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This article is concerned with the asymptotic behavior, at infinity and at the origin, of Green functions of operators of the form $Lu = -\text{div}(A\nabla u)$, where A is a periodic, coercive, and bounded matrix.

1 Introduction

The study of Green's functions for elliptic operators is an important research subject. It is linked with many different fields such as homogenization [1–4, 16, 18, 21] or the study of singular points [11, 23]. In particular, in [16], the Green function associated with a highly oscillatory elliptic operator is shown to converge to the Green function of the corresponding homogenized operator (rates of convergence are also established).

The aim of the present article is to provide explicit bounds at infinity for the Green function G of a divergence-type elliptic operator with periodic coefficients. Many arguments in this paper are already present in the literature in a scattered manner, and our main contribution is to put them together in a clear way. Our arguments also provide us with explicit bounds on G in the neighborhood of the origin, where G is singular. These latter results are already described in a comprehensive way in the literature.

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In all the article, we assume that $d \geq 2$ is the dimension of the ambient space and that (here, $\mathbb{R}^{d\times d}$ is the space of square matrices of size d) the field $A : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ satisfies

$$A \text{ is } \mathbb{Z}^d \text{ periodic}, \tag{1.1}$$

$$A \text{ is } \delta - \text{Hölder continuous for some } \delta > 0, \tag{1.2}$$

$$\exists \alpha > 0, \ \ \forall \xi \in \mathbb{R}^d, \ \ \forall x \in \mathbb{R}^d, \quad \xi^{\mathrm{T}} A(x)\xi \geq \alpha |\xi|^2, \tag{1.3}$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^d$, and

$$A \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d}). \tag{1.4}$$

We want to study the behavior at infinity of the Green function G associated with the operator

$$L = -\mathrm{div}(A\nabla\cdot),$$

that is, the function $G : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$$-\,\mathrm{div}_x(A(x)\nabla_x G(x, y)) = \delta_y(x). \tag{1.5}$$

See (2.1) for a more precise formulation. By behavior at infinity, we mean the asymptotic of $G(x, y)$ as $|x - y|$ goes to infinity. This question has been widely studied in the literature. According to [1, Theorem 13] (see also [20]), we have, if $d \geq 3$,

$$\exists C, \ \ \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |G(x, y)| \leq C|x - y|^{2-d}. \tag{1.6}$$

In addition (see [1, Theorem 13]), we have, in the case $d = 2$,

$$\exists C, \ \ \forall (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad |G(x, y)| \leq C(1 + \log|x - y|). \tag{1.7}$$

Note that these estimates characterize both the asymptotic behavior of G at infinity (when $|x - y| \to \infty$) and at the origin (when $|x - y| \to 0$). An important point here is that many papers consider only the case of Green functions for operators L defined in a *bounded* domain. (problem (1.5) is then complemented by appropriate boundary conditions.) This is the case for instance of Dolzmann and Müller [5; 11, Theorems 1.1 and 3.3]. This is also the case of Avellaneda and Lin [1, Theorem 13], although a remark following the theorem indicates that the constant in the estimate can be chosen independent of the domain. In [11, Theorem 3.3], bounds are provided on G, its gradient and the second derivatives $\nabla_x\nabla_y G$, in the case $d \geq 3$. A remark following that result points out that the constant in the estimate of G is independent of the domain, whereas the constants in the estimates of the derivatives of G a priori depend on the domain. The articles [1, 5, 11] all consider the case of homogeneous Dirichlet boundary conditions. The case of Neumann boundary conditions is considered in [17], where estimates on G, its gradient and the second derivatives $\nabla_x \nabla_y G$ are given in the case $d \geq 3$.

In this article, we also address the question of the decay of the *derivatives* of G at infinity. We have, as proved in Propositions 5 and 7 (the material is present in [1], and also in [3]), for any $d \geq 2$,

$$\exists C > 0, \ \ \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |\nabla_x G(x, y)| + |\nabla_y G(x, y)| \leq C |x - y|^{1-d} \tag{1.8}$$

and

$$\exists C > 0, \ \ \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |\nabla_x \nabla_y G(x, y)| \leq C |x - y|^{-d}. \tag{1.9}$$

Such estimates may also be found in [11] for $d \geq 3$ and bounded domains. In [9], similar results are proved for a domain which is a half-space. Dolzmann and Müller [5] prove (1.6)–(1.9) for $d \geq 2$ and bounded domains, and for systems of PDEs rather than a scalar PDE. In [6], (1.7) is proved in the case of systems, and domains $\Omega \subset \mathbb{R}^2$ which are either of finite volume, or of finite width, or which are of the form $\Omega = \{x_2 > \varphi(x_1)\}$, where φ is Lipschitz. Finally, (1.6) is proved in [14] for the case of systems and $d \geq 3$.

A preliminary question, before showing (1.6)–(1.9), is the existence and uniqueness of G defined by (1.5). This question is addressed in [11, Theorem 1.1], for the Green function in a bounded domain $\Omega \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary conditions. An existence proof is then provided for G such that $\nabla_x G(\cdot, y) \in L^p(\Omega \setminus B_r(y))$, for any $p > d/(d-1)$ and $r > 0$. Actually, in [11], only the case $d \geq 3$ is studied, but the existence proof carries through to the case $d = 2$. The uniqueness of G, under the assumption that $G \geq 0$, is also proved in [11, Theorem 1.1] for $d \geq 3$. The case $d = 2$ is not covered by their proof. A proof of uniqueness when $d = 2$ can be found in the appendix of Kenig and Ni [19], both for a bounded domain and for the whole space.

We finally mention that the case of nondivergence form operators (of parabolic and elliptic type) has also been considered, see, for example, [7].

The article is organized as follows. In Section 2, we discuss existence and uniqueness theorems for Green functions. In Section 3, we state asymptotic properties on G and its derivatives. Finally, Section 4 outlines some remarks about possible extensions of the results stated in the present article.

2 Definition of Green Function

In order to state the existence and uniqueness result for G solution of (1.5), we first write a weak formulation: we look for $G:\mathbb{R}^d\times\mathbb{R}^d\mapsto\mathbb{R}$ such that

$$\forall y\in\mathbb{R}^d,\ \ \forall\varphi\in\mathcal{D}(\mathbb{R}^d),\qquad \int_{\mathbb{R}^d}(\nabla\varphi(x))^{\mathrm{T}}A(x)\nabla_x G(x,y)\,\mathrm{d}x=\varphi(y). \tag{2.1}$$

In the sequel, we will need the definition of weak L^p spaces, which are special cases of Lorentz spaces: for any open subset $\Omega\subset\mathbb{R}^d$, for any $p\in[1,\infty]$,

$$L^{p,\infty}(\Omega)=\{f:\Omega\to\mathbb{R},\ \ f\text{ measurable},\ \|f\|_{L^{p,\infty}(\Omega)}<\infty\},$$

where

$$\|f\|_{L^{p,\infty}(\Omega)}=\sup_{t\geq 0}\{t\,\mu(\{x\in\Omega,\ |f(x)|\geq t\})^{1/p}\},$$

where μ is the Lebesgue measure. We recall that, for any $0<\beta\leq p-1$,

$$C(p,\beta,\Omega)\|f\|_{L^{p-\beta}(\Omega)}\leq\|f\|_{L^{p,\infty}(\Omega)}\leq\|f\|_{L^p(\Omega)}, \tag{2.2}$$

with $C(p,\beta,\Omega)=\frac{1}{2}\,\frac{(2^\beta-1)^{1/p}}{(2^\beta+1)^{1/(p-\beta)}}\,(\mu(\Omega))^{\frac{-\beta}{p(p-\beta)}}$. For the sake of completeness, a proof of this result is given in the Appendix below.

