d\epsilon}{4} |u|^{\beta+\gamma+2} - \epsilon P |u'|^{\alpha+2} - \rho_1.$$  (4.20)

By using Young’s inequality in the last term with exponents $\alpha + 2$ and $\frac{\alpha+2}{\alpha+1}$, we obtain

$$|u|^\gamma |u'|^{\alpha+2} \leq \delta |u|^{(\gamma+1)(\alpha+2)} + c'(\delta) |u'|^{\alpha+2},$$

we assume that

$$(\gamma + 1)(\alpha + 2) \leq \beta + \gamma + 2,$$

which reduces to the condition

$$\gamma \leq \frac{\beta - \alpha}{\alpha + 1}.$$  (4.21)

Then, we have

$$\forall u \in \mathbb{R}, |u|^{(\gamma+1)(\alpha+2)} \leq |u|^{\beta+\gamma+2} + 1.$$

Taking $\delta$ small enough, we have for some $P' > 0$ and $\rho_2 > 0$

$$-\epsilon |u|^\gamma |u'|^{\alpha+2} \geq -\frac{d\epsilon}{4} |u|^{\beta+\gamma+2} - \epsilon P' |u'|^{\alpha+2} - \rho_2.$$  (4.22)

Using (4.20) and (4.22), we have from (4.17)

$$F'(t) \geq (c - P\epsilon - P'\epsilon) |u'|^{\alpha+2} + \frac{d\epsilon}{2} |u|^{\beta+\gamma+2} - M$$

$$\geq (c - Q\epsilon) |u'|^{\alpha+2} + \frac{\epsilon}{2} |u|^{\beta+\gamma+2} - M,$$

where $Q = P + P'$.

we have for $\epsilon$ small enough,

$$F'(t) \geq \frac{\epsilon}{2} (|u'|^{\alpha+2} + |u|^{\beta+\gamma+2}) - M,$$
set
\[ \gamma = \min \left\{ \frac{(\beta + 2)\alpha - l}{l + 2}, \frac{\beta - \alpha}{\alpha + 1}, \frac{\beta - l}{l + 2} \right\}, \]
and
\[ \sigma = \min \left\{ \frac{\alpha + 2}{l + 2}, 1 + \frac{\beta - \alpha}{(\beta + 2)(\alpha + 1)} \right\}. \]

Then, by using (4.16) and the inequality \((x + y)^\sigma \leq c(\sigma)(x^\sigma + y^\sigma)\) for \(x, y \geq 0\), we have

\[
F'(t) \geq \frac{\epsilon}{2} c^{-1}(\sigma)c_1 E(t)^\sigma - M \\
\geq \frac{\epsilon}{4} c_2 F(t)^\sigma - M',
\]
where \(c_2 = c^{-1}(\sigma)c_1\) and \(M' > 0\).

First \(T_{\text{max}} < \infty\). Assuming \(T_{\text{max}} = \infty\), since \(E\) is unbounded and nondecreasing, \(E\) tends to infinity as \(t \to T_{\text{max}}\) and by (4.16) so is \(F\), thus there exists \(T^*\) for which \(\frac{\epsilon}{4} c_2 F(t)^\sigma > 2M'\) for \(t \geq T^*\). Therefore,

\[
F'(t) \geq \frac{\epsilon}{4} c_3 F(t)^\sigma,
\]
a contradiction. Then \(T_{\text{max}} = T < \infty\).

