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November 5, 2014

Abstract

A particle method with linear transformation of the particle shape functions is studied for the 1d-1v Vlasov-Poisson equation, and a priori error estimates are proven which show that the approximated densities converge in the uniform norm. When compared to standard fixed-shape particle methods, the present approach can be seen as a way to gain one order in the convergence rate of the particle trajectories at the cost of linearly transforming each particle shape. It also allows to compute strongly convergent densities with particles that overlap in a bounded way.

1 Introduction

The particle method is the basis of a variety of popular schemes to transport densities in computational plasma physics and fluid mechanics. It is conceptually simple and straightfoward to implement. However it suffers from weak convergence properties which in practice often result in "noisy", i.e., oscillating solutions. At the theoretical level this is expressed in classical a priori error estimates [4, 21] by the fact that for particle schemes to converge in the uniform norm, the particles radii must tend to zero at a slower rate than the average distance between their centers. In practice this constraint is very expensive as it would dramatically increase the number of overlapping particles, and for that reason it is usually not met.

To improve the accuracy of the method several variants have been developped over the past decades, such as particle remappings [17, 20, 11, 15], weight correction schemes [3, 12, 22] or wavelet filtering [8]. Successful results were also obtained by transforming the particles shape functions to better follow the local variations of the transport flow, see [19, 13, 10, 5]. In [10] for instance, Cohen and Perthame observed that by linearizing the flow around the particle trajectories one obtains a convergent method (in L^1) with particles scaled with their initialization grid and hence bounded particle overlapping. In [6] it was then shown that with linearly transformed particles, numerical densities transported along a given velocity field would also converge in the uniform norm with a bounded particle overlapping. In this article we extend this result to the case of a transport (Vlasov) equation coupled with a Poisson potential. The resulting a priori estimates can be seen as an improvement on those established in [14] when applied to fixed-shape particles with bounded overlapping.

The plan is as follows. In Section 2 we recall some known properties of the periodic solutions to the 1d-1v Vlasov-Poisson system, and in Section 3 we describe the main steps of the linearly-transformed particle scheme when coupled with an (exact) Poisson equation for the force field. Section 4 is then devoted to the statement and the proof of the a priori error estimates.

2 The 1d-1v Vlasov Poisson system

We consider periodic solutions of the 1d-1v Vlasov-Poisson system,

$$\{\partial_t + v\partial_x - E(t,x)\partial_v\}f(t,x,v) = 0, \qquad (t,x,v) \in [0,T]\times\mathbb{R}\times\mathbb{R} \tag{1}$$

$$\partial_x E(t,x) = 1 - \int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v, \qquad (t,x) \in [0,T]\times\mathbb{R} \tag{2}$$

supplemented with an initial condition $f(0,\cdot,\cdot) = f^0 \geq 0$ and a zero-mean field condition

$$\int_0^L E(t,x)\,\mathrm{d}x = 0, \qquad t \geq 0.$$

To establish a priori error estimates for the proposed particle method we shall consider that the initial data f^0

(i) satisfies a global neutrality relation

$$\int_0^L \left(\int_{\mathbb{R}} f^0(x,v)\,\mathrm{d}v - 1\right)\mathrm{d}x = 0,$$

(ii) is L-periodic with respect to x,

$$f^0(x+L,v) = f^0(x,v),$$

(iii) has bounded derivatives up to second order,

$$f^0 \in W^{2,\infty}(\mathbb{R}^2),$$

(iv) has a bounded support in the v-dimension,

$$\mathrm{supp}(f^0) \subset \mathbb{R}\times[-Q_0,Q_0] \quad \text{for some } Q_0 > 0. \tag{3}$$

Using standard arguments from [23] (see, e.g., [14, Theorem 1]) we easily verify that with the above conditions the system (1)-(2) has a unique solution (f,E). This solution is L-periodic with respect to x and has the same order of smoothness than f^0, namely

$$(f,E) \in W^{2,\infty}([0,T]\times\mathbb{R}^2)\times W^{2,\infty}([0,T]\times\mathbb{R}).$$

In particular, for all $(s,x,v) \in [0,T]\times\mathbb{R}^2$, there exists a unique characteristic trajectory $(X,V)(t)$ defined on $[0,T]$, solution to

$$\begin{cases} X(s) = x \\ V(s) = v \end{cases} \quad \text{and} \quad \begin{cases} X'(t) = V(t) \\ V'(t) = -E(t,X(t)) \end{cases} \quad \text{on} \quad [0,T]. \tag{4}$$

It is then straightforward to verify that f is transported along the associated characteristic flow $F_{s,t} : \mathbb{R}^2 \to \mathbb{R}^2,\ (x,v) \mapsto (X,V)(t)$, i.e.,

$$f(t,x,v) = f(s,F_{s,t}^{-1}(x,v)) =: T_{\mathrm{ex}}[F_{s,t}]f(s,\cdot)(x,v), \qquad x,v \in \mathbb{R}^2. \tag{5}$$

Note that here $F_{s,t}^{-1} = F_{t,s}$, moreover

$$F_{s,s} = I_{\mathbb{R}^2} \tag{6}$$

and by differentiating (4) with respect to x, v a Gronwall argument gives

$$|F_{s,t}|_{W^{1,\infty}(\mathbb{R}^2)} \leq \exp\big(CT(1+\|\partial_x E\|_{L^\infty([0,T]\times\mathbb{R})})\big) \tag{7}$$

with a constant C that depends on the specific definition of the $W^{1,\infty}$ semi-norm, see (13) below. The above properties also give an explicit bound on the support size,

$$\sup\{|v| : \exists\, x \in \mathbb{R},\ f(t,x,v) \neq 0\} \leq Q_T := Q_0 + T\|E\|_{L^\infty([0,T]\times\mathbb{R})}, \qquad t \in [0,T], \tag{8}$$

(with $\|E\|_{L^\infty([0,T]\times\mathbb{R})} \le L(1+Q_0\|f^0\|_{L^\infty(\mathbb{R}^2)})$) as well as on f,

$$0 \le f(t,x,v) \le \|f^0\|_{L^\infty(\mathbb{R}^2)}, \qquad (t,x,v) \in [0,T]\times\mathbb{R}^2. \tag{9}$$

Following [14] it will be convenient to write the electric field as

$$E(t,x) = \int_0^L K(x,y)\Big(1 - \int_{\mathbb{R}} f(t,y,v)\,\mathrm{d}v\Big)\,\mathrm{d}y \tag{10}$$

where K is defined on $[0,L[^2\backslash\{x=y\}$ by

$$K(x,y) = \begin{cases} \frac{y}{L} - 1 & \text{for} \quad 0 \le x < y \\ \frac{y}{L} & \text{for} \quad y < x < L \end{cases}$$

and is extended by periodicity almost everywhere in $\mathbb{R}^2$. In particular, K satisfies

$$\|K\|_{L^\infty(\mathbb{R}^2)} \le 1 \tag{11}$$

and

$$y \mapsto K(x,y) \quad \text{is Lipschitz with constant } \frac{1}{L} \text{ on } \mathbb{R}\setminus(x+L\mathbb{Z}), \quad \text{for all } x \in \mathbb{R}. \tag{12}$$

In the sequel we shall use the maximum norm $\|x\|_\infty := \max_i |x_i|$ for vectors and the associated $|||A|||_\infty := \max_i \sum_j |A_{i,j}|$ for matrices. For functions in Sobolev spaces $W^{m,\infty}(\omega)$ with $\omega \subset \mathbb{R}^d$ and integer index $m > 0$, we will use the semi-norm

$$|u|_{W^{m,\infty}(\omega)} := \max_i \Big\{ \sum_{l_1=1}^d \cdots \sum_{l_m=1}^d \|\partial_{l_1}\cdots\partial_{l_m} u_i\|_{L^\infty(\omega)} \Big\}. \tag{13}$$

We will say that a constant depends on the exact solution if it depends only on L, T and on the L^∞ norms of f, E and their derivatives over the respective domains $[0,T]\times\mathbb{R}^2$ and $[0,T]\times\mathbb{R}$. It will be convenient to use the letter C to denote such a constant (which value may vary at each occurence), and to write

$$a \lesssim b \tag{14}$$

as a short-hand to $a \le Cb$.

3 The linearly-transformed particle method

In this section we describe the main steps of the method. We observe that similar schemes have been developed by several authors to solve the Vlasov-Poisson system. In the Complex Particle Kinetic (CPK) scheme introduced by Bateson and Hewett [2, 18], particles have a Gaussian shape that is transformed by the local shearing of the flow. Moreover they can be fragmented to probe for emerging features, and merged where fine particles are no longer needed. In the Cloud in Mesh (CM) scheme of Alard and Colombi [1] particles also have Gaussian shapes, and they are deformed by local linearizations of the force field, in a manner similar to ours. Finally, the approach considered here is numerically tested in a Linearly Transformed Particle-In-Cell (LTPIC) scheme in [7].

3.1 General form of LTP solutions

As in standard particle methods, we represent the phase-space density f with weighted collections of finite-size particles $w_k\varphi^n_{h,k}(z)$, $k \in \mathbb{Z}^2$, that are pushed along their trajectories z^n_k on the discrete times $t_n = n\Delta t$, $n = 0, 1\ldots, N_t$ where $\Delta t = T/N_t$. However, in our method the particles also have their shape transformed to better represent the local shear and rotation flows in phase space, as illustrated in Figure 1. The particles can either be structured or unstructured. The first case corresponds to the initialization and remapping steps (if any, see Remark 3.1 below), where particles are defined as tensor-product B-splines and centered on regular nodes

$$z_k^0 = (x_k^0, v_k^0) := hk, \qquad k \in \mathbb{Z}^2. \tag{15}$$

Specifically, the univariate (i.e., one-dimensional) centered B-spline $\mathcal{B}_p$ is recursively defined as the piecewise polynomial of degree p satisfying

$$\mathcal{B}_0(x) := \begin{cases} 1, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathcal{B}_p(x) := \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \mathcal{B}_{p-1}(\tilde{x})\,\mathrm{d}\tilde{x}, \qquad \text{for} \quad p \ge 1.$$

Thus $\mathcal{B}_1(x) = \max\{1-|x|, 0\}$ is the traditional "hat-function", $\mathcal{B}_3$ is the well-known cubic B-spline supported on $[-2, 2]$, and so on, see e.g. Ref. [16]. The fundamental shape function is then defined on the $2d$ phase space as a tensor product

$$\varphi(z) := \mathcal{B}_p(x)\mathcal{B}_p(v) \quad \text{with support} \quad \mathrm{supp}(\varphi) =]-c_p, c_p[^2, \quad c_p := \tfrac{p+1}{2}, \tag{16}$$

from which we derive a normalized, grid-scaled shape function $\varphi_h(z) := h^{-2}\varphi(h^{-1}z)$. Structured particles are then defined as translated versions of the latter,

$$\varphi_{h,k}^0(z) := \varphi_h(z - z_k^0) = h^{-2}\varphi(h^{-1}z - k) \tag{17}$$

with square supports (here $B_\infty(z, \rho)$ denotes the open $\ell^\infty(\mathbb{R}^2)$ ball of center z and radius ρ).

$$B_{h,k}^0 := \mathrm{supp}(\varphi_{h,k}^0) = B_\infty(z_k^0, hc_p), \qquad k \in \mathbb{Z}^2. \tag{18}$$

When transported by our method, particles become unstructured in the sense that their centers z_k^n leave the nodes of the structured phase-space grid and their shapes are linearly transformed. That is, the different parts of the "cloud" associated with a single particle move with their own particular velocities and the cloud distorts, but the distortion is constrained to be linear. Generic particles are then characterized by the 2×2 deformation matrices D_k^n (initialized with $D_k^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$) which determine the linear transformation of their shape, and numerical solutions take the form

$$f_h^n(z) = \sum_{k \in \mathbb{Z}^2} w_k \varphi_{h,k}^n(z) \quad \text{with} \quad \varphi_{h,k}^n(z) := \varphi_h(D_k^n(z - z_k^n)).$$

