Kinetic models coupled with Gaussian thermostats: macroscopic frameworks

Carlo Bianca, Christian Dogbe

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Carlo Bianca[1,2], Christian Dogbe[3]

(1) Sorbonne Universités, UPMC Univ Paris 06, UMR 7600,
Laboratoire de Physique Théorique de la Matière Condensée,
4, place Jussieu, case courrier 121, 75252 Paris Cedex 05, France

(2) CNRS, UMR 7600 LPTMC, France

(3) Université de Caen Basse-Normandie, Département de Mathématiques,
Laboratoire de Mathématiques Nicolas Oresme
CNRS, UMR 6139, 14032 Caen Cedex, France

Abstract

This paper deals with the modelling of complex systems composed by a large number of elements grouped into different functional subsystems. The modelling framework is that of the thermostatted kinetic theory which consists in a set of nonlinear integro-differential equations. A source of nonlinearity is also the presence of the mathematical thermostat that ensures the control of the global energy of the system. Specifically this paper is devoted to the derivation of evolution equations for the macroscopic variables (density and momentum) from the underlying description at the microscopic scale delivered by the thermostatted kinetic models. To this aim, hyperbolic and parabolic type scalings of the thermostatted kinetic for active particles model are performed and the resulting macroscopic equations are obtained. Finally the asymptotic methods are applied to the relaxation model.

Keywords: Functional subsystems, kinetic theory, hyperbolic scaling, parabolic scaling, integro-differential equation

1 Introduction

The evolution of the macroscopic state of a physical or living system is described by means of suitable partial differential equations for the macroscopic variables (typically mass, momentum and energy). These equations are usually derived by following the guidelines offered by continuum mechanics approach. Indeed mass, momentum, and energy conservation equations are properly closed by phenomenological models corresponding to the material behavior of the system. In particular, the continuum mechanics approach has been recently employed

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for deriving macroscopic equations for biological systems viewed as large systems of interacting cells. Accordingly, macroscopic models are based on the interactions between cells and extracellular matrix, as cells may synthesize and undergo degradation, which may affect cell properties and orientation. The evolution of both cells and extracellular matrix is mediated by some chemical growth factors, which regulate both cell proliferation and extracellular matrix reorganization, see books [1, 2], the review paper [3] and the references cited therein. Macroscopic equations delivered by the continuum mechanics approach have been also derived in an attempt to describe crowds and swarms dynamics, see [4, 5] and the perspective section of the review paper [6].

An alternative method for deriving macroscopic equations able to take into account the dynamics at a lower scale (usually the mesoscopic), has been proposed by the kinetic theory. This method, called asymptotic method, consists in deriving macroscopic equations by suitable limits of Boltzmann-type equations related to the statistical microscopic description, see [37, 7, 8, 9, 10, 11, 13]. In particular Hillen and Othmer [14] and Hillen [15] deal with the moment closure of kinetic equations with stochastic jump velocity to derive hyperbolic equations modeling chemosentitive movement; Lachowicz and Wrzosek [9] derive diffusion equations for a coagulation model with nonlocal interactions by a generalization of the Boltzmann equation.

At the basic of the asymptotic methods is the time-space scaling. Classically different types of scalings lead to different types of equations: parabolic and hyperbolic [16, 18]. The low-field (or parabolic) limit of kinetic equations leads to a drift-diffusion type system in which the diffusion processes dominate the behavior of the solutions, see papers [19, 20] and the references therein. In the high-field (or hyperbolic) limit the influence of the diffusion terms is of lower (or equal) order of magnitude in comparison with other convective or interaction terms and the models consist of linear or nonlinear hyperbolic equations for the local density, see [21, 22, 23, 24, 26]. After the assessment of the scaling, the distribution function is expanded in terms of a small dimensionless parameter (the Knudsen number, the time- or the space-scale dimensionless parameter); the asymptotic limit is singular and convergence is guaranteed under suitable technical assumptions.

The low-field and the high-field limits have been applied to the kinetic theory for active particles models for the derivation of macroscopic equations that can be proposed as models for analyzing the complex phenomena occurring in a tumor tissue in competition with the immune system, see the review paper [29]. Different combinations of parabolic and hyperbolic scales are used, according on the dispersive or non-dispersive nature of the system under consideration. The macroscopic (tissue) models show that interactions which change the biological functions of cells may substantially modify the structure of the macroscopic equations. In particular, it has been shown the onset of linear and nonlinear diffusion terms and hyperbolic behavior, departing from the simple mass conservation equation. An interesting result is the appearance of source terms in the case of proliferative phenomena for systems consisting in, at least, two populations.

Recently, the kinetic theory for active particles (KTAP) has been generalized in [30] in order to model complex systems in physics and life sciences that are subjected to external force fields. Generalizations have been obtained by coupling the KTAP with the time-reversible Gaussian thermostats with the

aim of capturing the nonequilibrium stationary states that appear in most real-world complex phenomena, see paper [6]. The introduction of the Gaussian thermostat allows to maintain constant the total energy of the system and is based on the Gauss principle of least constraint. The Cauchy problem of the resulting framework, called *thermostatted kinetic framework* (briefly TKF), has been analyzed in [31] and in [32]. The TKF has been further generalized for taking into account systems where the microscopic state variables of the particles of the system can attain discrete values, e.g. in vehicular traffic dynamics, see [33]. In particular the microscopic state of the particles includes, in addition to classical space and velocity variables, the activity variable which models the ability of the particles to perform autonomous strategies.

This paper deals with the high-field and low-field asymptotic limits of the thermostatted kinetic theory for active particles framework which has been proposed for the modelling of complex systems composed by a large number of particles grouped in subsystems. The framework is further generalized in order to include the space structure by adding the stochastic velocity-jump process proposed in [16] and subsequently developed in various papers, see, among others, [27, 28, 29, 34]. The time evolution of each subsystem is depicted by a distribution function, over the microscopic state of the particles, satisfying an evolution equation.

It is worth precising that a high-field asymptotic analysis has been recently developed for a thermostatted kinetic model. Specifically Degond and Wennberg in [35] consider the thermostatted Boltzmann equation with the one-dimensional Bhatnagar-Gross-Krook relaxation type operator [36] (where the distribution function relaxes to a Maxwellian with same density and temperature), and the three dimensions Fokker-Planck collision operator [38] (where the density and energy are preserved). The classical Boltzmann equation for elastic collisions, see Cercignani [39], is subjected to a large force field and coupled with Gaussian thermostats. By adding a thermostat correction, it is possible to expand the solutions about a high-field equilibrium obtained when balancing the thermostatted field drift operator with the elastic collision operator. Degond and Wennberg derive coupled mass and energy balance laws that are hyperbolic, thereby indicating that they might be appropriate for a use in physically realistic situations.

The asymptotic method developed in the present paper is very different by that of Degond-Wennberg and also more general because we do not consider a kinetic model with an appropriate choice of the interaction operator (its terms are generic functions). Under suitable assumptions on the turning operator, that as already mentioned models the velocity-jump process, we derive macroscopic hyperbolic and parabolic equations. Specifically, the macroscopic equations model the evolution of the gross local variables which describe the state of the system, i.e. the density and the momentum. These equations are generally derived by heuristic methods of continuum mechanics.

It is worth stressing that, to the best of our knowledge, this is the first time that the high-field and low-field limit is performed for the thermostatted kinetic theory for active particles models.

The present paper is organized into four more sections and an appendix that follow this introduction. In detail, Section 2 deals with the thermostatted kinetic framework for active particles which models complex systems divided into different functional subsystems where the activity of the particles is modelled

by conservative interactions (namely the number of particles is preserved) and the velocity variable by a jump-process. The decomposition of the overall system into functional subsystems is performed in order to reduce its complexity. The framework acts as a background paradigm for the derivation of specific models in the applied sciences. Section 3 is devoted the low-field limit of the thermostatted kinetic framework by considering the parabolic scaling. Section 4 is concerned with the assessment of the hyperbolic scaling and the derivation of the rescaled thermostatted framework. Furthermore after the definition of the technical assumptions on the turning operator, the hyperbolic limit is performed and the macroscopic equations for the density and the momentum of the system are derived. The section is concluded with the derivation of the macroscopic equations for the relaxation model. Section 5 concludes the paper with a discussion on the applicability of the macroscopic models obtained by asymptotic analysis performed in the previous sections, with special attention to research perspectives and applications to biological phenomena, including tumor growth, cancer modelling, chemotaxis. Finally, the Appendix collects some technical computations.

2 The underlying mathematical framework

This section is devoted the derivation of the thermostatted kinetic framework that can be proposed for the modelling of most real-world complex systems. The section is presented through two sequential subsections. First we deal with the complexity problem, the decomposition of system into subsystems and their statistical representation, then with the modeling of the microscopic interactions among the selected subsystems, and finally with the derivation of the relative class of evolution equations that act as a general paradigm for the derivation of specific models.

2.1 Complexity, functional subsystems decomposition and representation

Complexity is an intrinsic characteristic of most living systems that makes the modelling of the system disputable. Indeed not only the large number of elements constituting the system but also the emergence of behaviour, that arises as result of the whole interactions among the elements that occur in nonlinear fashion, are key issues of the complexity. The complex behavior is also due to the fact that the living entities (cells, animals, pedestrians), differently from the inert matter entities (ball billiard, gas particles, electrons), have the ability to perform specific strategies and functions such that small variations in their will, could modify the overall asymptotic dynamics. Moreover external factors influence their strategies and consequently the occurrence of their emerging behaviour. Finally the strategy of entities at the lower scale (microscopic or mesoscopic) determines the behaviors of the system at the upper scale (macroscopic). The interested reader in a general and more deeper treatment of the complexity is referred to the book [41].

From the modelling viewpoint, the onset of complexity indicates the needing to develop tools and methods which allow to reduce the perplexity. In the biological systems case, several authors suggest the approach of system biology as an essential tool to achieve this objective, see, among others, papers [42, 43].

Principles that originate from physics and biology have also be proposed for the modelling of complex economic systems [44, 45, 46, 47].

In what follow we use the systems approach proposed in [32]. Accordingly the overall system $\mathcal{S}$ is viewed as composed by a large number of elements (particles of the inert matter, animals, cells, pedestrians) that interact in nonlinear matter and under the effect of macroscopic external force fields. In this approach the elements of the system are called *active particles* because in addition to their possibility to be allocated, at time t, in the position $\mathbf{x}$ with velocity $\mathbf{v}$, they also have an active role in the evolution of the system (the ability to express strategy) modelled by the scalar variable $u \in D_u$. The triplet $(\mathbf{x}, \mathbf{v}, u)$ is called the *microscopic state* of the particles. The active particles are of heterogeneous type and they are selected for the ability they express and not for their size and shape. Particles having the same ability are grouped into a subsystem, called *functional subsystem*.

Therefore the overall complex system $\mathcal{S}$ is decomposed into different functional subsystems $\mathcal{S}_i$, for $i \in \{1, 2, \ldots, n\}$, such that:

$$\mathcal{S}_i \cap \mathcal{S}_j = \emptyset, \quad \forall i \neq j, \quad \text{and} \quad \mathcal{S} = \bigcup_{i=1}^{n} \mathcal{S}_i. \tag{2.1}$$

It is worth precising that the decomposition into functional subsystems is a flexible approach to be adapted at each system and is related to the activity variable. Finally a functional subsystem could be itself a complex system and therefore can be decomposed into subsystems.

The evolution of each functional subsystem $\mathcal{S}_i$ is represented by the statistical mechanics approach. Accordingly, we assume that the active particle during a time interval $[0, T]$, or $[0, +\infty)$, is located in $\mathbf{x}$ which attains values in a (bounded or unbounded) domain $D_{\mathbf{x}} \subset \mathbb{R}^d$ (usually $d = 3$), with velocity variable $\mathbf{v}$ that attains values in a domain $D_{\mathbf{v}} \subset \mathbb{R}^d$ and activity variable u which varies in a domain $D_u \subset \mathbb{R}$. Then the kinetic of the i-th functional subsystem, for $i \in \{1, 2, \ldots, n\}$, is depicted by the evolution of the distribution function $f_i(t, \mathbf{x}, \mathbf{v}, u)$ defined on $[0, T] \times D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_u$. For any fixed time t, the quantity $f_i(t, \mathbf{x}, \mathbf{v}, u) d\mathbf{x}\, d\mathbf{v}\, du$ stands for the density of particles in the volume element $d\mathbf{x}\, d\mathbf{v}\, du$ centered at $(\mathbf{x}, \mathbf{v}, u)$. Let $\Omega = D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_u$ be the domain of the all possible microscopic states and $d\Omega = d\mathbf{x}\, d\mathbf{v}\, du$ the Lebesgue measure on Ω, then the minimal assumption that one can make on f is that for all $t \geqslant 0$,

$$f_i(t, \cdot, \cdot, \cdot) \in L^1_{loc}(\Omega, d\Omega).$$

If the number of particles is constant in time, the distribution function f_i of the functional subsystem $\mathcal{S}_i$ can be normalized with respect to such a number and acquires the structure of a probability function.

Let $\mathbf{f} = \mathbf{f}(t, \mathbf{x}, \mathbf{v}, u) = (f_1(t, \mathbf{x}, \mathbf{v}, u), \ldots, f_n(t, \mathbf{x}, \mathbf{v}, u))$ be the vector whose components are the distribution functions of the functional subsystems. The global density of the system $\mathcal{S}$ thus reads:

$$\Xi[\mathbf{f}](t) = \sum_{i=1}^{n} \int_{\Omega} f_i(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{x}\, d\mathbf{v}\, du = \int_{\Omega} \widetilde{f}(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{x}\, d\mathbf{v}\, du, \tag{2.2}$$

where

$$\widetilde{f}(t, \mathbf{x}, \mathbf{v}, u) = \sum_{i=1}^{n} f_i(t, \mathbf{x}, \mathbf{v}, u). \tag{2.3}$$

It worth noting that if f_i represents the joint distribution function, marginal distribution functions of f_i refer either to the distribution over the mechanical state or to distribution over the microscopic activity. These marginal distribution functions define the local quantities. For instance, the local linear activation moment at time t in $\mathbf{x}$ is computed as follows:

$$\mathbb{A}[f_i](t,\mathbf{x}) = \int_{\Sigma} u\, f_i(t,\mathbf{x},\mathbf{v},u)\, d\mathbf{v}\, du, \tag{2.4}$$

where $\Sigma = D_{\mathbf{v}} \times D_u$.

It is worth stressing that every function defined in this paper is assumed to be measurable in all variables.

2.2 The thermostatted kinetic framework

This subsection is devoted to the derivation of the evolution equation for the distribution function f_i of the ith functional subsystem, for $i \in \{1, 2, \ldots, n\}$. Each functional subsystem is subjected to the external force field $\mathcal{F}_i = \mathcal{F}_i(u) : D_u \to \mathbb{R}$ whose magnitude modifies the asymptotic behavior and move out of equilibrium the system. The function $\mathcal{F}_i$ is assumed to be a known function of u.

The evolution equation for the distribution function f_i is obtained by equating the time derivative of f_i to the balance of the inlet and outlet flows in the elementary volume $d\Omega$ of the space of the microscopic states and under the following assumptions.

($\mathbf{A}_1$) The interactions between the particles of the ith functional subsystem and the jth functional subsystem are binary and instantaneous in time, for $i, j \in \{1, 2, \ldots, n\}$. Moreover interactions occur when two particles are very close each other.

($\mathbf{A}_2$) The interactions among the particles of the functional subsystems occur with rate η. Specifically η_{ij} denote the probability that a particle of the ith functional subsystem with activity u_* interacts instantaneously with a particle of the jth functional subsystem with activity u^*.