Theorem 1 (Existence and uniqueness of G, $d\geq 3$). Let $d\geq 3$, and assume that A satisfies (1.3) and (1.4). Then, Equation (2.1) has a unique solution in $L^\infty_y(\mathbb{R}^d, W^{1,1}_{x,\mathrm{loc}}(\mathbb{R}^d))$ such that

$$\lim_{|x-y|\to\infty}G(x,y)=0. \tag{2.3}$$

Moreover, G satisfies the following estimate:

$$\forall q<\frac{d}{d-1},\ \ \forall y\in\mathbb{R}^d,\quad G(\cdot,y)\in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^d)\cap W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d\setminus\{y\}) \tag{2.4}$$

and

$$\exists C,\ \ \forall(x,y)\in\mathbb{R}^d\times\mathbb{R}^d,\quad 0\leq G(x,y)\leq C|x-y|^{2-d}. \tag{2.5}$$

□

Proof. First, note that, according to [10, Theorem 8.24], the function G is Hölder continuous with respect to x and y whenever $x\neq y$. The same property holds for G_R defined below.

Let $R > 0$. We first define G_R as the Green function of the operator $-\mathrm{div}(A\nabla\cdot)$ on the ball $B_R = B_R(0)$ with homogeneous Dirichlet boundary conditions, that is,

$$\forall y \in B_R, \ \ \forall \varphi \in \mathcal{D}(B_R), \quad \int_{B_R} (\nabla\varphi(x))^{\mathrm{T}} A(x) \nabla_x G_R(x, y)\, \mathrm{d}x = \varphi(y), \tag{2.6}$$

and $G_R(x, y) = 0$ if $|x| = R$. Applying [11, Theorem 1.1], we know that such a G_R exists, and satisfies

$$\forall y \in B_R, \quad \|G_R(\cdot, y)\|_{L^{\frac{d}{d-2},\infty}(B_R)} \leq C, \tag{2.7}$$

$$\forall y \in B_R, \quad \|\nabla_x G_R(\cdot, y)\|_{L^{\frac{d}{d-1},\infty}(B_R)} \leq C, \tag{2.8}$$

and

$$\forall (x, y) \in B_R \times B_R, \quad 0 \leq G_R(x, y) \leq \frac{C}{|x-y|^{d-2}}, \tag{2.9}$$

where $C > 0$ does not depend on R and y.

Next, we note that if $R' > R$, then, due to the maximum principle, we have $G_{R'} \geq G_R$ in $B_R \times B_R$. Thus, G_R is a nondecreasing function of R. With the help of (2.9), this implies that the function G_R converges almost everywhere to some function G, defined on $\mathbb{R}^d \times \mathbb{R}^d$, and that satisfies (2.5). This implies (2.3). In addition, we deduce from (2.9) that G_R converges to G in $L^p_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$, for any $p < d/(d-2)$, and that, for any $y \in \mathbb{R}^d$, the function $G_R(\cdot, y)$ converges to $G(\cdot, y)$ in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, for any $p < d/(d-2)$.

In view of (2.8) and (2.2), we see that, for any bounded domain $\Omega \subset \mathbb{R}^d$, and for any $q < d/(d-1)$, there exists $C(\Omega, q, d)$ such that

$$\forall R \text{ s.t. } \Omega \subset B_R, \ \ \forall y \in B_R, \quad \|\nabla_x G_R(\cdot, y)\|_{L^q(\Omega)} \leq C(\Omega, q, d).$$

Hence, extracting a subsequence if necessary, $\nabla_x G_R(\cdot, y)$ converges weakly in $(L^q(\Omega))^d$ to some $T \in (L^q(\Omega))^d$. Recall now that $G_R(\cdot, y)$ converges to $G(\cdot, y)$ in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, for any $p < d/(d-2)$. Hence $T = \nabla_x G_{|\Omega}$, and $\nabla_x G_R(\cdot, y)$ converges to $\nabla_x G$ weakly in $(L^q(\Omega))^d$, for any bounded domain Ω and any $q < d/(d-1)$. Passing to the limit in (2.6), we see that G is a solution to (2.1).

Finally, the bounds (2.7) and (2.8) imply, together with (2.2), that $G \in L^\infty_y(\mathbb{R}^d, W^{1,1}_{x,\mathrm{loc}}(\mathbb{R}^d))$. We have thus proved the existence of G.

Property (2.4) is proved in [11, Theorem 1.1], and its proof does not depend on the fact that the domain used there is bounded. Note that we have already proved part of this property. Indeed, as pointed above, for any $y \in \mathbb{R}^d$, we have $G(\cdot, y) \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for any $p < d/(d-2)$ and $\nabla_x G(\cdot, y) \in (L^q_{\mathrm{loc}}(\mathbb{R}^d))^d$ for any $q < d/(d-1)$, thus $G(\cdot, y) \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^d)$ for any $q < d/(d-1)$.

In order to prove uniqueness, we assume that G_1 and G_2 are two solutions, and point out that $H = G_1 - G_2$ satisfies $\operatorname{div}_x(A\nabla_x H) = 0$ for any $y \in \mathbb{R}^d$. Fixing y, we apply the corollary of Moser [26, Theorem 4], which implies that, if H is not constant, then $\sup\{H(x, y), |x - y| = r\} - \inf\{H(x, y), |x - y| = r\}$ must grow at least like a positive power of r as $r \to \infty$. This latter behavior is in contradiction with (2.3). Thus $H = G_1 - G_2$ is constant, and (2.3) implies that $G_1 \equiv G_2$.

Note finally that the corollary of Moser [26, Theorem 4] is stated in the case when A is symmetric, but the same result holds in the nonsymmetric case. Indeed, Harnack's inequality is still valid in such a case, see, for example, [10, Theorem 8.20; 25, Theorem 5.3.2] or [15]. ∎

Theorem 2 (Existence and uniqueness of G, $d = 2$). Let $d = 2$, and assume that A satisfies (1.3) and (1.4). Then, Equation (2.1) has a unique (up to the addition of a constant) solution in $L^\infty_{y,\mathrm{loc}}(\mathbb{R}^d, W^{1,1}_{x,\mathrm{loc}}(\mathbb{R}^d))$ such that

$$\exists C > 0, \quad \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |G(x, y)| \le C(1 + |\log|x - y||). \tag{2.10}$$

Moreover, G satisfies the following estimate:

$$\forall q < 2, \quad \forall y \in \mathbb{R}^d, \quad G(\cdot, y) \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^d) \cap W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d \setminus \{y\}). \tag{2.11}$$

□

Proof. The proof of this result may be found in the appendix of Kenig and Ni [19]. However, for the sake of completeness, we provide an alternative proof. This proof, in contrast to that of Kenig and Ni [19], relies on basic tools of analysis of PDEs.