Then, we distinguish two cases:

i) \(l < \alpha \leq \frac{\beta(l+1)+l}{\beta+2}\), so that \((\beta + 2)\frac{\alpha - l}{l + 2} \leq \frac{\beta - l}{l + 2}\) and

\[
\frac{\beta - \alpha}{\alpha + 1} - \frac{\beta - l}{l + 2} = \frac{(\beta - \alpha)(l + 2) - (\beta - l)(\alpha + 1)}{(\alpha + 1)(l + 2)} = \frac{\beta(l + 1) + l - \alpha(\beta + 2)}{(\alpha + 1)(l + 2)} \geq 0.
\]

We choose
\[
\gamma = \frac{(\beta + 2)(\alpha - l)}{l + 2} \quad \text{and} \quad \sigma = \frac{\alpha + 2}{l + 2}.
\]

By using (4.23), we obtain
\[
\frac{d}{dt} (F(t))^{\frac{-\sigma l}{l+2}} = -\frac{\alpha - l}{l + 2} F'(t) F(t)^{-\frac{\alpha + l}{l+2}} \\
\leq -\frac{\alpha - l}{4(l + 2)} \epsilon c_3,
\]
by integrating the above inequality from \(t\) to \(\tau\), we obtain

\[
F(\tau)^{-\frac{\alpha l}{l+2}} - F(t)^{-\frac{\alpha l}{l+2}} \leq -\epsilon c_4 (\tau - t),
\]
where \(c_4 = \frac{\alpha - l}{4(l + 2)} c_3\).

Since \(F(\tau) \to +\infty\) if \(\tau \to T\), then \(F(\tau)^{-\frac{\alpha l}{l+2}} \to 0\). Therefore by letting \(\tau \to T\), we obtain

\[
F(t) \leq e^{-\frac{l+2}{\alpha l} c'_4 (T - t)^{-\frac{l+2}{\alpha l}}},
\]
where \(c'_4 = \frac{\alpha - l}{4(l + 2)} c_3\).
assuming \(c_5 = \epsilon^{-\frac{l+2}{\alpha+1}}c_4'\), we have
\[ F(t) \leq c_5(T-t)^{-\frac{l+2}{\alpha+1}}, \]
using (4.16), we have
\[ E(t) \leq C_1(T-t)^{-\frac{l+2}{\alpha+1}}, \tag{4.24} \]
with \(C_1 > 2c_5\).
For the converse inequality, we have
\[
E'(t) = c|u'|^{\alpha+2} \leq cKE(t)^{\frac{\alpha+2}{l+2}}.
\]
Then
\[
\frac{d}{dt}E(t)^{-\frac{\alpha+l}{l+2}} = -\frac{\alpha-l}{l+2} \frac{d}{dt}E(t)E(t)^{-\frac{\alpha+l}{l+2}} \geq -\frac{\alpha-l}{l+2}cK,
\]
by integrating the above inequality from \(t\) to \(\tau\), we obtain
\[
E(\tau)^{-\frac{\alpha+l}{l+2}} - E(t)^{-\frac{\alpha+l}{l+2}} \geq -K\frac{\alpha-l}{l+2}c(\tau-t).
\]
Since \(E(\tau) \to +\infty\) if \(\tau \to T\), we have
\[ E(t) \geq C_0(T-t)^{-\frac{l+2}{\alpha+1}}. \tag{4.25} \]
Therefore by (4.24) and (4.25), we obtain
\[ C_0(T-t)^{-\frac{l+2}{\alpha+1}} \leq E(t) \leq C_1(T-t)^{-\frac{l+2}{\alpha+1}}, \forall t \in [0,T]. \]