($\mathbf{A}_3$) The particles of the ith functional subsystem with activity u_* interacting with the particles of the jth functional subsystem with activity u^* have a probability density to reach the activity u given by $\mathcal{A} = \mathcal{A}(u_*, u^*, u)$.

The kernel $\mathcal{A}(u_*, u^*, u) : D_u \times D_u \times D_u \to \mathbb{R}^+$ satisfies

$$\forall\, u_*, u^* \in D_u, \qquad \int_{D_u} \mathcal{A}(u_*, u^*, u)\, du = 1$$

(density or number of cells conservation).

($\mathbf{A}_4$) The velocity variable $\mathbf{v} \in D_{\mathbf{v}}$ changes according to the stochastic velocity-jump process: a particle moves with constant velocity in a straight line, stops after a certain time, chooses a new direction, continues running and so on.

($\mathbf{A}_5$) The domain $D_{\mathbf{v}}$ is bounded and spherically symmetric with respect to origin (i.e. $\mathbf{v}$ and $-\mathbf{v} \in D_{\mathbf{v}}$).

Therefore, denoting by ∂_t, $\nabla_{\mathbf{x}}$ the time derivative and the space gradient of f, respectively, the mathematical framework of thermostatted kinetic evolution equation for f_i, $i \in \{1, 2, \ldots, n\}$, thus reads:

$$(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}})\, f_i + \partial_u \left(\mathcal{F}_i(u) \left(1 - u \int\limits_{\Omega} u\, \widetilde{f}\, d\mathbf{x}\, d\mathbf{v}\, du \right) f_i \right) = J_i[\mathbf{f}] + \nu\, V_i[f_i], \quad (2.5)$$

where $\widetilde{f}$ is given by Eq. (2.3) and

- $\mathbf{v} \cdot \nabla_{\mathbf{x}} f_i$ is the usual transport operator;
- ν is the turning rate or turning frequency of the velocity-jump, hence $1/\nu$ is the mean run time;
- The operator $J_i[\mathbf{f}] = J_i[\mathbf{f}](t, \mathbf{x}, \mathbf{v}, u)$, which models the interactions in the activity, reads:

$$\begin{aligned} J_i[\mathbf{f}] \;=\; & \sum_{j=1}^{n} \int\limits_{D_u \times D_u} \eta_{ij}\, \mathcal{A}_{ij}(u_*, u^*, u) f_i(t, \mathbf{x}, \mathbf{v}, u_*)\, f_j(t, \mathbf{x}, \mathbf{v}, u^*)\, du_*\, du^* \\ & - f_i(t, \mathbf{x}, \mathbf{v}, u) \sum_{j=1}^{n} \int_{D_u} \eta_{ij}\, f_j(t, \mathbf{x}, \mathbf{v}, u^*)\, du^*. \end{aligned} \quad (2.6)$$

  The first term in the right-hand side of Eq. (2.6) models the gain due to transitions from other states: *a particle with state u_*, after an encounter with the particle with state u^* will reach the activity u.* The second term in the right-hand side of the Eq. (2.6) is the loss due to transitions into other states: *a particle with state u having an encounter with a particle with state u^* will change its activity as a result of the interaction and thus will leave the state u.*

- The operator $V_i[f_i] \equiv V_i[f_i](t, \mathbf{x}, \mathbf{v}, u)$, which models the velocity-jump process is defined as follows:

$$V_i[f_i] = \int_{D_{\mathbf{v}}} \left[T_i(\mathbf{v}^*, \mathbf{v}) f_i(t, \mathbf{x}, \mathbf{v}^*, u) - T_i(\mathbf{v}, \mathbf{v}^*) f_i(t, \mathbf{x}, \mathbf{v}, u) \right] d\mathbf{v}^*, \quad (2.7)$$

  where $T_i(\mathbf{v}^*, \mathbf{v})$ is the turning kernel which gives the probability that the velocity $\mathbf{v}^* \in D_{\mathbf{v}}$ jumps into the velocity $\mathbf{v} \in D_{\mathbf{v}}$ (if a jump occurs).

- The term

$$\mathcal{T}_{\mathcal{F}_i}[f_i] := \partial_u \left(\mathcal{F}_i(u) \left(1 - u \int_{\Omega} u\, \widetilde{f}(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{x}\, d\mathbf{v}\, du \right) f_i(t, \mathbf{x}, \mathbf{v}, u) \right), \quad (2.8)$$

  is a transport term due to the activity of the particles. This term models the Gaussian thermostat, which is based on the mathematical thermostats [48, 49] and the Gauss's principle of the least constraint [50].

Remark 2.1.

- Usually the turning event is governed by a Poisson process [51, 52].

- In general the interaction rate is a function of the activity u_* and u^*, namely $\eta_{ij} = \eta(u_*, u^*)$.
- Implicit in the above formulation is the assumption that the choice of a new velocity is independent of the run time. In general ν may be space dependent and may depend on internal and external variables as well.

Remark 2.2.
The function $\mathcal{A}_{ij}(u_*, u^*, u)$ is not symmetrical with respect to u and has the structure of a probability density only for mass conservative systems. Thus the micro-reversibility property $\mathcal{A}_{ij}(u_*, u^*, u) = \mathcal{A}_{ij}(u^*, u_*, u)$ is false.

We conclude this section by defining the local macroscopic quantities that we will study in the next sections. Specifically we introduce, for $i \in \{1, 2, \ldots, n\}$, the local density $\varrho[f_i](t, \mathbf{x}, u)$ of the ith functional subsystem defined at time t in the position $\mathbf{x}$ and activity u as follows:

$$\varrho_i := \varrho[f_i](t, \mathbf{x}, u) = \int_{D_\mathbf{v}} f_i(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}, \tag{2.9}$$

and the relative mass velocity of particles $\mathbb{U}_i(t, \mathbf{x}, u)$ defined on $[0, \infty[\times D_\mathbf{x} \times D_u$ by

$$\mathbb{U}_i := \mathbb{U}_i(t, \mathbf{x}, u) = \frac{1}{\varrho[f_i](t, \mathbf{x}, u)} \int\limits_{D_\mathbf{v}} \mathbf{v}\, f_i(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v} \tag{2.10}$$

and the pressure term

$$\mathbb{P}_i := \mathbb{P}_i(t, \mathbf{x}, u) = \int_{D_\mathbf{v}} (\mathbf{v} - \mathbb{U}_i) \otimes (\mathbf{v} - \mathbb{U}_i) f_i(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}. \tag{2.11}$$

Notice that, by definition of $\mathbb{U}$, one has

$$\int_{D_\mathbf{v}} \mathbf{v} \otimes \mathbf{v}\, f_i(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v} = \varrho_i \mathbb{U}_i \otimes \mathbb{U}_i + \mathbb{P}_i. \tag{2.12}$$

It worth stressing that high order local quantities are calculated if motivated by some interest for the modelling, see [34].

2.3 On the macroscopic description and equilibrium state

The goal of this paper is to derive by Eq. (2.5) the macroscopic (fluid) dynamics. Let $\psi(\mathbf{v}) \in \{1, \mathbf{v}\}$, then the transition from microscopic to macroscopic description is realized multiplying Eq. (2.5) by ψ and by integrating over the velocity variable $\mathbf{v}$ to get

$$\begin{aligned} &\partial_t \int_{D_\mathbf{v}} \psi\, f_i\, d\mathbf{v} + \nabla_\mathbf{x} \cdot \int_{D_\mathbf{v}} \mathbf{v}\, \psi\, f_i\, d\mathbf{v} + \partial_u \int_{D_\mathbf{v}} \psi\, \mathcal{T}_{\mathcal{F}_i}[f_i]\, d\mathbf{v} \\ &\quad = \int_{D_\mathbf{v}} \psi\, J_i[\mathbf{f}_i]\, d\mathbf{v} + \nu \int_{D_\mathbf{v}} \psi\, V_i[f_i]\, d\mathbf{v}. \end{aligned} \tag{2.13}$$

Making the identification (2.9)-(2.10)-(2.11), we obtain formally at least, the system of conservation laws:

$$\partial_t \varrho_i + \nabla_{\mathbf{x}} \cdot (\varrho_i \mathbb{U}_i) \, + \, \partial_u \int_{D_{\mathbf{v}}} \mathcal{T}_{\mathcal{F}_i}[f_i] \, d\mathbf{v}$$
$$= \int_{D_{\mathbf{v}}} J_i[\mathbf{f}_i] \, d\mathbf{v} + \nu \int_{D_{\mathbf{v}}} V_i[f_i] \, d\mathbf{v} \tag{2.14a}$$
$$\partial_t (\varrho_i \mathbb{U}_i) + \nabla_{\mathbf{x}} \cdot \int_{D_{\mathbf{v}}} \mathbf{v} \otimes \mathbf{v} f_i \, d\mathbf{v} \, + \, \partial_u \int_{D_{\mathbf{v}}} \mathbf{v} \, \mathcal{T}_{\mathcal{F}_i}[f_i] \, d\mathbf{v}$$
$$= \int_{D_{\mathbf{v}}} \mathbf{v} \, J_i[\mathbf{f}_i] \, d\mathbf{v} + \int_{D_{\mathbf{v}}} \mathbf{v} \, V_i[f_i] \, d\mathbf{v}. \tag{2.14b}$$

Recalling (2.12), the equation (2.14b) of the system of conservation laws can be rewritten as follows:

$$\partial_t (\varrho_i \mathbb{U}) + \nabla_{\mathbf{x}} \cdot (\varrho_i \mathbb{U}_i \otimes \mathbb{U}_i + \mathbb{P}_i) + \partial_u \int_{D_{\mathbf{v}}} \mathbf{v} \, \mathcal{T}_{\mathcal{F}_i}[f_i] \, d\mathbf{v}$$
$$= \int_{D_{\mathbf{v}}} \mathbf{v} \, J_i[\mathbf{f}_i] \, d\mathbf{v} + \int_{D_{\mathbf{v}}} \mathbf{v} \, V_i[f_i] \, d\mathbf{v}. \tag{2.15}$$

The equations (2.14a) and (2.14b) provide the fluid dynamic description of the system (that is not closed).

It is well known that in the kinetic theory approach the equilibrium is, in general, a Maxwellian. When an equilibrium state is known it is possible to avoid the physical description of the local interactions. A typical equilibrium state has the following form (see [17]):

$$\frac{\varrho}{\vartheta^{d/2}(\varrho)} F\left(\frac{\mathbf{v} - \mathbb{U}}{\vartheta^{1/2}(\varrho)}\right), \tag{2.16}$$

where $F : \mathbb{R} \to [0, \infty)$ is a given smooth function, $\vartheta : [0, \infty) \to [0, \infty)$ is a power-like function and the velocity domain $D_{\mathbf{v}} = \mathbb{R}^d$ (according to Galilean invariance). Therefore in order to fulfill basic conservation laws (particle density and momentum) one assumes that

$$\int_{D_{\mathbf{v}}} F(\eta) \, d\eta = 1, \quad \text{i.e.} \quad \int_{D_{\mathbf{v}}} \frac{\varrho}{\vartheta^{d/2}(\varrho)} F\left(\frac{\mathbf{v} - \mathbb{U}}{\vartheta^{1/2}(\varrho)}\right) d\mathbf{v} = \varrho$$

$$\int_{D_{\mathbf{v}}} \eta F(\eta) \, d\eta = 0, \quad \text{i.e.} \quad \int_{D_{\mathbf{v}}} \mathbf{v} \, \frac{\varrho}{\vartheta^{d/2}(\varrho)} F\left(\frac{\mathbf{v} - \mathbb{U}}{\vartheta^{1/2}(\varrho)}\right) d\mathbf{v} = \varrho \mathbb{U}.$$

Usually, in the asymptotic regimes (see e.g. [39, 25, 57] and the references therein), the density of particles f, when the mean free path (Knudsen number) goes to zero, is very close to a Maxwellian $\mathcal{M}$, i.e. there exist a density ϱ, a bulk velocity $u \in \mathbb{R}^3$ and a temperature θ such that $f = \mathcal{M}_{\varrho,u,\theta}$, where

$$\mathcal{M}_{\varrho,u,\theta} := \frac{\varrho}{(2\pi\theta)^{3/2}} \, e^{-\frac{|v-u|^2}{2\theta}}, \quad \text{for each } v \in \mathbb{R}^3.$$

In particular one has

$$\int_{\mathbb{R}^3} \mathcal{M}_{\varrho,u,\theta}\, d\mathbf{v} = \varrho, \quad \int_{\mathbb{R}^3} \mathbf{v}\mathcal{M}_{\varrho,u,\theta}\, d\mathbf{v} = \varrho u, \quad \int_{\mathbb{R}^3} \frac{|\mathbf{v}|^2}{2}\mathcal{M}_{\varrho,u,\theta}\, d\mathbf{v} = \varrho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right).$$

Nevertheless, in our framework, we do not need to know θ because it does not have sense in our model. Our asymptotic limit will be only in function of ϱ and u.

It is worth stressing that in [58] the asymptotic limits of kinetic-type equation are performed in the case where the thermodynamical equilibrium is not given by a Maxwellian but by a heavy tail distribution function.

3 Derivation of macroscopic frameworks by parabolic scaling

This section is concerned with the rescaled thermostatted kinetic theory for active particles framework (2.5) which states that the time evolution and the interactions among the activity variable u of the active particles of the functional subsystems is modelled by the operator J_i and the evolution of the velocity variable $\mathbf{v}$ is ruled by the velocity jump-process modelled by V_i.

Specifically this section deals with the macroscopic equation obtained applying the low-field asymptotic limit to the thermostatted framework (2.5). The solutions of the rescaled equation are assumed to be bounded in a space of functions where all the convergence results needed will be true.

Let ε be the ratio between the microscopic length scale (mean free path) and macroscopic length scale. In order to obtain a relation between the kinetic-type Eq. (2.5) and the fluid dynamic equations, we introduce the following scaling:

$$(t, \mathbf{x}, \mathbf{v}, u, F) \to \left(\frac{t}{\varepsilon}, \mathbf{x}, \mathbf{v}, u, \varepsilon^\ell \mathcal{F}\right), \tag{3.1}$$

and the following choice of the parameters is setted:

$$\eta = \varepsilon^r, \quad \nu = \frac{1}{\varepsilon^p}, \tag{3.2}$$

where $\ell, p, r \geq 1$. Applying these transformations in (2.5) and setting for all $i \in \{1, 2, \ldots, n\}$

$$f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u) = f_i\left(\frac{t}{\varepsilon}, \mathbf{x}, \mathbf{v}, u\right), \quad \text{and} \quad \widetilde{f}_\varepsilon(t, \mathbf{x}, \mathbf{v}, u) = \sum_{i=1}^{n} f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u),$$

the thermostatted framework (2.5) thus reads:

$$(\varepsilon\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{x}})\, f_{i,\varepsilon} + \varepsilon^\ell \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] = \varepsilon^r J_i[\mathbf{f}_\varepsilon] + \frac{1}{\varepsilon^p} V_i[f_{i,\varepsilon}], \tag{3.3}$$

where $J_i[\mathbf{f}_\varepsilon]$ and $V_i[f_{i,\varepsilon}]$ are the operators given by the following formulas

$$\begin{aligned} J_i[\mathbf{f}_\varepsilon](t,\mathbf{x},\mathbf{v},u) &= \sum_{j=1}^{n} \int_{D_u\times D_u} \mathcal{A}_{ij}(u_*,u^*,u) f_{i,\varepsilon}(t,\mathbf{x},\mathbf{v},u_*)\, f_{j,\varepsilon}(t,\mathbf{x},\mathbf{v},u^*)\, du_*\, du^* \\ &\quad - f_{i,\varepsilon}(t,\mathbf{x},\mathbf{v},u) \sum_{j=1}^{n} \int_{D_u} f_{j,\varepsilon}(t,\mathbf{x},\mathbf{v},u^*)\, du^*, \end{aligned} \tag{3.4}$$

$$V_i[f_{i,\varepsilon}](t,\mathbf{x},\mathbf{v},u) = \int_{D_\mathbf{v}} \left[T_i(\mathbf{v}^*,\mathbf{v}) f_{i,\varepsilon}(t,\mathbf{x},\mathbf{v}^*,u) - T_i(\mathbf{v},\mathbf{v}^*) f_{i,\varepsilon}(t,\mathbf{x},\mathbf{v},u)\right] d\mathbf{v}^*, \tag{3.5}$$

for $\mathbf{f}_\varepsilon = (f_{1,\varepsilon}, f_{2,\varepsilon}, \ldots, f_{n,\varepsilon})$.