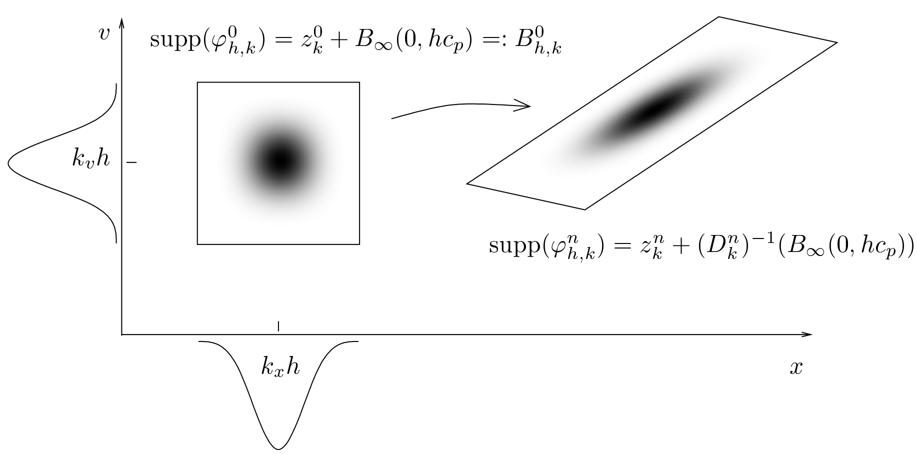

Figure 1: Structured particles (left) are defined at initialization and remapping steps, as tensor-product B-splines centered on regular nodes $z_k^0 = hk$, see (17). Unstructured particles (right) are obtained by pushing the particle centers along their trajectories z_k^n and transforming their shapes with a matrix $(D_k^n)^{-1}$ representing the local Jacobian of the characteristic flow.

3.2 Particle initialization with local B-spline quasi-interpolation

Since our particles are scaled with their initialization grid, their weights can be computed with standard approximation schemes that rely on the fact that the space $\mathcal{P}_p$ of polynomials with coordinate degree less or equal to p is spanned by the structured particles (17) derived by shifting φ_h on the grid, see, e.g., [16]. Specifically, we may consider quasi-interpolation schemes described by [9] and [24], where high-order B-spline approximations are locally obtained by pointwise evaluations of the target function. In the univariate case they take the form (writing $\varphi^0_{h,k}(x) = h^{-1}\mathcal{B}_p(h^{-1}x - k)$)

$$A_h^{(1d)} : g(x) \mapsto \sum_{k\in\mathbb{Z}} w_k(g)\varphi^0_{h,k}(x) \quad \text{with normalized weights} \quad w_k(g) := h \sum_{|l|\le m_p} a_l\, g(x^0_{k+l})$$

and symmetric coefficients $a_l = a_{-l}$ defined in such a way that $A_h^{(1d)}$ reproducts the space $\mathcal{P}_p^{(1d)}$. They can be computed with the iterative algorithm in [9, Section 6]: for the first odd degrees (which may be preferred for symmetry reasons) we obtain

- $m_p = 0$ and $a_0 = 1$ for $p = 1$,
- $m_p = 1$ and $(a_0, a_1) = (\frac{8}{6}, -\frac{1}{6})$ for $p = 3$,
- $m_p = 4$ and $(a_0, a_1, a_2, a_3, a_4) = (\frac{503}{288}, -\frac{1469}{3600}, \frac{7}{225}, \frac{13}{3600}, \frac{1}{14400})$ for $p = 5$.

In the bivariate case we can tensorize the above, as it is easily checked that the operator

$$A_h : g(z) \mapsto \sum_{k\in\mathbb{Z}^2} w_k(g)\varphi^0_{h,k}(z) \quad \text{with} \quad w_k(g) := h^2 \sum_{\|l\|_\infty \le m_p} a_l\, g(z^0_{k+l}), \quad a_l := a_{l_1}a_{l_2} \tag{19}$$

reproducts any polynomial $\pi \in \mathcal{P}_p$. Moreover, we have

$$\|A_h g\|_{L^\infty(\mathbb{R}^2)} \le (2c_p)^2 \|\varphi_h\|_{L^\infty(\mathbb{R}^2)} \sup_{k\in\mathbb{Z}^2} |w_k(g)| \le (2c_p)^2 \|\varphi\|_{L^\infty(\mathbb{R}^2)} \|a\|_{\ell^1} \|g\|_{L^\infty(\mathbb{R}^2)} \tag{20}$$

with $\|a\|_{\ell^1} = \sum_{\|l\|_\infty \le m_p} |a_l|$, by using the fact that no more than $(2c_p)^2$ B-splines overlap. It follows that A_h is uniformly bounded in L^∞, with

$$\|A_h\|_{\mathcal{L}(L^\infty(\mathbb{R}^2))} := \sup_{g\neq 0} \frac{\|A_h g\|_{L^\infty(\mathbb{R}^2)}}{\|g\|_{L^\infty(\mathbb{R}^2)}} \le (2c_p)^d \|\varphi\|_{L^\infty(\mathbb{R}^2)} \|a\|_{\ell^1}.$$

Using a localized version of (20), we write that for an arbitrary $\pi \in \mathcal{P}_q$, $q \le p$,

$$\begin{aligned} \|A_h g - g\|_{L^\infty(B_\infty(z^0_k,h))} &\le \|A_h(g-\pi)\|_{L^\infty(B_\infty(z^0_k,h))} + \|g-\pi\|_{L^\infty(B_\infty(z^0_k,h))} \\ &\le (\|A_h\|_{\mathcal{L}(L^\infty(\mathbb{R}^2))} + 1)\|g-\pi\|_{L^\infty(B_\infty(z^0_k,h(m_p+c_p)))}. \end{aligned}$$

Taking for π the q-th Taylor expansion of g around z^0_k, we thus find for all $q \le p$,

$$\|A_h g - g\|_{L^\infty(\mathbb{R}^2)} \le h^{q+1} c_A |g|_{W^{q+1}(\mathbb{R}^2)} \quad \text{with} \quad c_A = (\|A_h\|_{\mathcal{L}(L^\infty(\mathbb{R}^2))} + 1)\frac{(m_p+c_p)^{q+1}}{(q+1)!}. \tag{21}$$

3.3 Particle transport with linear transformations

Our scheme is based on a linearly-transformed particle (LTP) transport operator $T_h[F]$ that transforms the particles through local linearizations of a given characteristic flow F that is assumed measure preserving. Applied to a generic particle $\varphi^n_{h,k}$ with deformation matrix D^n_k, the LTP transport operator reads

$$T_h[F] : \ \varphi^n_{h,k} := \varphi_h(D^n_k(\cdot - z^n_k)) \ \mapsto \ \varphi^{n+1}_{h,k} := \varphi_h(D^{n+1}_k(\cdot - z^{n+1}_k)) \quad \text{with} \quad \begin{cases} z^{n+1}_k := F(z^n_k) \\ D^{n+1}_k := D^n_k(J^n_k)^{-1} \end{cases}$$

where J_k^n is an approximation of the flow Jacobian matrix at z_k^n. Note that this amounts to applying to the particle $\varphi_{h,k}^n$ the *exact* transport operator associated to the linearized flow at z_k^n,

$$F_{h,z_k^n} : z \mapsto F(z_k^n) + J_k^n(z - z_k^n)$$

indeed we have $\varphi_{h,k}^{n+1} = \varphi_{h,k}^n \circ (F_{h,z_k^n})^{-1}$. In other terms, the LTP transport operator (when applied to a particle with associated node z_k^n) can be equivalently defined as

$$T_h[F]\varphi_{h,k}^n = T_{\rm ex}[F_{h,z_k^n}]\varphi_{h,k}^n, \tag{22}$$

see (5). In [6] a generic scheme was used for the approximated Jacobian matrix, that we shall slightly modify in the next section. It consisted of computing first a finite difference approximation of the Jacobian matrix,

$$(\tilde{J}_k^n)_{i,j} := (2h)^{-1}\big(F_i(z_k^n + he_j) - F_i(z_k^n - he_j)\big) \approx \partial_j F_i(z_k^n) \quad \text{where} \quad e_j = (\delta_{i,j})_{1\le i\le 2},$$

and of defining next J_k^n by a normalization step

$$J_k^n := \det(\tilde{J}_k^n)^{-\frac{1}{2}}\tilde{J}_k^n, \tag{23}$$

so as to obtain a measure preserving flow ($\det(J_{F_{h,z_k^n}}(z)) = \det(J_k^n) = 1$), hence a conservative transport operator

$$\int \varphi_{h,k}^{n+1}(z)\,\mathrm{d}z = \int \varphi_{h,k}^n\big((F_{h,z_k^n})^{-1}(z)\big)\,\mathrm{d}z = \int \varphi_{h,k}^n(z)\,\mathrm{d}z.$$

3.4 The numerical scheme

In order to have periodic numerical solutions we consider the following grid spacing for the particles,

$$h = L/N_x \tag{24}$$

for some positive integer N_x. We begin with a particle approximation of the initial data, i.e.,

$$f_h^0 = \sum_{k\in\mathbb{Z}^2} w_k\varphi_{h,k}^0 = \sum_{k\in\mathbb{Z}^2} w_k\varphi_h(\cdot - z_k^0) := A_h f^0 \quad \text{where} \quad z_k^0 = (x_k^0, v_k^0) = hk, \tag{25}$$

see (15), (17), (19). Since f^0 is assumed to have a bounded support in the v dimension (3), so is the case for its numerical approximation. Specifically, using (19) and $h \le L$ we see that the non-vanishing weights w_k have their indices in the set

$$\mathbb{Z}_w^2 := \{k = (k_x, k_v) \in \mathbb{Z}^2 : |k_v h| \le Q_0 + m_p L\}. \tag{26}$$

Thus, from the localized supports (18) of the initial particles we infer that

$$\sup\{|v| : \exists\, x \in \mathbb{R},\ f_h^0(x,v) \neq 0\} \le Q_0^* = Q_0 + (m_p + c_p)L. \tag{27}$$

For $n = 0, \dots N_t - 1$, we then advance the solutions with a leap-frog scheme involving two LTP transports per time step, as follows.

