Oscillatory Behavior near Blow-up of the Solutions to Some Nonlinear Singular Second Order ODE's

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Abstract

In this paper, we study the oscillation properties of solutions for the scalar second order nonlinear ODE: $(|u'|^l u')' + d|u|^\beta u = c|u'|^\alpha u'$, where α, β, l, c, d are positive constants.

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1 Introduction

We consider the scalar second order nonlinear ODE

$$(|u'|^l u')' + d|u|^\beta u = c|u'|^\alpha u', \tag{1.1}$$

where l, d, c, β, α are positive constants.

For dissipative ordinary differential equation of the type

$$(|u'|^l u')' + d|u|^\beta u + c|u'|^\alpha u' = 0, \tag{1.2}$$

In **[1]** Abdelli and Haraux proved the existence and uniqueness of a global solution $u(t)$ of (1.2) with initial data $(u_0, u_1) \in \mathbb{R}^2$. They established the decay rate and used a method introduced by Haraux **[4]** to study the oscillatory or non-oscillatory properties of nontrivial solutions. This method is based on a polar coordinate system and the oscillation properties appear to depend on the relation between α and $\frac{\beta(l+1)+l}{\beta+2}$.
The results of **[1]** can be summarized as follows:
Let (A_1), (A_2), (A_3) and (A_4) be the assumptions defined as follows:

(A_1) $\alpha > \frac{\beta(l+1)+l}{\beta+2}$

(A_2) $\alpha = \frac{\beta(l+1)+l}{\beta+2}$ and $c < d(\beta+2)(\frac{(\beta+2)(l+1)}{d(\beta+1)(l+2)})^{\frac{\beta+1}{\beta+2}}$

(A_3) $\alpha < \frac{\beta(l+1)+l}{\beta+2}$

(A_4) $\alpha = \frac{\beta(l+1)+l}{\beta+2}$ and $c \geq d(\beta+2)(\frac{(\beta+2)(l+1)}{d(\beta+1)(l+2)})^{\frac{\beta+1}{\beta+2}}$

i) If (A_1) or (A_2) is satisfied, then any non-null solution $u(t)$ of (1.2) and its derivative $u'(t)$ have non-constant sign on each interval (T, ∞).

ii) If (A_3) is satisfied, any non-null solution $u(t)$ of (1.2) has a finite number of zeroes on $(0, \infty)$. Moreover, for t large enough, $u(t)$ and $u'(t)$ have opposite sign and $u(t)$ and $u''(t)$ have the same sign.

iii) If (A_4) is satisfied, then any non-null solution $u(t)$ of (1.2) has at most one zero on $(0, \infty)$.

We can also consider the equation

$$u'' + |u|^\beta u = \widetilde{g}(u'), \tag{1.3}$$

where $\widetilde{g}$ is a locally Lipschitz continuous function satisfying the following hypotheses

$$\exists c > 0, \qquad \forall v, \qquad |g(v)| \leq c|v|^{\alpha+1} \tag{1.4}$$

$$\exists \eta > 0, \qquad \forall v, \qquad g(v)v \geq \eta|v|^{\alpha+1}, \tag{1.5}$$

The equation (1.3) has been studied by Aloui [**2**]. By using a method different from the ones from Souplet [**5**] and Balabane, Jazar and Souplet [**6**], the author recovers the oscillation (or non-oscillation) properties of the solution of (1.3) near the blow-up time T by the same method as [**4**] when $1 < \alpha < \beta$. Moreover, the author generalized the results to (1.3) with g a general function satisfying (1.4)-(1.5).
The results of [**2**] can be summarized as follows:

i) The energy defined by $E(t) = \frac{u'^2}{2} + \frac{|u|^{\beta+2}}{\beta+2}$ blows-up as soon as $u \not\equiv 0$ and we have, denoting by T the blow-up time

   a) If $0 < \alpha \leq \frac{\beta}{\beta+2}$, $\quad C_0(T-t)^{-\frac{2}{\alpha}} \leq E(t) \leq C_1(T-t)^{-\frac{2}{\alpha}}$,

   b) If $\frac{\beta}{\beta+2} < \alpha < \beta$, $\quad E(t) \leq C'(T-t)^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}}$.

   as $t \to T$, for some $C_0,\ C_1,\ C' > 0$.

ii) If $0 < \alpha < \frac{\beta}{\beta+2}$ or $\alpha = \frac{\beta}{\beta+2}$, $c < (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}}$, then all nontrivial solutions have an oscillatory finite-time blow-up T and

$$\liminf_{t\to T} u(t) = \liminf_{t\to T} u'(t) = -\infty, \quad \limsup_{t\to T} u(t) = \limsup_{t\to T} u'(t) = +\infty$$

iii) If $\frac{\beta}{\beta+2} < \alpha < \beta$, $\ g \in \mathcal{C}^1$ and $g' > 0$, then all nontrivial solutions have a non-oscillatory finite-time blow-up T and $u,\ u'$ have the same sign as $t \to T$.

iv) If $\alpha = \frac{\beta}{\beta+2}$, $\ c \geq c_0 = (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}}$. Then any solution u(t) of (1.3) blows-up in finite time T and has a finite number of zeroes in $[0, T]$.

Note that (1.3) with $\widetilde{g}(v) = c|v|^{\alpha}v$ is a special case of (1.1) when $l = 0$.

The objective of this paper is to recover the oscillatory (or non-oscillatory) properties of solutions of (1.1) when $t \in [0, T]$ by the same method as in [**1**] when $l < \alpha < \beta$. Moreover, we use the techniques from [**2**].

The plan of the paper is as follows. In section 2 we prove the local existence of the solution of (1.1). In section 3, we show that any solution has an unbounded energy for any nontrivial initial data. In section 4 we show that, under natural conditions, all nontrivial solutions are blowing up and we obtain precise energy estimates of solutions when $t \to T$, with T the blow-up time. Finally, oscillatory and non-oscillatory behavior's are delimited in section 5 and 6.