ii) if \(\frac{\beta(l+1)+l}{\beta+2} < \alpha < \beta\), we have \(\frac{\beta-\alpha}{\alpha+1} < \frac{\beta-l}{l+2}\) and \((\beta+2)\frac{\alpha-l}{l+2} > \frac{\beta-l}{l+2}\).
We choose
\[ \gamma = \frac{\beta-\alpha}{\alpha+1} \quad \text{and} \quad \sigma = 1 + \frac{\beta-\alpha}{(\beta+2)(\alpha+1)}. \]
From (4.23), we obtain
\[ F'(t) \geq \frac{\epsilon}{4}c_3(\alpha, \beta)F(t)^{1+\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}}, \tag{4.26} \]
by (4.26), we have
\[ \frac{d}{dt}F(t)^{-\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}} = -\frac{\beta-\alpha}{(\beta+2)(\alpha+1)} \frac{d}{dt}F(t)F(t)^{-1-\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}} \leq -\epsilon c_6, \]
by integrating the above inequality from \(t\) to \(\tau\), we have
\[ F(\tau)^{-\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}} - F(t)^{-\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}} \leq -\epsilon c_6(\tau-t), \]
if \(\tau \to T\), we obtain
\[ F(t) \leq \epsilon^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}}c_6'(T-t)^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}}, \]
assuming \(C' = \epsilon^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}}c_6'\)
\[ E(t) \leq C'(T-t)^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}}. \]
The proof of Theorem 4.1 is now completed. \(\Box\)
5 Oscillatory blow-up of solutions to (1.1) for $\alpha$ small

In this section, we establish the oscillatory blow-up of nontrivial solutions of (1.1). We can use the method from [2], we obtain the following result.

**Theorem 5.1.** Assume that

$$l < \alpha < \frac{\beta(l+1) + l}{\beta + 2}$$

or

$$\alpha = \frac{\beta(l+1) + l}{\beta + 2}, \quad c < (\beta + 2)\left(\frac{(\beta + 2)(l+1)}{d(\beta+1)(l+2)}\right)^{\frac{\beta+1}{\beta+2}},$$

then, all nontrivial solutions of (1.1) have oscillatory blow-up at time $T < \infty$ and

$$\limsup_{t \to T} u(t) = \limsup_{t \to T} u'(t) = +\infty, \quad \liminf_{t \to T} u(t) = \liminf_{t \to T} u'(t) = -\infty.$$

**Proof.** We proceed in 2 steps.

Step 1. For $T > 0$, $u'(t)$ has at least a zero on $[0, T]$. Assume the contrary, which means that $u'(t)$ has a constant sign on $[0, T]$.

For $t \in [0, T]$, we introduce the polar coordinate as follows

$$\left(\frac{d(l+2)}{(\beta+2)(l+1)}\right)^\frac{1}{2} |u|^\frac{\beta}{2} u = r(t) \cos \theta(t), \quad |u'|^{\frac{\beta}{2}} u' = r(t) \sin \theta(t), \quad (5.27)$$

where $r$ and $\theta$ are two $C^1$ functions and $r(t) = \left(\frac{t+2}{t+1} E(t)\right)^\frac{1}{2} > 0$.

A simple calculations shows that $\theta$ satisfies at any non-singular point, the differential equation

$$\theta' = Ar^{\frac{2(\alpha-l)}{l+2}} \sin \theta \cos \theta |\sin \theta|^{\frac{2(\alpha-l)}{l+2}} - Br^{\frac{2(\beta-l)}{(\beta+2)(l+2)}} |\cos \theta|^{\frac{\beta}{l+2}} |\sin \theta|^{\frac{\beta}{l+2}}, \quad (5.28)$$

where

$$A = c \frac{l + 2}{2(l+1)}, \quad B = \left(\frac{(\beta + 2)(l+1)}{d(l+2)}\right)^{\frac{\beta+1}{\beta+2}} \frac{l + 2}{2(l+1)}.$$

Since $l < \alpha < \frac{\beta(l+1)+l}{\beta+2}$, we have $\frac{2(\alpha-l)}{l+2} < \frac{2(\beta-l)}{(\beta+2)(l+2)}$ and if $t \to T$, $r(t) \sim C(T-t)^{-\frac{l+2}{2(\alpha-l)}}$.