1. An intermediate solution
$$f_h^{n,1} = \sum_{k\in\mathbb{Z}^2} w_k\varphi_{h,k}^{n,1} := T_h[F_{\frac{1}{2}}]f_h^n \tag{28}$$
is first obtained by transporting the particles along the flow
$$F_{\frac{1}{2}} : (x,v) \mapsto \big(x + \tfrac{\Delta t}{2}v, v\big).$$
Since this flow is linear, $T_h[F_{\frac{1}{2}}]$ coincides with the exact transport $T_{\rm ex}[F_{\frac{1}{2}}]$. In particular we have $\varphi_{h,k}^{n,1} = \varphi_{h,k}^n \circ (F_{\frac{1}{2}})^{-1}$ for all $k \in \mathbb{Z}^2$. Note that (28) also involves a forward transport of the particle centers, namely
$$z_k^{n,1} := F_{\frac{1}{2}}(z_k^n) = \big(x_k^n + \tfrac{\Delta t}{2}v_k^n, v_k^n\big). \tag{29}$$

Mimicking (10) and using the explicit bound (8) for the exact solution support, the numerical electric field is then defined as

$$E_h^{n,1}(x) := \int_0^L K(x,y)\Big(1 - \int_{-Q_T}^{Q_T} f_h^{n,1}(y,v)\,\mathrm{d}v\Big)\,\mathrm{d}y. \tag{30}$$

2. The time step is then completed by transporting the particles along the flow

$$F_h^{n,1} : (x,v) \mapsto \big(x + \tfrac{\Delta t}{2}\tilde{v}, \tilde{v} := v - \Delta t E_h^{n,1}(x)\big), \tag{31}$$

i.e., by computing

$$f_h^{n+1} = \sum_{k\in\mathbb{Z}^2} w_k \varphi_{h,k}^{n+1} := T_h[F_h^{n,1}]f_h^{n,1}. \tag{32}$$

Consistent with the definition (22), this amounts to transporting the particles with

$$\varphi_{h,k}^{n+1} := T_h[F_h^{n,1}]\varphi_{h,k}^{n,1} = T_{\mathrm{ex}}[F_{h,k}^{n,1}]\varphi_{h,k}^{n,1} = \varphi_{h,k}^{n,1} \circ (F_{h,k}^{n,1})^{-1}, \qquad k \in \mathbb{Z}^2,$$

where the *local linearization* of the flow around the particle center $z_k^{n,1}$ is given by

$$F_{h,k}^{n,1} : z \mapsto F_h^{n,1}(z_k^{n,1}) + J_k^{n,1}(z - z_k^{n,1}) \tag{33}$$

and where the numerical Jacobian matrix is defined as follows. Denoting the finite difference operator by $\Delta_h u(x) = (2h)^{-1}\big(u(x+h) - u(x-h)\big)$ and motivated by the fact that the exact electric field satisfies an a priori estimate

$$1 - 2Q_T\|f^0\|_{L^\infty} \le \partial_x E(t,x) \le 1 \qquad (t,x) \in [0,T]\times[0,L]$$

(easily derived from the Poisson equation (2) and the bounds (8), (9)), we introduce a truncated finite difference operator

$$\Delta_h^* u(x) := \min\big\{\max\{1 - 2Q_T\|f^0\|_{L^\infty}, \Delta_h u(x)\}, 1\big\}. \tag{34}$$

The numerical Jacobian matrix is then the truncated finite differentiation of (31), i.e.,

$$J_k^{n,1} := \begin{pmatrix} 1 - \frac{\Delta t^2}{2}\Delta_h^* E_h^{n,1}(x_k^{n,1}) & \frac{\Delta t}{2} \\ -\Delta t \Delta_h^* E_h^{n,1}(x_k^{n,1}) & 1 \end{pmatrix}. \tag{35}$$

Note that we have $\det(J_k^{n,1}) = 1$, hence the normalization step (23) is not needed here. Moreover, we observe that (32) also involves a forward transport of the particle centers, namely

$$z_k^{n+1} := F_h^{n,1}(z_k^{n,1}) = \big(x_k^{n,1} + \tfrac{\Delta t}{2}v_k^{n+1}, v_k^{n+1} := v_k^{n,1} - \Delta t E_h^{n,1}(x_k^{n,1})\big)$$

which, given (29), gives

$$z_k^{n+1} = \big(x_k^n + \tfrac{\Delta t}{2}(v_k^n + v_k^{n+1}), v_k^{n+1} := v_k^n - \Delta t E_h^{n,1}(x_k^n + \tfrac{\Delta t}{2}v_k^n)\big).$$

Remark 3.1. *In practice the accuracy of the method is greatly improved by remapping the particles when their shapes deform too much. The scheme reads then (with an ad-hoc remapping criterion)*

$$f_h^{n+1} := T_h[F_h^{n,1}]T_h[F_{\frac{1}{2}}]\tilde{f}_h^n \quad \textit{with} \quad \tilde{f}_h^n := \begin{cases} A_h f_h^n & \textit{if } n \textit{ is a remapping step,} \\ f_h^n & \textit{otherwise.} \end{cases}$$

Remark 3.2. *Transporting the particles with the electric field* (30) *is of course a simplification, similar to the one considered in [14]. In practice one does rather use a piecewise polynomial approximation of the field, obtained by solving numerically the Poisson equation* (2).

Proposition 3.3. *For $n = 0, \dots, N_t$, the numerical densities f_h^n and $f_h^{n,1}$ are L-periodic with respect to x.*

Proof. Using (25) and (19), we easily derive from (24) that

$$w_k = w_{k-k_L} \quad \text{for} \quad k \in \mathbb{Z}^2, \qquad \text{where} \quad k_L := (N_x, 0).$$

Turning next to the particles we claim that

$$z_k^n = z_{k-k_L}^n + (L, 0) \qquad \text{and} \qquad \varphi_{h,k}^n(x + L, v) = \varphi_{h,k-k_L}^n(x, v) \tag{36}$$

hold for all k and n, as well as the analog property on the intermediate centers $z_k^{n,1}$ and particles $\varphi_{h,k}^{n,1}$. Note that one readily infers the desired result from the second half of claim, indeed

$$f_h^n(x + L, v) = \sum_{k \in \mathbb{Z}^2} w_k \varphi_{h,k}^n(x + L, v) = \sum_{k \in \mathbb{Z}^2} w_{k-k_L} \varphi_{h,k-k_L}^n(x, v) = f_h^n(x, v).$$

For $n = 0$ the property (36) is easily checked from the definition $z_k^0 = hk$ and the tensor-product structure (16)-(17) of the initial particles, so let us assume that it holds for some $n \geq 0$. Since the flow $F_{\frac{1}{2}}$ satisfies

$$F_{\frac{1}{2}}(x + L, v) = \big(x + L + \tfrac{\Delta t}{2} v, v\big) = F_{\frac{1}{2}}(x, v) + (L, 0), \tag{37}$$

we have

$$z_k^{n,1} = F_{\frac{1}{2}}(z_k^n) = F_{\frac{1}{2}}(z_{k-k_L}^n + (L, 0)) = F_{\frac{1}{2}}(z_{k-k_L}^n) + (L, 0) = z_{k-k_L}^{n,1} + (L, 0). \tag{38}$$

Moreover, we also have $(F_{\frac{1}{2}})^{-1}(x+L, v) = (F_{\frac{1}{2}})^{-1}(x, v)+(L, 0)$, hence using the transport structure $\varphi_{h,k}^{n,1} = \varphi_{h,k}^n \circ (F_{\frac{1}{2}})^{-1}$ we derive that

$$\begin{aligned} \varphi_{h,k}^{n,1}(x + L, v) &= \varphi_{h,k}^n\big((F_{\frac{1}{2}})^{-1}(x + L, v)\big) \\ &= \varphi_{h,k}^n\big((F_{\frac{1}{2}})^{-1}(x, v) + (L, 0)\big) \\ &= \varphi_{h,k-k_L}^n\big((F_{\frac{1}{2}})^{-1}(x, v)\big) \\ &= \varphi_{h,k-k_L}^{n,1}(x, v). \end{aligned} \tag{39}$$

To verify that (36) also holds for the centers and particles on the time step $(n + 1)$, we use the L-periodicity of the field $E_h^{n,1}$ (infered from that of the kernel K) to observe that

$$F_h^{n,1}(x + L, v) = F_h^{n,1}(x, v) + (L, 0).$$

Firstly, this shows that the centers z_k^{n+1} do satisfy the first half of the claim. Secondly, together with (38) it allows to compute that the linearized flows (33) satisfy

$$\begin{aligned} F_{h,k}^{n,1}(x + L, v) &= F_h^{n,1}(z_k^{n,1}) + J_k^{n,1}((x + L, v) - z_k^{n,1}) \\ &= F_h^{n,1}(z_{k-k_L}^{n,1} + (L, 0)) + J_k^{n,1}((x, v) - z_{k-k_L}^{n,1}) \\ &= F_h^{n,1}(z_{k-k_L}^{n,1}) + (L, 0) + J_{k-k_L}^{n,1}((x, v) - z_{k-k_L}^{n,1}) \\ &= F_{h,k-k_L}^{n,1}(x, v) + (L, 0) \end{aligned} \tag{40}$$

where we have used again (38) together with the periodicity of $E_h^{n,1}$ to identify $J_k^{n,1} = J_{k-k_L}^{n,1}$ in the third equality. Finally, computing as in (39) shows that the second half of our claim also holds for the intermediate particles $\varphi_{h,k}^{n,1}$, and the result follows by induction on n. □

4 A priori error estimates

In this section we establish several a priori estimates for the errors resulting from the above scheme. Our analysis goes as follows. We first establish an a priori bound for the numerical electric field $E_h^{n,1}$ and for the *flow error* e_F^n corresponding to the approximation of the exact flow F_{0,t_n} by the local linearizations involved in the particle transformations, see (43) and (44) below. We then control the error on the electric field by this flow error. The main task is to bound this error flow according to h and Δt; to do so we distinguish the error resulting from the approximation of Jacobian matrix $J_{F_{\rm ex}^{n,1}}(z_k^{n,1})$ and the one from the splitting scheme. Finally, we relate the error done on the phase space density with that on the flow.

This eventually allows us to prove that, under a mild condition $\Delta t \lesssim \sqrt{h}$, the particle centers and the (intermediate) approximate electric field satisfy the a priori estimate

$$\sup_{k\in\mathbb{Z}^2} \|z_k^n - F_{0,t_n}(z_k^0)\| + \|E_h^{n,1} - E(t_{n+\frac{1}{2}})\|_{L^\infty(\mathbb{R})} \le C(T)(h^2 + \Delta t^2), \qquad n \le T/\Delta t \tag{41}$$

whereas the particle density satisfies

$$\|f_h^n - f_{\rm ex}^n\|_{L^\infty(\mathbb{R}^2)} \le C(T)(h + h^{-1}\Delta t^2), \qquad n \le T/\Delta t \tag{42}$$

for constants independent of h, see Corollary 4.7 and Theorem 4.2 below.

We may compare the above estimates with those established for classical (fixed-shape) particles in [14]. In this work the authors consider a time-continuous method similar to ours, where smooth particles of fixed radius ε follow trajectories $Z_k^h = (X_k^h, V_k^h)$, $k \in \mathbb{Z}^2$, accelerated by the exact electrostatic field E_h they create, i.e.,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X_k^h(t) = V_k^h(t), \qquad \dfrac{\mathrm{d}}{\mathrm{d}t} V_k^h(t) = -E^h(t, X_k^h(t)) \\ (X_k^h, V_k^h)(0) = z_k^0 = hk \\ E^h(t,x) = \displaystyle\int_0^L K(x,y)\rho_\varepsilon^h(t,y)\,\mathrm{d}y. \end{cases}$$

Here,

$$\rho_\varepsilon^h(t,x) := 1 - \sum_{k\in\mathbb{Z}^2} w_k \zeta_\varepsilon(x - X_k^h(t))$$

corresponds to the space charge density carried by smooth particles with shape function $\varphi_\varepsilon(z) = \zeta_\varepsilon(x)\zeta_\varepsilon(y)$ where $\zeta_\varepsilon := \frac{1}{\varepsilon}\zeta(\frac{\cdot}{\varepsilon})$. Thus, in the case considered here where the particle radius ε is proportional to the initial grid size h, the error estimate stated in [14, Th. 3] reads

$$\|E_h(t) - E(t)\|_{L^\infty(\mathbb{R})} + \sup_{k\in\mathbb{Z}^2} \|Z_k^h(t) - F_{0,t}(z_k^0)\| \le C(T)h, \qquad t \le T.$$

In particular we see that by transporting the smooth particles along local linearizations of the flow, we gain one order of convergence for the particle trajectories and the electric field, and in addition the resulting phase space density converges in L^∞.