It is worth stressing the singular character of the problem because the small parameter ε basically multiplies all derivatives in the equation (3.3).

The limit of Eq. (3.3), when ϵ goes to zero, strongly depends on the properties of the turning operator $V_i[f_i]$, as it appears in the leading order term in (3.3).

Before proceeding further, the turning operator $V_i[f_i]$ has to be recast in a suitable form: a gain term $K[f_i]$ (global operator) and a loss term $\nu_i f_i$ (local operator).

$$\begin{aligned} V_i[f_i] &= \int_{D_\mathbf{v}} \left[T_i(\mathbf{v},\mathbf{v}^*) f_i(t,\mathbf{x},\mathbf{v}^*,u) - T_i(\mathbf{v}^*,\mathbf{v}) f_i(t,\mathbf{x},\mathbf{v},u)\right] d\mathbf{v}^* \\ &:= K[f_i] - \nu_i f_i, \end{aligned}$$

where

$$K[f_i] = \int_{D_\mathbf{v}} T_i(\mathbf{v},\mathbf{v}^*)\, f_i(t,\mathbf{x},\mathbf{v}^*,u)\, d\mathbf{v}^*, \qquad \nu_i(\mathbf{v}) = \int_{D_\mathbf{v}} T_i(\mathbf{v}^*,\mathbf{v})\, d\mathbf{v}^*.$$

We consider the following assumptions:

($\mathbf{A}_6$) The turning operator V_i satisfies, for all f_i and $i \in \{1,2,\ldots,n\}$, the following solvability conditions:

$$\int_{D_\mathbf{v}} V_i[f_i](t,\mathbf{x},\mathbf{v},u)\, d\mathbf{v} = 0; \qquad \int_{D_\mathbf{v}} \mathbf{v}\, V_i[f_i](t,\mathbf{x},\mathbf{v},u)\, d\mathbf{v} = 0. \tag{3.6}$$

($\mathbf{A}_7$) (i) There exists a bounded equilibrium velocity distribution, denoted by $F_i(\mathbf{v}) : D_\mathbf{v} \to \mathbb{R}^+$, independent of t and $\mathbf{x}$, such that the following detailed balance assumption holds:

$$T_i(\mathbf{v}^*,\mathbf{v}) F_i(\mathbf{v}) = T_i(\mathbf{v},\mathbf{v}^*) F_i(\mathbf{v}^*), \quad i \in \{1,2,\ldots,n\}. \tag{3.7}$$

Furthermore, the flow produced by this equilibrium distribution F_i vanishes, and F_i is normalized:

$$\int_{D_\mathbf{v}} \mathbf{v} F_i(\mathbf{v})\, d\mathbf{v} = 0, \quad \int_{D_\mathbf{v}} F_i(\mathbf{v})\, d\mathbf{v} = 1, \qquad V_i[F_i] = 0. \tag{3.8}$$

(ii) The kernel $T_i(\mathbf{v}, \mathbf{v}^*)$ is locally integrable on $D_\mathbf{v} \times D_\mathbf{v}$, bounded, and there exists a constant $\sigma_i > 0$ such that

$$T_i(\mathbf{v}, \mathbf{v}^*) \geq \sigma_i F_i(\mathbf{v}), \quad \forall (\mathbf{v}, \mathbf{v}^*) \in D_\mathbf{v} \times D_\mathbf{v}. \tag{3.9}$$

($\mathbf{A}_8$) (i) The collision frequency ν_i is a function locally integrable on $D_\mathbf{v}$ and even with respect to $\mathbf{v}$, namely

$$\nu_i(-\mathbf{v}) = \nu_i(\mathbf{v}) > 0 \quad \text{forall}\, \mathbf{v} \in D_\mathbf{v}.$$

(ii) The function F_i defined by the assumption ($\mathbf{A}_7$) is such that $F_i \in L^1(D_\mathbf{v})$, $|\mathbf{v}|^2 \nu_i(\mathbf{v})^{-1} F_i$ is locally integrable and

$$\nu_i(\mathbf{v}) F_i(\mathbf{v}) = K_i[F_i](\mathbf{v}) = \int_{D_\mathbf{v}} T_i(\mathbf{v}, \mathbf{v}^*)\, F_i(\mathbf{v}^*)\, d\mathbf{v}^*.$$

Remark 3.1.

- The assumption (A_6) is trivially satisfied when T is symmetric with respect to $\mathbf{v}$ and $\mathbf{v}'$ (then the equilibrium is the constant).
- Furthermore, it is easy to check that (3.8) implies

$$\nu_i(\mathbf{v}) = \int_{D_\mathbf{v}} T_i(\mathbf{v}, \mathbf{v}^*) d\mathbf{v}^* = \int_{D_\mathbf{v}} T_i(\mathbf{v}^*, \mathbf{v}) d\mathbf{v}^*.$$

Preliminary to the asymptotic analysis, it follows a detailed qualitative analysis of the operator $V_i[f_i]$. Specifically it is proven that for the operator $V_i[f_i]$, the H-Theorem is very simple and can be expressed in the $H := L^2(D_\mathbf{v}, F_i^{-1}\, d\mathbf{v})$ norm. The interaction processes have a relaxation effect which forces the solution of (3.3) to resembles an equilibrium. This is related to dissipation mechanisms that we rephrase here as a spectral gap property and we give the Fredholm alternative type property which we need as well. These properties are summarized in the following Lemmas.

Lemma 3.1 *Assume that the Assumptions ($\boldsymbol{A}_6$)-($\boldsymbol{A}_7$)-($\boldsymbol{A}_8$) hold. Then the following statements on the operator V_i, for $i \in \{1, 2, \ldots, n\}$, hold true.*

(i) (H-Theorem). The entropy equality

$$-\int_{D_\mathbf{v}} V_i[f_i]\, f_i F_i^{-1}\, d\mathbf{v} = \frac{1}{2} \int\limits_{D_\mathbf{v} \times D_\mathbf{v}} T_i(\mathbf{v}, \mathbf{v}^*)\, F_i^* \left(\frac{f_i^*}{F_i^*} - \frac{f_i}{F_i} \right)^2 d\mathbf{v}\, d\mathbf{v}^* \geqslant 0 \tag{3.10}$$

holds.

(ii) The null-space of V_i is spanned by a unique normalized and nonnegative function $F_i(\mathbf{v})$:

$$Ker(V_i) = \text{Span}\{F_i\}.$$

(iii) There exists a constant $\kappa > 0$ such that

$$-\int_{D_{\mathbf{v}}} V_i[f_i]\, f_i F_i^{-1}\, d\mathbf{v} \geqslant \kappa \int_{D_{\mathbf{v}}} |f_i - \langle f_i\rangle F_i|^2\, \frac{\nu d\mathbf{v}}{F_i(\mathbf{v})} = \kappa \|f_i - \langle f_i\rangle\|_H^2 \quad (3.11)$$

being $\langle g\rangle := \displaystyle\int_{D_{\mathbf{v}}} g\, d\mathbf{v}$.

(iv) The following Fredholm alternative holds: For any $h_i \in H$ satisfying $\displaystyle\int_{D_{\mathbf{v}}} h_i d\mathbf{v} = 0$, there exists a unique $f_i \in H$ such that $V_i[f_i] = h_i$ and $\displaystyle\int_{D_{\mathbf{v}}} f_i d\mathbf{v} = 0$.

The proof of Lemma 3.1 essentially relies on Fredholm's alternative and Krein-Rutmann's Theorem (see [59]). It is inspired by [60]. We postpone it to the appendix.

Next the diffusion regime requires some conditions involving the velocity function.

($\mathbf{A}_9$) The matrix

$$\mathbb{D}_i := \int_{D_{\mathbf{v}}} \mathbf{v} \otimes \mathbf{v}\, F_i(\mathbf{v})\, d\mathbf{v} \quad (3.12)$$

is positive-definite.

Remark 3.2.

- The null flux assumption ($\mathbf{A}_7$)-(i) is crucial. Indeed it makes relevant the scaling in our problem. As a matter of fact, it allows to define $\chi_i \in L^2(D_{\mathbf{v}}, d\mathbf{v}/F_i)$ as the unique solution of the equation:

$$V_i[\chi_i] = -\mathbf{v}\, F_i(\mathbf{v}), \qquad \int_{D_{\mathbf{v}}} \chi_i\, d\mathbf{v} = 0.$$

- According to Lemma 3.1, the last assumption ($\mathbf{A}_9$) leads to the positiveness of the coefficients in the limit (macroscopic) equation.

The positiveness of the coefficients $\mathbb{D}_i$ of the limit equation for the relaxation model (see (3.35) in Section 3.1) is guaranteed by the following lemma:

Lemma 3.2 *Let Assumptions ($\boldsymbol{A}_7$)-($\boldsymbol{A}_9$) be satisfied and set*

$$\mathcal{D}_i = \int_{D_{\mathbf{v}}} \mathbf{v} \otimes \chi_i\, d\mathbf{v}.$$

Then there exists $\alpha > 0$ such that for all $\xi \in D_{\mathbf{v}}$, we have $\mathcal{D}_i\xi \cdot \xi \geqslant \alpha|\xi|^2$.

Proof of Lemma 3.2. From the definition of $\mathcal{D}_i$, we have

$$\mathcal{D}_i\xi \cdot \xi = -\int_{D_{\mathbf{v}}} V_i[\chi_i \cdot \xi]\chi_i \cdot \xi \frac{d\mathbf{v}}{F_i(\mathbf{v})} \geqslant \kappa \int_{D_{\mathbf{v}}} |\chi_i \cdot \xi|^2 \frac{d\mathbf{v}}{F_i(\mathbf{v})} \geqslant 0.$$

Next observe that, from $\mathcal{D}_i\xi\cdot\xi = 0$ we deduce that $\chi_i(\mathbf{v})\cdot\xi = 0$ for a.e. $\mathbf{v}\in D_{\mathbf{v}}$, and accordingly, $V_i(\chi_i\cdot\xi) = -\mathbf{v}F_i(\mathbf{v})\cdot\xi = 0$. Thus we have

$$\mathbb{D}_i\xi\cdot\xi = \int_{D_{\mathbf{v}}} |\mathbf{v}\cdot\xi|^2 F(\mathbf{v})d\mathbf{v} = 0.$$

Under the assumption $(\mathbf{A}_7)$, this happens for $\xi = 0$ only. Consequently $\xi\mapsto \mathcal{D}_i\xi\cdot\xi$ is a continuous and positive function on $D_{\mathbf{v}}$ and we therefore conclude our lemma by defining $\alpha > 0$ as to be its minimum. ■

The function $\chi_i(\mathbf{v})$ gives the coefficient in the parabolic limit equation.
The Hilbert space $L^2(D_{\mathbf{v}}, d\mathbf{v})$ will be used systematically in the sequel; in particular, we denote by

$$\langle\psi\rangle = \int_{D_{\mathbf{v}}} \psi(\mathbf{v})\, d\mathbf{v}$$

the average of the function ψ with respect to variable $\mathbf{v}$.

Bearing all above in mind, the evolution equation for the function ϱ_i can be obtained as stated in the following theorem.

Theorem 3.1 *Let $f_{i,\varepsilon}(t,\mathbf{x},\mathbf{v},u)$ be a sequence of solutions of the scaled thermostatted kinetic equation (3.3). Assume that, for $i\in\{1,2,\dots,n\}$, the turning operator V_i satisfies the assumptions $(\boldsymbol{A}_6$-$\boldsymbol{A}_7$-$\boldsymbol{A}_8$-$\boldsymbol{A}_9)$ and, when $\varepsilon\to 0$, the following statements hold true:*

$$f_{i,\varepsilon} \longrightarrow f_i \qquad a.e. \tag{3.13}$$

$$V_i[f_{i,\varepsilon}] \longrightarrow V_i[f_i] \tag{3.14}$$

and the following quantities

$$\langle f_{i,\varepsilon}\rangle\,,\quad \langle \mathbf{v}f_{i,\varepsilon}\rangle\,,\quad \langle \mathbf{v}\otimes\mathbf{v}f_{i,\varepsilon}\rangle\,,\quad \big\langle J_i[f_{i,\varepsilon}]\big\rangle\,,\ \big\langle \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}]\big\rangle\,,\ \big\langle \mathbf{v}J_i[f_{i,\varepsilon}]\big\rangle\,,\ \big\langle \mathbf{v}\mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}]\big\rangle$$

converge, in the sense of distributions on $\mathbb{R}^*_+\times D_{\mathbf{x}}\times D_u$, to the corresponding quantities

$$\langle f_i\rangle\,,\quad \langle \mathbf{v}f_i\rangle\,,\quad \langle \mathbf{v}\otimes\mathbf{v}f_i\rangle\,,\quad \big\langle J_i[f_i]\big\rangle\,,\quad \big\langle \mathcal{T}_{\mathcal{F}_i}[f_i]\big\rangle\,,\quad \big\langle \mathbf{v}J_i[f_i]\big\rangle\,,\quad \big\langle \mathbf{v}\mathcal{T}_{\mathcal{F}_i}[f_i]\big\rangle$$

and that every formally small term in ε vanishes. Then the asymptotic limit f_i of the sequence $f_{i,\varepsilon}$ (modulo the extraction of a subsequence) admits the following factorization:

$$f_i(t,\mathbf{x},\mathbf{v},u) = \varrho_i(t,\mathbf{x},u)\, F_i(\mathbf{v}),\quad \forall i\in\{1,2,\dots,n\}, \tag{3.15}$$

where ϱ_i is the weak solution of the following equation

$$\partial_t\varrho_i + \delta_{\ell,1}\,\partial_u\left(\mathcal{F}_i(u)(1-u\mathbb{A}[\boldsymbol{\varrho}](t))\varrho_i\right) = \delta_{p,1}\mathrm{div}_{\mathbf{x}}(\mathbb{D}_{\varrho_i}\cdot\nabla_{\mathbf{x}}\varrho_i) + \delta_{r,1}H_i[\boldsymbol{\varrho}], \tag{3.16}$$

and

- *$\mathbb{A}[\boldsymbol{\varrho}](t)$ is the following operator:*

$$\mathbb{A}[\boldsymbol{\varrho}](t) = \sum_{j=1}^{n}\int_{D_{\mathbf{x}}\times D_u} u\,\varrho_j(t,\mathbf{x},u)\,d\mathbf{x}\,du, \tag{3.17}$$

  with $\boldsymbol{\varrho} = (\varrho_1,\varrho_2,\dots,\varrho_n)$;

- $\mathbb{D}_{\varrho_i}$ *is the following tensor:*

$$\mathbb{D}_{\varrho_i} = -\int_{D_{\mathbf{v}}} \mathbf{v} \otimes \chi_i(\mathbf{v})\, d\mathbf{v}, \tag{3.18}$$

- $H_i[\boldsymbol{\varrho}](t)$ *is the following operator:*

$$\begin{aligned} H_i[\boldsymbol{\varrho}] &= \sum_{j=1}^{n} \langle F_i(\mathbf{v}) F_j(\mathbf{v}) \rangle \int_{D_u \times D_u} \mathcal{A}_{ij}(u_*, u^*, u)\, \varrho_i(t, \mathbf{x}, u_*)\, \varrho_j(t, \mathbf{x}, u^*)\, du_*\, du^* \\ &- \varrho_i(t, \mathbf{x}, u) \sum_{j=1}^{n} \langle F_i(\mathbf{v}) F_j(\mathbf{v}) \rangle \int_{D_u} \varrho_j(t, \mathbf{x}, u^*)\, du^*. \end{aligned} \tag{3.19}$$

Proof. To prove Theorem 3.1 an integral approach would be more suitable than Chapman-Enskog method. Indeed, employing test functions directly on conservation equations for the velocity moments of distribution function and balancing with care the various terms, we can pass to the limit of vanishing mean free path and hope to derive our fluid description.
In order to avoid confusion we split the proof into three steps.