We use the same strategy as in the proof of Theorem 1, defining first the Green function G_R of the operator L on B_R. However, we cannot simply apply the results of Grüter and Widman [11] to define G_R, as those results are stated in dimension $d \ge 3$. It is possible to adapt the proof of Grüter and Widman [11, Theorem 1.1] to the two-dimensional case, but a simpler proof consists in following the approach of Dolzmann and Müller [5, Section 6]. These results give the existence and uniqueness of G_R solution to (2.6) in the ball $B_R = B_R(0) \subset \mathbb{R}^2$, with the homogeneous Dirichlet boundary conditions $G_R(x, y) = 0$ if $|x| = R$, in $W^{1,p}(B_R)$ for any $p < 2$. In addition, it is shown in [5, Section 6] that estimate (2.8) holds, namely

$$\forall y \in B_R, \quad \|\nabla_x G_R(\cdot, y)\|_{L^{2,\infty}(B_R)} \le C \tag{2.12}$$

for a constant C independent of R and y.

Step 1: *Passing to the limit* $R \to \infty$ *on* G_R. Consider the domain $\Omega = B_{R'}$, with R' fixed, and consider next $R > R'$. Applying (2.2) to $\nabla_x G_R(\cdot, y)$ on Ω, we see that (2.12) implies that $\nabla_x G_R$ is bounded in $(L^q(B_{R'} \times B_{R'}))^2$ for any $q < 2$, independently of R. Hence, extracting a subsequence if necessary, $\nabla_x G_R$ converges weakly in $(L^q(B_{R'} \times B_{R'}))^2$ to $T \in (L^q(B_{R'} \times B_{R'}))^2$. Now, we have, in the sense of distribution,

$$\partial_{x_1} \partial_{x_2} G_R = \partial_{x_2} \partial_{x_1} G_R.$$

This property passes to the limit, so that $\partial_{x_1} T_2 = \partial_{x_2} T_1$. This implies that $T = \nabla_x G$ for some $G \in W^{1,q}(B_{R'} \times B_{R'})$. Next, we point out that this limit does not depend on R' in the sense that if $R'' > R'$, then $\nabla_x G'$ obtained in $B_{R'}$ is equal to $\nabla_x G''_{|B_{R'}}$, where $\nabla_x G''$ is obtained in $B_{R''}$. Hence $G \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}^2) \subset L^q_{y,\mathrm{loc}}(\mathbb{R}^2, W^{1,1}_{x,\mathrm{loc}}(\mathbb{R}^2))$. Passing to the limit in (2.6), we obtain that G is a solution to (2.1). Until now, the function $G(\cdot, y)$ is only determined up to a constant. We fix this constant by choosing $G(\cdot, y)$ such that

$$\int_{B_1(y)} G(x, y)\, \mathrm{d}x = 0. \tag{2.13}$$

To prove the existence of a function G satisfying the claimed properties, we are now left with showing that the function G that we have built satisfies (2.10) and (2.11).

Step 2: *Proving that* G *satisfies* (2.11). By construction, we have $G(\cdot, y) \in W^{1,q}(\Omega)$, for any $q < 2$ and any bounded domain Ω. The proof of the fact that $G(\cdot, y) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus y)$ follows the same lines as the proof given in [11, Theorem 1.1], which does not depend on the fact that the domain used there is bounded, nor on the fact that the dimension there is $d \geq 3$. We thus have proved (2.11).

Step 3: *Proving that* G *satisfies* (2.10). We first infer from (2.12) and (2.2) that, for any bounded domain $\Omega \subset B_R$ and any $y \in B_R$, we have

$$\frac{1}{\sqrt{\mu(\Omega)}} \|\nabla_x G_R(\cdot, y)\|_{L^1(\Omega)} \leq C$$

for a constant C independent of R, Ω, and y. Since $\nabla_x G_R(\cdot, y)$ weakly converges to $\nabla_x G(\cdot, y)$, we deduce that

$$\frac{1}{\sqrt{\mu(\Omega)}} \|\nabla_x G(\cdot, y)\|_{L^1(\Omega)} \leq C \tag{2.14}$$

for a constant C independent of Ω and y. Note that this implies that $G \in L^\infty_{y,\mathrm{loc}}(\mathbb{R}^d, W^{1,1}_{x,\mathrm{loc}}(\mathbb{R}^d))$, as claimed in the theorem.

Second, we apply Poincaré–Wirtinger inequality to $G(\cdot, y)$ on the ball $B_1(y)$: using (2.13), we have

$$\int_{B_1(y)} |G(x, y)|\, \mathrm{d}x \leq C \int_{B_1(y)} |\nabla_x G(x, y)|\, \mathrm{d}x.$$

Applying (2.14) with $\Omega = B_1(y)$, we deduce that

$$\int_{B_1(y)} |G(x,y)|\,\mathrm{d}x \le C, \tag{2.15}$$

where C does not depend on y.

We next define, for any $R>0$, the function

$$f(R) = \frac{1}{2\pi R}\int_{\partial B_R(y)} |G(x,y)|\,\mathrm{d}x,$$

where $\mathrm{d}x$ denotes the Lebesgue measure on the circle $\partial B_R(y)$. Note that f depends on y, but we keep this dependency implicit in our notation. In the sequel of the proof, we first show a bound on f (step 3a), and then deduce from that bound a bound on G (step 3b).

Step 3a: *Bound on* f. We have, for any $R > R' > 0$:

$$\begin{aligned} |f(R) - f(R')| &\le \int_{R'}^{R} |f'(r)|\,\mathrm{d}r \le \int_{R'}^{R} \frac{1}{2\pi r}\int_{\partial B_r(y)} |\nabla_x G(x,y)|\,\mathrm{d}x\,\mathrm{d}r \\ &\le \frac{1}{2\pi R'}\int_{B_R(y)\setminus B_{R'}(y)} |\nabla_x G(x,y)|\,\mathrm{d}x \le C\frac{\sqrt{R^2 - R'^2}}{R'} = C\frac{R}{R'}, \end{aligned} \tag{2.16}$$

where we have again used (2.14) and where the constant C does not depend on y. This implies that $f(R)$ is bounded independently of R and y for $R\in(\frac{1}{2},1)$. Indeed, for such an R, we rewrite (2.16) as $f(R)\le f(R') + CR/R'$ (recall that f is nonnegative), and integrate with respect to R' between $\frac{1}{4}$ and $\frac{1}{2}$, finding

$$\frac{1}{4}f(R) \le \int_{1/4}^{1/2} f(R')\,\mathrm{d}R' + CR.$$

Using (2.15), we infer

$$\forall R\in[\tfrac{1}{2},1],\quad f(R)\le C, \tag{2.17}$$

for some constant C independent of R and y. Next, we consider two different cases: $R>1$ and $R<\frac{1}{2}$.

1. Case $R>1$: in such a case, we define $p\in\mathbb{N}$ such that

$$\frac{1}{2} < \frac{R}{2^p} \le 1,$$

that is, p is the integer part of $\frac{\log R}{\log 2}$, which reads $\frac{\log R}{\log 2}\le p < \frac{\log R}{\log 2}+1$. We then apply (2.16) with $R = 2^{-j}R$, $R' = 2^{-j-1}R$, finding

$$\left| f\left(\frac{R}{2^j}\right) - f\left(\frac{R}{2^{j+1}}\right)\right| \le C,$$

where C is a constant which does not depend on R, j, nor on y. We sum up all these inequalities for $0 \le j \le p-1$ and obtain

$$f(R) \le f\left(\frac{R}{2^p}\right) + Cp.$$

Recalling (2.17) and the definition of p, we infer

$$|f(R)| \le C(1 + |\log(R)|), \tag{2.18}$$

where C is independent of R and y.