2 Local existence

In this section, we shall discuss the local existence for the initial value problem associated to equation (1.1)

Proposition 2.1. *assume that $l \leq \inf\{\alpha, \beta\}$. Then for any $(u_0, u_1) \in \mathbb{R}^2$, there exists $T > 0$ for which problem* (1.1) *has a solution on* $[0, T]$ *in the following sense:*

$$u \in \mathcal{C}^1[0,T], \quad |u'|^l u' \in \mathcal{C}^1[0,T] \quad \text{and} \;\; u_0 = u(0), \quad u_1 = u'(0). \tag{2.6}$$

Proof. To show the existence of the solution for (2.6), we consider for $\varepsilon \in (0,1)$

$$\begin{cases} (\varepsilon + (l+1)|u'_\varepsilon|^l)u''_\varepsilon + d|u_\varepsilon|^\beta u_\varepsilon = c|u'_\varepsilon|^\alpha u'_\varepsilon \\ u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = u_1. \end{cases} \tag{2.7}$$

The existence and uniqueness of u_ε in the class $\mathcal{C}^2[0,T]$ for some $T > 0$ is classical. Multiplying (2.7) by u'_ε, we have the following energy identity

$$\frac{d}{dt}\Big[\frac{\varepsilon}{2}|u'_\varepsilon(t)|^2 + \frac{l+1}{l+2}|u'_\varepsilon(t)|^{l+2} + \frac{d}{\beta+2}|u_\varepsilon(t)|^{\beta+2}\Big] = c|u'_\varepsilon|^{\alpha+2}. \tag{2.8}$$

Introducing

$$E_\varepsilon(t) = \frac{\varepsilon}{2}|u'_\varepsilon(t)|^2 + \frac{l+1}{l+2}|u'_\varepsilon(t)|^{l+2} + \frac{d}{\beta+2}|u_\varepsilon(t)|^{\beta+2},$$

we have as a consequence of (2.8)

$$\begin{aligned} \frac{d}{dt}E_\varepsilon(t) &\leq c\frac{l+2}{l+1}E_\varepsilon(t)^{\frac{\alpha+2}{l+2}} \\ &\leq c\frac{l+2}{l+1}E_\varepsilon(t)^{1+\frac{\alpha-l}{l+2}}. \end{aligned}$$

Then

$$-\frac{l+2}{\alpha-l}\frac{d}{dt}\Big[E_\varepsilon(t)\Big]^{-\frac{\alpha-l}{l+2}} \leq c\frac{l+2}{l+1}$$

$$\frac{d}{dt}\Big[E_\varepsilon(t)\Big]^{-\frac{\alpha-l}{l+2}} \geq -c\frac{\alpha-l}{l+1}$$

By integrating over $(0,t)$, we have

$$E_\varepsilon(t)^{-\frac{\alpha-l}{l+2}} \geq -c\frac{\alpha-l}{l+1}t + \frac{1}{E_\varepsilon(0)^{\frac{\alpha-l}{l+2}}}.$$

Hence, we can estimate an existence time for u_ε as a consequence of the inequality

$$E_\varepsilon(t) \leq \Big(-c\frac{\alpha-l}{l+1}t + \frac{1}{E_\varepsilon(0)^{\frac{\alpha-l}{l+2}}}\Big)^{-\frac{l+2}{\alpha-l}}, \;\; \forall 0 \leq t \leq T_\varepsilon = \frac{l+1}{c(\alpha-l)E_\varepsilon(0)^{\frac{\alpha-l}{l+2}}}.$$

Introducing $T_0 = \dfrac{l+1}{c(\alpha-l)E_\varepsilon(0)^{\frac{\alpha-l}{l+2}}}$, it is clear that $T_0 < T_\varepsilon$ and for ε small enough, we have

$$\forall t \in [0, T_0], \;\; |u_\varepsilon(t)| \leq M_1, \;\; |u'_\varepsilon(t)| \leq M_2. \tag{2.9}$$

where M_1, M_2 are positive constants independent of ε. Then $u_\varepsilon,\ u'_\varepsilon$ are uniformly bounded. From (2.7), we obtain $\forall t \in [0, T_0]$,

$$\begin{aligned}\left|\left(|u'_\varepsilon(t)|^l u'_\varepsilon(t)\right)'\right| &= (l+1)|u'_\varepsilon(t)|^l |u''_\varepsilon(t)| \\ &\leq \left|(\varepsilon + (l+1)|u'_\varepsilon(t)|^l)u''_\varepsilon(t)\right|,\end{aligned}$$

by using (2.9), we deduce

$$\forall t \in [0, T_0], \quad \left|\left(|u'_\varepsilon(t)|^l u'_\varepsilon(t)\right)'\right| \leq M_3. \tag{2.10}$$

Therefore the function $w_\varepsilon(t) := |u'_\varepsilon(t)|^l u'_\varepsilon(t)$ is uniformly Lipshitz on $[0, T_0]$ independently of ε. Then the family of functions $u'_\varepsilon(t) = |w_\varepsilon(t)|^{\frac{1}{l+1}} \mathrm{sgn} w_\varepsilon(t)$ is uniformly equicontinous (actually Hölder continuous) on $[0, T_0]$.
We can now pass to the limit as $\varepsilon \to 0$. As a consequence of Ascoli's theorem and a priori estimate (2.9), we may extract a subsequence which is still denoted for simplicity by (u_ε) for which

$$u_\varepsilon \to u \quad \text{in } \mathcal{C}^1[0, T_0]$$

as ε tends to 0. Integrating (2.7) over $(0, t)$, we get

$$\begin{aligned}|u'_\varepsilon(t)|^l u'_\varepsilon(t) - |u'_\varepsilon(0)|^l u'_\varepsilon(0) &= c\int_0^t |u'_\varepsilon(s)|^\alpha u'_\varepsilon(s)\,ds - d\int_0^t |u_\varepsilon(s)|^\beta u_\varepsilon(s)\,ds - \varepsilon\int_0^t u''_\varepsilon(s)\,ds \\ &= c\int_0^t |u'_\varepsilon(s)|^\alpha u'_\varepsilon(s)\,ds - d\int_0^t |u_\varepsilon(s)|^\beta u_\varepsilon(s)\,ds - \varepsilon(u'_\varepsilon(t) - u_1).\end{aligned} \tag{2.11}$$