Step 1. (Factorization of the asymptotic limit of $f_{i,\epsilon}$).
Firstly we multiply Eq. (3.3) by ε^p and we let ε go to zero. Since $f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u)$ is, for $T > 0$, uniformly bounded in

$$L^\infty[0, T; L^2(D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_u)]$$

it weakly converges (modulo the extraction of a subsequence) to a solution $f_i(t, \mathbf{x}, \mathbf{v}, u)$ of the equation

$$V_i[f_i] = 0. \tag{3.20}$$

Therefore we deduce from Lemma 3.1 the existence of a function $\varrho_i = \varrho_i(t, \mathbf{x}, u) : [0, \infty[\times D_{\mathbf{x}} \times D_u \to \mathbb{R}^+$ independent of $\mathbf{v}$ and such that the relation (3.15) holds true.

Taking the average of the equation (3.3) w.r.t. $\mathbf{v}$, using $\mathbf{A}_6$ and dividing by ϵ, one obtains:

$$\partial_t \left\langle f_{i,\varepsilon} \right\rangle + \frac{1}{\varepsilon} \left\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} f_{i,\varepsilon} \right\rangle + \varepsilon^{\ell - 1} \left\langle \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \right\rangle = \varepsilon^{r-1} \left\langle J[f_{i,\varepsilon}] \right\rangle. \tag{3.21}$$

The assumption of convergence for $f_{i\epsilon}$ entails the convergence in the sense of distributions for its moments, so that one can pass to the limit in the equation (3.21).

Assume that $\ell > 1$ and $r > 1$. When $\varepsilon \to 0$, the third and the fourth term in Eq. (3.21) goes to zero, therefore just the limit of the term:

$$\frac{1}{\varepsilon} \left\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u) \right\rangle \tag{3.22}$$

has to be evaluated.

Step 2. (The limit of the transport term).
Next, we perform the limit when ε goes to zero in Eq (3.21). In order to calculate this limit, we study the asymptotic limit of the term

$$\frac{1}{\varepsilon} \left\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} f_{i,\varepsilon} \right\rangle. \tag{3.23}$$

According to Lemma 3.1, the term (3.23) can be rewritten as follows:

$$\begin{aligned}\frac{1}{\varepsilon}\left\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} f_{i,\varepsilon}\right\rangle &= \nabla_{\mathbf{x}}\cdot\left\langle \frac{\mathbf{v} f_{i,\varepsilon}}{\varepsilon}\right\rangle \\ &= \frac{1}{\varepsilon}\mathrm{div}_{\mathbf{x}}\left\langle \mathbf{v}\, f_{i,\varepsilon}\,\frac{F_i(\mathbf{v})}{F_i(\mathbf{v})}\right\rangle = \mathrm{div}_{\mathbf{x}}\left\langle \frac{V_i[f_{i,\varepsilon}]}{\varepsilon}\,\frac{\chi_i(\mathbf{v})}{F_i(\mathbf{v})}\right\rangle. \qquad (3.24)\end{aligned}$$

Moreover, multiplying the right-hand side and the left-hand side of Eq. (3.3) by ε^{p-1}, we obtain

$$\frac{1}{\varepsilon}V_i[f_{i,\varepsilon}] = \varepsilon^p\partial_t f_{i,\varepsilon} + \varepsilon^{p+\ell-1}\mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] + \varepsilon^{p-1}\mathbf{v}\cdot\nabla_{\mathbf{x}} f_{i,\varepsilon} - \varepsilon^{p+r-1}J[f_{i,\varepsilon}] := \mathbf{L}[f_{i\varepsilon}].$$

Therefore, plugging the above term into Eq. (3.24) gives

$$\frac{1}{\varepsilon}\left\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} f_{i,\varepsilon}\right\rangle = \mathrm{div}_{\mathbf{x}}\left\langle \mathbf{L}[f_{i,\varepsilon}]\frac{\chi_i(\mathbf{v})}{F_i(\mathbf{v})}\right\rangle.$$

The assumption of convergence of $f_{i,\varepsilon}$ ensures the convergence in the sense of distributions of its moments, so that one can pass to the limit in the equation. Note that if $p=1$ then, when $\varepsilon\to 0$, a similar procedure yields

$$\varepsilon^p\partial_t f_{i,\varepsilon} + \varepsilon^{p+\ell-1}\mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] - \varepsilon^{p+r-1}J[f_{i,\varepsilon}] \to 0 \quad \text{and} \quad \varepsilon^{p-1}\mathbf{v}\cdot\nabla_{\mathbf{x}} f_{i,\varepsilon} \to \mathbf{v}\cdot\nabla_{\mathbf{x}} f_i.$$

If $p>1$ then $\mathbf{L}[f_{i,\varepsilon}]\to 0$, when $\varepsilon\to 0$.
Bearing all above in mind, we have the following limit:

$$\frac{1}{\varepsilon}\left\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} f_{i,\varepsilon}\right\rangle \xrightarrow[\varepsilon\to 0]{} \delta_{p,1}\,\nabla_{\mathbf{x}}\cdot\left\langle \left(\frac{\chi_i(\mathbf{v})}{F_i(\mathbf{v})}\otimes\mathbf{v}\right)\nabla_{\mathbf{x}} f_i\right\rangle = \delta_{p,1}\,\nabla_{\mathbf{x}}\cdot\left\langle \left(\chi_i(\mathbf{v})\otimes\mathbf{v}\right)\nabla_{\mathbf{x}}\varrho_i\right\rangle.$$

Step 3. (The limit of the thermostat term).
The term related to the Gaussian isokinetic thermostat reads:

$$\begin{aligned}\varepsilon^{\ell-1}K_i[f_{i,\varepsilon}] &= \varepsilon^{\ell-1}\left\langle \partial_u\left(\mathcal{F}_i(u)\left(1-u\sum_{i=1}^n\int_\Omega u\, f_{i,\varepsilon}\,d\mathbf{x}\,d\mathbf{v}\,du\right) f_{i,\varepsilon}\right)\right\rangle \\ &\xrightarrow[\varepsilon\to 0]{} \delta_{l,1}\left\langle \partial_u\left(\mathcal{F}_i(u)\left(1-u\sum_{i=1}^n\int_\Omega u\,\varrho_i F_i\,d\mathbf{x}\,d\mathbf{v}\,du\right)\varrho_i F_i\right)\right\rangle \\ &= \delta_{l,1}\left\langle \partial_u\left(\mathcal{F}_i(u)\left(1-u\sum_{i=1}^n\int_\Sigma u\,\varrho_i\,d\mathbf{x}\,du\right)\varrho_i F_i\right)\right\rangle \\ &= \delta_{l,1}\,\partial_u\Big(\mathcal{F}_i(u)\left(1-u\,\mathbb{A}[\boldsymbol{\varrho}](t)\right)\varrho_i\Big). \qquad (3.25)\end{aligned}$$

Bearing all above in mind, it is an easy task to show that

$$\varepsilon^{r-1}\langle J_i[f_{i,\varepsilon}]\rangle \xrightarrow[\varepsilon\to 0]{} \delta_{r,1}H_i[\boldsymbol{\varrho}]$$

where

$$H_i[\boldsymbol{\varrho}] = \sum_{j=1}^{n} \langle F_i(\mathbf{v}) F_j(\mathbf{v}) \rangle \int\limits_{D_u \times D_u} \mathcal{A}_{ij}(u_*, u^*, u)\, \varrho_i(t, \mathbf{x}, u_*)\, \varrho_j(t, \mathbf{x}, u^*)\, du_*\, du^*$$
$$- \varrho_i(t, \mathbf{x}, u) \sum_{j=1}^{n} \langle F_i(\mathbf{v}) F_j(\mathbf{v}) \rangle \int\limits_{D_u} \varrho_j(t, \mathbf{x}, u^*)\, du^*. \tag{3.26}$$

Therefore the proof is concluded. □

Remark 3.3.
In general the tensor $\mathbb{D}_{\varrho_i}$ is non-isotropic.

3.1 On the relaxation model

In this subsection, we restrict our results to the classical case of the relaxation model. We illustrate the above limit in the simple version of a transport process as expressed by (2.5), that is an isotropic case. By isotropic we mean as usual that the diffusion coefficient $\mathbb{D}$ is invariant under all orthogonal transformations. For this analysis we assume that the set of velocities $D_{\mathbf{v}}$ is symmetric with respect to the rotation group $\mathtt{SO}(d)$. Since directional changes are described by the kernel T_i this assumption is not a restriction. Assume that the velocity domain $D_{\mathbf{v}} = s\,\mathbb{S}^{d-1}$, $s > 0$, where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and for convenience we denote

$$\omega := |D_{\mathbf{v}}| = s^{d-1}|\mathbb{S}^{d-1}| = s^{d-1}\omega_0, \quad \text{where} \quad \omega_0 = |\mathbb{S}^{d-1}|.$$

The kernel distribution is assumed to be constant; that is the redistribution is uniform in velocity space:

$$T_i(\mathbf{v}, \mathbf{v}^*) = \frac{1}{\omega} = \frac{s^{1-d}}{|\mathbb{S}^{d-1}|}.$$

In particular we take $F_i(\mathbf{v}) = \frac{1}{\sigma\omega}$. Then the operator $V_i[f_i]$ is the relaxation operator:

$$V_i[f_i] = \frac{\sigma}{\omega}\left(\langle f_i \rangle - \omega\, f_i\right), \quad \sigma > 0, \quad i \in \{1, 2, \ldots, n\}, \tag{3.27}$$

and we can write the model as follows:

$$f_i = \frac{1}{\omega}\langle f_i \rangle - \frac{1}{\sigma}\mathbf{v} \cdot \nabla_{\mathbf{x}} f_i - \frac{1}{\sigma}\partial_t f_i. \tag{3.28}$$

If we assume that $\sigma \gg 1$, then the relation (3.28) shows that f_i is isotropic, i.e. independent in $\mathbf{v}$:

$$f_i(t, \mathbf{x}, \mathbf{v}, u) \simeq \frac{1}{\omega}\,\langle f_i \rangle(t, \mathbf{x}, u) = \frac{1}{\omega}\,\varrho_i(t, \mathbf{x}). \tag{3.29}$$

This approximation suggests that

$$f_i \simeq \frac{1}{\omega^2}\varrho_i - \frac{1}{\sigma\omega}\mathbf{v} \cdot \nabla_{\mathbf{x}}\varrho_i - \frac{1}{\sigma\omega}\partial_t \varrho_i. \tag{3.30}$$

Then, in the sense of distributions on $\mathbb{R}_+^* \times D_{\mathbf{x}}$, we deduce the following relation:

$$\bar{\mathbf{J}} := \langle \mathbf{v} f_i \rangle \simeq \frac{1}{\omega^2} \langle \mathbf{v} \varrho_i \rangle - \frac{1}{\sigma\omega} \langle (\mathbf{v} \otimes \mathbf{v}) \nabla_{\mathbf{x}} \varrho_i) \rangle - \frac{1}{\sigma\omega} \partial_t \langle \mathbf{v} \varrho_i \rangle. \tag{3.31}$$

We consider now the following term:

$$\bar{\mathbf{J}}_k = -\frac{1}{\sigma\omega} \sum_{l=1}^{n} \langle \mathbf{v}_k \mathbf{v}_l \rangle \, \partial_{\mathbf{x}_l} \varrho_i. \tag{3.32}$$

Simple computations give

$$\int_{D_{\mathbf{v}}} \mathbf{v} \otimes \mathbf{v} \, d\mathbf{v} = \int_{\mathbb{S}^{d-1}} (s\zeta)(s\zeta)^T s^{d-1} d\zeta = \frac{s^2 \omega}{d} \mathbb{I}_d$$

and one gets

$$\bar{\mathbf{J}}_k \simeq -\frac{1}{\sigma\omega} \sum_{l=1}^{n} \langle \mathbf{v}_k \mathbf{v}_l \rangle \, \partial_{\mathbf{x}_l} \varrho_i = -\frac{s^2}{\sigma d} \, \mathbb{I}_d \nabla_{\mathbf{x}_l} \varrho_i. \tag{3.33}$$

Thus, under the assumption that $\sigma \gg 1$, we arrive at Fick's law

$$\bar{\mathbf{J}} \simeq -\mathbb{D}_i \, \nabla_{\mathbf{x}} \varrho$$

where according to our analysis, the diffusion tensor $\mathbb{D}$ is defined by

$$\mathbb{D}_i = \int_{D_{\mathbf{v}}} \mathbf{v} \otimes \mathbf{v} F_i(\mathbf{v}) \, d\mathbf{v} = \frac{1}{\sigma\omega} \int_{D_{\mathbf{v}}} \mathbf{v} \otimes \mathbf{v} \, d\mathbf{v} = \frac{s^2}{\sigma d} \, \mathbb{I}_d. \tag{3.34}$$

Therefore the macroscopic equation (3.16) reads:

$$\partial_t \boldsymbol{\varrho}_i + \delta_{l,1} \partial_u (\mathcal{F}_i \left(1 - u\mathbb{A}[\boldsymbol{\varrho}](t)\right) \boldsymbol{\varrho}_i) = \delta_{p,1} \mathbb{D}_i \, \Delta_{\mathbf{x}} \boldsymbol{\varrho}_i + \delta_{r,1} H_i[\boldsymbol{\varrho}], \tag{3.35}$$

where $\mathbb{D}_i$ is defined by (3.34) and

$$H_i[\boldsymbol{\varrho}] = \frac{1}{\omega} \, J_i[\boldsymbol{\varrho}].$$

4 Derivation of macroscopic frameworks by hyperbolic scaling

When the flux of the equilibrium function does not vanish, it is common to deal with another scaling involving a slower time scale, see the analysis developed in paper [11] in the case of chemosensitive movement and in paper [12] for a simple fluid or for mixtures.

The hyperbolic asymptotic limit technically consists in expanding the distribution function f_i in terms of a small dimensionless parameter ε by using the following scaling for the microscopic state variables:

$$(t, \mathbf{x}, \mathbf{v}\,, u) \to \left(\frac{t}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon}, \mathbf{v}, u \right), \tag{4.1}$$

together with a rescaling of the external force field:

$$\mathcal{F}_i(u) \to \varepsilon^{\ell} \mathcal{F}_i(u), \quad \forall i \in \{1, 2, \ldots, n\}, \qquad l \geq 1, \tag{4.2}$$

and letting the parameter ε go toward zero. The hyperbolic scaling given by Eq. (4.1) (this terminology will get more clear later) implies that the turning time (the inverse of the turning frequency $1/\nu$) is small compared with the typical mechanical time τ of the system. In particular the following relation holds:

$$\tau\nu = \frac{1}{\varepsilon}. \tag{4.3}$$

Accordingly the parameter ε models the ratio between these two times.