Remark 4.1. *In fact the error estimate given in [14, Th. 3] covers arbitrary orders of convergence, but for particle radii ε that tend to 0 at a much slower rate than h. This results in an extended particle overlapping and yields expensive computations for the particle interactions. In the present study the particle overlapping is bounded by only considering shape functions of radius $\varepsilon \sim h$.*

As explained just above, a central piece of our error analysis consists of estimating the error between the exact (non-linear) flow and its local linearizations over $[0, t_n]$, namely

$$\bar{F}_{\rm ex}^n := F_{0,t_n} \qquad \text{and} \qquad \bar{F}_{h,k}^n := F_{h,k}^{n-1,1} F_{\frac{1}{2}} \bar{F}_{h,k}^{n-1}, \qquad k \in \mathbb{Z}^2 \tag{43}$$

where $\bar{F}_{h,k}^0 := I_{\mathbb{R}^2}$ consistent with (6) (here and in what follows we will use a bar to distinguish time-integrated flows). More precisely, what we are interested in is to estimate this error where the numerical flows are actually used in the scheme, which is on the initial particle supports $B_{h,k}^0$, see (18). Restricting ourselves to the particles with non vanishing weights, (26), we thus let

$$e_F^n := \sup_{k\in\mathbb{Z}_w^2} \|\bar{F}_{h,k}^n - \bar{F}_{\rm ex}^n\|_{L^\infty(B_{h,k}^0)}. \tag{44}$$

In order to give a first a priori bound for e_F^n (in Corollary 4.3 below) it will be convenient to reduce this supremum to a finite subset of $\mathbb{Z}_w^2$. Indeed, from the periodicity (37), (40) of the numerical flows we derive that

$$\bar{F}_{h,k}^n(x + L, v) = \bar{F}_{h,k-k_L}^n(x,v) + (L, 0),$$

hence

$$\|\bar{F}^n_{h,k+k_L} - I\|_{L^\infty(B^0_{h,k+k_L})} = \|\bar{F}^n_{h,k} - I\|_{L^\infty(B^0_{h,k})}$$

holds with $k_L = (N_x, 0)$, under the assumption (24). Moreover, as is easily checked using (4) and the periodicity of E the exact flow satisfies $\bar{F}^n_{\rm ex}(x+L, v) = \bar{F}^n_{\rm ex}(x, v) + (L, 0)$, therefore $\|\bar{F}^n_{\rm ex} - I\|_{L^\infty(B^0_{h,k+mk_L})}$ does not depend on $m \in \mathbb{Z}$. In particular, we have

$$e^n_F = \sup_{k \in \mathbb{Z}^2_{w,L}} \|\bar{F}^n_{h,k} - \bar{F}^n_{\rm ex}\|_{L^\infty(B^0_{h,k})}$$

with (compare with (26))

$$\mathbb{Z}^2_{w,L} = \{k = (k_x, k_v) \in \mathbb{Z}^2 : 0 \le k_x h < L \text{ and } |k_v h| \le Q_0 + m_p L\}. \tag{45}$$

4.1 Preliminary estimates

To study the numerical errors it will be convenient to define reference entities parallel to the ones involved in the splitted numerical scheme (28)-(31). Thus, for all n we let

$$f^{n,1}_{\rm ex}(x, v) := T_{\rm ex}[F_{\frac{1}{2}}] f^n_{\rm ex}(x, v) = f^n_{\rm ex}({F_{\frac{1}{2}}}^{-1}(x, v)) = f(t_n, x - \tfrac{\Delta t}{2} v, v) \tag{46}$$

where $f^n_{\rm ex} = f(t_n, \cdot, \cdot)$ is the exact solution of (1). Accordingly we define an auxiliary reference field

$$E^{n,1}_{\rm ex}(x) := \int_0^L K(x, y)\Big(1 - \int_{\mathbb{R}} f^{n,1}_{\rm ex}(y, v)\,\mathrm{d}v\Big)\,\mathrm{d}y \tag{47}$$

and the associated reference flow

$$F^{n,1}_{\rm ex} : (x, v) \mapsto \big(x + \tfrac{\Delta t}{2}\tilde{v}, \tilde{v} := v - \Delta t E^{n,1}_{\rm ex}(x)\big). \tag{48}$$

From $(E^{n,1}_{\rm ex})'(x) = 1 - \int_{\mathbb{R}} f^{n,1}_{\rm ex}(x, v)\,\mathrm{d}v = 1 - \int_{\mathbb{R}} f(t_n, x - \frac{\Delta t}{2} v, v)\,\mathrm{d}v$, we easily derive

$$\|E^{n,1}_{\rm ex}\|_{W^{1,\infty}(\mathbb{R})} \le 1 + 2Q_T\|f^0\|_{L^\infty(\mathbb{R}^2)} \quad \text{and} \quad \|E^{n,1}_{\rm ex}\|_{W^{2,\infty}(\mathbb{R})} \le 2Q_T\|\partial_x f(t_n)\|_{L^\infty(\mathbb{R}^2)}. \tag{49}$$

Moreover, the Jacobian matrix of the reference flow (48) reads

$$J_{F^{n,1}_{\rm ex}}(z) = \begin{pmatrix} 1 - \frac{\Delta t^2}{2}(E^{n,1}_{\rm ex})'(x) & \frac{\Delta t}{2} \\ -\Delta t (E^{n,1}_{\rm ex})'(x) & 1 \end{pmatrix}. \tag{50}$$

Therefore

$$|F^{n,1}_{\rm ex}|_{W^{1,\infty}(\mathbb{R}^2)} \le 1 + C\Delta t(1 + \|E^{n,1}_{\rm ex}\|_{W^{1,\infty}(\mathbb{R})}) \le 1 + C\Delta t \tag{51}$$

and

$$|F^{n,1}_{\rm ex}|_{W^{2,\infty}(\mathbb{R}^2)} \le C\Delta t \|E^{n,1}_{\rm ex}\|_{W^{2,\infty}(\mathbb{R})} \le C\Delta t. \tag{52}$$

Lemma 4.2. *The approximate electric field satisfies*

$$\|E^{n,1}_h\|_{L^\infty(\mathbb{R})} \le L(1 + C\|f^0\|_{L^\infty(\mathbb{R}^2)}) \tag{53}$$

for $n = 0, \dots, N_t$, with a constant C that only depends on p and will be specified in the proof.

Proof. According to Proposition 3.3 we know that both f^0_h and $f^{n,1}_h$ are L-periodic in x, and the same arguments show that this is also the case for the functions $\hat{f}^0_h(z) := \sum_{k \in \mathbb{Z}^2} |w_k \varphi^0_{h,k}(z)|$ and $\hat{f}^{n,1}_h(z) := \sum_{k \in \mathbb{Z}^2} |w_k \varphi^{n,1}_{h,k}(z)|$. In particular, the average integrals on $\Omega_m := [-mL, mL] \times \mathbb{R}$,

$$M^0_h(m) := \frac{1}{2m} \iint_{\Omega_m} \hat{f}^0_h(z)\,\mathrm{d}z \quad \text{and} \quad M^{n,1}_h(m) := \frac{1}{2m} \iint_{\Omega_m} \hat{f}^{n,1}_h(z)\,\mathrm{d}z$$

do not depend on m. We claim that they actually coincide, i.e., $M_h^0(1) = M_h^{n,1}(1)$. Or equivalently,

$$\iint_{[0,L]\times\mathbb{R}} \hat{f}_h^0(z)\,\mathrm{d}z = \iint_{[0,L]\times\mathbb{R}} \hat{f}_h^{n,1}(z)\,\mathrm{d}z \tag{54}$$

(the same result holds for f_h^0 and $f_h^{n,1}$ as well, but we shall not need it). To show our claim we first derive from (37) and (40) that

$$F_{\frac{1}{2}}\bar{F}_{h,k}^n(x+L,v) = F_{\frac{1}{2}}\bar{F}_{h,k-k_L}^n(x,v) + (L,0) \tag{55}$$

holds for all n and k, and it follows that

$$\|F_{\frac{1}{2}}\bar{F}_{h,k+k_L}^n - I\|_{L^\infty(B_{h,k+k_L}^0)} = \|F_{\frac{1}{2}}\bar{F}_{h,k}^n - I\|_{L^\infty(B_{h,k}^0)}, \quad k \in \mathbb{Z}^2.$$

This implies that for a given value of h, the supremum

$$\eta_h^n := \sup_{k\in\mathbb{Z}_w^2} \|F_{\frac{1}{2}}\bar{F}_{h,k}^n - I\|_{L^\infty(B_{h,k}^0)}$$

involves a finite number of indices and hence it is finite. We thus observe that any $z = (x,v) \in B_{h,k}^0 \cap \Omega_m$ satisfies

$$-mL - \eta_h^n \le x - \eta_h^n \le F_{\frac{1}{2}}\bar{F}_{h,k}^n(z)_x \le x + \eta_h^n \le -mL - \eta_h^n,$$

i.e.,

$$F_{\frac{1}{2}}\bar{F}_{h,k}^n(B_{h,k}^0 \cap \Omega_m) \subset \Omega_{m'} \quad \text{with} \quad m' = m + \lceil \eta_h^n L^{-1} \rceil.$$

This allows to compute

$$\begin{aligned}
M_h^0(m) &= \frac{1}{2m}\sum_{k\in\mathbb{Z}^2} \iint_{B_{h,k}^0\cap\Omega_m} |w_k\varphi_{h,k}^0(z^0)|\,\mathrm{d}z^0 \\
&= \frac{1}{2m}\sum_{k\in\mathbb{Z}^2} \iint_{\bar{F}_{h,k}^{n,1}(B_{h,k}^0\cap\Omega_m)} |w_k\varphi_{h,k}^{n,1}(z)|\,\mathrm{d}z \\
&\le \frac{1}{2m}\iint_{\Omega_{m'}} \sum_{k\in\mathbb{Z}^2} |w_k\varphi_{h,k}^{n,1}(z)|\,\mathrm{d}z \\
&\le \frac{m'}{m} M_h^{n,1}(m),
\end{aligned}$$

from which we infer that $\lim_m M_h^0(m) \le \lim_m M_h^{n,1}(m)$. Or, since these quantities do not depend on m, that $M_h^0(1) \le M_h^{n,1}(1)$. Finally we observe that (55) readily holds for $(\bar{F}_{h,k}^{n,1})^{-1}$, and by symmetry this gives $M_h^{n,1}(1) \le M_h^0(1)$ which proves our claim. Using the L^∞ bound (11) on K, our claim (54) and the bounded support (27) we compute then

$$|E_h^{n,1}(x)| \le \int_0^L \Big(1 + \int_{\mathbb{R}} \hat{f}_h^{n,1}(y,v)\,\mathrm{d}v\Big)\,\mathrm{d}y \le L + \iint_{[0,L]\times\mathbb{R}} \hat{f}_h^0(z)\,\mathrm{d}z \le L(1 + 2Q_0^*\|\hat{f}_h^0\|_{L^\infty(\mathbb{R}^2)})$$

Now, using the same arguments as in (20) we find

$$\|\hat{f}_h^0\|_{L^\infty(\mathbb{R}^2)} \le (2c_p)^2\|\varphi\|_{L^\infty(\mathbb{R}^2)}\|a\|_{\ell^1}\|f^0\|_{L^\infty(\mathbb{R}^2)}, \tag{56}$$

which completes the proof with $C = 2Q_0^*(2c_p)^2\|\varphi\|_{L^\infty(\mathbb{R}^2)}\|a\|_{\ell^1}$. □

Corollary 4.3. *The numerical flow error satisfies an a priori bound*

$$e_F^n \le C \tag{57}$$

with a constant C that depends only on the exact solution.