Then, if $t \to T$, we have

$$r(t)^{\frac{2(\alpha-l)}{l+2}} |\sin \theta|^{\frac{2(\alpha-l)}{l+2} + 1} \cos \theta = r(t)^{\frac{2(\alpha-l)}{l+2}} |\sin \theta|^{\frac{2(\alpha-l)}{l+2} - \frac{l}{l+2}} \cos \theta$$

$$\leq gr(t)^{\frac{2(\beta-l)}{(l+2)(\beta+2)}} |\sin \theta|^{\frac{\beta}{l+2}} |\cos \theta|^{\frac{\beta}{l+2}},$$

then

$$\theta' \leq -\xi (T-t)^{-\gamma} |\sin \theta|^{-\frac{l}{l+2}} |\cos \theta|^{\frac{\beta}{l+2}}, \text{ if } t \to T,$$

where $\xi > 0$ and

$$\gamma = \frac{l + 2}{\alpha - l (\beta + 2)(l+2)} > 1.$$
In the case \( \alpha = \frac{\beta(l+1)+l}{\beta+2} \), we have

\[
\theta' = -\frac{l + 2}{2(l + 1)} r'(t) \left| \frac{(\beta + 2)(l + 1)}{d(l + 2)} \right| \sin \theta \left| \frac{\beta}{\beta+2} \right| \left\{ \left( \frac{(\beta + 2)(l + 1)}{d(l + 2)} \right)^{\frac{\beta+1}{\beta+2}} - c \left| \sin \theta \right|^{\frac{2\alpha-l}{\beta+2}} \cos \theta \left|^{1-\frac{\beta}{\beta+2}} \right. \right. ,
\]

since \( \alpha = \frac{\beta(l+1)+l}{\beta+2} \), we have \( \frac{\beta}{\beta+2} = \frac{2\beta-1}{l+2} \).

Then

\[
\theta' \leq -\frac{l + 2}{2(l + 1)} r'(t) \left| \frac{(\beta + 2)(l + 1)}{d(l + 2)} \right| \sin \theta \left| \frac{\beta}{\beta+2} \right| \left\{ \left( \frac{(\beta + 2)(l + 1)}{d(l + 2)} \right)^{\frac{\beta+1}{\beta+2}} - c \left| \sin \theta \right|^{\frac{2\alpha-l}{\beta+2}} \cos \theta \left|^{1-\frac{\beta}{\beta+2}} \right. \right. ,
\]

assuming \( f(\theta) = \left| \sin \theta \right|^{\frac{2\alpha-l}{\beta+2}} \left| \cos \theta \right|^{1-\frac{2\alpha-l}{\beta+2}}, \theta \in \mathbb{R} \).

Then, we have

\[
\max_{\theta \in \mathbb{R}} f(\theta) = \left( \frac{1}{\beta + 2} \right) \left( \frac{\beta + 1}{\beta + 2} \right)^{\frac{\beta+1}{\beta+2}} . \tag{5.29}
\]

Hence

\[
\left( \frac{(\beta + 2)(l + 1)}{d(l + 2)} \right)^{\frac{\beta+1}{\beta+2}} - c \left| \sin \theta \right|^{\frac{2\alpha-l}{\beta+2}} \cos \theta \left|^{1-\frac{\beta}{\beta+2}} \right. \right. \left. \right. + c \left( \frac{1}{\beta + 2} \right) \left( \frac{\beta + 1}{\beta + 2} \right)^{\frac{\beta+1}{\beta+2}} > 0 \Leftrightarrow c < (\beta + 2) \left( \frac{(\beta + 2)(l + 1)}{d(\beta + 1)(l + 2)} \right)^{\frac{\beta+1}{\beta+2}},
\]

then, we find in all cases for \( t \to T \),

\[
\theta' \leq -\xi(T-t)^{-1} \left| \sin \theta \right|^{-\frac{l}{\beta+2}} \left| \cos \theta \right|^{\frac{\beta}{\beta+2}} .
\]

We introduce the function

\[
H(s) = \int_a^s \left| \frac{\sin u}{\cos u} \right|^{\frac{1}{\beta+2}} du,
\]

suppose that \( u \) does not vanish if \( t \to T \) and for \( t \in [t_0, T] \), we may assume for instance \( \theta(t) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( H(\theta(t)) = F(t) \)

\[
\forall t_0 \leq t \leq T, \quad F'(t) \leq -\xi(T-t)^{-1},
\]

we integrate from \( t_0 \) to \( t \)

\[
H(\theta(t)) \leq H(\theta(t_0)) - \xi \log(T-t_0) + \xi \log(T-t),
\]

if \( t \to T \), we find \( H(\theta(t)) \to -\infty \). Or \( H(\theta(t)) \) is non-negative, then, we obtain a contradiction. Therefore, \( u' \) has a zero on each half-line.