Moreover in paper [51] the magnitude of the interaction rate η related to the activity variable is assumed to be small compared with the turning frequency ν related to the velocity variable. Therefore the following choice for the interaction rate η and the turning rate ν is performed:

$$\eta = \varepsilon^{r-1}, \quad \nu = \frac{1}{\varepsilon} \tag{4.4}$$

where $r \geq 1$ and ε is a small real parameter. Inserting the hyperbolic scaling (4.1) and the various rates set in (4.4), we are interested in the following rescaled thermostatted kinetic framework:

$$\varepsilon \left(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}}\right) f_{i,\varepsilon} + \varepsilon^{\ell} \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] = \varepsilon^{r} J_i[\mathbf{f}_\varepsilon] + V_i[f_{i,\varepsilon}], \tag{4.5}$$

where $\mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}]$ is defined in (2.8) and where, as already mentioned in the previous section, with a slight abuse of notation, we have set for all $i \in \{1, 2, \ldots, n\}$

$$f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u) = f_i\left(\frac{t}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon}, \mathbf{v}, u\right), \quad \text{and} \quad \widetilde{f}_\varepsilon(t, \mathbf{x}, \mathbf{v}, u) = \sum_{i=1}^{n} f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u),$$

where $J_i[\mathbf{f}_\varepsilon]$ and $V_i[f_{i,\varepsilon}]$ are the operators defined by formulas (3.4) and (3.5), respectively.

It is worth mentioning that, as usually, the scaling in time allows us to choose the phenomenon we want to emphasize. By varying ℓ and r we can formally derive the following systems.

4.1 Hyperbolic equations for macroscopic variables

The aim of this section is the derivation of macroscopic equations for the local density and the first velocity momentum obtained by performing the asymptotic limit of Eq. (4.5) as ε goes to zero.

The formal analysis is made under some additional assumptions regarding the turning operator V_i. According to the function (2.16), we consider the following hypotheses about the equilibrium state of $V_i[f_i]$.

($\mathbf{A}_{10}$) For all $\varrho_i \in [0, +\infty)$ and $\mathbb{U}_i \in \mathbb{R}^d$ there exists a unique function $F_{\varrho_i,\mathbb{U}_i} = F_{\varrho_i,\mathbb{U}_i}(\mathbf{v}) \in L^1(D_{\mathbf{v}}, (1 + |\mathbf{v}|)\, d\mathbf{v})$ such that:

$$V_i[F_{\varrho_i,\mathbb{U}_i}] = 0, \quad \int_{D_{\mathbf{v}}} F_{\varrho_i,\mathbb{U}_i}(\mathbf{v}) d\mathbf{v} = \varrho_i, \quad \int_{D_{\mathbf{v}}} \mathbf{v}\, F_{\varrho_i,\mathbb{U}_i}(\mathbf{v})\, d\mathbf{v} = \varrho_i\, \mathbb{U}_i. \tag{4.6}$$

Remark 4.1.
Since the above balance is not Maxwellian, we prefer to indicate it by F. Moreover, the assumption ($\mathbf{A}_{10}$) simply means that the kernel of V_i is $(d+1)$-dimensional.

Let $\psi \in \{1, \mathbf{v}\}$. Then multiplying the rescaled Eq. (4.5) by ψ and integrating with respect to velocity variable $\mathbf{v}$ we have

$$\varepsilon \int_{D_{\mathbf{v}}} (\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}})\, f_{i,\varepsilon}\, \psi \, d\mathbf{v} + \varepsilon^{\ell} \partial_u \int_{D_{\mathbf{v}}} \psi \, \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}]\, d\mathbf{v} = \varepsilon^{r} \int_{D_{\mathbf{v}}} \psi \, J_i[\mathbf{f}_{i,\varepsilon}]\, d\mathbf{v} + \int_{D_{\mathbf{v}}} \psi \, V_i[f_{i,\varepsilon}]\, d\mathbf{v}. \tag{4.7}$$

The system of conservation law equations will be derived from Eq. (4.7) by performing the limit when ϵ goes to zero. Specifically in this section we show that the moments of $f_{i,\varepsilon}$, the weak limit of the sequence $f_{i,\varepsilon}$, solve a class of hyperbolic equations.

The local macroscopic laws equations (mass conservation and momentum) are obtained multiplying (4.7) by 1 and $\mathbf{v}$ respectively and averaging over velocity. Therefore, according to the assumptions on the turning operator, these equations read:

$$\partial_t \left\langle f_{i,\varepsilon} \right\rangle + \left\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} f_{i,\varepsilon} \right\rangle + \varepsilon^{\ell-1} \left\langle \partial_u \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \right\rangle = \varepsilon^{r-1} \left\langle J_i[\mathbf{f}_{\varepsilon}] \right\rangle, \tag{4.8a}$$

$$\partial_t \left\langle \mathbf{v} f_{i,\varepsilon} \right\rangle + \operatorname{div}_{\mathbf{x}} \left\langle \mathbf{v} \otimes \mathbf{v} f_{i,\varepsilon} \right\rangle + \varepsilon^{\ell-1} \left\langle \mathbf{v} \partial_u \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \right\rangle = \varepsilon^{r-1} \left\langle \mathbf{v} J_i[\mathbf{f}_{\varepsilon}] \right\rangle. \tag{4.8b}$$

In what follows we set $K_i[f_{i,\varepsilon}] = \left\langle \partial_u \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \right\rangle$ and $S_i[f_{i,\varepsilon}] = \left\langle \mathbf{v} \partial_u \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \right\rangle$.

We are now able to enunciate the statement of the main result of this section, which it will be split in two parts: the first is devoted to the limit of the rescaled distribution function $f_{i,\varepsilon}$ while the second part refers to the limit of the number density and mass velocity. The proofs of technical details are postponed to the Appendix.

Part I. The main result is the following theorem.

Theorem 4.1 *Let $f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u)$, for $i \in \{1, 2, \dots, n\}$, be a sequence of non-negative solutions to the rescaled thermostatted kinetic framework (4.5) such that $f_{i,\varepsilon}$ converges, in the sense of distributions theory, to a function $f_i \in L^2([0,T]; L^2(\Omega))$ as ε goes to zero. Furthermore, assume that as $\varepsilon \to 0$ the terms*

$$\left\langle f_{i,\varepsilon} \right\rangle, \ \left\langle \mathbf{v} f_{i,\varepsilon} \right\rangle, \ \left\langle \mathbf{v} \otimes \mathbf{v} f_{i,\varepsilon} \right\rangle, \quad \left\langle J_i[f_{i,\varepsilon}] \right\rangle, \ \left\langle \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \right\rangle \ \left\langle \mathbf{v} J_i[f_{i,\varepsilon}] \right\rangle, \ \left\langle \mathbf{v} \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \right\rangle$$

converge, in the sense of distributions, to the corresponding terms

$$\left\langle f_i \right\rangle, \quad \left\langle \mathbf{v} f_i \right\rangle, \quad \left\langle \mathbf{v} \otimes \mathbf{v} f_i \right\rangle, \quad \left\langle J_i[f_i] \right\rangle, \quad \left\langle \mathcal{T}_{\mathcal{F}_i}[f_i] \right\rangle \quad \left\langle \mathbf{v} J_i[f_i] \right\rangle, \quad \left\langle \mathbf{v} \mathcal{T}_{\mathcal{F}_i}[f_i] \right\rangle$$

and that every formally small term in ε vanishes. Then the asymptotic limit f_i admits the following form

$$f_i(t, \mathbf{x}, \mathbf{v}, u) = F_{\varrho_i(t,\mathbf{x},u), \mathbb{U}_i(t,\mathbf{x},u)}(\mathbf{v}), \qquad i \in \{1, 2, \dots, n\}. \tag{4.9}$$

where ϱ_i and $\varrho_i \mathbb{U}_i$ are the weak solutions of the following macroscopic equations:

$$\partial_t \varrho_i + \operatorname{div}_{\mathbf{x}} (\varrho_i \mathbb{U}_i) + \delta_{\ell,1} K_i[\rho_i] = \delta_{1,r} \left\langle J_i[\mathbf{f}] \right\rangle \tag{4.10a}$$

$$\partial_t (\varrho_i \mathbb{U}_i) + \operatorname{div}_{\mathbf{x}} (\varrho_i \mathbb{U}_i \otimes \mathbb{U}_i + \mathbb{P}_i) + \delta_{\ell,1} S_i[\varrho_i \mathbb{U}_i] = \delta_{1,r} \left\langle \mathbf{v} J_i[\mathbf{f}] \right\rangle, \tag{4.10b}$$

where

$$K_i[\rho_i] = \partial_u \left(\mathcal{F}_i(u) \left(1 - u\, \mathbb{A}[\boldsymbol{\varrho}](t)\right) \varrho_i \right)$$

$$S_i[\varrho_i \mathbb{U}_i] = \partial_u \left(\mathcal{F}_i(u) \left(1 - u\, \mathbb{A}[\boldsymbol{\varrho}](t)\right) \varrho_i \mathbb{U}_i \right)$$

being $\mathbb{P}_i$ positive-definite matrix (the stress tensor) and $\delta_{h,k}$ the standard Kronecker delta.

Remark 4.2.
The first equation (4.10a) looks like a "mass conservation equation"; in fact, let us point out that the right-hand side $\delta_{1,r}\langle J_i[\mathbf{f}]\rangle$ is unknown, because it depends (in particular) on the arbitrary $\eta_{ij}\mathcal{A}_{ij}$. The production terms $\langle J_i[\mathbf{f}]\rangle$ and $\langle \mathbf{v} J_i[\mathbf{f}]\rangle$ can be specified if J_i is given.

Proof. We make use of a technique similar to the one employed in [8]. The proof of this theorem is divided into three steps.

Step 1. (Expansion of $f_{i,\epsilon}$).
The convergence hypothesis for $f_{i,\varepsilon}$ entails the convergence in the sense of distributions of its moments so that one can pass to the limit in Eq (4.5) when $\varepsilon \to 0$, which yields $V_i[f_{i,0}] = 0$.
Now we consider, for $i \in \{1, \dots, n\}$, the following moments of $f_{i,\varepsilon}$:

$$\varrho_{i,\varepsilon}(t, \mathbf{x}, u) = \int_{D_{\mathbf{v}}} f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}, \tag{4.11}$$

$$\varrho_{i,\varepsilon}(t, \mathbf{x}, u)\mathbb{U}_{i,\varepsilon}(t, \mathbf{x}, u) = \int_{D_{\mathbf{v}}} \mathbf{v} f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}. \tag{4.12}$$

Assumption $(\mathbf{A}_{10})$ implies that there exists the unique function $F_{\varrho_i, \mathbb{U}_i}$, where $\mathbb{U}_i$ depends on $(t, \mathbf{x}, u)$, verifying the conditions (4.6). Therefore

$$f_{i,0}(t, \mathbf{x}, \mathbf{v}, u) = F_{\varrho_i(t,\mathbf{x},u), \mathbb{U}_i(t,\mathbf{x},u)}(\mathbf{v}), \qquad i \in \{1, 2, \dots, n\}. \tag{4.13}$$

As a consequence, following [11], we introduce the function g_i (uniformly bounded in a suitable functional space) such that:

$$f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u) = F_{\varrho_i, \mathbb{U}_i}(\mathbf{v}) + \varepsilon g_i(t, \mathbf{x}, \mathbf{v}, u). \tag{4.14}$$

Step 2. Recalling (4.8a) and (4.8b) we obtain

$$\partial_t \left\langle f_{i,\varepsilon} \right\rangle + \left\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} f_{i,\varepsilon} \right\rangle \longrightarrow \partial_t \varrho_i + \operatorname{div}_{\mathbf{x}} (\varrho_i \mathbb{U}_i)$$

$$\partial_t \left\langle \mathbf{v} f_{i,\varepsilon} \right\rangle + \operatorname{div}_{\mathbf{x}} \left\langle \mathbf{v} \otimes \mathbf{v} f_{i,\varepsilon} \right\rangle \longrightarrow \partial_t(\varrho_i \mathbb{U}_i) + \nabla_{\mathbf{x}} \cdot \left(\int_{D_{\mathbf{v}}} \mathbf{v} \otimes \mathbf{v} F_{\varrho_i, \mathbb{U}_i} \right).$$

As previously mentioned

$$\int (\mathbf{v} \otimes \mathbf{v}) F_{\varrho_i, \mathbb{U}}\, d\mathbf{v} = \varrho_i \mathbb{U}_i \otimes \mathbb{U}_i + \mathbb{P}_i,$$

therefore

$$\partial_t \left\langle \mathbf{v} f_{i,\varepsilon} \right\rangle + \operatorname{div}_{\mathbf{x}} \left\langle \mathbf{v} \otimes \mathbf{v} f_{i,\varepsilon} \right\rangle \longrightarrow \partial_t(\varrho_i \mathbb{U}_i) + \nabla_{\mathbf{x}} \cdot \left(\varrho_i \mathbb{U}_i \otimes \mathbb{U}_i + \mathbb{P}_i \right),$$

which leads us to infer that the asymptotic limit depends on the parameter ℓ and r as follows:

- If $\ell > 1$ and $r > 1$, then the pair $(\varrho_i, \varrho_i \mathbb{U}_i)$ is weak solution of the following hyperbolic system without source term:

$$\partial_t \varrho_i + \operatorname{div}_{\mathbf{x}} (\varrho_i \mathbb{U}_i) = 0 \tag{4.15a}$$
$$\partial_t (\varrho_i \mathbb{U}_i) + \nabla_{\mathbf{x}} \cdot (\varrho_i \mathbb{U}_i \otimes \mathbb{U}_i + \mathbb{P}_i) = 0. \tag{4.15b}$$

- If $\ell = 1$ and $r > 1$. Letting $\varepsilon \to 0$, we get

$$\partial_t \left\langle f_{i,\varepsilon} \right\rangle + \left\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} f_{i,\varepsilon} \right\rangle + K_i[f_{i,\varepsilon}] \longrightarrow \partial_t \varrho_i + \operatorname{div}_{\mathbf{x}} (\varrho_i \mathbb{U}_i) + K_i[f_i]$$
$$\partial_t \left\langle \mathbf{v} f_{i,\varepsilon} \right\rangle + \operatorname{div}_{\mathbf{x}} \left\langle \mathbf{v} \otimes \mathbf{v} f_{i,\varepsilon} \right\rangle + S_i[f_{i,\varepsilon}] \longrightarrow \partial_t (\varrho_i \mathbb{U}_i) + \nabla_{\mathbf{x}} \cdot (\varrho_i \mathbb{U}_i \otimes \mathbb{U}_i + \mathbb{P}_i) + S_i[f_i].$$

It remains to identify $K_i[f_i] = \lim_{\varepsilon \to 0} K_i[f_{i,\varepsilon}]$ and $S_i[f_i] = \lim_{\varepsilon \to 0} S_i[f_{i,\varepsilon}]$.