2. Case $R < \frac{1}{2}$: the approach is similar to the previous case. We define $p \in \mathbb{N}$ such that

$$\tfrac{1}{2} \le 2^p R < 1,$$

that is, p is the integer part of $-\frac{\log R}{\log 2} - 1$. We apply (2.16) with $R' = 2^j R$ and $R = 2^{j+1} R$, finding

$$|f(2^j R) - f(2^{j+1} R)| \le C.$$

We sum this with respect to $0 \le j \le p-1$ and find that (2.18) is again valid in this case.

Collecting the result of the above two cases, we find that

$$\forall R > 0, \quad |f(R)| \le C(1 + |\log(R)|), \tag{2.19}$$

where the constant C does not depend on R nor on y.

Step 3b: *Bound on* G. We first make use of (2.19) to obtain a bound on the L^1 norm of G in any annulus. For any $\beta \le \gamma$, we indeed have

$$\|G(\cdot, y)\|_{L^1(B_\gamma(y)\setminus B_\beta(y))} = 2\pi \int_\beta^\gamma r f(r)\, \mathrm{d}r,$$

hence, using (2.19), we obtain

$$\|G(\cdot, y)\|_{L^1(B_\gamma(y)\setminus B_\beta(y))} \le C \int_\beta^\gamma r(1 + |\log(r)|)\, \mathrm{d}r. \tag{2.20}$$

Consider now $R \ge \frac{1}{2}$. Then $3R \ge 2R \ge 1$, and (2.20) implies

$$\begin{aligned} \forall R \ge \frac{1}{2}, \quad \|G(\cdot, y)\|_{L^1(B_{3R}(y)\setminus B_{2R}(y))} &\le C \int_{2R}^{3R} r(1 + \log(r))\, \mathrm{d}r \\ &\le 3C R^2 (1 + \log(3R)) \le \mathcal{C} R^2 (1 + |\log(R)|), \end{aligned} \tag{2.21}$$

for some $\mathcal{C}$ independent of R and y. In turn, if $R \le \frac{1}{3}$, then (2.20) implies

$$\forall R \le \frac{1}{3}, \quad \|G(\cdot, y)\|_{L^1(B_{3R}(y)\setminus B_{2R}(y))} \le C \int_{2R}^{3R} r(1 - \log(r))\, \mathrm{d}r$$
$$\le 3C R^2(1 - \log(3R)) \le \mathcal{C} R^2(1 + |\log(R)|), \tag{2.22}$$

for some $\mathcal{C}$ independent of R and y.

Next, we recall that, according to Sobolev imbeddings (see for instance [10, Theorem 7.10]), we have

$$\forall p < 2, \ \forall u \in W^{1,p}(\mathbb{R}^2), \quad \|u\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)} \le C_p \|\nabla u\|_{L^p(\mathbb{R}^2)}.$$

We apply this inequality to $u = G(\cdot, y)\chi_R$, where χ_R is a cutoff function satisfying

$$\chi_R \in \mathcal{D}(\mathbb{R}^2), \quad |\nabla \chi_R| \le \frac{C}{R}, \quad \chi_R = 0 \text{ outside } B_{3R}(y), \quad \chi_R = 1 \text{ in } B_{2R}(y).$$

We find, for $p = 1$, that

$$\|G(\cdot, y)\|_{L^2(B_{2R}(y)\setminus B_R(y))} \le \|u\|_{L^2(\mathbb{R}^2)} \le C\|\nabla u\|_{L^1(\mathbb{R}^2)}$$
$$\le C\|\nabla_x G(\cdot, y)\|_{L^1(B_{3R}(y))} + \frac{C}{R}\|G(\cdot, y)\|_{L^1(B_{3R}(y)\setminus B_{2R}(y))}. \tag{2.23}$$

The first term of the right-hand side is bounded using (2.14), which yields

$$\|\nabla_x G(\cdot, y)\|_{L^1(B_{3R}(y))} \le C R. \tag{2.24}$$

The second term is bounded using (2.20)–(2.22). If $R \ge \frac{1}{2}$ or $R \le \frac{1}{3}$, we indeed see from (2.21) and (2.22) that

$$\frac{C}{R}\|G(\cdot, y)\|_{L^1(B_{3R}(y)\setminus B_{2R}(y))} \le \mathcal{C} R(1 + |\log(R)|). \tag{2.25}$$

In turn, if $\frac{1}{3} \le R \le \frac{1}{2}$, then we deduce from (2.20) that

$$\frac{C}{R}\|G(\cdot, y)\|_{L^1(B_{3R}(y)\setminus B_{2R}(y))} \le 3C\|G(\cdot, y)\|_{L^1(B_{3/2}(y)\setminus B_{2/3}(y))} \le \mathcal{C},$$

and hence (2.25) is again valid.

Collecting (2.23)–(2.25), we obtain

$$\|G\|_{L^2(B_{2R}(y)\setminus B_R(y))} \le C R + C R|\log(R)|.$$

Finally, we apply [26, Theorem 2] (see also [10, Theorem 8.15]), which implies that, for any $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ such that $Lv = 0$ in $B_{4R}(y) \setminus B_{R/2}(y)$, we have

$$\sup_{B_{2R}(y)\setminus B_R(y)} v \le \frac{C}{R}\|v\|_{L^2(B_{2R}(y)\setminus B_R(y))}.$$

Applying this to $G(\cdot, y)$ and $-G(\cdot, y)$, we find

$$\sup_{B_{2R}(y)\setminus B_R(y)} |G(\cdot, y)| \leq C(1 + |\log(R)|). \tag{2.26}$$

The function G hence satisfies (2.10). This concludes the proof of the existence of a function G satisfying the properties claimed in Theorem 2.

To prove the uniqueness of G (up to a constant), we follow the same argument as in the case $d \geq 3$ (see Theorem 1). Assume that G_1 and G_2 are two solutions. We point out that $H = G_1 - G_2$ satisfies $\mathrm{div}_x(A\nabla_x H) = 0$ for any $y \in \mathbb{R}^2$. Fixing y, we apply the corollary of Moser [26, Theorem 4], which implies that, if H is not constant, then $\sup\{H(x, y), |x - y| = r\} - \inf\{H(x, y), |x - y| = r\}$ must grow at least like a positive power of r as $r \to \infty$. This latter behavior is in contradiction with (2.10). Thus, $H = G_1 - G_2$ is constant. This concludes the proof of Theorem 2. ■

Remark 3. The above proof can be adapted to the case of the Green function G_R on the bounded domain B_R, that is, the solution to (2.6). We hence obtain

$$\forall (x, y) \in B_R \times B_R, \quad |G_R(x, y)| \leq \mathcal{C}_R + \mathcal{C}|\log|x - y||,$$

where $\mathcal{C}$ is independent of R, thus recovering the result of Dolzmann and Müller [5, Section 6]. Note that the constant $\mathcal{C}_R$ in the above bound *a priori* depends on R. Think indeed for instance of the case $L = -\Delta$, where $G_R(x, 0) = -\log|x| + \log R$. □

3 Asymptotic Behavior

We now give some results about the asymptotic behavior (at infinity and at the origin) of the Green function G. First, we note that, collecting (2.5) and (2.10), we have the following proposition.