Step 2. Applying Step 1, we know that \( u' \) has an infinite sequence of zeroes tending to infinity.
We claim that between two successive zeroes of \( u' \) there is a zero of \( u \). Indeed let \( u'(a) = u'(b) = 0 \) with \( a < b \) and \( u' \neq 0 \) in \((a, b)\). If \( u \) has a constant sign in \((a, b)\), by the equation \((|u'|u')'\) has the same sign for \( t = a \) and \( t = b \), which implies that \((|u'|u')'\) have opposite signs on \((a, a + \eta)\) and \((b - \eta, b)\) for \( \eta > 0 \) small enough, a contradiction with \( u' \neq 0 \) in \((a, b)\). Finally, by (4.13) we have \( \lim_{t \to T} u^2(t) + u'(t) = +\infty \). From the existence of infinitely many zeroes of \( u(t) \) and \( u'(t) \) as \( t \to T \) it is easy to deduce that

\[
\lim_{t \to T} u(t) = \lim_{t \to T} u'(t) = +\infty,
\]

and

\[
\lim_{t \to T} u(t) = \lim_{t \to T} u'(t) = -\infty.
\]

The proof of Theorem 5.1 is now completed.

\[\square\]

6 Non-oscillatory blow-up of solutions to (1.1) for \( \alpha \) large

**Theorem 6.1.** Assume \( l \leq \alpha \) and \( \frac{\beta(l+1)+l}{\beta+2} < \alpha < \beta \). Then any solution \( u(t) \) has a finite number of zeroes in \((T - \epsilon, T)\), for some \( \epsilon > 0 \) and blows-up as \( t \to T \), where \( T \) is the blow-up time.

**Proof.** We introduce

\[
G(s) = \int_0^s |\sin v|^{\frac{2\alpha+1}{\beta+2}} \sin v \cos v \, dv.
\]

First we observe that \( G \circ \theta \) is \( C^1 \) on any interval where \( u' \) does not vanish. Indeed on such an interval, \( \theta \) is \( C^1 \) and

\[
(G(\theta(t)))' = Ar(t)^{\frac{2(\alpha-l)}{\beta+2}} \cos^2 \theta |\sin \theta|^{\frac{4(\alpha+1)+l}{\beta+2}} - Br(t)^{\frac{2(\beta-l)}{\beta+2}} |\cos \theta|^{\frac{\beta}{\beta+2}} |\sin \theta|^{\frac{2\alpha}{\beta+2}} \sin \theta \cos \theta.
\]

Then we observe that when \( \sin \theta \) vanishes, the right hand side of the above equality is 0. Actually it is also continuous at points where \( \sin \theta \) vanishes, so that finally \( G \circ \theta \) is \( C^1 \) everywhere. Now using Cauchy-Schwarz inequality, we obtain

\[
Br(t)^{\frac{2(\beta-l)}{\beta+2}} |\cos \theta|^{\frac{\beta}{\beta+2}} |\sin \theta|^{\frac{2\alpha}{\beta+2}} \sin \theta \cos \theta \\
\leq \frac{B^2}{A} r(t)^{\frac{4(\beta-l)}{(\beta+2)(\beta+2)}} - \frac{2(\alpha-l)}{\beta+2} + Ar(t)^{\frac{2(\alpha-l)}{\beta+2}} |\sin \theta|^{\frac{2(\alpha+1)+l}{\beta+2}} |\cos \theta|^2,
\]

then

\[
(G(\theta(t)))' \geq -Cr(t)^{\frac{4(\beta-l)}{(\beta+2)(\beta+2)}} - \frac{2(\alpha-l)}{\beta+2}.
\]