Step 3. (The asymptotic limit of the thermostat term).
The velocity average of the term related to the Gaussian thermostat reads:

$$K_i[f_{i,\varepsilon}] = \left\langle \partial_u \left(\mathcal{F}_i(u) \left(1 - u \int_\Omega u \, \widetilde{f}_\varepsilon \, d\mathbf{x} \, d\mathbf{v} \, du \right) f_{i,\varepsilon} \right) \right\rangle. \tag{4.16}$$

Letting ε go to zero, and according to the (4.9), one has the following:

$$\begin{aligned} K_i[f_{i,\varepsilon}] \quad \xrightarrow[\varepsilon \to 0]{} \quad K_i[\varrho_i] &= \left\langle \partial_u \left(\mathcal{F}_i(u) \left(1 - u \sum_{j=1}^{n} \int_\Omega u \, F_{\varrho_j, \mathbb{U}_j} \, d\mathbf{x} \, d\mathbf{v} \, du \right) F_{\varrho_i, \mathbb{U}_i} \right) \right\rangle \\ &= \left\langle \partial_u \left(\mathcal{F}_i(u) \left(1 - u \sum_{j=1}^{n} \int_\Sigma u \, \varrho_j \, d\mathbf{x} \, du \right) F_{\varrho_i, \mathbb{U}_i} \right) \right\rangle. \end{aligned} \tag{4.17}$$

Fubini' Theorem implies that

$$\begin{aligned} K_i[\varrho_i] &= \partial_u \left(\mathcal{F}_i(u) \left(1 - u \sum_{j=1}^{n} \int_\Sigma u \, \varrho_j \, d\mathbf{x} \, du \right) \int_{D_{\mathbf{v}}} F_{\varrho_i, \mathbb{U}_i}(\mathbf{v}) d\mathbf{v} \right) \\ &= \partial_u \left(\mathcal{F}_i(u) \left(1 - u \, \mathbb{A}[\boldsymbol{\varrho}](t) \right) \varrho_i \right), \end{aligned} \tag{4.18}$$

where $\mathbb{A}[\boldsymbol{\varrho}]$ is given by Eq. (3.17).
The term $S_i[f_{i,\varepsilon}]$ reads:

$$S_i[f_{i,\varepsilon}] = \left\langle \mathbf{v} \partial_u \left(\mathcal{F}_i(u) \left(1 - u \int_\Omega u \, \widetilde{f}_\varepsilon \, d\mathbf{x} \, d\mathbf{v} \, du \right) f_{i,\varepsilon} \right). \right\rangle \tag{4.19}$$

Letting ε go to zero, and according to the (4.9), one has the following:

$$\begin{aligned} S_i[f_{i,\varepsilon}] \quad \xrightarrow[\varepsilon \to 0]{} \quad S_i[\varrho_i \mathbb{U}_i] &= \left\langle \mathbf{v} \partial_u \left(\mathcal{F}_i(u) \left(1 - u \sum_{j=1}^{n} \int_\Omega u \, F_{\varrho_j, \mathbb{U}_j} \, d\mathbf{x} \, d\mathbf{v} \, du \right) F_{\varrho_i, \mathbb{U}_i} \right) \right\rangle \\ &= \left\langle \partial_u \left(\mathcal{F}_i(u) \left(1 - u \sum_{j=1}^{n} \int_\Sigma u \, \varrho_j \, d\mathbf{x} \, du \right) \mathbf{v} F_{\varrho_i, \mathbb{U}_i} \right) \right\rangle \\ &= \partial_u \left(\mathcal{F}_i(u) \left(1 - u \, \mathbb{A}[\boldsymbol{\varrho}](t) \right) \varrho_i \mathbb{U}_i \right). \end{aligned} \tag{4.20}$$

Finally, ϱ_i and $\mathbb{U}_i$ satisfy the relations:

$$\partial_t \varrho_i + \mathrm{div}_{\mathbf{x}} (\varrho_i \mathbb{U}_i) + K_i[\varrho_i] = \delta_{1,r} \left\langle J_i[\mathbf{f}] \right\rangle \tag{4.21a}$$

$$\partial_t (\varrho_i \mathbb{U}_i) + \nabla_{\mathbf{x}} \cdot (\varrho_i \mathbb{U}_i \otimes \mathbb{U}_i + \mathbb{P}_i) + S_i[\varrho_i \mathbb{U}_i] = \delta_{1,r} \left\langle \mathbf{v} J_i[\mathbf{f}] \right\rangle, \tag{4.21b}$$

where $K_i[\mathbf{f}]$ and $S_i[\mathbf{f}]$ are given by relations (4.18) and (4.20). Therefore the proof is concluded.

It is worth observing that the influence of the turning operator V_i on the macroscopic equations comes into play through the equilibrium state $F_{\varrho_i, \mathbb{U}_i}$ and the pressure tensor $\mathbb{P}_i$. □

Part II. The next step is to obtain the asymptotic limit for the global density and mass velocity. Accordingly, we define

$$n[\mathbf{f}_\varepsilon](t, \mathbf{x}) = \sum_{i=1}^{n} \int_\Sigma f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}\, du \tag{4.22}$$

and

$$q[\mathbf{f}_\varepsilon](t, \mathbf{x}) = n[\mathbf{f}_\varepsilon](t, \mathbf{x}) \mathbb{U}[\mathbf{f}_\varepsilon](t, \mathbf{x}) = \sum_{i=1}^{n} \int_\Sigma \mathbf{v} f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}\, du. \tag{4.23}$$

From (4.14) and (4.6) one has

$$\begin{aligned} \sum_{i=1}^{n} \int f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}\, du &= \sum_{i=1}^{n} \int F_{\varrho_i, \mathbb{U}_i}\, d\mathbf{v}\, du + \varepsilon \sum_{i=1}^{n} \int g_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}\, du \\ &= \sum_{i=1}^{n} \int \varrho_i(t, \mathbf{x}, u)\, du + \varepsilon \sum_{i=1}^{n} \int g_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}\, du \\ &= n[\mathbf{f}](t, \mathbf{x}) + O(\varepsilon) \end{aligned}$$

which means that $n[\mathbf{f}](t, \mathbf{x})$ is the first approximation of the global density given by

$$n[\mathbf{f}](t, \mathbf{x}) = \sum_{i=1}^{n} \int_{D_u} \varrho_i(t, \mathbf{x}, u)\, du. \tag{4.24}$$

Similarly one obtains that

$$n[\mathbf{f}_\varepsilon](t, \mathbf{x}) \mathbb{U}[\mathbf{f}_\varepsilon](t, \mathbf{x}) = q[\mathbf{f}](t, \mathbf{x}) + O(\varepsilon), \tag{4.25}$$

where

$$q[\mathbf{f}](t, \mathbf{x}) = \sum_{i=1}^{n} \int_{D_u} \varrho_i(t, \mathbf{x}, u) \mathbb{U}_i(t, \mathbf{x}, u)\, du. \tag{4.26}$$

The quantity $d\mathbf{v}\, du$ is a nonnegative unit measure on $D_{\mathbf{v}} \times D_u$, so we denote by $\langle\langle \psi \rangle\rangle$ the average over this measure of any integrable function $\psi(\mathbf{v}, u)$, namely

$$\langle\langle \psi \rangle\rangle = \int_{D_{\mathbf{v}} \times Du} \psi(\mathbf{v}, u)\, d\mathbf{v}\, du. \tag{4.27}$$

The next step is to integrate Eq. (4.5) with respect to $\mathbf{v}$ and u, then one obtains

$$\partial_t \langle\!\langle f_{i,\varepsilon} \rangle\!\rangle + \langle\!\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} f_{i,\varepsilon} \rangle\!\rangle + \varepsilon^{\ell-1} \langle\!\langle \partial_u \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \rangle\!\rangle = \varepsilon^{r-1} \langle\!\langle J_i[\mathbf{f}_\varepsilon] \rangle\!\rangle \tag{4.28a}$$

$$\partial_t \langle\!\langle \mathbf{v} f_{i,\varepsilon} \rangle\!\rangle + \operatorname{div}_{\mathbf{x}} \langle\!\langle \mathbf{v} \otimes \mathbf{v} f_{i,\varepsilon} \rangle\!\rangle + \varepsilon^{\ell-1} \langle\!\langle \mathbf{v} \partial_u \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \rangle\!\rangle = \varepsilon^{r-1} \langle\!\langle \mathbf{v} J_i[\mathbf{f}_\varepsilon] \rangle\!\rangle. \tag{4.28b}$$

From the identities

$$\sum_{i=1}^{n} \int_{D_u} J_i[F_{\varrho_i,\mathbb{U}_i}, F_{\varrho_i,\mathbb{U}_i}]\, du = 0 \quad \text{and} \quad \sum_{i=1}^{n} \int_{D_u} \mathbf{v} J_i[F_{\varrho_i,\mathbb{U}_i}, F_{\varrho_i,\mathbb{U}_i}]\, du = 0 \tag{4.29}$$

and taking the sum of (4.28a)-(4.28b) over i, the problem reduces to

$$\begin{cases} \partial_t n[\mathbf{f}_{i,\varepsilon}] + \operatorname{div}_{\mathbf{x}} q[\mathbf{f}_{i,\varepsilon}] = -\varepsilon^{\ell-1} \displaystyle\int_{D_{\mathbf{v}} \times D_u} \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}]\, d\mathbf{v}\, du + \varepsilon^{r-1} \langle\!\langle J_i[\mathbf{f}_\varepsilon] \rangle\!\rangle \\ \partial_t q[\mathbf{f}_{i,\varepsilon}] + \operatorname{div}_{\mathbf{x}} \left(\displaystyle\sum_{i=1}^{n} \int_{D_{\mathbf{v}} \times D_u} \mathbf{v} \otimes \mathbf{v} F_{\varrho_i \mathbb{U}_i}\, d\mathbf{v}\, du \right) = -\varepsilon^{\ell-1} \displaystyle\int_{D_{\mathbf{v}} \times D_u} \mathbf{v} \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}]\, d\mathbf{v}\, du + \\ \qquad\qquad = \varepsilon^{r-1} \langle\!\langle \mathbf{v} J_i[\mathbf{f}_\varepsilon] \rangle\!\rangle. \end{cases}$$

The following theorem holds true.

Theorem 4.2 *Let $f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u)$, for $i \in \{1, 2, \ldots, n\}$, be a sequence of non-negative solutions to the rescaled thermostatted kinetic framework (4.5) such that $f_{i,\varepsilon}$ converges, in the sense of distributions theory, to a function $f_i \in L^2([0,T]; L^2(\Omega))$ as ε goes to zero. Assume that $f_{i,\varepsilon}$ has the following form*

$$f_{i,\varepsilon}(t, \mathbf{x}, \mathbf{v}, u) = F_{\varrho_i,\mathbb{U}_i}(\mathbf{v}) + \varepsilon g_i(t, \mathbf{x}, \mathbf{v}, u),$$

that as $\varepsilon \to 0$ the following terms

$$\langle\!\langle f_{i,\varepsilon} \rangle\!\rangle,\ \langle\!\langle \mathbf{v} f_{i,\varepsilon} \rangle\!\rangle,\ \langle\!\langle \mathbf{v} \otimes \mathbf{v} f_{i,\varepsilon} \rangle\!\rangle,\ \langle\!\langle \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \rangle\!\rangle,\ \langle\!\langle \mathbf{v} \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \rangle\!\rangle,\ \langle\!\langle J_i[f_{i,\varepsilon}] \rangle\!\rangle,\ \langle\!\langle \mathbf{v} J_i[f_{i,\varepsilon}] \rangle\!\rangle$$

converge, in the sense of the distributions theory, to the corresponding terms

$$\langle\!\langle f_i \rangle\!\rangle, \quad \langle\!\langle \mathbf{v} f_i \rangle\!\rangle, \quad \langle\!\langle \mathbf{v} \otimes \mathbf{v} f_i \rangle\!\rangle, \quad \langle\!\langle \mathcal{T}_{\mathcal{F}_i}[f_i] \rangle\!\rangle, \quad \langle\!\langle \mathbf{v} \mathcal{T}_{\mathcal{F}_i}[f_i] \rangle\!\rangle, \quad \langle\!\langle J_i[f_i] \rangle\!\rangle, \quad \langle\!\langle \mathbf{v} J_i[f_i] \rangle\!\rangle$$

and that every formally small term in ε vanishes. Then the density $n[\mathbf{f}_\varepsilon]$ defined by (4.24) and the mass velocity $n[\mathbf{f}_\varepsilon]\mathbb{U}[\mathbf{f}_\varepsilon]$ defined by (4.26) converge respectively, in the sense of distribution, to $n[\mathbf{f}]$ and $q[\mathbf{f}]$ where

- *if $\ell > 1$ and $r > 1$, then*

$$\begin{cases} \partial_t n + \operatorname{div}_{\mathbf{x}} q = 0 \\ \partial_t q + \operatorname{div} \left(\displaystyle\sum_{i=1}^{n} \int_{D_{\mathbf{v}} \times D_u} \mathbf{v} \otimes \mathbf{v} F_{\varrho_i,\mathbb{U}_i}\, d\mathbf{v}\, du \right) = 0 \end{cases} \tag{4.30}$$

- *if* $\ell = 1$ *and* $r > 1$, *then*

$$\begin{cases} \partial_t n + \operatorname{div}_{\mathbf{x}} q = -\displaystyle\int_{D_{\mathbf{v}}\times D_u} \mathcal{T}_{\mathcal{F}_i}[\mathbf{f}]\, d\mathbf{v}\, du \\ \partial_t q + \operatorname{div}\left(\displaystyle\sum_{i=1}^{n} \int_{D_{\mathbf{v}}\times D_u} \mathbf{v}\otimes\mathbf{v} F_{\varrho_i,\mathbb{U}_i} d\mathbf{v} du\right) = -\displaystyle\int_{D_{\mathbf{v}}\times D_u} \mathbf{v}\mathcal{T}_{\mathcal{F}_i}[\mathbf{f}]\, d\mathbf{v}\, du. \end{cases} \tag{4.31}$$

Proof. The proof is gained by following the same steps in the proof of Theorem 4.1. It just remains the computation of the term related to the thermostat; in particular note that

$$\begin{aligned} &\int_{D_{\mathbf{v}}\times D_u} \partial_u \left(\mathcal{F}_i(u)\left(1 - u\int_\Omega u\,\widetilde{f}_\varepsilon\, d\mathbf{x}\, d\mathbf{v}\, du\right) f_{i,\varepsilon}\right) d\mathbf{v}\, du \\ &\quad = \int_{D_u} \partial_u \left(\mathcal{F}_i(u)\left(1 - u\,\mathbb{A}[\boldsymbol{\varrho}](t)\right)\varrho_i\mathbb{U}_i\right) du = \mathcal{F}_i(u)\left(1 - u\,\mathbb{A}[\boldsymbol{\varrho}](t)\right)\varrho_i\mathbb{U}_i. \end{aligned}$$

The rest of the proof is straightforward and therefore omitted. □

It is worth observing that the specific form of $V_i[f_i]$ will depend on the choice of the turning kernels and will be discussed in the next paragraph.

4.2 Macroscopic equations for the relaxation model

In order to obtain convergence results corresponding to our formal results, we restrict the scope of our study. We deal with a specific example obtained by choosing a concrete turning operator. Specifically we assume that the velocity domain $D_{\mathbf{v}}$ is the $(d-1)-$sphere of radius $s > 0$, namely

$$D_{\mathbf{v}} = sS^{d-1} = \{\mathbf{v} \in \mathbb{R}^d : |\mathbf{v}| = s\}.$$

Remember that $|\mathbf{v}| = s$, $D_{\mathbf{v}} = s\mathbb{S}^{d-1}$ and $\omega = |D_{\mathbf{v}}|$. If we choose an appropriate turning kernel, then some models with constant coefficients can be also obtained.