From (2.11), we then have, as ε tends to 0

$$|u'_\varepsilon|^l u'_\varepsilon \to c\int_0^t |u'(s)|^\alpha u'(s)\,ds - d\int_0^t |u(s)|^\beta u(s)\,ds + |u'(0)|^l u'(0) \quad \text{in } \mathcal{C}^0[0, T_0].$$

Hence

$$|u'|^l u' = c\int_0^t |u'(s)|^\alpha u'(s)\,ds - d\int_0^t |u(s)|^\beta u(s)\,ds + |u'(0)|^l u'(0), \tag{2.12}$$

and $|u'|^l u' \in \mathcal{C}^1[0, T_0]$. Finally by differentiating (2.12) we conclude that u is a solution of (1.1). Hence, the result with $T = T_0$. □

3 The maximal solution

In this section, we still assume $0 \leq l \leq \inf\{\alpha, \beta\}$. Then as a consequence of **[1]** the solution u of (1.1) with $u(0) = u_0$ and $u'(0) = u_1$ is unique on $[0, T_0]$. Moreover, if v is another solution of the same problem on $[0, T_1]$ with $T_1 > T_0$, then $u = v$ on $[0, T_0]$. This allows us to obtain a maximal solution on $[0, T^*)$ with $0 < T^* \leq +\infty$.

Remark 3.1. *Integrating (2.8) over* $(0,t)$*, we then have, by passing to the limit as* ε *tends to* 0

$$E(t) - E(0) = c\int_0^t |u'(s)|^{\alpha+2}\,ds,$$

where

$$E(t) = \frac{l+1}{l+2}|u'(t)|^{l+2} + \frac{d}{\beta+2}|u(t)|^{\beta+2}.$$

It follows that E *is differentiable at any point* $t \in [0, T^*)$ *and*

$$\frac{d}{dt}E(t) = c|u'(t)|^{\alpha+2}.$$

Proposition 3.2. *Let* $(u_0, u_1) \neq (0,0)$ *be such that the unique solution of (2.6) is global. Then,* u *is unbounded and* $E(t) \to \infty$ *as* t *tends to* ∞.

Proof. Assuming u to be global and bounded, we can introduce the compact metric space $\mathbb{Z} = \overline{\bigcup_{t\geq 0}\{u(t), u'(t)\}}^{\mathbb{R}^2}$ endowed with the distance associated to the euclidian norm in $\mathbb{R}^2$. Let $\{S(t)\}_{t\geq 0}$ be the dynamical system such that

$$\begin{aligned} S(t) &: \mathbb{Z} \to \mathbb{Z} \\ (v_0, v_1) &\mapsto (v(t), v'(t)) \end{aligned}$$

.

where v is the solution of problem (1.1) with $v_0 = v(0)$ and $v'(0) = v_1$.
For $(\varphi, \psi) \in \mathbb{Z}$, we set

$$\Phi(\varphi,\psi) = -E(\varphi,\psi) = -\Big(\frac{l+1}{l+2}|\psi|^{l+2} + \frac{d}{\beta+2}|\varphi|^{\beta+2}\Big).$$

Then if $(\varphi(t), \psi(t)) = S(t)(\varphi_0, \psi_0)$, we have as previously shown in remark 3.1:

$$\frac{d}{dt}\Phi(\varphi(t),\psi(t)) = -c|\psi(t)|^{\alpha+2} \leq 0.$$

In particular

$$\Phi(S(t)(\varphi,\psi)) \leq \Phi(\varphi,\psi),\ \ \forall(\varphi,\psi)\in\mathbb{Z},\ \ \forall t\geq 0.$$

Let $\omega(u_0, u_1)$ be the ω-limit set of the $(u(t), u'(t))$ as $t \to +\infty$. It is clear that

$$\omega(u_0,u_1) \subset \{(v_0,v_1)\in\mathbb{Z},\ \ (v(t),v'(t)) \text{ is global and bounded where } (v(t),v'(t)) = S(t)(v_0,v_1)\}.$$

Since $\Phi(u(t), u'(t))$ is non-increasing and bounded, it has a limit L as $t \to \infty$.
Hence,

$$\forall(v_0,v_1)\in\omega(u_0,u_1),\ \ \ \Phi(S(t)(v_0,v_1)) = L,\ \ \forall t\geq 0.$$

Because

$$\frac{d}{dt}\Phi(v(t),v'(t)) = -c|v'(t)|^{\alpha+2} = 0,\ \ \forall t\geq 0,$$

this implies $v' \equiv 0$ on $\mathbb{R}^+$ and by the equation (1.1) we derive $v \equiv 0$.
We now know that $\omega(u_0, u_1) = \{0, 0\}$. In particular, as $t \to \infty$ $\Phi(u(t), u'(t)) \to \Phi(0,0) = 0$. But by hypothesis $\Phi(u(t), u'(t))$ is non-increasing and $\Phi(u(0), u'(0)) < 0$. This is contradictory hence $E(t)$ cannot be bounded. □

4 Blow-up of nontrivial solutions and energy estimates near blow-up

Theorem 4.1. *Let $\beta > \alpha > l$ and let $u \neq 0$ be a solution of* (1.1)*, then u blows-up in a finite time. Moreover, if $T > 0$ denotes the blow-up time,*

i) If $l < \alpha \leq \frac{\beta(l+1)+l}{\beta+2}$, then there exist $C_0,\ C_1 > 0$ such that

$$C_0(T-t)^{-\frac{l+2}{\alpha-l}} \leq E(t) \leq C_1(T-t)^{-\frac{l+2}{\alpha-l}}, \quad \text{as } t \to T, \tag{4.13}$$

ii) If $\frac{\beta(l+1)+l}{\beta+2} < \alpha < \beta$, then there exists $C' > 0$ such that

$$E(t) \leq C'(T-t)^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}}, \quad \text{as } t \to T. \tag{4.14}$$