Proof. Using the L^∞ estimate (53) on the electric field, one finds that the approximate flow (31) satisfies

$$\|F_h^{n,1}(z)\|_\infty \le (1+\tfrac{\Delta t}{2})\|z\|_\infty + C\Delta t$$

with a constant C that only depends on p and on the exact solution. Since one clearly has

$$\|F_{\frac12}(z)\|_\infty \le (1+\tfrac{\Delta t}{2})\|z\|_\infty, \tag{58}$$

a discrete Gronwall lemma gives

$$\|F_h^{n-1,1}F_{\frac12}\cdots F_h^{0,1}F_{\frac12}(z)\|_\infty \le C(\|z\|_\infty+1)$$

for $n \le N_t$, with another such constant C. In particular, applying the above estimate to $z = z_k^0$ and considering only the particle centers with indices in the finite set (45) gives (up to changing the constant)

$$\|z_k^n\|_\infty \le C \tag{59}$$

for all $n \le N_t$ and $k \in \mathbb{Z}^2_{w,L}$. Turning to the linearized flow (43), we next compute

$$\begin{aligned}
\|\bar F^n_{h,k}(z) - z_k^n\|_\infty &= \|F^{n-1,1}_{h,k}F_{\frac12}\bar F^{n-1}_{h,k}(z) - z^n_k\|_\infty\\
&= \|J_k^{n-1,1}(F_{\frac12}\bar F^{n-1}_{h,k}(z) - z_k^{n-1,1})\|_\infty\\
&= \|J_k^{n-1,1}F_{\frac12}(\bar F^{n-1}_{h,k}(z) - z_k^{n-1})\|_\infty\\
&\le |||J_k^{n-1,1}|||_\infty(1+\tfrac{\Delta t}{2})|||\bar F^{n-1}_{h,k}(z) - z_k^{n-1}|||_\infty\\
&\le (1+C\Delta t)(1+\tfrac{\Delta t}{2})|||\bar F^{n-1}_{h,k}(z) - z_k^{n-1}|||_\infty.
\end{aligned}$$

Here we have used (33) and $z_k^n = F_h^{n-1,1}(z_k^{n-1,1})$ in the second equality, $z_k^{n-1,1} = F_{\frac12}z_k^{n-1}$ in the third one, (58) in the first inequality and the definition of the truncated finite difference operator (34) on the last one. For $n \le N_t$ this readily gives $\|\bar F^n_{h,k}(z) - z^n_k\|_\infty \le C\|z - z^0_k\|_\infty$, and with (59) we find that

$$\|\bar F^n_{h,k}\|_{L^\infty(B^0_{h,k})} \le C + \|z_k^n\|_\infty \le C$$

holds again with constants that only depend on p and on the exact solution. Clearly this is also the case for the exact flow, hence the a priori bound (57). ☐

4.2 Field error estimate

An important step is to control the error on the electric field by the flow error.

Lemma 4.4. *We have*

$$\|E^{n,1}_{\mathrm{ex}} - E^{n,1}_h\|_{L^\infty(\mathbb{R})} \le C(e^n_F + h^2)$$

with a constant that only depends on the spline degree $p \ge 1$ and on the exact solution.

Proof. For conciseness we write $F = F_{\frac12}\bar F^n_{\mathrm{ex}}$ and $\tilde F_k = F_{\frac12}\bar F^n_{h,k}$ in this proof. The main argument relies on the transport structure of the auxiliary particle solution, namely

$$f_h^{n,1}(z) = \sum_{k\in\mathbb{Z}^2} w_k\varphi^{n,1}_{h,k}(z) = \sum_{k\in\mathbb{Z}^2} w_k\varphi^0_{h,k}(\tilde F_k^{-1}(z)), \qquad z\in\mathbb{R}^2 \tag{60}$$

and on that of the reference one (readily derived from (5), (46)),

$$f^{n,1}_{\mathrm{ex}}(z) = f^0(F^{-1}(z)), \qquad z \in \mathbb{R}^2. \tag{61}$$

Given $x\in\mathbb{R}$ and $\varepsilon > 0$, it will be convenient to define $\Omega_x :=]x, x+L[\times]-Q_T, Q_T[$ and

$$\Omega_x^\varepsilon := \Omega_x \setminus (\partial\Omega_x + B_\infty(0,\varepsilon)) = [x+\varepsilon, x+L-\varepsilon]\times[-Q_T+\varepsilon, Q_T-\varepsilon]. \tag{62}$$

For later use we take $\varepsilon := (1+\Delta t)e^n_F$, so that

$$\|F - \tilde F_k\|_{L^\infty(B^0_{h,k})} \le (1+\tfrac{\Delta t}{2})e^n_F < \varepsilon.$$

In particular, if some $z^0 \in B^0_{h,k}$ is such that its image $\tilde F_k(z^0)$ by the linearized flow is in a given domain $\omega \subset \mathbb{R}^2$, then its image $F(z^0)$ by the reference flow is in $\omega + B_\infty(0,\varepsilon)$. In other terms, we have

$$B^0_{h,k} \cap (\tilde F_k)^{-1}(\omega) \subset F^{-1}(\omega + B_\infty(0,\varepsilon)). \tag{63}$$

With the above tools we handle the electric field as follows:

$$\begin{aligned}
\int_x^{x+L} K(x,\tilde y)\,\mathrm{d}\tilde y - E_h^{n,1}(x) &= \iint_{\Omega_x} K(x,\tilde y) f_h^{n,1}(\tilde y,\tilde v)\,\mathrm{d}\tilde v\,\mathrm{d}\tilde y \\
&= \iint_{\Omega_x} K(x,\tilde y) \sum_{k\in\mathbb{Z}^2} w_k \varphi^0_{h,k}(\tilde F_k^{-1}(\tilde y,\tilde v))\,\mathrm{d}\tilde v\,\mathrm{d}\tilde y \\
&= \sum_{k\in\mathbb{Z}^2} w_k \iint\limits_{\tilde F_k^{-1}(\Omega_x)} K(x,\tilde F_k(z^0)_x)\varphi^0_{h,k}(z^0)\,\mathrm{d}z^0 \\
&= \mathbf{R_1} + \sum_{k\in\mathbb{Z}^2} w_k \iint\limits_{\tilde F_k^{-1}(\Omega_x^\varepsilon)} K(x,\tilde F_k(z^0)_x)\varphi^0_{h,k}(z^0)\,\mathrm{d}z^0 \\
&= \mathbf{R_1} + \mathbf{R_2} + \sum_{k\in\mathbb{Z}^2} w_k \iint\limits_{\tilde F_k^{-1}(\Omega_x^\varepsilon)} K(x,F(z^0)_x)\varphi^0_{h,k}(z^0)\,\mathrm{d}z^0 \\
&= \mathbf{R_1} + \mathbf{R_2} + \mathbf{R_3} + \iint\limits_{F^{-1}(\Omega_x)} K(x,F(z^0)_x)\sum_{k\in\mathbb{Z}^2} w_k\varphi^0_{h,k}(z^0)\,\mathrm{d}z^0 \\
&= \mathbf{R_1} + \mathbf{R_2} + \mathbf{R_3} + \mathbf{R_4} + \iint\limits_{F^{-1}(\Omega_x)} K(x,F(z^0)_x) f^0(z^0)\,\mathrm{d}z^0 \\
&= \mathbf{R_1} + \mathbf{R_2} + \mathbf{R_3} + \mathbf{R_4} + \iint_{\Omega_x} K(x,y) f_{\mathrm{ex}}^{n,1}(y,v)\,\mathrm{d}v\,\mathrm{d}y \\
&= \mathbf{R_1} + \mathbf{R_2} + \mathbf{R_3} + \mathbf{R_4} + \int_0^L K(x,y)\,\mathrm{d}y - E_{\mathrm{ex}}^{n,1}(x).
\end{aligned}$$

Here we have used the periodicity of K and $f_h^{n,1}$ (see Prop. 3.3) in the first equality, the transport structure (60) in the second one, the measure preserving change of variable $(\tilde y,\tilde v) = \tilde F_k(z^0)$ in the third one, the change of variable $(y,v) = F(z^0)$ (also measure preserving) together with the transport structure (61) in the eighth one, and finally the periodicity of $f_{\mathrm{ex}}^{n,1}$ in the last one. The remainders are estimated as follows. From the definition (62) we see that $\Omega_x^\varepsilon \subset \Omega_x$ and also $(\Omega_x \setminus \Omega_x^\varepsilon) \subset \partial\Omega_x + B_\infty(0,\varepsilon)$. Using the embedding (63) (with $\omega = \partial\Omega_x + B_\infty(0,\varepsilon)$), this gives

$$\begin{aligned}
|\mathbf{R_1}| &= \Big| \sum_{k\in\mathbb{Z}^2} \iint\limits_{\tilde F_k^{-1}(\Omega_x\setminus\Omega_x^\varepsilon)} K(x,\tilde F_k(z^0)_x) w_k \varphi^0_{h,k}(z^0)\,\mathrm{d}z^0 \Big| \\
&\le \sum_{k\in\mathbb{Z}^2} \iint\limits_{\tilde F_k^{-1}(\partial\Omega_x + B_\infty(0,\varepsilon))} |w_k\varphi^0_{h,k}(z^0)|\,\mathrm{d}z^0 \le \iint\limits_{F^{-1}(\partial\Omega_x + B_\infty(0,2\varepsilon))} \hat f_h^0(z^0)\,\mathrm{d}z^0 \le C\varepsilon \|\hat f_h^0\|_{L^\infty(\mathbb{R}^2)}.
\end{aligned}$$

where we have denoted $\hat f_h^0 = \sum_{k\in\mathbb{Z}^2} |w_k\varphi^0_{h,k}|$ as in the proof of Lemma 4.2. Here we have used the L^∞ bound (11) on K, the fact that $\mathrm{supp}(\varphi^0_{h,k}) = B^0_{h,k}$ and the estimate $|\partial\Omega_x + B_\infty(0,2\varepsilon)| \le C(\varepsilon+\varepsilon^2) \le C\varepsilon$, derived from the stability bound (57). To estimate $\mathbf{R_2}$ we next observe that taking $\omega = \Omega_x^\varepsilon$ in (63) yields

$$B^0_{h,k} \cap (\tilde F_k)^{-1}(\Omega_x^\varepsilon) \subset F^{-1}(\Omega_x). \tag{64}$$

Thus for all $z^0 \in B^0_{h,k} \cap (\tilde F_k)^{-1}(\Omega_x^\varepsilon)$, we find that both $F(z^0)_x$ and $\tilde F_k(z^0)_x$ are in the interval $]x, x+L[$ where $K(x,\cdot)$ is $\frac{1}{L}$-Lipschitz, see (12). Hence,

$$|K(x,\tilde F_k(z^0)_x) - K(x,F(z^0)_x)| \le \tfrac{1}{L}\|\tilde F_k(z^0) - F(z^0)\|_\infty \le \frac{\varepsilon}{L} \quad \text{for } z^0 \in B^0_{h,k} \cap \tilde F_k^{-1}(S_x(\varepsilon)),$$

and it follows that

$$|\mathbf{R_2}| = \Big| \sum_{k\in\mathbb{Z}^2} w_k \iint\limits_{\tilde F_k^{-1}(\Omega_x^\varepsilon)} [K(x,\tilde F_k(z^0)_x) - K(x,F(z^0)_x)]\varphi^0_{h,k}(z^0)\,\mathrm{d}z^0\Big|$$
$$\le \frac{\varepsilon}{L} \iint\limits_{F^{-1}(\Omega_x)} \sum_{k\in\mathbb{Z}^2} |w_k\varphi^0_{h,k}(z^0)|\,\mathrm{d}z^0 \le \frac{\varepsilon}{L}|F^{-1}(\Omega_x)|\|\hat f^0_h\|_{L^\infty(\mathbb{R}^2)} \le 2Q_T\varepsilon\|\hat f^0_h\|_{L^\infty(\mathbb{R}^2)},$$

where we have used $|F^{-1}(\Omega_x)| = |\Omega_x| = 2LQ_T$. Turning to $\mathbf{R_3}$, we use again (64) to write that

$$\mathbf{R_3} = -\sum_{k\in\mathbb{Z}^2} w_k \iint\limits_{B^0_{h,k}\cap(F^{-1}(\Omega_x)\setminus\tilde F_k^{-1}(\Omega_x^\varepsilon))} K(x,F(z^0)_x)\varphi^0_{h,k}(z^0)\,\mathrm{d}z^0, \tag{65}$$

and we observe that the embedding (63) with $\omega = (\Omega_x^\varepsilon)^c := \mathbb{R}^2\setminus\Omega_x^\varepsilon$ now gives

$$B^0_{h,k}\cap\tilde F_k^{-1}((\Omega_x^\varepsilon)^c) \subset F^{-1}((\Omega_x^\varepsilon)^c + B_\infty(0,\varepsilon)) \subset F^{-1}\big(((\Omega_x)^c + B_\infty(0,\varepsilon))\cup(\partial\Omega_x + B_\infty(0,2\varepsilon))\big).$$