Assume that the probability of a change of velocity $\mathbf{v}^*$ to $\mathbf{v}$ depends on the angle between these two velocities. Then the turning kernel (2.7) is replaced by

$$T_i(\mathbf{v}, \mathbf{v}^*) = \frac{1}{\omega}\left(1 + \frac{a}{s^2}(\mathbf{v}\cdot\mathbf{v}^*)\right) \qquad \text{with} \quad a < d. \tag{4.32}$$

For the choice (4.32), simple computations (see Appendix A) give:

$$V[f_i] = \frac{\varrho_i}{\omega}\left(1 + \frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_i\right) - f_i(\mathbf{v}).$$

Before going on we set

$$F_{\varrho_i,\mathbb{U}_i}(\mathbf{v}) = \frac{\varrho_i}{\omega}\left(1 + \frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_i\right). \tag{4.33}$$

(for notational convenience, we do not mention explicitly the dependence of ϱ_i and $\mathbb{U}_i$ on $(t, \mathbf{x}, u)$).

According to the above choice we have

$$F_{\varrho_{i_*},\mathbb{U}_{i_*}}(\mathbf{v}) = \frac{\varrho_{i_*}}{\omega}\left(1 + \frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_{i_*}\right)$$

and (see Appendix A) the following identities hold true:

$$\int\limits_{D_{\mathbf{v}}} F_{\varrho_{i_*},\mathbb{U}_{i_*}}(\mathbf{v})\, F_{\varrho_{j^*},\mathbb{U}_{j^*}}(\mathbf{v})\, d\mathbf{v} = \frac{\varrho_{i_*}\varrho_{j^*}}{\omega}\left(1 + \frac{a^2}{ds^2}\,\mathbb{U}_{i_*}\cdot\mathbb{U}_{j^*}\right),$$

and

$$\int\limits_{D_{\mathbf{v}}} \mathbf{v}\, F_{\varrho_{i_*},\mathbb{U}_{i_*}}(\mathbf{v})\, F_{\varrho_{j^*},\mathbb{U}_{j^*}}(\mathbf{v})\, d\mathbf{v} = \quad \frac{\varrho_{i_*}\varrho_{j^*}}{\omega}\,\frac{a^2}{ds^2}\,(\mathbb{U}_{i_*} + \mathbb{U}_{j^*}).$$

Moreover we have

$$\begin{aligned}
\int\limits_{D_{\mathbf{v}}} \mathcal{T}_{\mathcal{F}_i}[f_i]d\mathbf{v} &= \int\limits_{D_{\mathbf{v}}} \partial_u\left(\mathcal{F}_i(u)\left(1 - u\int\limits_{\Omega} u\,\widetilde{f}(t,\mathbf{x},\mathbf{v},u)\,d\mathbf{x}\,d\mathbf{v}\,du\right) f_i(t,\mathbf{x},\mathbf{v},u)\right)\\
&= \int\limits_{D_{\mathbf{v}}} \partial_u\left(\mathcal{F}_i(u)\left(1 - u\sum_{j=1}^{n}\int_{\Omega} u\,\frac{\varrho_j}{\omega}\left(1 + \frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_j\right)\,d\mathbf{x}\,d\mathbf{v}\,du\right)\right.\\
&\qquad\qquad \left.\times\frac{\varrho_i}{\omega}\left(1 + \frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_i\right)\right)d\mathbf{v}.
\end{aligned}$$

By using Fubini's Theorem, one has

$$u\sum_{j=1}^{n}\int\limits_{\Omega} u\,\frac{\varrho_j}{\omega}\left(1 + \frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_j\right)d\mathbf{x}\,d\mathbf{v}\,du$$

$$= \frac{u}{\omega}\sum_{j=1}^{n}\int\limits_{D_{\mathbf{x}}\times D_u} u\,\varrho_j\,d\mathbf{x}\,du\int\limits_{D_{\mathbf{v}}} d\mathbf{v} + \frac{u}{\omega}\sum_{j=1}^{n}\frac{a}{s^2}\int\limits_{D_{\mathbf{x}}\times D_u}\varrho_j\mathbb{U}_j\,d\mathbf{x}\,du\cdot\int\limits_{D_{\mathbf{v}}}\mathbf{v}\,d\mathbf{v}$$

namely

$$u\sum_{j=1}^{n}\int\limits_{\Omega} u\,\frac{\varrho_j}{\omega}\left(1 + \frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_j\right)d\mathbf{x}\,d\mathbf{v}\,du = u\mathbb{A}[\boldsymbol{\varrho}],$$

and

$$\int\limits_{D_{\mathbf{v}}}\frac{\varrho_i}{\omega}\left(1 + \frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_i\right)d\mathbf{v} = \varrho_i + \frac{a\varrho_i}{\omega s^2}\,\mathbb{U}_i\cdot\int\limits_{D_{\mathbf{v}}}\mathbf{v}d\mathbf{v} = \varrho_i. \tag{4.34}$$

Thus

$$\int\limits_{D_{\mathbf{v}}} \mathcal{T}_{\mathcal{F}_i}[f_i]d\mathbf{v} = \partial_u(\mathcal{F}_i(u)(1 - u\mathbb{A}[\boldsymbol{\varrho}]\varrho_i)). \tag{4.35}$$

Finally

$$\begin{aligned}\int\limits_{D_{\mathbf{v}}} \mathbf{v}\,\frac{\varrho_i}{\omega}\left(1 + \frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_i\right)d\mathbf{v} &= \frac{\varrho_i}{\omega}\int\limits_{D_{\mathbf{v}}}\mathbf{v}\,d\mathbf{v} + \frac{a\varrho_i}{\omega s^2}\,\mathbb{U}_i\cdot\int\limits_{D_{\mathbf{v}}} v_k v_l d\mathbf{v}\\ &= \frac{a\varrho_i}{\omega s^2}\,\mathbb{U}_i\cdot\frac{\omega}{d}s^2\mathbb{I} = \frac{a}{d}\,\varrho_i\mathbb{U}_i,\end{aligned} \tag{4.36}$$

and

$$\int\limits_{D_{\mathbf{v}}} \mathbf{v}\mathcal{T}_{\mathcal{F}_i}[f_i]\,d\mathbf{v} = \partial_u\left(\mathcal{F}_i(u)\left(1 - \frac{a}{d}\,u\mathbb{A}[\boldsymbol{\varrho}]\,\varrho_i\mathbb{U}_i\right)\right).$$

The following two lemmas, which provide the computation of the integrals of equilibrium function and the operator $J_i[f]$, are key tools for the derivation of the macroscopic equation.

Lemma 4.1 *Let $u^*, u_* \in D_u$ and $\varrho^* = \varrho(t,\mathbf{x},u^*)$, $\mathbb{U}^* = \mathbb{U}(t,\mathbf{x},u^*)$, $\varrho_* = \varrho(t,\mathbf{x},u_*)$, $\mathbb{U}_* = \mathbb{U}(t,\mathbf{x},u_*)$. Then*

$$\int_{D_{\mathbf{v}}} F_{\varrho_{i_*},\mathbb{U}_{i_*}}(\mathbf{v})\,F_{\varrho_{j^*},\mathbb{U}_{j^*}}(\mathbf{v})\,d\mathbf{v} = \frac{\varrho_{i_*}\varrho_{j^*}}{\omega}\left(1 + \frac{a^2}{ds^2}\,\mathbb{U}_{i_*}\mathbb{U}_{j^*}\right) \tag{4.37}$$

$$\int_{D_{\mathbf{v}}} \mathbf{v}\,F_{\varrho_{i_*},\mathbb{U}_{i_*}}(\mathbf{v})\,F_{\varrho_{j^*},\mathbb{U}_{j^*}}(\mathbf{v})\,d\mathbf{v} = \frac{\varrho_{i_*}\varrho_{j^*}}{\omega}\,\frac{a^2}{ds^2}\,(\mathbb{U}_{i_*} + \mathbb{U}_{j^*}). \tag{4.38}$$

Lemma 4.2 *If $\mathbf{f} = (F_{\varrho_1,\mathbb{U}_1}, F_{\varrho_2,\mathbb{U}_2}, \ldots, F_{\varrho_n,\mathbb{U}_n})$, where $F_{\varrho_i,\mathbb{U}_i}$ is given by Eq. (4.33), then the following equalities hold true:*

$$\int_{D_{\mathbf{v}}} J_i[\mathbf{f}]\,d\mathbf{v} = \frac{1}{|D_{\mathbf{v}}|}\left(J_i[\boldsymbol{\varrho}] + \frac{a^2}{ds^2}\,J_i[\boldsymbol{\varrho}\mathbf{U}]\right), \tag{4.39}$$

$$\int_{D_{\mathbf{v}}} \mathbf{v}\,J_i[\mathbf{f}]\,d\mathbf{v} = \frac{1}{|D_{\mathbf{v}}|}\left(A_i[\boldsymbol{\varrho},\boldsymbol{\varrho}\mathbf{U}] + B_i[\boldsymbol{\varrho},\boldsymbol{\varrho}\mathbf{U}]\right), \tag{4.40}$$

hold true, where $\boldsymbol{\varrho} = (\varrho_1,\varrho_2,\ldots,\varrho_n)$, $\boldsymbol{\varrho}\mathbf{U} = (\varrho_1\mathbb{U}_1, \varrho_2\mathbb{U}_2, \ldots, \varrho_n\mathbb{U}_n)$ and

$$\begin{aligned}A_i(\boldsymbol{\varrho},\boldsymbol{\varrho}\mathbf{U}) &= \sum_{j=1}^{n}\int\limits_{D_u\times D_u}\mathcal{A}_{ij}(u_*,u^*,u)\,\varrho_{i*}\,\mathbb{U}_{i*}\,\varrho_j^*\,du_*\,du^*\\ &\quad -\varrho_i(u)\mathbb{U}_i(u)\sum_{j=1}^{n}\int_{D_u\times D_u}\mathcal{A}_{ij}(u_*,u^*,u)\,\varrho_j^*\,du_*\,du^*,\end{aligned} \tag{4.41}$$

$$\begin{aligned}B_i(\boldsymbol{\varrho},\boldsymbol{\varrho}\mathbf{U}) &= \sum_{j=1}^{n}\int\limits_{D_u\times D_u}\mathcal{A}_{ij}(u_*,u^*,u)\,\varrho_j^*\,\mathbb{U}_j^*\,\varrho_{i*}\,du_*\,du^*\\ &\quad -\varrho_i(u)\sum_{j=1}^{n}\int\limits_{D_u\times D_u}\mathcal{A}_{ij}(u_*,u^*,u)\,\varrho_j^*\,\mathbb{U}_j^*\,du_*\,du^*.\end{aligned} \tag{4.42}$$

The proof of these lemmas is postponed to Appendix A.

By gathering all the above results we obtain the main result for the relaxation model.

Theorem 4.3 *Let $f_{i,\varepsilon}(t,\mathbf{x},\mathbf{v},u)$ be a sequence of solutions to the rescaled kinetic equation (4.5) and V_i the relaxation operator given by Eq. (4.33) such that $f_{i,\varepsilon}$ converges, in the sense of distributions, to a function f_i as ε goes to zero. Furthermore, assume that for $\psi \in \{1,\mathbf{v}\}$, the moments*

$$\left\langle \psi f_{i,\varepsilon} \right\rangle, \qquad \left\langle \psi J_i[\mathbf{f}_\varepsilon] \right\rangle, \qquad \left\langle \psi \mathcal{T}_{\mathcal{F}_i}[f_{i,\varepsilon}] \right\rangle,$$

converge in the sense of distributions to the corresponding moments

$$\left\langle \psi f_i \right\rangle, \qquad \left\langle \psi J_i[\mathbf{f}] \right\rangle, \qquad \left\langle \psi \mathcal{T}_{\mathcal{F}_i}[f_i] \right\rangle.$$

Then the asymptotic limit f_i, for $i \in \{1,2,\dots,n\}$, has the form (4.9), where ϱ_i and $\varrho_i \mathbb{U}_i$ are the weak solutions of the following macroscopic equations:

$$\begin{cases} \partial_t \varrho_i + \operatorname{div}_{\mathbf{x}}\left(\varrho_i \mathbb{U}_i\right) + \delta_{l,1}\partial_u\left(\mathcal{F}_i(u)\left(1 - u\mathbb{A}[\boldsymbol{\varrho}](t)\right)\varrho_i\right) \\ \qquad\qquad = \dfrac{\delta_{1,r}}{|D_{\mathbf{v}}|}\left(J_i[\boldsymbol{\varrho}] + \dfrac{d}{s^2} J_i[\boldsymbol{\varrho}\mathbf{U}]\right), \\ \partial_t(\varrho_i \mathbb{U}_i) + \dfrac{s^2}{d}\nabla_{\mathbf{x}}\varrho_i + \delta_{l,1}\partial_u\left(\mathcal{F}_i(u)\left(1 - u\mathbb{A}_1[\boldsymbol{\varrho}](t)\right)\varrho_i \mathbb{U}_i\right) \\ \qquad\qquad = \dfrac{\delta_{1,r}}{|D_{\mathbf{v}}|}\left(A_i[\boldsymbol{\varrho},\boldsymbol{\varrho}\mathbf{U}] + B_i[\boldsymbol{\varrho},\boldsymbol{\varrho}\mathbf{U}]\right), \end{cases} \tag{4.43}$$

being the operators A_i and B_i given by (4.41) and (4.42) respectively and $\delta_{h,k}$ the standard Kronecker delta.

Proof. Starting with Eq. (4.5) one replaces the turning kernel $V_i[f_{i,\varepsilon}]$ by (4.33). Next letting $\varepsilon \to 0$, we obtain the equations of the moments. The rest of the proof is straightforward and therefore omitted. □

We derive now the limit of the density and mass velocity in the case of the relaxation model.

Theorem 4.4 *Let $f_{i,\varepsilon}$ be a smooth solution of the rescaled thermostatted kinetic framework (4.5). Assume that, for any $t \in [0,T]$*

$$\int_{D_{\mathbf{x}}\times D_{\mathbf{v}}\times D_u} f_{i,\varepsilon}(t,\mathbf{x},\mathbf{v},u)\, d\mathbf{x}\, d\mathbf{v}\, du < \infty, \qquad i \in \{1,2,\dots,n\} \tag{4.44}$$

and $f_{i,\varepsilon}$ converges a.e. in $[0,T]\times D_{\mathbf{x}} \times s\mathbb{S}^{d-1}\times D_u$ for some $T>0$. Furthermore, assume that the kernel $\mathcal{A}(u_*,u^*,u)$ of the operator $J_i[f_i]$ is in $L^2(D_u)^3$. Then, the pointwise limit of $f_{i,\varepsilon}$ is the function $F_{\varrho_i,\mathbb{U}_i}$ given by (4.32) where

$$\varrho_i \equiv \lim_{\varepsilon\to 0}\varrho_{i,\varepsilon}, \qquad \mathbb{U}_i \equiv \lim_{\varepsilon\to 0}\mathbb{U}_{i,\varepsilon},$$

that is, the weak moment and pointwise limit of the moments (4.11)-(4.12) of $f_{i,\varepsilon}$. Then

- *if $\ell > 1$, one obtains, as $\varepsilon \to 0$ the following limit:*

$$\begin{cases} \partial_t \varrho_i + \operatorname{div}_{\mathbf{x}} (\varrho_i \mathbb{U}_i) = 0 \\ \partial_t (\varrho_i \mathbb{U}_i) + \dfrac{s^2}{d} \nabla_{\mathbf{x}} \varrho_i = 0 \end{cases} \tag{4.45}$$

- *if $\ell = 1$, one obtains:*

$$\begin{cases} \partial_t \varrho_i + \operatorname{div}_{\mathbf{x}} (\varrho_i \mathbb{U}_i) = \partial_u \left(\mathcal{F}_i(u) \left(1 - u\, \mathbb{A}[\boldsymbol{\varrho}](t)\right) \varrho_i \right) \\ \partial_t (\varrho_i \mathbb{U}_i) + \dfrac{s^2}{d} \nabla_{\mathbf{x}} \varrho_i = \partial_u \left(\mathcal{F}_i(u) \left(1 - u\, \mathbb{A}[\boldsymbol{\varrho}](t)\right) \varrho_i \mathbb{U}_i \right). \end{cases} \tag{4.46}$$

Proof. The proof is based on the Dunford-Pettis Criterion. The assumption (4.44) ensures that any solution $f_{i,\varepsilon}$ is contained in a relatively weakly compact set of $L^1_{loc}(\mathbb{R}^*_+ \times D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_u)$. From now on, weak convergence at least means convergence in this sense.