Proof. We consider the functional:

$$F(t) = E(t) - \epsilon|u|^\gamma u|u'|^l u',$$

where $l > 0,\ \gamma > 0$ and $\epsilon > 0$.
By using Young's inequality with exponents $l+2$ and $\frac{l+2}{l+1}$, we obtain

$$\Big||u|^\gamma u|u'|^l u'\Big| \leq c_1|u|^{(\gamma+1)(l+2)} + c_2|u'|^{l+2},$$

we assume that

$$(\gamma+1)(l+2) \leq \beta+2,$$

which reduces to the condition

$$\gamma \leq \frac{\beta - l}{l+2}, \tag{4.15}$$

therefore

$$\forall u \in \mathbb{R}, \quad |u|^{(\gamma+1)(l+2)} \leq \max\{|u|^{\beta+2}, 1\} \leq |u|^{\beta+2} + 1.$$

Then, we obtain the existence of $K > 0$ such that

$$-C_1 + E(t)(1 - K\epsilon) \leq F(t) \leq E(t)(1 + K\epsilon) + C_2,$$

for ϵ small enough, we have

$$\frac{1}{2}E(t) - C_1 \leq F(t) \leq 2E(t) + C_2, \quad \forall t \in [0, T]. \tag{4.16}$$

On the other hand

$$\begin{aligned} F'(t) &= \frac{d}{dt}E(t) - \epsilon(|u|^\gamma u)'|u'|^l u' - \epsilon|u|^\gamma u(|u'|^l u')' \\ &= c|u'|^{\alpha+2} + d\epsilon|u|^{\gamma+\beta+2} - \epsilon(\gamma+1)|u|^\gamma|u'|^{l+2} - c\epsilon|u|^\gamma u|u'|^\alpha u'. \end{aligned} \tag{4.17}$$

By using Young's inequality in the third term with exponents $\frac{\alpha+2}{\alpha-l}$ and $\frac{\alpha+2}{l+2}$, we obtain

$$|u|^{\gamma}|u'|^{l+2} \leq \delta|u|^{\gamma(\frac{\alpha+2}{\alpha-l})} + c(\delta)|u'|^{\alpha+2}, \tag{4.18}$$

we assume that

$$\gamma\Big(\frac{\alpha+2}{\alpha-l}\Big) \leq \gamma+\beta+2,$$

this is equivalent to the condition

$$\gamma \leq (\beta+2)(\frac{\alpha-l}{l+2}), \tag{4.19}$$

in order that

$$\forall u \in \mathbb{R}, \ \ |u|^{\gamma(\frac{\alpha+2}{\alpha-l})} \leq |u|^{\beta+\gamma+2}+1.$$

Taking δ small enough, we have for some $P>0$ and $\rho_1>0$

$$-\epsilon(\gamma+1)|u|^{\gamma}|u'|^{l+2} \geq -\frac{d\epsilon}{4}|u|^{\beta+\gamma+2} - \epsilon P|u'|^{\alpha+2} - \rho_1. \tag{4.20}$$

By using Young's inequality in the last term with exponents $\alpha+2$ and $\frac{\alpha+2}{\alpha+1}$, we obtain

$$|u|^{\gamma}u|u'|^{\alpha}u' \leq \delta|u|^{(\gamma+1)(\alpha+2)} + c'(\delta)|u'|^{\alpha+2},$$

we assume that

$$(\gamma+1)(\alpha+2) \leq \beta+\gamma+2,$$

which reduces to the condition

$$\gamma \leq \frac{\beta-\alpha}{\alpha+1}. \tag{4.21}$$

Then, we have

$$\forall u \in \mathbb{R}, \ \ |u|^{(\gamma+1)(\alpha+2)} \leq |u|^{\beta+\gamma+2}+1.$$

Taking δ small enough, we have for some $P'>0$ and $\rho_2>0$

$$-\epsilon|u|^{\gamma}u|u'|^{\alpha}u' \geq -\frac{d\epsilon}{4}|u|^{\beta+\gamma+2} - \epsilon P'|u'|^{\alpha+2} - \rho_2. \tag{4.22}$$

Using (4.20) and (4.22), we have from (4.17)

$$\begin{aligned} F'(t) &\geq (c-P\epsilon-P'\epsilon)|u'|^{\alpha+2} + \frac{d\epsilon}{2}|u|^{\beta+\gamma+2} - M \\ &\geq (c-Q\epsilon)|u'|^{\alpha+2} + \frac{\epsilon}{2}|u|^{\beta+\gamma+2} - M, \end{aligned}$$

where $Q=P+P'$.
we have for ϵ small enough,

$$F'(t) \geq \frac{\epsilon}{2}(|u'|^{\alpha+2} + |u|^{\beta+\gamma+2}) - M,$$

set

$$\gamma = \min\Big\{(\beta+2)\frac{\alpha-l}{l+2}, \frac{\beta-\alpha}{\alpha+1}, \frac{\beta-l}{l+2}\Big\},$$

and

$$\sigma = \min\Big\{\frac{\alpha+2}{l+2},\ \ 1+\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}\Big\}.$$

Then , by using (4.16) and the inequality $(x+y)^\sigma \le c(\sigma)(x^\sigma + y^\sigma)$ for $x,\ \ y\ge 0$, we have

$$\begin{aligned} F'(t) &\ge \frac{\epsilon}{2}c^{-1}(\sigma)c_1 E(t)^\sigma - M \\ &\ge \frac{\epsilon}{4}c_2 F(t)^\sigma - M', \end{aligned}$$

where $c_2 = c^{-1}(\sigma)c_1$ and $M'>0$.
First $T_{\max}<\infty$. Assuming $T_{\max}=\infty$, since E is unbounded and nondecreasing, E tends to infinity as $t\to T_{\max}$ and by (4.16) so is F, thus there exists T^* for which $\frac{\epsilon}{4}c_2F(t)^\sigma > 2M'$ for $t\ge T^*$. Therefore,

$$F'(t) \ge \frac{\epsilon}{4}c_3 F(t)^\sigma, \tag{4.23}$$

a contradiction. Then $T_{\max}=T<\infty$.