Since \( \beta > \alpha > \frac{\beta(l+1)+l}{\beta+2} \), from (4.14), we have \( r(t) \leq C'(T - t)^{-\frac{(\beta+2)(\alpha+1)}{\beta - \alpha}} \) for \( t \) close enough to \( T \), then

\[
(G(\theta(t)))' \geq -C'(T - t)^{-\lambda},
\]
with
\[
\lambda = \left( \frac{4(\beta - l)}{(\beta + 2)(l + 2)} - \frac{2(\alpha - l)}{l + 2} \right) \frac{(\alpha + 1)(\beta + 2)}{\beta - \alpha} = 1 + \alpha \left[ \frac{\beta(l + 1) + l - \alpha(\beta + 2)}{(\beta - \alpha)(l + 2)} \right] < 1.
\]

To finish the proof we shall use the following Lemma (cf. [4] for proof).

**Lemma 6.2.** Let \( \theta \in C^1(a,T) \) and \( G \) be a non constant \( \tau \)-periodic function. We assume \( G \circ \theta \in C^1(a,T) \) and for some \( h \in L^1(a,T) \)
\[
[G(\theta(t))]' \geq h(t), \quad \forall t \in [a,T].
\]
Then, for \( t_1 \leq t < T \), \( \theta(t) \) remains in some interval of length \( \leq \tau \). In addition, if \( G' \) has finite number of zeroes on \([0,\tau]\), then \( \theta(t) \) has a limit for \( t \to T \).

**The proof of Theorem 6.1.** From Lemma 6.2, \( \theta(t) \to \Theta \) as \( t \to T \). We distinguish two cases:

- **Case 1:** If \( \Theta \neq \frac{\pi}{2} \mod [\pi] \), \( u \sim Cr^{\pi/2} > 0 \) if \( t \to T \), then \( u \) has a constant sign.
- **Case 2:** If \( \Theta = \frac{\pi}{2} \mod [\pi] \), \( |u'| \sim r(t)^{2} > 0 \) if \( t \to T \), then \( u'(t) \) does not vanish and \( u(t) \) has a constant sign if \( t \to T \).

Let \( t_0 \) be such that \( u \) has a constant sign on \((t_0, T)\), if \( u'(t) \) has several zeroes in \((T - \epsilon, T)\) for \( \epsilon > 0 \) small enough, then \((|u'(t)|u'(t))'\) must have different signs at two successive zeroes \(|u'(t)|u'(t)\). From equation (1.1) \( u \) must have different signs also, which is impossible. Thus, \( u'(t) \) has a constant sign as \( t \to T \).

\( E(t) \) is unbounded, then \( E(t) \to \infty \) as \( t \) tends to \( T \). Then
\[
\lim_{t \to T} u(t) = \lim_{t \to T} u'(t) = \pm \infty.
\]
Since \( u(t) \) and \( u'(t) \) have the same sign if \( t \to T \). \( \square \)

**Theorem 6.3.** Assuming \( l \leq \alpha \) and
\[
\alpha = \frac{\beta(1+l)+l}{\beta+2}, \quad c \geq c_0 = (\beta + 2)\left( \frac{(\beta + 2)(l + 1)}{d(\beta + 1)(l + 2)} \right)^{\frac{\beta+1}{\beta+2}},
\]
then any solution \( u(t) \) of (1.1) blows-up in finite time \( T \) and has a finite number of zeroes in \([0,T]\).