Proposition 4. Assume that A satisfies (1.3) and (1.4). Then, the Green function G of the operator $-\mathrm{div}(A\nabla\cdot)$ (namely, the solution to (2.1)) satisfies

$$\exists C > 0, \ \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |G(x, y)| \leq \begin{cases} C(1 + |\log|x - y||) & \text{if } d = 2, \\ C|x - y|^{2-d} & \text{if } d > 2. \end{cases} \tag{3.1}$$

□

As we pointed out in Section 1, this result is well known for bounded domains [1, 5, 11, 20, 23]. However, almost all results are limited to this case, except for [1, Theorem 13], for which “in spirit”, the domain is infinite due to the scaling with respect to $\varepsilon \to 0$. Kozlov [20] and Littman *et al.* [23, Section 10] also consider the case of unbounded domains (see also a remark following [11, Theorem 3.3]), but do not consider the case $d=2$. Finally, the appendix of [19] treats the case of $\mathbb{R}^2$.

Next, we give results on the gradient of G.

Proposition 5. Assume that A satisfies (1.1)–(1.4). Then, the Green function G associated with $L = -\mathrm{div}(A\nabla\cdot)$ satisfies the following estimates:

$$\exists C > 0,\ \ \forall x \in \mathbb{R}^d,\ \ \forall y \in \mathbb{R}^d, \quad |\nabla_x G(x,y)| \le \frac{C}{|x-y|^{d-1}}, \tag{3.2}$$

$$\exists C > 0,\ \ \forall x \in \mathbb{R}^d,\ \ \forall y \in \mathbb{R}^d, \quad |\nabla_y G(x,y)| \le \frac{C}{|x-y|^{d-1}}. \tag{3.3}$$

□

Similar results are given in [11, Theorem 3.3], in the case of bounded domains.

Proof. We start with the case $d \ge 3$, and apply [1, Lemma 16] to G as a function of x, which implies that

$$\forall x \in \mathbb{R}^d,\ \ \forall y \in \mathbb{R}^d,\ \ \forall r < |x-y|, \quad \|\nabla_x G(\cdot,y)\|_{L^\infty(B_{r/2}(x))} \le \frac{C}{r}\, \|G(\cdot,y)\|_{L^\infty(B_r(x))}, \tag{3.4}$$

where C depends only on $\|A\|_{C^{0,\delta}}$, δ, α, and d. Using (3.1), we thus obtain

$$|\nabla_x G(x,y)| \le \frac{C}{r} \sup_{z \in B_r(x)} \frac{1}{|z-y|^{d-2}}. \tag{3.5}$$

Note that we have used $|\nabla_x G(x,y)| \le \|\nabla_x G(\cdot,y)\|_{L^\infty(B_{r/2}(x))}$. This comes from the fact that, on $B_{r/2}(x)$, we have

$$-\mathrm{div}_x(A(x)\nabla_x G(x,y)) = 0,$$

and since A is Hölder continuous, we know that $\nabla_x G$ is also Hölder continuous (see, e.g., [10, Theorem 8.22 and Corollary 8.36]).

Taking $r = |x-y|/2$, we have, for any $z \in B_r(x)$,

$$|x-y| \le |x-z| + |z-y| \le r + |z-y| = \tfrac{1}{2}|x-y| + |z-y|.$$

We hence deduce from (3.5) that

$$|\nabla_x G(x,y)| \le \frac{C}{r}\left(\frac{2}{|x-y|}\right)^{d-2} = \frac{2^{d-1}C}{|x-y|^{d-1}}.$$

This proves (3.2).

Next, in order to prove (3.3), we point out that $G^\star(x,y) := G(y,x)$ is the Green function of the operator $L^\star$ defined by

$$L^\star u = -\mathrm{div}(A^{\mathrm{T}}\nabla u). \tag{3.6}$$

A proof of this fact is given in [5, Theorem 1; 11, Theorem 1.3] in the case $d \geq 3$, and this proof carries over to the case $d=2$. (The main idea of the proof consists in choosing the test function $\varphi(x) = G(z,x)$ in (2.1), for any $z \in \mathbb{R}^d$, and next multiplying (2.1) by an arbitrary function $f(y)$ and integrating over y. However, as the function $G(z,\cdot)$ does not belong to $\mathcal{D}(\mathbb{R}^d)$, some regularization arguments are in order.) Hence, applying (3.2) to $G^\star$, we deduce (3.3).

We turn to the case $d=2$. The estimate (3.4) is not sufficient here, since $G(x,y)$ is not bounded as $|x-y| \to \infty$. Instead, we use the same trick as in the proof of Avellaneda and Lin [1, Theorem 13], using (3.2) for $d=3$. For this purpose, we introduce the operator $\tilde{L}$ defined on $H^1(\mathbb{R}^3)$ by

$$\tilde{L}u = -\mathrm{div}_x(A(x)\nabla_x u) - \partial_t^2 u, \tag{3.7}$$

where $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Let $\tilde{G}$ be the associated Green function. According to the above proof and to (3.1), we have

$$|\tilde{G}(x,t,y,s)| \leq \frac{C}{|x-y| + |t-s|}$$

and

$$|\nabla_x \tilde{G}(x,t,y,s)| + |\partial_t \tilde{G}(x,t,y,s)| \leq \frac{C}{|x-y|^2 + (t-s)^2}. \tag{3.8}$$

Next, we set, for any x and y in $\mathbb{R}^2$, with $x \neq y$,

$$G_\kappa(x,y) = \int_{-\kappa}^{\kappa} \tilde{G}(x,t,y,0)\,\mathrm{d}t.$$

We deduce from (3.8) that

$$|\nabla_x G_\kappa(x,y)| \leq C \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{|x-y|^2 + t^2} = \frac{C\pi}{|x-y|}, \tag{3.9}$$

for a constant C independent of κ, x, and y. Hence, $\nabla_x G_\kappa$ is bounded in $L^p_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}^2)$, uniformly with respect to κ, for any $p < 2$. Thus, for any $R > 0$, extracting a subsequence if necessary, $\nabla_x G_\kappa$ converges weakly in $(L^p(B_R \times B_R))^2$ to some $T \in (L^p(B_R \times B_R))^2$. Now, we have, in the sense of distribution,

$$\partial_{x_1}\partial_{x_2}G_\kappa = \partial_{x_2}\partial_{x_1}G_\kappa.$$

This property passes to the limit, so that $\partial_{x_1}T_2 = \partial_{x_2}T_1$. This implies that $T = \nabla_x \bar{G}$ for some $\bar{G} \in W^{1,p}(B_R \times B_R)$. We next point out that the limit $\bar{G}$ does not depend on R, in the sense that if $R' > R$, then $\bar{G}'$ defined on $B_{R'} \times B_{R'}$ as above satisfies $\nabla_x \bar{G}' = \nabla_x \bar{G}$ on $B_R \times B_R$. We thus have $\bar{G} \in L^\infty_{y,\mathrm{loc}}(\mathbb{R}^2, W^{1,1}_{x,\mathrm{loc}}(\mathbb{R}^2))$.

Note also that (3.9) implies that, for any $y \in \mathbb{R}^2$, the function $\nabla_x G_\kappa(\cdot, y)$ is bounded in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$, uniformly with respect to κ, for any $p < 2$. Thus, for any bounded domain B_R, extracting a subsequence if necessary, $\nabla_x G_\kappa(\cdot, y)$ converges weakly in $(L^p(B_R))^2$, and, by uniqueness, $\nabla_x G_\kappa(\cdot, y)$ converges to $\nabla_x \bar{G}(\cdot, y)$ weakly in $(L^p(B_R))^2$.