Then, we distinguich two cases:

i) $l<\alpha\le \frac{\beta(l+1)+l}{\beta+2}$, so that $(\beta+2)\frac{\alpha-l}{l+2}\le\frac{\beta-l}{l+2}$ and

$$\frac{\beta-\alpha}{\alpha+1}-\frac{\beta-l}{l+2} = \frac{(\beta-\alpha)(l+2)-(\beta-l)(\alpha+1)}{(\alpha+1)(l+2)} = \frac{\beta(l+1)+l-\alpha(\beta+2)}{(\alpha+1)(l+2)} \ge 0.$$

We choose

$$\gamma = \frac{(\beta+2)(\alpha-l)}{l+2} \ \text{ and }\ \sigma = \frac{\alpha+2}{l+2}.$$

By using (4.23), we obtain

$$\begin{aligned} \frac{d}{dt}(F(t))^{-\frac{\alpha-l}{l+2}} &= -\frac{\alpha-l}{l+2}F'(t)F(t)^{-\frac{\alpha+2}{l+2}} \\ &\le -\frac{\alpha-l}{4(l+2)}\epsilon c_3, \end{aligned}$$

by integrating the above inequality from t to τ, we obtain

$$F(\tau)^{-\frac{\alpha-l}{l+2}} - F(t)^{-\frac{\alpha-l}{l+2}} \le -\epsilon c_4(\tau - t),$$

where $c_4 = \frac{\alpha-l}{4(l+2)}c_3$.
Since $F(\tau)\to+\infty$ if $\tau\to T$, then $F(\tau)^{-\frac{\alpha-l}{l+2}}\to 0$. Therefore by letting $\tau\to T$, we obtain

$$F(t)\le \epsilon^{-\frac{l+2}{\alpha-l}}c_4'(T-t)^{-\frac{l+2}{\alpha-l}},$$

assuming $c_5 = \epsilon^{-\frac{l+2}{\alpha-l}} c_4'$, we have

$$F(t) \leq c_5 (T-t)^{-\frac{l+2}{\alpha-l}},$$

using (4.16), we have

$$E(t) \leq C_1 (T-t)^{-\frac{l+2}{\alpha-l}}, \tag{4.24}$$

with $C_1 > 2c_5$.

For the converse inequality, we have

$$E'(t) = c|u'|^{\alpha+2} \leq cKE(t)^{\frac{\alpha+2}{l+2}}.$$

Then

$$\frac{d}{dt}E(t)^{-\frac{\alpha-l}{l+2}} = -\frac{\alpha-l}{l+2}\frac{d}{dt}E(t)E(t)^{-\frac{\alpha+2}{l+2}} \geq -\frac{\alpha-l}{l+2}cK,$$

by integrating the above inequality from t to τ, we obtain

$$E(\tau)^{-\frac{\alpha-l}{l+2}} - E(t)^{-\frac{\alpha-l}{l+2}} \geq -K\frac{\alpha-l}{l+2}c(\tau-t).$$

Since $E(\tau) \to +\infty$ if $\tau \to T$, we have

$$E(t) \geq C_0 (T-t)^{-\frac{l+2}{\alpha-l}}. \tag{4.25}$$

Therefore by (4.24) and (4.25), we obtain

$$C_0 (T-t)^{-\frac{l+2}{\alpha-l}} \leq E(t) \leq C_1 (T-t)^{-\frac{l+2}{\alpha-l}}, \quad \forall t \in [0,T].$$

ii) if $\frac{\beta(l+1)+l}{\beta+2} < \alpha < \beta$, we have $\frac{\beta-\alpha}{\alpha+1} < \frac{\beta-l}{l+2}$ and $(\beta+2)\frac{\alpha-l}{l+2} > \frac{\beta-l}{l+2}$.

We choose

$$\gamma = \frac{\beta-\alpha}{\alpha+1} \quad \text{and} \quad \sigma = 1 + \frac{\beta-\alpha}{(\beta+2)(\alpha+1)}.$$

From (4.23), we obtain

$$F'(t) \geq \frac{\epsilon}{4}c_3(\alpha,\beta)F(t)^{1+\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}}, \tag{4.26}$$

by (4.26), we have

$$\frac{d}{dt}F(t)^{-\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}} = -\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}\frac{d}{dt}F(t)F(t)^{-1-\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}} \leq -\epsilon c_6,$$

by integrating the above inequality from t to τ, we have

$$F(\tau)^{-\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}} - F(t)^{-\frac{\beta-\alpha}{(\beta+2)(\alpha+1)}} \leq -\epsilon c_6 (\tau-t),$$

if $\tau \to T$, we obtain

$$F(t) \leq \epsilon^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}} c_6' (T-t)^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}},$$

assuming $C' = \epsilon^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}} c_6'$

$$E(t) \leq C'(T-t)^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}}.$$

The proof of Theorem 4.1 is now completed. □

5 Oscillatory blow-up of solutions to (1.1) for α small

In this section, we establish the oscillatory blow-up of nontrivial solutions of (1.1). We can use the method from **[2]**, we obtain the following result.