Hence, writing $F^{-1}(\Omega_x)\setminus\tilde F_k^{-1}(\Omega_x^\varepsilon) = F^{-1}(\Omega_x)\cap\tilde F_k^{-1}((\Omega_x^\varepsilon)^c)$ we obtain

$$B^0_{h,k}\cap(F^{-1}(\Omega_x)\setminus\tilde F_k^{-1}(\Omega_x^\varepsilon)) \subset F^{-1}\big(\Omega_x\cap((\Omega_x^\varepsilon)^c + B_\infty(0,\varepsilon))\big) \subset F^{-1}(\partial\Omega_x + B_\infty(0,2\varepsilon)).$$

As a result we proceed with our estimate (65) as follows,

$$|\mathbf{R_3}| \le \iint\limits_{F^{-1}(\partial\Omega_x + B_\infty(0,2\varepsilon))} \sum_{k\in\mathbb{Z}^2} |w_k\varphi^0_{h,k}(z^0)|\,\mathrm{d}z^0 \le C\varepsilon\|\hat f^0_h\|_{L^\infty(\mathbb{R}^2)}$$

where again C is a constant depending on p and on the exact solution. The remaining term $\mathbf{R_4}$ is finally bounded using the initial approximation estimate (21),

$$|\mathbf{R_4}| = \Big| \iint\limits_{F^{-1}(\Omega_x)} K(x,F(z^0)_x)(f^0_h(z^0) - f^0(z^0))\,\mathrm{d}z^0\Big| \le |F^{-1}(\Omega_x)|\|f^0_h - f^0\|_{L^\infty(\mathbb{R}^2)} \le Ch^2$$

with a constant that depends on p and on the exact solution. Gathering the above estimates with the L^∞ bound (56) on $\hat f^0_h$ and observing that $\varepsilon \le Ce^n_F$ yields the announced result. □

Lemma 4.5. *We have*

$$\|J^{n,1}_k - J_{F^{n,1}_{\mathrm{ex}}}(z^{n,1}_k)\| \le C\Delta t\min\{1, h^{-1}e^n_F + h\}$$

with a constant that only depends on p and on the exact solution.

Proof. By definition of the numerical (35) and reference (50) Jacobian matrices, i.e.,

$$J^{n,1}_k = \begin{pmatrix} 1 - \frac{\Delta t^2}{2}\Delta^*_h E^{n,1}_h(x^{n,1}_k) & \frac{\Delta t}{2} \\ -\Delta t\Delta^*_h E^{n,1}_h(x^{n,1}_k) & 1\end{pmatrix}, \qquad J_{F^{n,1}_{\mathrm{ex}}}(z) = \begin{pmatrix} 1 - \frac{\Delta t^2}{2}(E^{n,1}_{\mathrm{ex}})'(x) & \frac{\Delta t}{2} \\ -\Delta t(E^{n,1}_{\mathrm{ex}})'(x) & 1\end{pmatrix},$$

the Jacobian error consists of an error on the derivative of the field,

$$\|J^{n,1}_k - J_{F^{n,1}_{\mathrm{ex}}}(z^{n,1}_k)\| \lesssim \Delta t|\Delta^*_h E^{n,1}_h(x^{n,1}_k) - (E^{n,1}_{\mathrm{ex}})'(x^{n,1}_k)|.$$

Here the use of a truncated finite difference (34) gives the first a priori estimate (see (49)),

$$\|\Delta^*_h E^{n,1}_h - (E^{n,1}_{\mathrm{ex}})'\|_{L^\infty} \le (2 + 2Q_T\|f^0\|_{L^\infty}) \lesssim 1.$$

Moreover, we see from (47) and (8)-(9) that $(E^{n,1}_{\mathrm{ex}})'$ is always in the interval $[1 - Q_T\|f^0\|_{L^\infty}, 1]$. Hence by using a truncated finite difference Δ^*_h we can only reduce the error, in the sense that $\|\Delta^*_h E^{n,1}_h - (E^{n,1}_{\mathrm{ex}})'\|_{L^\infty} \le \|\Delta_h E^{n,1}_h - (E^{n,1}_{\mathrm{ex}})'\|_{L^\infty}$. Using the mean value theorem we then obtain

$$\begin{aligned}\|\Delta_h E^{n,1}_h - (E^{n,1}_{\mathrm{ex}})'\|_{L^\infty} &\le \|\Delta_h(E^{n,1}_h - E^{n,1}_{\mathrm{ex}})\|_{L^\infty} + \|\Delta_h E^{n,1}_{\mathrm{ex}} - (E^{n,1}_{\mathrm{ex}})'\|_{L^\infty} \\ &\le h^{-1}\|E^{n,1}_h - E^{n,1}_{\mathrm{ex}}\|_{L^\infty} + h\|(E^{n,1}_{\mathrm{ex}})''\|_{L^\infty} \\ &\lesssim h^{-1}e^n_F + h\end{aligned}$$

where the last inequality follows from Lemma 4.4 and (49). This completes the proof. □

Lemma 4.6. *The splitting scheme satisfies*

$$\|F_{t_n,t_{n+1}} - F^{n,1}_{\mathrm{ex}} F_{\frac{1}{2}}\|_{L^\infty(\bar{F}^n_{\mathrm{ex}}(B^0_{h,k}))} \lesssim \Delta t^3$$

for every $k \in \mathbb{Z}^2_w$, with a constant that only depends on p and on the exact solution.

Proof. For (X_n, V_n) fixed in $\bar{F}^n_{\mathrm{ex}}(B^0_{h,k})$, we denote

$$(X,V)(t) = F_{t_n,t}(X_n, V_n), \qquad t \in [t_n, t_{n+1}]$$

(that is, $(X,V)'(t) = \big(V(t), -E(t, X(t))\big)$ and $(X,V)(t_n) = (X_n, V_n)$), and define

$$(X_{n+1}, V_{n+1}) := F^{n,1}_{\mathrm{ex}} F_{\frac{1}{2}}(X_n, V_n) = \big(X_n + \Delta t V_n - \tfrac{\Delta t^2}{2} E^{n,1}_{\mathrm{ex}}(X_n + \tfrac{\Delta t}{2} V_n), V_n - \Delta t E^{n,1}_{\mathrm{ex}}(X_n + \tfrac{\Delta t}{2} V_n)\big).$$

Thus, we need to prove that $|X_{n+1} - X(t_{n+1})|$, $|V_{n+1} - V(t_{n+1})| \lesssim \Delta t^3$. Letting $E_X(t) := E(t, X(t))$ be the exact field along the characteristic curve, we derive from the definition of (X,V) that

$$\begin{cases} \|E_X\|_{L^\infty(t_n,t_{n+1})} \le \|E\|_{L^\infty} \\ \|E'_X\|_{L^\infty(t_n,t_{n+1})} \le \|\partial_t E\|_{L^\infty} + \|V\|_{L^\infty}\|\partial_x E\|_{L^\infty} \\ \|E''_X\|_{L^\infty(t_n,t_{n+1})} \le \|\partial^2_{tt} E\|_{L^\infty} + 2\|V\|_{L^\infty}\|\partial^2_{tx} E\|_{L^\infty} + \|V^2\|_{L^\infty}\|\partial^2_{xx} E\|_{L^\infty} + \|E\|_{L^\infty}\|\partial_x E\|_{L^\infty}. \end{cases}$$

Here the L^∞ norms of V are taken over $[t_n, t_{n+1}]$ and those of E and its derivatives over the whole $([0,T] \times \mathbb{R})$. Now, since $(X_n, V_n) = \bar{F}^n_{\mathrm{ex}}(X_0, V_0) = F_{0,t_n}(X_0, V_0)$ for some $(X_0, V_0) \in B^0_{h,k}$ we have $(X,V)(t) = F_{0,t}(X_0, V_0)$ and hence

$$|V(t)| \le |V_0| + \int_0^t |E_X(s)|\,\mathrm{d}s \le Q^*_0 + T\|E\|_{L^\infty([0,T]\times\mathbb{R})}$$

for $t \le T$, see (26), (27). It follows that

$$\|E_X\|_{W^{2,\infty}(t_n,t_{n+1})} \le C \tag{66}$$

holds with a constant depending only on p and on the exact solution. We next decompose

$$\begin{cases} X_{n+1} - X(t_{n+1}) = \mathbf{E_1} + \frac{\Delta t^2}{2}(\mathbf{E_2} + \mathbf{E_3}) \\ V_{n+1} - V(t_{n+1}) = \mathbf{E_4} + \Delta t(\mathbf{E_2} + \mathbf{E_3}) \end{cases}$$

with error terms defined as follows (writing $t_{n+\frac{1}{2}} := (n + \frac{1}{2})\Delta t$),

$$\begin{cases} \mathbf{E_1} = X_n + \Delta t V_n - \frac{\Delta t^2}{2} E_X(t_{n+\frac{1}{2}}) - X(t_{n+1}) \\ \mathbf{E_2} = E_X(t_{n+\frac{1}{2}}) - E(t_{n+\frac{1}{2}}, X_n + \frac{\Delta t}{2} V_n) \\ \mathbf{E_3} = E(t_{n+\frac{1}{2}}, X_n + \frac{\Delta t}{2} V_n) - E^{n,1}_{\mathrm{ex}}(X_n + \frac{\Delta t}{2} V_n) \\ \mathbf{E_4} = V(t_n) - \Delta t E_X(t_{n+\frac{1}{2}}) - V(t_{n+1}). \end{cases}$$