At this point, $\bar{G}(\cdot, y)$ is only determined up to an additive constant. We now fix this constant (and hence uniquely define $\bar{G}(\cdot, y)$) by assuming that

$$\int_{B_1(y)} \bar{G}(x, y)\,\mathrm{d}x = 0.$$

In the sequel, we show that $\bar{G}$ satisfies all the properties of Theorem 2. By uniqueness of the Green function G up to an additive constant, we will obtain that $\bar{G} = G$ up to a constant. We will then deduce bounds on ∇G from the bounds we have on $\nabla \bar{G}$.

We first show that $\bar{G}$ satisfies (2.1). Consider $\varphi \in \mathcal{D}(\mathbb{R}^2)$ and $\psi \in \mathcal{D}(\mathbb{R})$. Considering the test function $\psi(t)\varphi(x)$ in (2.1), we see that the Green function $\tilde{G}$ satisfies the weak formulation

$$\int_{\mathbb{R}^3} \psi(t)(\nabla\varphi(x))^{\mathrm{T}} A(x)\nabla_x \tilde{G}(x,t,y,0)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^3} \varphi(x)\psi'(t)\partial_t \tilde{G}(x,t,y,0)\,\mathrm{d}x\,\mathrm{d}t = \varphi(y)\psi(0).$$

Consider ψ such that $\psi(t) = 1$ whenever $|t| \le \kappa$, $\psi(t) = 0$ whenever $|t| \ge 1+\kappa$, and $\max(\|\psi\|_{L^\infty(\mathbb{R})}, \|\psi'\|_{L^\infty(\mathbb{R})}) \le 1$. We have

$$\int_{\mathbb{R}^2} (\nabla\varphi(x))^{\mathrm{T}} A(x)\nabla_x G_\kappa(x,y)\,\mathrm{d}x + e_1(\kappa) + e_2(\kappa) = \varphi(y), \tag{3.10}$$

with

$$e_1(\kappa) = \int_{\mathbb{R}^2}\int_{\kappa\le|t|\le 1+\kappa} \psi(t)(\nabla\varphi(x))^{\mathrm{T}} A(x)\nabla_x \tilde{G}(x,t,y,0)\,\mathrm{d}x\,\mathrm{d}t,$$
$$e_2(\kappa) = \int_{\mathbb{R}^2}\int_{\kappa\le|t|\le 1+\kappa} \varphi(x)\,\psi'(t)\,\partial_t \tilde{G}(x,t,y,0)\,\mathrm{d}x\,\mathrm{d}t.$$

Let us now bound from above e_1 and e_2. Using (3.8), and introducing a compact $K \subset \mathbb{R}^2$ containing the support of φ, we have

$$\begin{aligned}|e_1(\kappa)| &\leq \|\psi\|_{L^\infty}\|A\|_{L^\infty}\|\nabla\varphi\|_{L^\infty}\int_K\int_{\kappa\leq|t|\leq 1+\kappa}|\nabla_x\tilde{G}(x,t,y,0)|\,\mathrm{d}x\,\mathrm{d}t\\ &\leq \|\psi\|_{L^\infty}\|A\|_{L^\infty}\|\nabla\varphi\|_{L^\infty}\int_K\int_{\kappa\leq|t|\leq 1+\kappa}\frac{C}{|x-y|^2+t^2}\,\mathrm{d}x\,\mathrm{d}t\\ &\leq \|\psi\|_{L^\infty}\|A\|_{L^\infty}\|\nabla\varphi\|_{L^\infty}\mu(K)\frac{C}{\kappa^2}.\end{aligned}$$

Hence, $e_1(\kappa)$ vanishes when $\kappa \to \infty$. Likewise, $e_2(\kappa)$ also vanishes when $\kappa \to \infty$. Passing to the limit $\kappa \to \infty$ in (3.10), and using that $\nabla_x G_\kappa(\cdot, y)$ weakly converges to $\nabla_x \bar{G}(\cdot, y)$, we deduce that, for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2}(\nabla\varphi(x))^{\mathrm{T}}A(x)\nabla_x\bar{G}(x,y)\,\mathrm{d}x = \varphi(y).$$

We have thus obtained that the function $\bar{G} \in L^\infty_{y,\mathrm{loc}}(\mathbb{R}^2, W^{1,1}_{x,\mathrm{loc}}(\mathbb{R}^2))$ satisfies (2.1). Assume now that $\bar{G}$ also satisfies (2.10). Then, according to the uniqueness of G (see Theorem 2), we have $\nabla_x G = \nabla_x \bar{G}$.

In turn, we deduce from (3.9) that

$$|\nabla_x G(x,y)| = |\nabla_x\bar{G}(x,y)| \leq \frac{C\pi}{|x-y|}. \tag{3.11}$$

This hence proves the estimate (3.2) in the case $d = 2$.

To prove (3.3) in the case $d = 2$, we again use the fact that $G(y, x)$ is the Green function of $L^\star$ defined by (3.6), so the estimate (3.2) that we have just shown implies (3.3).

There only remains to prove that $\bar{G}$ satisfies (2.10). To this end, we note that (3.11) implies the estimate (2.14), for Ω a ball or an annulus of the form $B_{2R} \setminus B_R$. Hence, the end of the proof of Theorem 2 applies here, leading from (2.14) to (2.26), which implies that $\bar{G}$ satisfies (2.10). ■

Remark 6. The above arguments indicate two different proofs for the existence of G in dimension two: the first one consists in defining the Green function on the bounded domain B_R, and then letting $R \to \infty$, as it is done in the proof of Theorem 2. The second strategy uses the three-dimensional Green function $\tilde{G}$ of the operator $\tilde{L}$ defined by (3.7). One integrates $\tilde{G}$ with respect to the third variable, finding a Green function for the operator L in dimension two. This approach is used in the proof of Proposition 5.

Note also that Proposition 5 is proved under stronger assumptions than Theorem 2. □

We next prove upper bounds on $\nabla_x \nabla_y G$.

Proposition 7. Assume that A satisfies (1.1)–(1.4). Then, the Green function G associated with $L = -\mathrm{div}(A\nabla\cdot)$ satisfies the following estimate:

$$\exists C > 0, \quad \forall x \in \mathbb{R}^d, \quad \forall y \in \mathbb{R}^d, \quad |\nabla_x \nabla_y G(x, y)| \le \frac{C}{|x - y|^d}. \tag{3.12}$$

□

Here again, similar results for the Green function in a *bounded* domain are given in the literature, for instance in [11, Theorem 3.3].

Proof. We have, in the sense of distribution,

$$-\mathrm{div}_x(A(x)\nabla_x \nabla_y G(x, y)) = 0 \quad \text{in } B_\delta(y)^C, \text{ for any } \delta > 0.$$

We can thus apply [1, Lemma 16] and obtain, as in (3.4), that

$$\forall x \in \mathbb{R}^d, \quad \forall y \in \mathbb{R}^d, \quad \forall r < |x - y|, \quad \|\nabla_x \nabla_y G(\cdot, y)\|_{L^\infty(B_{r/2}(x))} \le \frac{C}{r} \|\nabla_y G(\cdot, y)\|_{L^\infty(B_r(x))}.$$

Using (3.3), we deduce (3.12). ■

Using arguments similar to those used to prove Propositions 5 and 7, we also show the following result on the Green function G_R of the operator $-\mathrm{div}(A\nabla\cdot)$ on the bounded domain B_R with homogeneous Dirichlet boundary conditions. The interest of this result is the independence of the obtained bounds with respect to the size of the domain B_R.