Theorem 5.1. *Assume that*

$$l < \alpha < \frac{\beta(l+1)+l}{\beta+2}$$

or

$$\alpha = \frac{\beta(l+1)+l}{\beta+2}, \qquad c < (\beta+2)\Big(\frac{(\beta+2)(l+1)}{d(\beta+1)(l+2)}\Big)^{\frac{\beta+1}{\beta+2}},$$

then, all nontrivial solutions of (1.1) *have oscillatory blow-up at time* $T < \infty$ *and*

$$\limsup_{t\to T} u(t) = \limsup_{t\to T} u'(t) = +\infty, \quad \liminf_{t\to T} u(t) = \liminf_{t\to T} u'(t) = -\infty.$$

Proof. We proceed in 2 steps.

Step 1. For $T > 0$, $u'(t)$ has at least a zero on $[0, T]$. Assume the contrary, which means that $u'(t)$ has a constant sign on $[0, T]$.

For $t \in [0, T]$, we introduce the polar coordinate as follows

$$\Big(\frac{d(l+2)}{(\beta+2)(l+1)}\Big)^{\frac{1}{2}} |u|^{\frac{\beta}{2}} u = r(t)\cos\theta(t), \qquad |u'|^{\frac{l}{2}} u' = r(t)\sin\theta(t), \tag{5.27}$$

where r and θ are two $\mathcal{C}^1$ functions and $r(t) = \Big(\frac{l+2}{l+1}E(t)\Big)^{\frac{1}{2}} > 0$.
A simple calculations shows that θ satisfies at any non-singular point, the differential equation

$$\theta' = Ar^{\frac{2(\alpha-l)}{l+2}} \sin\theta\cos\theta|\sin\theta|^{\frac{2(\alpha-l)}{l+2}} - Br^{\frac{2(\beta-l)}{(\beta+2)(l+2)}}|\cos\theta|^{\frac{\beta}{\beta+2}}|\sin\theta|^{\frac{-l}{l+2}}, \tag{5.28}$$

where

$$A = c\frac{l+2}{2(l+1)}, \qquad B = \Big(\frac{(\beta+2)(l+1)}{d(l+2)}\Big)^{\frac{\beta+1}{\beta+2}} \frac{l+2}{2(l+1)}.$$

Since $l < \alpha < \frac{\beta(l+1)+l}{\beta+2}$, we have $\frac{2(\alpha-l)}{l+2} < \frac{2(\beta-l)}{(\beta+2)(l+2)}$ and if $t \to T$, $\;r(t) \sim C(T-t)^{-\frac{l+2}{2(\alpha-l)}}$.

Then, if $t \to T$, we have

$$\begin{aligned} r(t)^{\frac{2(\alpha-l)}{l+2}}|\sin\theta|^{\frac{2(\alpha-l)}{l+2}+1}\cos\theta &= r(t)^{\frac{2(\alpha-l)}{l+2}}|\sin\theta|^{\frac{2(\alpha+1)}{l+2}-\frac{l}{l+2}}\cos\theta \\ &\le \varrho r(t)^{\frac{2(\beta-l)}{(\beta+2)(l+2)}}|\sin\theta|^{-\frac{l}{l+2}}|\cos\theta|^{\frac{\beta}{\beta+2}}, \end{aligned}$$

then

$$\theta' \le -\xi(T-t)^{-\gamma}|\sin\theta|^{-\frac{l}{l+2}}|\cos\theta|^{\frac{\beta}{\beta+2}}, \text{ if } t \to T,$$

where $\xi > 0$ and

$$\gamma = \frac{l+2}{\alpha-l}\frac{\beta-l}{(\beta+2)(l+2)} > 1.$$

In the case $\alpha = \frac{\beta(l+1)+l}{\beta+2}$, we have

$$\theta' = -\frac{l+2}{2(l+1)} r(t)^{\frac{2(\alpha-l)}{l+2}} |\sin\theta|^{\frac{-l}{l+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} \Big\{ \Big(\frac{(\beta+2)(l+1)}{d(l+2)} \Big)^{\frac{\beta+1}{\beta+2}} - c|\sin\theta|^{\frac{2\alpha-l}{l+2}+1} |\cos\theta|^{1-\frac{\beta}{\beta+2}} \Big\},$$

since $\alpha = \frac{\beta(l+1)+l}{\beta+2}$, we have $\frac{\beta}{\beta+2} = \frac{2\alpha-l}{l+2}$.
Then

$$\theta' \leq -\frac{l+2}{2(l+1)} r(t)^{\frac{2(\alpha-l)}{l+2}} |\sin\theta|^{\frac{-l}{l+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} \Big\{ \Big(\frac{(\beta+2)(l+1)}{d(l+2)} \Big)^{\frac{\beta+1}{\beta+2}} - c|\sin\theta|^{\frac{2\alpha-l}{l+2}+1} |\cos\theta|^{1-\frac{2\alpha-l}{l+2}} \Big\},$$

assuming $f(\theta) = |\sin\theta|^{\frac{2\alpha-l}{l+2}+1} |\cos\theta|^{1-\frac{2\alpha-l}{l+2}}, \ \ \theta \in \mathbb{R}$.
Then, we have

$$\max_{\theta\in\mathbb{R}} f(\theta) = \Big(\frac{1}{\beta+2}\Big)^{\frac{1}{\beta+2}} \Big(\frac{\beta+1}{\beta+2}\Big)^{\frac{\beta+1}{\beta+2}}. \tag{5.29}$$

Hence

$$\begin{aligned} \Big(\frac{(\beta+2)(l+1)}{d(l+2)} \Big)^{\frac{\beta+1}{\beta+2}} & - c|\sin\theta|^{\frac{2\alpha-l}{l+2}+1} |\cos\theta|^{1-\frac{2\alpha-l}{l+2}} \\ & \geq \Big(\frac{(\beta+2)(l+1)}{d(l+2)} \Big)^{\frac{\beta+1}{\beta+2}} - c\Big(\frac{1}{\beta+2}\Big)^{\frac{1}{\beta+2}} \Big(\frac{\beta+1}{\beta+2}\Big)^{\frac{\beta+1}{\beta+2}} \end{aligned}$$

$$\Big(\frac{(\beta+2)(l+1)}{d(l+2)} \Big)^{\frac{\beta+1}{\beta+2}} - c\Big(\frac{1}{\beta+2}\Big)^{\frac{1}{\beta+2}} \Big(\frac{\beta+1}{\beta+2}\Big)^{\frac{\beta+1}{\beta+2}} > 0 \Leftrightarrow c < (\beta+2)\Big(\frac{(\beta+2)(l+1)}{d(\beta+1)(l+2)} \Big)^{\frac{\beta+1}{\beta+2}},$$

then, we find in all cases for $t \to T$,

$$\theta' \leq -\xi(T-t)^{-1} |\sin\theta|^{-\frac{l}{l+2}} |\cos\theta|^{\frac{\beta}{\beta+2}}.$$