It thus remains to show that $|\mathbf{E_1}|$, $|\mathbf{E_4}| \lesssim \Delta t^3$ and $|\mathbf{E_2}|$, $|\mathbf{E_3}| \lesssim \Delta t^2$. For the first term we have

$$\mathbf{E_1} = \int_{t_n}^{t_{n+1}} (V(t_n) - V(t))\,\mathrm{d}t - \tfrac{\Delta t^2}{2} E_X(t_{n+\frac{1}{2}}) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (E_X(s) - E_X(t_{n+\frac{1}{2}}))\,\mathrm{d}s\,\mathrm{d}t,$$

hence $|\mathbf{E_1}| \le C\Delta t^3$ readily follows from (66). For the second term, we compute

$$\begin{aligned} |\mathbf{E_2}| &= \big|E(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) - E(t_{n+\frac{1}{2}}, X_n + \tfrac{\Delta t}{2} V_n)\big| \\ &\le \|\partial_x E\|_{L^\infty([0,T]\times\mathbb{R})} |X(t_{n+\frac{1}{2}}) - X(t_n) - \tfrac{\Delta t}{2} V(t_n)| \\ &\le C \int_{t_n}^{t_{n+\frac{1}{2}}} |V(t) - V(t_n)|\,\mathrm{d}t \\ &\le C \int_{t_n}^{t_{n+\frac{1}{2}}} \int_{t_n}^{t} |E_X(s)|\,\mathrm{d}s\,\mathrm{d}t \lesssim \Delta t^2 \end{aligned}$$

where the last inequality follows from (66). Turning to $\mathbf{E_3}$, we next use the integral formulations (10) and (47) of $E(t_{n+\frac{1}{2}},\cdot)$ and $E^{n,1}_{\rm ex}$ and the L^∞ bound (11) on K to write

$$|\mathbf{E_3}| = \Big|\int_0^L K(X_n + \tfrac{\Delta t}{2}V_n, y)\int_{\mathbb{R}}\big[f(t_{n+\frac{1}{2}}, y, v) - f(t_n, y - \tfrac{\Delta t}{2}v, v)\big]\,\mathrm{d}v\,\mathrm{d}y\Big| \le \int_0^L \Big|\int_{\mathbb{R}} \Phi(y,v)\,\mathrm{d}v\Big|\,\mathrm{d}y$$

with $\Phi(y,v) := f(t_{n+\frac{1}{2}}, y, v) - f(t_n, y - \frac{\Delta t}{2}v, v)$. Denoting $\tilde{t}_s = t_n + s$ and $\tilde{y}_s(v) = y + (s - \frac{\Delta t}{2})v)$, we then derive from the Vlasov equation that

$$\Phi(y,v) = \int_0^{\frac{\Delta t}{2}} \frac{\mathrm{d}}{\mathrm{d}s}\{f(\tilde{t}_s, \tilde{y}_s(v), v)\}\,\mathrm{d}s = \int_0^{\frac{\Delta t}{2}} (\partial_t f + v\partial_x f)(\tilde{t}_s, \tilde{y}_s(v), v)\,\mathrm{d}s = \int_0^{\frac{\Delta t}{2}} \Theta(s,y,v)\,\mathrm{d}s$$

with $\Theta(s,y,v) := E(\tilde{t}_s, \tilde{y}_s(v))\partial_v f(\tilde{t}_s, \tilde{y}_s(v), v)$. Now, instead of writing a straightforward bound $|\mathbf{E_3}| \lesssim \Delta t$ (which is not enough for our purposes), we integrate by parts using $\tilde{y}_s'(v) = s$ (and the bounded support of f in the v dimension),

$$\begin{aligned}
\int_{\mathbb{R}} s\partial_x E(\tilde{t}_s, \tilde{y}_s(v)) f(\tilde{t}_s, \tilde{y}_s(v), v)\,\mathrm{d}v &= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}v}\{E(\tilde{t}_s, \tilde{y}_s(v))\} f(\tilde{t}_s, \tilde{y}_s(v), v)\,\mathrm{d}v \\
&= -\int_{\mathbb{R}} E(\tilde{t}_s, \tilde{y}_s(v))\frac{\mathrm{d}}{\mathrm{d}v}\{f(\tilde{t}_s, \tilde{y}_s(v), v)\}\,\mathrm{d}v \\
&= -\int_{\mathbb{R}} E(\tilde{t}_s, \tilde{y}_s(v))\big[(s\partial_x f + \partial_v f)(\tilde{t}_s, \tilde{y}_s(v), v)\big]\,\mathrm{d}v.
\end{aligned}$$

This gives $\int_{\mathbb{R}} \Theta(s,y,v)\,\mathrm{d}v = -s\int_{\mathbb{R}} \partial_x(Ef)(\tilde{t}_s, \tilde{y}_s(v), v)\,\mathrm{d}v$, hence

$$|\mathbf{E_3}| \le \int_0^L \Big|\int_{\mathbb{R}} \Phi(y,v)\,\mathrm{d}v\Big|\,\mathrm{d}y = \int_0^L \Big|\int_0^{\frac{\Delta t}{2}}\int_{\mathbb{R}} \Theta(s,y,v)\,\mathrm{d}v\,\mathrm{d}s\Big|\,\mathrm{d}y \le C\Delta t^2 Q_T \|\partial_x(Ef)\|_{L^\infty}$$

which yields $|\mathbf{E_3}| \lesssim (\Delta t)^2$ with a constant that only depends on the exact solution. Finally, the last term $\mathbf{E_4}$ is a midpoint formula ($E_X = -V'$), hence it is bounded by

$$|\mathbf{E_4}| \le \frac{\Delta t^3}{24}\|V^{(3)}\|_{L^\infty} = \frac{\Delta t^3}{24}\|E_X''\|_{L^\infty} \lesssim \Delta t^3$$

according to (66). This completes the proof. □

4.3 Main estimates

We are now in position to state and prove the error estimates announced in (41) and (42).

Theorem 4.1. *Provided*

$$\Delta t \lesssim \sqrt{h}, \tag{67}$$

the numerical flow error $e^n_F := \sup_{k\in\mathbb{Z}^2_w}\|\bar{F}^n_{h,k} - \bar{F}^n_{\rm ex}\|_{L^\infty(B^0_{h,k})}$ *satisfies*

$$e^n_F \lesssim h^2 + \Delta t^2$$

for $n \le T/\Delta t$*, with a constant that only depends on the spline degree* $p \ge 1$ *and on the exact solution, see* (14).

Corollary 4.7. *The particle centers approximate the exact trajectories* $\bar{F}^n_{\rm ex}(z^0_k) := F_{0,t_n}(z^0_k)$ *according to the estimate*

$$\sup_{k\in\mathbb{Z}^2}\|z^n_k - \bar{F}^n_{\rm ex}(z^0_k)\| \lesssim h^2 + \Delta t^2, \tag{68}$$

and the auxiliary (leap-frog) approximate field satisfies

$$\|E^{n,1}_h - E(t_{n+\frac{1}{2}})\|_{L^\infty(\mathbb{R})} \lesssim h^2 + \Delta t^2. \tag{69}$$

Theorem 4.2. *Under the assumption* (67)*, the particle approximation of the phase space density satisfies*

$$\|f_h^n - f_{\mathrm{ex}}^n\|_{L^\infty(\mathbb{R}^2)} \lesssim h + h^{-1}\Delta t^2$$

with a constant only depending on the spline degree $p \geq 1$ and on the exact solution.

Proof of Theorem 4.1. Let $n \in \{0, \dots, N_t - 1\}$, $k \in \mathbb{Z}^2$ and $z^0 \in B_{h,k}^0$. For conciseness we denote

$$z^{n,1} := F_{\frac{1}{2}}\bar{F}_{\mathrm{ex}}^n(z^0) \qquad \text{and} \qquad \tilde{z}^{n,1} := F_{\frac{1}{2}}\bar{F}_{h,k}^n(z^0).$$

Keeping in mind that the numerical (linearized) flow for the k-th trajectory reads $\bar{F}_{h,k}^{n+1} = F_{h,k}^{n,1}F_{\frac{1}{2}}\bar{F}_{h,k}^n$ and using the expression (33) for $F_{h,k}^{n,1}$, we decompose the flow error according to

$$\begin{aligned}
(\bar{F}_{h,k}^{n+1} - \bar{F}_{\mathrm{ex}}^{n+1})(z^0) &= F_{h,k}^{n,1}(\tilde{z}^{n,1}) - \bar{F}_{\mathrm{ex}}^{n+1}(z^0) \\
&= F_h^{n,1}(z_k^{n,1}) + J_k^{n,1}(\tilde{z}^{n,1} - z_k^{n,1}) - \bar{F}_{\mathrm{ex}}^{n+1}(z^0) \\
&= \underbrace{(F_h^{n,1} - F_{\mathrm{ex}}^{n,1})(z_k^{n,1})}_{=:\,\mathbf{A}} + \underbrace{F_{\mathrm{ex}}^{n,1}(z_k^{n,1}) + J_{F_{\mathrm{ex}}^{n,1}}(z_k^{n,1})(\tilde{z}^{n,1} - z_k^{n,1}) - F_{\mathrm{ex}}^{n,1}(\tilde{z}^{n,1})}_{=:\,\mathbf{B}} \\
&\quad + \underbrace{\big(J_k^{n,1} - J_{F_{\mathrm{ex}}^{n,1}}(z_k^{n,1})\big)(\tilde{z}^{n,1} - z_k^{n,1})}_{=:\,\mathbf{C}} + \underbrace{F_{\mathrm{ex}}^{n,1}(\tilde{z}^{n,1}) - \bar{F}_{\mathrm{ex}}^{n+1}(z^0)}_{=:\,\mathbf{D}}.
\end{aligned}$$

From the definitions (32) and (48) of $F_h^{n,1}$ and $F_{\mathrm{ex}}^{n,1}$, and using Lemma 4.4 we find then

$$\|\mathbf{A}\| \leq \|F_h^{n,1} - F_{\mathrm{ex}}^{n,1}\|_{L^\infty} \lesssim \Delta t\|E_h^{n,1} - E_{\mathrm{ex}}^{n,1}\|_{L^\infty} \lesssim \Delta t(e_F^n + h^2) \tag{70}$$

with a constant that only depends on p and on the exact solution. Using (52) we bound

$$\|\mathbf{B}\| \lesssim |F_{\mathrm{ex}}^{n,1}|_{W^{2,\infty}}\|\tilde{z}^{n,1} - z_k^{n,1}\|^2 \lesssim \Delta t\|\tilde{z}^{n,1} - z_k^{n,1}\|^2, \tag{71}$$

and we estimate $\|\tilde{z}^{n,1} - z_k^{n,1}\| = \|F_{\frac{1}{2}}\bar{F}_{h,k}^n(z^0) - F_{\frac{1}{2}}\bar{F}_{h,k}^n(z_k^0)\|$ as

$$\begin{aligned}
\|\tilde{z}^{n,1} - z_k^{n,1}\| &\leq \|F_{\frac{1}{2}}(\bar{F}_{h,k}^n - \bar{F}_{\mathrm{ex}}^n)(z^0)\| + \|F_{\frac{1}{2}}(\bar{F}_{\mathrm{ex}}^n(z^0) - \bar{F}_{\mathrm{ex}}^n(z_k^0))\| + \|F_{\frac{1}{2}}(\bar{F}_{\mathrm{ex}}^n - \bar{F}_{h,k}^n)(z_k^0)\| \\
&\leq 2(1 + \tfrac{\Delta t}{2})\|\bar{F}_{h,k}^n - \bar{F}_{\mathrm{ex}}^n\|_{L^\infty(B_{h,k}^0)} + \|\bar{F}_{\mathrm{ex}}^n(z^0) - \bar{F}_{\mathrm{ex}}^n(z_k^0)\| \\
&\lesssim e_F^n + h
\end{aligned} \tag{72}$$

where we have used that $z^0, z_k^0 \in B_{h,k}^0$, and that the exact flow $\bar{F}_{\mathrm{ex}}^n = F_{0,t_n}$ has a Lipschitz constant that only depends on T and on the norm $\|\partial_x E\|_{L^\infty([0.T]\times\mathbb{R})}$. Combined with (57) and (71), this gives

$$\|\mathbf{B}\| \lesssim \Delta t(e_F^n + h)^2 \lesssim \Delta t(e_F^n + h^2). \tag{73}$$

The term $\mathbf{C}$ is bounded using Lemma 4.5 and (72),

$$\|\mathbf{C}\| \leq \|J_k^{n,1} - J_{F_{\mathrm{ex}}^{n,1}}(z_k^{n,1})\|\|\tilde{z}^{n,1} - z_k^{n,1}\| \lesssim \Delta t \min\{1, h^{-1}e_F^n + h\}(e_F^n + h). \tag{74}$$