Proposition 8. Assume that A satisfies (1.1)–(1.4). Let G_R be the Green function of the operator $-\mathrm{div}(A\nabla\cdot)$ on B_R with homogeneous Dirichlet boundary conditions (namely, G_R is the unique solution to (2.6) with the boundary condition $G_R(x, y) = 0$ if $|x| = R$).

Then, there exists a constant C such that, for any $R > 0$,

$$\forall (x, y) \in B_R \times B_R, \quad |\nabla_x G_R(x, y)| \le \frac{C}{|x - y|^{d-1}}, \tag{3.13}$$

$$\forall (x, y) \in B_R \times B_R, \quad |\nabla_y G_R(x, y)| \le \frac{C}{|x - y|^{d-1}}, \tag{3.14}$$

$$\forall (x, y) \in B_R \times B_R, \quad |\nabla_x \nabla_y G_R(x, y)| \le \frac{C}{|x - y|^{d}}. \tag{3.15}$$

□

4 Counter-Examples and Extensions

We collect in this section some remarks about possible extensions of the results stated above. We also discuss the periodicity and regularity assumptions (1.1) and (1.2) on the matrix A, under which we have shown asymptotic properties for $\nabla_x G$, $\nabla_y G$, and $\nabla_x \nabla_y G$.

First, it should be noted that, assuming further regularity on the coefficients of the matrix A, it is possible to prove more precise decaying properties of the Green function. This was proved in [14, 28]. Likewise, it should be possible to extend our results to the case of periodic, piecewise Hölder coefficients. For instance, gradient estimates for elliptic equations with such discontinuous coefficients are derived in [22]. It is probably possible to use them in the setting of the current article, although we have not pursued in that direction.

Next, it is clearly possible to adapt the technique of [5, 8] (see also [12, 13]) to treat the case of *systems* of elliptic PDEs. This case is also considered in [1, 14].

4.1 Nonsmooth matrix A

Another question is the extension of the present results to the case of coefficients which are not Hölder continuous (i.e., that do not satisfy (1.2)). Here, we first point out that our crucial estimate, namely (3.4), is no more valid. A counter-example may be found in [24, Section 5; 27, p. 400]. Following [24, Section 5], let $d=2$, and assume that A is given by

$$A(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1-\mu^2}{|x|^2} \begin{pmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{pmatrix}, \tag{4.1}$$

where $\mu \in (0,1)$ is fixed. For any x, the eigenvalues of $A(x)$ are equal to 1 and μ^2, hence A satisfies (1.3) and (1.4) for any $\mu > 0$. A simple computation shows that the function

$$u(x_1, x_2) = x_1 |x|^{\mu-1} \tag{4.2}$$

satisfies

$$-\operatorname{div}(A \nabla u) = 0. \tag{4.3}$$

However, u cannot satisfy (3.4) since $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ and $\nabla u \notin L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$.

Furthermore, it is possible to use this example to prove that the Green function G_A of the operator $L = -\operatorname{div}(A\nabla \cdot)$ on $\mathbb{R}^2$ does not satisfy (3.2) and (3.12). More precisely, assume that G_A satisfies (3.2) and (3.12) for any x and y with $r \le |x-y| \le R$ and $|y| < (R-r)/2$, for some $R > r > 0$. Then, fix $y \in B_{(R-r)/2}(0)$, multiply (4.3) by $G_A(x, y)$ and integrate over $B_{(r+R)/2}(0)$ with respect to x. After two integrations by parts, and using the symmetry of A, we obtain

$$u(y)=\int_{\partial B_{(r+R)/2}(0)} G_A(x,y)(A(x)\nabla_x u(x))\cdot n(x)\,\mathrm{d}x-\int_{\partial B_{(r+R)/2}(0)} u(x)(A(x)\nabla_x G_A(x,y))\cdot n(x)\,\mathrm{d}x.$$

Taking the gradient of this expression with respect to y, we have

$$\begin{aligned}\nabla u(y)=&\int_{\partial B_{(r+R)/2}(0)} \nabla_y G_A(x,y)(A(x)\nabla_x u(x))\cdot n(x)\,\mathrm{d}x\\ &-\int_{\partial B_{(r+R)/2}(0)} u(x)(A(x)\nabla_y\nabla_x G_A(x,y))\cdot n(x)\,\mathrm{d}x.\end{aligned}\tag{4.4}$$

Since $|x|=(r+R)/2$ and $y\in B_{(R-r)/2}(0)$, we have $r\le|x-y|\le R$. Using (4.2) and the assumptions on G_A to bound the right-hand side, we find

$$|\nabla u(y)|\le C\left(\frac{(r+R)^{\mu}}{r}+\frac{(r+R)^{\mu+1}}{r^2}\right)\quad \text{a.e. on } B_{(R-r)/2}(0).$$

We thus get $\nabla u\in L^{\infty}(B_{(R-r)/2}(0))$, which is in contradiction with (4.2). Thus, we have proved the following lemma.

Lemma 9. Let the matrix A be given by (4.1), and let G_A be the corresponding Green function defined by Theorem 2. Let $R>r>0$. Then, G_A cannot satisfy both (3.2) and (3.12) on the set

$$\mathcal{A}_{r,R}:=\{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2 \text{ such that } r\le|x-y|\le R \text{ and } |y|<(R-r)/2\}.$$ □

Remark 10. Since the matrix A is symmetric, (3.3) is equivalent to (3.2). Thus, we may replace the assumption (3.2) by (3.3) in Lemma 9. □

Remark 11. The above results are closely linked with Liouville-type theorems. Indeed, (4.1) and (4.2) give a counter-example to the property "Let A be a matrix satisfying (1.3) and (1.4), and let u satisfy $-\mathrm{div}(A\nabla u)=0$ on $\mathbb{R}^d$. If u is sublinear at infinity, that is, $|u(x)|/|x|\to 0$ as $|x|\to\infty$, then u is a constant." □

4.2 Smooth and nonperiodic matrix A

In the above counter-example, we have used the matrix A defined by (4.1), which is neither Hölder continuous nor periodic. We are now going to use it to construct a counter-example in which the matrix is smooth (Hölder continuous), but not periodic. For this purpose, we first define the matrix B as follows:

$$B(x) = \begin{cases} A(x) & \text{if } |x| \le 2, \\ I & \text{if } |x| \ge 4, \\ \eta(|x|)I + (1-\eta(|x|))A(x) & \text{if } 2 < |x| < 4, \end{cases} \tag{4.5}$$

where $A(x)$ is defined by (4.1), I is the identity matrix, and where η is a smooth function satisfying

$$\eta(r) = \begin{cases} 0 & \text{if } r \le 2, \\ 1 & \text{if } r \ge 4, \end{cases}$$

and $0 \le \eta \le 1$ everywhere. Hence, the matrix B is smooth away from the origin, and the matrix $B(x)$ has all its eigenvalues in the interval $[\mu^2, 1]$ for any x. Moreover, the function u defined by (4.2) satisfies $-\mathrm{div}(B\nabla u) = 0$ on $B_2(0)$. Hence, repeating the argument of the proof of Lemma 9, we see that, for any $0 < r < R$ with $r + R < 4$, the Green function G_B associated with B cannot satisfy both (3.2) and (3.12) on the set $\mathcal{A}_{r,R}$.