We introduce the function

$$H(s) = \int_a^s \frac{|\sin u|^{\frac{l}{l+2}}}{|\cos u|^{\frac{\beta}{\beta+2}}}\, du,$$

suppose that u does not vanish if $t \to T$ and for $t \in [t_0, T]$, we may assume for instance $\theta(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $H(\theta(t)) = F(t)$

$$\forall t_0 \leq t \leq T, \ F'(t) \leq -\xi(T-t)^{-1},$$

we integrate from t_0 to t

$$H(\theta(t)) \leq H(\theta(t_0)) - \xi \log(T-t_0) + \xi \log(T-t),$$

if $t \to T$, we find $H(\theta(t)) \to -\infty$. Or $H(\theta(t))$ is non-negative, then, we obtain a contradiction. Therefore, u' has a zero on each half-line.
Step 2. Applying Step 1, we know that u' has an infinite sequence of zeroes tending to infinity.

We claim that between two successive zeroes of u' there is a zero of u. Indeed let $u'(a) = u'(b) = 0$ with $a < b$ and $u' \neq 0$ in (a, b). If u has a constant sign in (a, b), by the equation $(|u'|^l u')'$ has the same sign for $t = a$ and $t = b$, which implies that $(|u'|^l u')'$ have opposite signs on $(a, a + \eta)$ and $(b - \eta, b)$ for $\eta > 0$ small enough, a contradiction with $u' \neq 0$ in (a, b). Finally, by (4.13) we have $\lim_{t\to T} u^2(t) + u'(t) = +\infty$. From the existence of infinitely many zeroes of $u(t)$ and $u'(t)$ as $t \to T$ it is easy to deduce that

$$\limsup_{t\to T} u(t) = \limsup_{t\to T} u'(t) = +\infty,$$

and

$$\liminf_{t\to T} u(t) = \liminf_{t\to T} u'(t) = -\infty.$$

The proof of Theorem 5.1 is now completed. □

6 Non-oscillatory blow-up of solutions to (1.1) for α large

Theorem 6.1. *Assume $l \le \alpha$ and $\frac{\beta(l+1)+l}{\beta+2} < \alpha < \beta$. Then any solution $u(t)$ has a finite number of zeroes in $(T - \epsilon, T)$, for some $\epsilon > 0$ and blows-up as $t \to T$, where T is the blow-up time.*

Proof. We introduce

$$G(s) = \int_0^s |\sin v|^{\frac{2\alpha+l}{l+2}} \sin v \cos v \, dv.$$

First we observe that $G \circ \theta$ is $\mathcal{C}^1$ on any interval where u' does not vanish. Indeed on such an interval, θ is $\mathcal{C}^1$ and

$$[G(\theta(t)]' = Ar(t)^{\frac{2(\alpha-l)}{l+2}} \cos^2\theta |\sin\theta|^{\frac{4(\alpha+1)}{l+2}+\frac{l}{l+2}} - Br(t)^{\frac{2(\beta-l)}{(\beta+2)(l+2)}} |\cos\theta|^{\frac{\beta}{\beta+2}} |\sin\theta|^{\frac{2\alpha}{l+2}} \sin\theta\cos\theta.$$

Then we observe that when $\sin\theta$ vanishes, the right hand side of the above equality is 0. Actually it is also continuous at points where $\sin\theta$ vanishes, so that finally $G \circ \theta$ is $\mathcal{C}^1$ everywhere. Now using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &Br(t)^{\frac{2(\beta-l)}{(\beta+2)(l+2)}-\frac{2(l+1)}{l+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} \sin\theta|^{\frac{2\alpha}{l+2}} \sin\theta\cos\theta \\ &\qquad\qquad \le \frac{B^2}{A} r(t)^{\frac{4(\beta-l)}{(\beta+2)(l+2)}-\frac{2(\alpha-l)}{l+2}} + Ar(t)^{\frac{2(\alpha-l)}{l+2}} |\sin\theta|^{\frac{4(\alpha+1)}{l+2}+\frac{l}{l+2}} \cos^2\theta, \end{aligned}$$

then

$$[G(\theta(t)]' \ge -Cr(t)^{\frac{4(\beta-l)}{(\beta+2)(l+2)}-\frac{2(\alpha-l)}{l+2}}.$$

Since $\beta > \alpha > \frac{\beta(l+1)+l}{\beta+2}$, from (4.14), we have $r(t) \le C'(T-t)^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}}$ for t close enough to T, then

$$[G(\theta(t)]' \ge -C'(T-t)^{-\lambda},$$

with

$$\lambda = \Big(\frac{4(\beta-l)}{(\beta+2)(l+2)} - \frac{2(\alpha-l)}{l+2}\Big)\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}$$
$$= 1+\alpha\left[\frac{\beta(l+1)+l-\alpha(\beta+2)}{(\beta-\alpha)(l+2)}\right] < 1.$$

To finish the proof we shall use the following Lemma (cf.[**4**] for proof).