Turning to the term $\mathbf{D}$ we infer from (51) that $|F_{\mathrm{ex}}^{n,1}|_{W^{1,\infty}} \leq 1 + C\Delta t$, hence

$$\begin{aligned}
\|\mathbf{D}\| &\leq \|F_{\mathrm{ex}}^{n,1}(\tilde{z}^{n,1}) - F_{\mathrm{ex}}^{n,1}(z^{n,1})\| + \|F_{\mathrm{ex}}^{n,1}(z^{n,1}) - \bar{F}_{\mathrm{ex}}^{n+1}(z^0)\| \\
&\leq |F_{\mathrm{ex}}^{n,1}|_{W^{1,\infty}}\|F_{\frac{1}{2}}(\bar{F}_{h,k}^n - \bar{F}_{\mathrm{ex}}^n)(z^0)\| + \|(F_{\mathrm{ex}}^{n,1}F_{\frac{1}{2}} - F_{t_n,t_{n+1}})(\bar{F}_{\mathrm{ex}}^n(z^0))\| \\
&\leq (1 + C\Delta t)(1 + \tfrac{\Delta t}{2})e_F^n + \|F_{\mathrm{ex}}^{n,1}F_{\frac{1}{2}} - F_{t_n,t_{n+1}}\|_{L^\infty(\bar{F}_{\mathrm{ex}}^n(B_{h,k}^0))} \\
&\leq e_F^n + C\Delta t(e_F^n + \Delta t^2)
\end{aligned} \tag{75}$$

where the last inequality follows from Lemma 4.6. Our error estimate is then obtained by a two-stage Gronwall argument. Gathering (70), (73), $\|\mathbf{C}\| \lesssim \Delta t(e_F^n + h)$ from (74) and (75) we first write

$$e_F^{n+1} \leq e_F^n + C\Delta t(e_F^n + h + \Delta t^2) \tag{76}$$

moreover we have $e_F^0 = 0$ since both $\bar F^n_{h,k}$ and $\bar F^n_{\rm ex}$ are initialized to $I_{\mathbb{R}^2}$, see (6). Thus, a discrete Gronwall Lemma and condition (67) on Δt yield

$$e_F^n \lesssim h + \Delta t^2 \lesssim h.$$

With this result we next see that (74) gives $\|\mathbf{C}\| \lesssim \Delta t(h^{-1}e_F^n + h)(e_F^n + h) \lesssim \Delta t(e_F^n + h^2)$. Combined with (70), (73) and (75) this allows to update (76) into

$$e_F^{n+1} \le e_F^n + C\Delta t(e_F^n + h^2 + \Delta t^2),$$

and a second use of the discrete Gronwall Lemma gives the desired estimate. □

Proof of Corollary 4.7. Estimate (68) is straightforward from the definition of the numerical trajectories, $z_k^n := F^{n,1}_{h,k} F_{\frac12}(z_k^{n-1}) = \cdots = \bar F^n_{h,k}(z_k^0)$, and estimate (69) follows from the observation that the bound $|\mathbf{E}_3| \lesssim \Delta t^2$ in the proof of Lemma 4.6 is easily established for any $(X_n, V_n) \in \mathbb{R} \times \{0\}$, yielding $\|E^{n,1}_{\rm ex} - E(t_{n+\frac12})\|_{L^\infty(\mathbb{R})} \lesssim \Delta t^2$. □

Proof of Theorem 4.2. Given $z \in \mathbb{R}^2$, we let $\tilde z^n = (\bar F^n_{\rm ex})^{-1}(z)$ and $\tilde z^n_k = (\bar F^n_{h,k})^{-1}(z)$. We have

$$\begin{aligned}
|(f^n_{\rm ex} - f^n_h)(z)| &= \big|f^n_{\rm ex}(z) - \sum_{k\in\mathbb{Z}^2} w_k \varphi^n_{h,k}(z)\big| \\
&= \big|f^0_{\rm ex}(\tilde z^n) - \sum_{k\in\mathbb{Z}^2} w_k \varphi_{h,k}(\tilde z^n_k)\big| \\
&\le |(f^0_{\rm ex} - f^0_h)(\tilde z^n)| + \sum_{k\in\mathbb{Z}^2} |w_k||\varphi_{h,k}(\tilde z^n) - \varphi_{h,k}(\tilde z^n_k)| \\
&\le \|f^0_{\rm ex} - f^0_h\|_{L^\infty} + \sum_{k\in\mathcal{K}^n(z)\cup\mathcal{K}^n_{\rm ex}(z)} |w_k||\varphi_{h,k}|_{W^{1,\infty}}\|\tilde z^n_k - \tilde z^n\| \\
&\lesssim h^2 + \#(\mathcal{K}^n(z)\cup\mathcal{K}^n_{\rm ex}(z))h^{-1}\|\tilde z^n_k - \tilde z^n\|
\end{aligned} \tag{77}$$

where $\mathcal{K}^n(z) = \{k : \tilde z^n_k \in B^0_{h,k}\} = \{k : z \in \bar F^n_{h,k}(B^0_{h,k})\}$ denotes the *overlapping set* of the numerical particles, while $\mathcal{K}^n_{\rm ex}(z) = \{k : \tilde z^n \in B^0_{h,k}\} = \{k : z \in \bar F^n_{\rm ex}(B^0_{h,k})\}$ corresponds to the one of the auxiliary particles transported along the exact flow. Note that in the last inequality we have used the initial spline approximation estimate (21) for $p \ge 1$, the fact that the spline weights satisfy $w_k \lesssim \|f^0_{\rm ex}\|_{L^\infty} h^2$ and the scaling $|\varphi_{h,k}|_{W^{1,\infty}} \lesssim h^{-3}$, see (17). Now, from (18) one easily derives that $\#(\mathcal{K}^n_{\rm ex}(z))$ is uniformly bounded by a constant depending on p, namely $C = (2c_p)^2$. We next observe that

$$\|\tilde z^n_k - \tilde z^n\| = \|\tilde z^n_k - (\bar F^n_{\rm ex})^{-1}(\bar F^n_{h,k}(\tilde z^n_k))\| \lesssim |(\bar F^n_{\rm ex})^{-1}|_{W^{1,\infty}}\|\bar F^n_{\rm ex}(\tilde z^n_k) - \bar F^n_{h,k}(\tilde z^n_k)\| \lesssim Ce^n_F$$

where we have used the Lipschitz estimate (7) for the exact flow. Thus any $k \in \mathcal{K}^n(z)$ satisfies

$$\|hk - \tilde z^n\| \le c_p h + \|\tilde z^n_k - \tilde z^n\| \lesssim h + e^n_F$$

which yields $\#(\mathcal{K}^n(z)) \lesssim (1 + h^{-1}e^n_F)^2$. Using (76) this shows that $\#(\mathcal{K}^n(z)\cup\mathcal{K}^n_{\rm ex}(z)) \lesssim 1$ for all z, and we obtain that (77) gives

$$\|f^n_{\rm ex} - f^n_h\|_{L^\infty} \lesssim h^2 + h^{-1}e^n_F.$$

The claimed estimate follows then from Theorem 4.1. □

5 Conclusion

In this article we have extended the analysis of the Linearly-Transformed Particle scheme (LTP) introduced in [6] to the case where the transport (Vlasov) equation is coupled with a 1d Poisson potential. This method can be seen as a variant of the standard fixed-shape particle method [14] where the particles radii coincide with the initial inter-particle distance, and where each particle is transported along the local linearization of the characteristic flow. By studying the error resulting from these local linearizations we have established that the numerical trajectories converge towards the exact ones with one order higer compared to the standard case. Moreover, we have shown that the phase space density converges in the uniform norm towards the exact one without resorting to an increasing number of overlapping (i.e., interacting) particles as h tends to 0. These a priori estimates can be seen as an improvement on those established in [14] when applied to fixed-shape particles with bounded overlapping.

References

[1] C. Alard and S. Colombi. A cloudy Vlasov solution. *Monthly Notices of the Royal Astronomical Society*, 359(1):123–163, May 2005.

[2] W.B. Bateson and D.W. Hewett. Grid and Particle Hydrodynamics. *Journal of Computational Physics*, 144:358–378, 1998.

[3] J.T. Beale. On the accuracy of vortex methods at large times. In *Computational fluid dynamics and reacting gas flows (Minneapolis, MN, 1986)*, pages 19–32. Springer, New York, 1988.

[4] J.T. Beale and A. Majda. Vortex methods. II. Higher order accuracy in two and three dimensions. *Mathematics of Computation*, 39(159):29–52, 1982.

[5] M. Bergdorf, G.-H. Cottet, and P. Koumoutsakos. Multilevel Adaptive Particle Methods for Convection-Diffusion Equations. *Multiscale Modeling & Simulation*, 4(1):328–357, 2005.

[6] M. Campos Pinto. Towards smooth particle methods without smoothing. *Journal of Scientific Computing (to appear)*, 2014.

[7] M. Campos Pinto, E. Sonnendrücker, A. Friedman, D.P. Grote, and S.M. Lund. Noiseless Vlasov–Poisson simulations with linearly transformed particles. *Journal of Computational Physics*, 275(C):236–256, October 2014.

[8] J.-P. Chehab, A. Cohen, D. Jennequin, J.J. Nieto, Ch. Roland, and J.R. Roche. An adaptive Particle-In-Cell method using multi-resolution analysis. In *Numerical methods for hyperbolic and kinetic problems, Numerical methods for hyperbolic and kinetic problems, CEMRACS 2003/IRMA Lectures in Math. and Theoretical Physics*, pages 29–42. 2005.

[9] C.K. Chui and H. Diamond. A characterization of multivariate quasi-interpolation formulas and its applications. *Numerische Mathematik*, 57(1):105–121, 1990.

[10] A. Cohen and B. Perthame. Optimal Approximations of Transport Equations by Particle and Pseudoparticle Methods. *SIAM Journal on Mathematical Analysis*, 32(3):616–636, October 2000.

[11] C.J. Cotter, J. Frank, and S. Reich. The remapped particle-mesh semi-Lagrangian advection scheme. *Quarterly Journal of the Royal Meteorological Society*, 133(622):251–260, 2007.

[12] G.-H. Cottet. Artificial Viscosity Models for Vortex and Particle Methods. *Journal of Computational Physics*, 1996.

[13] G.-H. Cottet, P. Koumoutsakos, and M.L.O. Salihi. Vortex Methods with Spatially Varying Cores. *Journal of Computational Physics*, 162(1):164–185, July 2000.

[14] G.-H. Cottet and P.-A. Raviart. Particle methods for the one-dimensional Vlasov-Poisson equations. *SIAM Journal on Numerical Analysis*, pages 52–76, 1984.

[15] N. Crouseilles, T. Respaud, and E. Sonnendrücker. A forward semi-Lagrangian method for the numerical solution of the Vlasov equation. *Computer Physics Communications*, 180(10):1730–1745, October 2009.

[16] C. de Boor. *A practical guide to splines*, volume 27 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1978.

[17] J. Denavit. Numerical Simulation of Plasmas with Periodic Smoothing in Phase Space. *Journal of Computational Physics*, 9:75–98, 1972.

[18] D.W. Hewett. Fragmentation, merging, and internal dynamics for PIC simulation with finite size particles. *Journal of Computational Physics*, 189(2):390–426, August 2003.

[19] T.Y. Hou. Convergence of a Variable Blob Vortex Method for the Euler and Navier-Stokes Equations. *SIAM Journal on Numerical Analysis*, 27(6):1387–1404, December 1990.

[20] R.D. Nair, J.S. Scroggs, and F.H.M. Semazzi. A forward-trajectory global semi-Lagrangian transport scheme. *Journal of Computational Physics*, 190(1):275–294, September 2003.

[21] P.-A. Raviart. An analysis of particle methods. In *Numerical methods in fluid dynamics (Como, 1983)*, pages 243–324. Lecture Notes in Mathematics, Berlin, 1985.

[22] J. Strain. 2D Vortex Methods and Singular Quadrature Rules. *Journal of Computational Physics*, 124(1):131–145, March 1996.

[23] S. Ukai and T. Okabe. On classical solutions in the large in time of two-dimensional Vlasov's equation. *Osaka Journal of Mathematics*, 1978.

[24] M. Unser and I. Daubechies. On the approximation power of convolution-based least squares versus interpolation. *Signal Processing, IEEE Transactions on*, 45(7):1697–1711, 1997.