We are now going to use an inversion technique similar to the one used in [19], that is, we apply the Kelvin transform $x \mapsto \frac{x}{|x|^2}$ to the equation. Introduce the function

$$\tilde{G}(x, y) := G_B\left(\frac{x}{|x|^2}, \frac{y}{|y|^2}\right). \tag{4.6}$$

A simple computation shows that $\tilde{G}$ is the Green function associated with the matrix $\tilde{B}$ defined by

$$\tilde{B}(x) = R(x) B\left(\frac{x}{|x|^2}\right) R(x) \tag{4.7}$$

with

$$R(x) = \begin{pmatrix} 1 - 2\dfrac{x_1^2}{x_1^2 + x_2^2} & -2\dfrac{x_1 x_2}{x_1^2 + x_2^2} \\ -2\dfrac{x_1 x_2}{x_1^2 + x_2^2} & 1 - 2\dfrac{x_2^2}{x_1^2 + x_2^2} \end{pmatrix}.$$

The matrix $R(x)$ satisfies $R(x) = R(x)^{\mathrm{T}} = R(x)^{-1}$, so that the eigenvalues of $\tilde{B}(x)$ are in $[\mu^2, 1]$ for any x. Moreover, $x \mapsto \tilde{B}(x)$ is smooth. Indeed, when $|x| \le \frac{1}{4}$, we have $B(\frac{x}{|x|^2}) = I$, hence $\tilde{B}(x) = I$. When $|x| > \frac{1}{4}$, the functions $x \mapsto R(x)$ and $x \mapsto B(\frac{x}{|x|^2})$ are smooth, hence $x \mapsto \tilde{B}(x)$ is also smooth.

We know from the above argument that, for any $0 < r < R$ with $r + R < 4$, the function G_B cannot satisfy both (3.2) and (3.12) on the set $\mathcal{A}_{r,R}$. Noting that, for $r = 1$ and

$R = \frac{3}{2}$, we have

$$\mathcal{A}_{r,R} \subset \{(x, y) \in B_2(0) \times B_2(0)\},$$

we deduce from (4.6) that $\tilde{G}$ cannot satisfy both (3.2) and (3.12) on the set $\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2,\ |x| > \frac{1}{2},\ |y| > \frac{1}{2}\}$. Hence, we have proved the following lemma.

Lemma 12. Let $\tilde{B}$ be the matrix defined by (4.7), which satisfies the assumptions (1.2)–(1.4). The Green function $\tilde{G}$ of the operator $\tilde{L} = -\mathrm{div}(\tilde{B}\nabla\cdot)$ on $\mathbb{R}^2$ cannot satisfy both (3.2) and (3.12) on the set

$$\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2,\ |x| > \tfrac{1}{2},\ |y| > \tfrac{1}{2}\}.$$

□

Remark 13. As we have pointed out above, the contradiction is reached only for large values of x and y. Indeed, for $y \in B_{1/4}(0)$, we have, for any $\varphi \in \mathcal{D}(B_{1/4}(0))$,

$$\varphi(y) = \int_{\mathbb{R}^2} (\nabla\varphi(x))^{\mathrm{T}} \tilde{B}(x) \nabla_x \tilde{G}(x, y)\, \mathrm{d}x = \int_{\mathbb{R}^2} (\nabla\varphi(x))^{\mathrm{T}} \nabla_x \tilde{G}(x, y)\, \mathrm{d}x,$$

since $\tilde{B}(x) = I$ for $|x| < \frac{1}{4}$. Letting $G_\Delta(x, y) = -\frac{1}{2\pi} \log|x - y|$ be the Green function of the Laplacian on $\mathbb{R}^2$, we have

$$\varphi(y) = \int_{\mathbb{R}^2} (\nabla\varphi(x))^{\mathrm{T}} \nabla_x G_\Delta(x, y)\, \mathrm{d}x.$$

We thus obtain that

$$\Delta_x \left(\tilde{G}(x, y) + \frac{1}{2\pi} \log|x - y| \right) = 0 \quad \text{in } \mathcal{D}'(B_{1/4}(0)).$$

Hence, $\tilde{G}(x, y) + \frac{1}{2\pi} \log|x - y|$ is smooth, and (3.2) and (3.12) are satisfied when $|x| < \frac{1}{4}$ and $|y| < \frac{1}{4}$. □

In the counter-example of Lemma 12, the matrix $\tilde{B}$ is smooth but not periodic. Of course, this does not prove that periodicity is necessary for (3.2) and (3.12) to hold (in the sense that alternative assumptions may be sufficient). However, the construction of the counter-example indicates that periodicity may be seen as a regularity condition at infinity: the singularity of the matrix A defined by (4.1) has been sent to infinity by the Kelvin transform. This is why $\tilde{G}$ cannot satisfy the gradient bounds at infinity.

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Appendix. Proof of the Estimate (2.2)

In this Appendix, we prove (2.2), which has been useful here, and which was already used in [3].

The estimate $\|f\|_{L^{p,\infty}(\Omega)} \leq \|f\|_{L^p(\Omega)}$ is a direct consequence of the definition of $\|f\|_{L^{p,\infty}(\Omega)}$. We now turn to the proof of the other estimate. For any $N \in \mathbb{Z}$, we write

$$\begin{aligned}
\|f\|^{p-\beta}_{L^{p-\beta}(\Omega)} &= \int_\Omega |f(x)|^{p-\beta}\, \mathrm{d}x \\
&= \int_\Omega |f(x)|^{p-\beta} 1_{|f(x)|\leq 2^{-N}}\, \mathrm{d}x + \sum_{n\geq -N} \int_\Omega |f(x)|^{p-\beta} 1_{2^n \leq |f(x)| \leq 2^{n+1}}\, \mathrm{d}x \\
&\leq \mu(\Omega) 2^{-N(p-\beta)} + \sum_{n\geq -N} 2^{(n+1)(p-\beta)-np} 2^{np}\, \mu(\{x;\, |f(x)| \geq 2^n\}) \\
&\leq \mu(\Omega) 2^{-N(p-\beta)} + \sum_{n\geq -N} 2^{(n+1)(p-\beta)-np} \|f\|^p_{L^{p,\infty}(\Omega)} \\
&= \mu(\Omega) 2^{-N(p-\beta)} + \|f\|^p_{L^{p,\infty}(\Omega)} \frac{2^{p+\beta N}}{2^\beta - 1}.
\end{aligned} \tag{A.1}$$

We now pick N to balance the two terms of the above right-hand side. This leads to choosing N such that

$$\frac{1}{2}[\mu(\Omega)]^{1/p} \frac{(2^\beta - 1)^{1/p}}{\|f\|_{L^{p,\infty}(\Omega)}} \leq 2^N \leq [\mu(\Omega)]^{1/p} \frac{(2^\beta - 1)^{1/p}}{\|f\|_{L^{p,\infty}(\Omega)}}.$$

Inserting this in (A.1), we obtain

$$\|f\|^{p-\beta}_{L^{p-\beta}(\Omega)} \leq \frac{2^p}{(2^\beta - 1)^{(p-\beta)/p}} [\mu(\Omega)]^{\beta/p} \|f\|^{p-\beta}_{L^{p,\infty}(\Omega)} (1 + 2^{-\beta}),$$

from which we deduce (2.2).

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