Lemma 6.2. *Let $\theta \in \mathcal{C}^1(a,T)$ and G be a non constant τ-periodic function. We assume $G\circ\theta \in \mathcal{C}^1(a,T)$ and for some $h \in L^1(a,T)$*

$$[G(\theta(t))]' \geq h(t), \quad \forall t \in [a,T].$$

Then, for $t_1 \leq t < T$, $\theta(t)$ remains in some interval of length $\leq \tau$. In addition, if G' has finite number of zeroes on $[0,\tau]$, then $\theta(t)$ has a limit for $t \to T$.

The proof of Theorem 6.1. From Lemma 6.2, $\theta(t) \to \Theta$ as $t \mapsto T$. We distinguish two cases:
Case 1: If $\Theta \neq \frac{\pi}{2} \mod [\pi]$, $u \sim Cr^{\frac{2}{\beta+2}} > 0$ if $t \to T$, then u has a constant sign.
Case 2: If $\Theta = \frac{\pi}{2} \mod [\pi]$, $|u'| \sim r(t)^{\frac{2}{l+2}} > 0$ if $t \to T$, then $u'(t)$ does not vanish and $u(t)$ has a constant sign if $t \to T$.
Let t_0 be such that u has a constant sign on (t_0,T), if $u'(t)$ has several zeroes in $(T-\epsilon,T)$ for $\epsilon > 0$ small enough, then $(|u'(t)|^l u'(t))'$ must have different signs at two successive zeroes $|u'(t)|^l u'(t)$. From equation (1.1) u must have different signs also, which is impossible. Thus, $u'(t)$ has a constant sign as $t \to T$.
$E(t)$ is unbounded, then $E(t) \to \infty$ as t tends to T. Then

$$\lim_{t\to T} u(t) = \lim_{t\to T} u'(t) = \pm\infty,$$

Since $u(t)$ and $u'(t)$ have the same sign if $t \to T$. □

Theorem 6.3. *Assuming $l \leq \alpha$ and*

$$\alpha = \frac{\beta(1+l)+l}{\beta+2}, \quad c \geq c_0 = (\beta+2)\Big(\frac{(\beta+2)(l+1)}{d(\beta+1)(l+2)}\Big)^{\frac{\beta+1}{\beta+2}},$$

then any solution $u(t)$ of (1.1) *blows-up in finite time T and has a finite number of zeroes in $[0,T]$.*

Proof. If $\alpha = \frac{\beta(1+l)+l}{\beta+2}$, then clearly $\frac{\beta}{\beta+2} = \frac{2\alpha-l}{l+2}$. In this case

$$\theta' = -\frac{l+2}{2(l+1)} r^{\frac{2(\alpha-l)}{l+2}} |\sin\theta|^{\frac{-l}{l+2}} \Big\{\Big(\frac{(\beta+2)(l+1)}{d(l+2)}\Big)^{\frac{\beta+1}{\beta+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} - c\sin\theta\cos\theta|\sin\theta|^{\frac{2\alpha-l}{l+2}}\Big\}$$
$$= -\frac{l+2}{2(l+1)} r^{\frac{2(\alpha-l)}{l+2}} |\sin\theta|^{\frac{-l}{l+2}} |\cos\theta|^{\frac{2\alpha-l}{l+2}} \Big\{\Big(\frac{(\beta+2)(l+1)}{d(l+2)}\Big)^{\frac{\beta+1}{\beta+2}} - c\sin\theta\cos\theta|\sin\theta|^{\frac{2\alpha-l}{l+2}} |\cos\theta|^{\frac{-\beta}{\beta+2}}\Big\}.$$

We set

$$K(\theta) = \frac{l+2}{2(l+1)} |\sin\theta|^{\frac{-l}{l+2}} \Big\{\Big(\frac{(\beta+2)(l+1)}{d(l+2)}\Big)^{\frac{\beta+1}{\beta+2}} |\cos\theta|^{\frac{2\alpha-l}{l+2}} - c\sin\theta\cos\theta|\sin\theta|^{\frac{2\alpha-l}{l+2}}\Big\}$$

i) If $c = c_0$, using (5.29), we have

$$\begin{aligned}
&\Big(\frac{(\beta+2)(l+1)}{d(l+2)}\Big)^{\frac{\beta+1}{\beta+2}} - c\sin\theta\cos\theta|\sin\theta|^{\frac{2\alpha-l}{l+2}}|\cos\theta|^{\frac{-\beta}{\beta+2}}\\
&= \Big(\frac{(\beta+2)(l+1)}{d(l+2)}\Big)^{\frac{\beta+1}{\beta+2}} - c_0|\sin\theta|^{1+\frac{2\alpha-l}{l+2}}|\cos\theta|^{1-\frac{2\alpha-l}{l+2}}\\
&\geq \Big(\frac{(\beta+2)(l+1)}{d(l+2)}\Big)^{\frac{\beta+1}{\beta+2}} - c_0\Big(\frac{1}{\beta+2}\Big)^{\frac{1}{\beta+2}}\Big(\frac{\beta+1}{\beta+2}\Big)^{\frac{\beta+1}{\beta+2}} = 0
\end{aligned}$$

$K(\theta) > 0$, so that θ is non-increasing. The distance of two consecutive zeroes of $K(\theta)$ other than $\frac{\pi}{2}$ (modπ) is not more than π, therefore we have two cases:
Case 1: if $\theta(t)$ remains in an interval of length less than π, then θ is bounded from above and is non-increasing thus it converges to a limit as $t \to T$ and achieves at most one a value for which u vanishes.
Case 2: if $\theta(t)$ coincides with one of these zeros for a finite value of t, due to existence and uniqueness for the ODE satisfies by $\theta(t)$ near the non-trivial equilibria, $\theta(t)$ is constant and u never vanishes.

ii) If $c > c_0$, $K(\theta) < 0$. We have two cases:
Case 1: if $\theta(t) \neq \frac{\pi}{2}$, then, it is bounded and since $K(\theta) < 0$ near the trivial zeros, $\theta(t)$ is monotone, and therefore it is convergent as $t \to T$.
Case 2: if $\theta(t) = \frac{\pi}{2}$, then it remains constant and u never vanishes.

□

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