**Proof.** If \( \alpha = \frac{\beta(1+l)+l}{\beta+2} \), then clearly \( \frac{\beta}{\beta+2} = \frac{2\alpha-l}{l+2} \). In this case
\[
\theta' = \frac{l + 2}{2(l + 1)} r^{2(\alpha - l)/\beta+2} \sin \theta \left\{ \left( \frac{(\beta + 2)(l + 1)}{d(l + 2)} \right)^{\frac{\beta+1}{\beta+2}} \cos \theta |\cos \theta|^{\frac{\beta}{\beta+2}} - c \sin \theta \cos \theta |\sin \theta|^{\frac{\beta}{\beta+2}} \right\},
\]
\[
= -\frac{l + 2}{2(l + 1)} r^{2(\alpha - l)/\beta+2} \sin \theta |\cos \theta|^{\frac{2\alpha-l}{\beta+2}} \left\{ \left( \frac{(\beta + 2)(l + 1)}{d(l + 2)} \right)^{\frac{\beta+1}{\beta+2}} \cos \theta |\cos \theta|^{\frac{2\alpha-l}{\beta+2}} - c \sin \theta \cos \theta |\sin \theta|^{\frac{\beta}{\beta+2}} \right\}.
\]
We set
\[
K(\theta) = \frac{l + 2}{2(l + 1)} |\sin \theta|^{\frac{1}{\beta+2}} \left\{ \left( \frac{(\beta + 2)(l + 1)}{d(l + 2)} \right)^{\frac{\beta+1}{\beta+2}} |\cos \theta|^{\frac{2\alpha-l}{\beta+2}} - c \sin \theta \cos \theta |\sin \theta|^{\frac{\beta}{\beta+2}} \right\}.
\]
i) If $c = c_0$, using (5.29), we have

$$
\left( \frac{\beta + 2}{d(l+2)} \right)^{\frac{\beta+2}{\beta+1}} - c \sin \theta \cos \theta | \sin \theta |^{\frac{2\alpha+1}{\beta+2}} | \cos \theta |^{\frac{2\alpha+1}{\pi^2}}
$$

$$
= \left( \frac{\beta + 2}{d(l+2)} \right)^{\frac{\beta+2}{\beta+1}} - c_0 | \sin \theta |^{1 + \frac{2\alpha-1}{\beta+1}} | \cos \theta |^{1 - \frac{2\alpha-1}{\beta+1}}
$$

$$
\geq \left( \frac{\beta + 2}{d(l+2)} \right)^{\frac{\beta+2}{\beta+1}} - c_0 \left( \frac{1}{\beta+2} \right)^{\frac{1}{\pi^2}} \left( \frac{\beta + 1}{\beta + 2} \right)^{\frac{\beta+1}{\pi^2}} = 0
$$

$K(\theta) > 0$, so that $\theta$ is non-increasing. The distance of two consecutive zeroes of $K(\theta)$ other than $\frac{\pi}{2}$ (mod $\pi$) is not more than $\pi$, therefore we have two cases:

**Case 1:** if $\theta(t)$ remains in an interval of length less than $\pi$, then $\theta$ is bounded from above and is non-increasing thus it converges to a limit as $t \to T$ and achieves at most one a value for which $u$ vanishes.

**Case 2:** if $\theta(t)$ coincides with one of these zeros for a finite value of $t$, due to existence and uniqueness for the ODE satisfies by $\theta(t)$ near the non-trivial equilibria, $\theta(t)$ is constant and $u$ never vanishes.

ii) If $c > c_0$, $K(\theta) < 0$. We have two cases:

**Case 1:** if $\theta(t) \neq \frac{\pi}{2}$, then, it is bounded and since $K(\theta) < 0$ near the trivial zeros, $\theta(t)$ is monotone, and therefore it is convergent as $t \to T$.

**Case 2:** if $\theta(t) = \frac{\pi}{2}$, then it remains constant and $u$ never vanishes.

\[ \square \]

**Acknowledgments** We thank very much Professor Alain Haraux for suggesting us to study this delicate problem and for his interesting remarks and comments.

**References**