From the statement (4.44) on $f_{i,\varepsilon}$, by virtue of Dunford-Pettis Theorem (see [68] Th. 4.21.2 p. 274), up to the extraction of a subsequence (still denoted with the index i, ε) such that $f_{i,\varepsilon} \rightharpoonup f_i$ weakly in $L^p([0,T] \times D_{\mathbf{x}} \times s\mathbb{S}^{d-1} \times D_u)$ to its pointwise limit and weakly in $L^1([0,T] \times D_{\mathbf{x}} \times s\mathbb{S}^{d-1} \times D_u)$ locally, so then strongly in $L^1_{loc}([0,T] \times D_{\mathbf{x}} \times s\mathbb{S}^{d-1} \times D_u)$. Then, there exists a function $f_{i,0}(t, \mathbf{x}, \mathbf{v}, u)$ such that, the subsequences

$$f_{i,\varepsilon} \rightharpoonup f_{i,0} \qquad \text{weakly in} \quad L^1_{loc}([0,T] \times D_{\mathbf{x}} \times s\mathbb{S}^{d-1} \times D_u),$$

$$V[f_{i,\varepsilon}] \rightharpoonup V[f_{i,0}] \qquad \text{weakly in} \quad L^1_{loc}([0,T] \times D_{\mathbf{x}} \times s\mathbb{S}^{d-1} \times D_u),$$

as $\varepsilon \to 0$. On the other hand, from the definition of J_i and the hypotheses under f_ε, $\mathcal{A}_{ij}$ and η_{ij}, we deduce similarly the convergence of

$$J_{i,\varepsilon} \rightharpoonup J_{i,0} \qquad \text{strongly in } L^1_{loc}([0,T] \times D_{\mathbf{x}} \times s\mathbb{S}^{d-1} \times D_u)$$

and

$$\mathcal{T}_{\mathcal{F}_i}(f_\varepsilon) \rightharpoonup \mathcal{T}_{\mathcal{F}_i}(f_0) \qquad \text{strongly in } L^1_{loc}([0,T] \times D_{\mathbf{x}} \times s\mathbb{S}^{d-1} \times D_u).$$

The limit f_0 is identified by taking the limit in (4.5), in distributional sense on $\mathbb{R}^*_+ \times D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_u$ to deduce that $V[f_0] = 0$ and then Lemma 4.1 ensures that $f_0 = F_{\varrho_0, \mathbb{U}_0}$.

Moreover the velocity space $s\mathbb{S}^{d-1}$ has finite measure, then the hypothesis (4.44) holds for the $\mathbf{v}$-moments of f_ε, therefore

$$\varrho_{i,\varepsilon} \rightharpoonup \varrho_{i,0}, \qquad \int_{D_{\mathbf{v}}} \mathbf{v} f_{i,\varepsilon}\, d\mathbf{v} \rightharpoonup \int_{D_{\mathbf{v}}} \mathbf{v} f_{i,0}\, d\mathbf{v} = \varrho_{i,0} \mathbb{U}_{i,0},$$

and

$$\int_{D_{\mathbf{v}}} \mathbf{v} \otimes \mathbf{v} f_{i,\varepsilon}\, d\mathbf{v} \rightharpoonup \int_{D_{\mathbf{v}}} \mathbf{v} \otimes \mathbf{v} f_{i,0}\, d\mathbf{v} = \int_{D_{\mathbf{v}}} \mathbf{v} \otimes \mathbf{v} F_{\varrho_0, \mathbb{U}_0}\, d\mathbf{v} = \frac{s^2}{d} \mathbb{I}$$

Combining the convergence of $J_i[f_{i,\varepsilon}]$ and $V_i[f_{i,\varepsilon}]$ with the weak convergence of $f_{i,\varepsilon}$, we can pass to the limit $\varepsilon \to 0$ in this relation, which proves that (n, q) is indeed solution, in the sense of distributions, of (4.28a)-(4.28b). If we successively apply Lemmas 4.1 and 4.2, the different regimes are straightforwardly obtained. This proves our claim and complete the proof. □

We are now able to formulate the derivation of macroscopic equations for the relaxation model. Let n_ε and $n_\varepsilon \mathbb{U}_\varepsilon$ be the number density and the main velocity defined in (4.24) and (4.26), then one gets the following result in the case of the relaxation model.

Theorem 4.5 *Let $f_{i,\varepsilon}$ be a smooth solution of the rescaled thermostatted kinetic framework (4.5). Under the assumptions of Theorem 4.2 the density n_ε and the mass velocity $n_\varepsilon \mathbb{U}_\varepsilon$ converge respectively, in the distributional sense, to n and q. In addition, every limit point is governed by a weak solution of the following hyperbolic equations:*

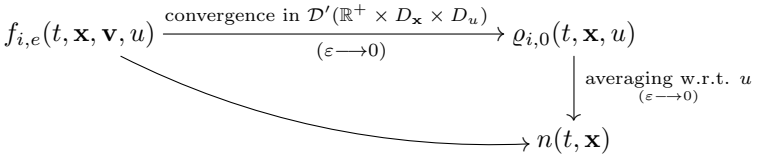
- *if $\ell > 1$, one obtains, as $\varepsilon \to 0$ the following limit:*

$$\begin{cases} \partial_t n + \operatorname{div}_{\mathbf{x}} q = 0 \\ \partial_t q + \dfrac{s^2}{d} \nabla_{\mathbf{x}} n = 0 \end{cases} \tag{4.47}$$

- *if $\ell = 1$, one obtains:*

$$\begin{cases} \partial_t n + \operatorname{div}_{\mathbf{x}} q + \mathcal{F}_i(u) \left(1 - \dfrac{a}{d} u \mathbb{A}[\boldsymbol{\varrho}] q \right) = 0 \\ \partial_t q + \dfrac{s^2}{d} \nabla_{\mathbf{x}} n + \mathcal{F}(u) \left(1 - u\, \mathbb{A}[\boldsymbol{\varrho}](t) \right) q = 0. \end{cases} \tag{4.48}$$

We can summarize our results in the following diagram:



where $n(t, \mathbf{x})$ is a weak solution to hyperbolic equations (4.47) and (4.48).

5 Applications and perspective

The present paper was concerned with the derivation of macroscopic equations for the modelling of macroscopic variables of complex systems composed by a large number of active particles. The macroscopic description has been obtained by asymptotic limits of the thermostatted kinetic for active particles models.

Therefore the linking between the mesoscopic (kinetic) and the macroscopic scale has been reached in a first attempt to obtain the multiscale description.

Applications refer to biological systems and specifically to the derivation of tissue scale models in the modelling of cancer phenomena where the interactions among the cells at the kinetic scale have a key role in the onset and growth of tumor and cancer metastasis [53, 54, 55, 56, 61, 62], capillary sprouts phenomena and angiogenesis [63, 64, 65]. A technical difficulty in treating biological systems is their evolutionary characteristics. Indeed genetic mutations may affect the dynamics at the cellular scale and subsequently the whole dynamics at the tissue scale [61].

Another application refers to the modelling of chemotaxis, which describes the motion of microorganisms induced by chemical signals that they are able to sense, transduce and eventually relay into the medium. This phenomenon plays a key role in a large number of homeostatic and pathological situations [66]. The thermostatted kinetic framework appears more appropriate for the modelling of this phenomenon because introduces the external force but ensures the conservation of the total energy and the reaching of nonequilibrium stationary steady states.

It is worth stressing that the analysis performed in this paper is brought to the attention of applied mathematicians involved in multiscale modeling and simulations of complex systems. Nevertheless we are aware that interesting problems are still open. Among others, as already mentioned, in the biological systems case, genetic mutations can modify the interactions among the cancer cells. Therefore the underlying description offered by the thermostatted kinetic theory models needs to be related to the evolution at the molecular (microscopic) scale.

Finally, as known, the collective behavior of biological, animal or human systems occurs in response to environmental factors that can affect the whole dynamics, e.g., the dynamics of swarms of insect can be modified by the attack of a predator, the tumor growth can be slacked by injections of vaccine. Therefore the environment role has to be taken into account in the models of the thermostatted kinetic theory not only by considering the external macroscopic force field $\mathcal{F}$ but also by modeling the interaction with the outer environment at the microscopic scale. Modelling external actions at the microscopic scale means representing the outer system as a specific functional subsystem with the ability to interact with the active particles of the inner system. This generalization appears to be essentially technical and has been proposed and developed in [6], while the formal asymptotic analysis appears definitively to be a hard problem.

It is worth noticing that a comparison between the parabolic and hyperbolic limit shows that a different choice (and the magnitude-predominant) of the interaction rate η and the turning rate ν modifies the structure of the macroscopic equations and correspondingly the macroscopic behavior. Therefore the scaling plays a crucial role in the treatment of different complex phenomena. Specifically the parabolic scaling allows the modelling of transport propriety instead the hyperbolic scaling is chosen consistently with the phenomenological behavior which requires models with finite speed of propagation. The interested reader in the phenomena described by the hyperbolic models is referred to the review [67].

It is worth stressing that the asymptotic analysis developed in this paper as-

sumes that the same time or space scaling (hyperbolic or parabolic) is performed in the distribution function of each functional subsystem. Mixed parabolic and hyperbolic limit can be considered with reference to the system to be modelled and according to the dispersive or non-dispersive nature of the subsystems (a subsystem is involved in a diffusion process and a subsystem has a dynamics that is dominated by the hyperbolic behavior). This is object of future analysis.

A Appendix

This Appendix is devoted to some computations that have been omitted in the main part of the paper.

A.1 Proof of Lemma 3.1

To prove the coercitivity inequality (3.10), we write

$$\begin{aligned}\int_{D_{\mathbf{v}}} V_i[f_i]\,\frac{f_i}{F_i}\,d\mathbf{v} &= \int\limits_{D_{\mathbf{v}}\times D_{\mathbf{v}}} T_i(\mathbf{v},\mathbf{v}^*) f_i^*\,\frac{f_i}{F_i}\,d\mathbf{v}\,d\mathbf{v}^* - \int_{D_{\mathbf{v}}} \nu_i(\mathbf{v})\,\frac{f_i^2}{F}\,d\mathbf{v} \\ &= \int\limits_{D_{\mathbf{v}}\times D_{\mathbf{v}}} T_i(\mathbf{v},\mathbf{v}^*) F_i^*\,\frac{f_i^*}{F_i^*}\,\frac{f_i}{F_i}\,d\mathbf{v}\,d\mathbf{v}^* - \int_{D_{\mathbf{v}}} \nu_i(\mathbf{v})\,\frac{f_i^2}{F_i}\,d\mathbf{v}, \qquad \text{(A.1)}\end{aligned}$$

where f^* denotes $f(v^*)$. We now consider the second term of the right-hand side of (A.1). On one hand we have:

$$\begin{aligned}\int_{D_{\mathbf{v}}} \nu_i(\mathbf{v})\,\frac{f_i^2}{F_i}\,d\mathbf{v} &= \int_{D_{\mathbf{v}}}\int_{D_{\mathbf{v}}} T_i(\mathbf{v}^*,\mathbf{v})\,F_i\,\frac{f_i^2}{F_i^2}\,d\mathbf{v}\,d\mathbf{v}^* \\ &= \int\limits_{D_{\mathbf{v}}\times D_{\mathbf{v}}} T_i(\mathbf{v},\mathbf{v}^*)\,F_i^*\,\frac{f_i^{*2}}{F_i^{*2}}\,d\mathbf{v}\,d\mathbf{v}^*. \qquad \text{(A.2)}\end{aligned}$$

On the other hand, considering that $\nu_i F_i = K[F_i]$, we have also:

$$\begin{aligned}\int_{D_{\mathbf{v}}} \nu_i(\mathbf{v})\,\frac{f_i^2}{F_i}\,d\mathbf{v} &= \int_{D_{\mathbf{v}}} [\nu_i(\mathbf{v})F_i]\,\frac{1}{F_i}\,\frac{f_i^2}{F_i}\,d\mathbf{v} = \int_{D_{\mathbf{v}}} K[F_i]\,\frac{f_i^2}{F_i^2}\,d\mathbf{v} \\ &= \int\limits_{D_{\mathbf{v}}\times D_{\mathbf{v}}} T_i(\mathbf{v},\mathbf{v}^*)\,F_i^*\,\frac{f_i^2}{F_i^2}\,d\mathbf{v}\,d\mathbf{v}^*. \qquad \text{(A.3)}\end{aligned}$$

Combining (A.1)-(A.2)-A.3) we get

$$\begin{aligned}2\int_{D_{\mathbf{v}}} V_i[f_i]\,\frac{f_i}{F_i}\,d\mathbf{v} &= 2\int\limits_{D_{\mathbf{v}}\times D_{\mathbf{v}}} T_i(\mathbf{v},\mathbf{v}^*)F_i^*\,\frac{f_i^*}{F_i^*}\,\frac{f_i}{F_i}\,d\mathbf{v}\,d\mathbf{v}^* - \int\limits_{D_{\mathbf{v}}\times D_{\mathbf{v}}} T_i(\mathbf{v},\mathbf{v}^*)\frac{F_i^* f_i^{*2}}{F_i^{*2}}\,d\mathbf{v}\,d\mathbf{v}^* \\ &\quad - \int\limits_{D_{\mathbf{v}}\times D_{\mathbf{v}}} T_i(\mathbf{v},\mathbf{v}^*)\,F_i^*\,\frac{f_i^2}{F_i^2}\,d\mathbf{v}\,d\mathbf{v}^* \\ &= \int\limits_{D_{\mathbf{v}}\times D_{\mathbf{v}}} T_i(\mathbf{v},\mathbf{v}^*)\,F_i^*\left[2\frac{f_i^*}{F_i^*}\,\frac{f_i}{F_i} - \frac{f_i^{*2}}{F_i^{*2}} - \frac{f_i^2}{F_i^2}\right] d\mathbf{v}\,d\mathbf{v}^*,\end{aligned}$$

so that (A.1) reads

$$-\int_{D_{\mathbf{v}}} V_i[f_i]\,\frac{f_i}{F_i}\,d\mathbf{v} = \frac{1}{2}\int\limits_{D_{\mathbf{v}}\times D_{\mathbf{v}}} T_i(\mathbf{v},\mathbf{v}^*)\,F_i^*\left(\frac{f_i^*}{F_i^*}-\frac{f_i}{F_i}\right)^2 d\mathbf{v}d\mathbf{v}^* \geqslant 0. \quad \text{(A.4)}$$

We have proved the dissipative property of the collision operator in the sense that an 'H- theorem' holds, i.e. $\displaystyle\int_{D_{\mathbf{v}}} V_i[f_i]\,\frac{f_i}{F_i}\,d\mathbf{v} \leqslant 0$. In particular, if $V_i[f_i]=0$. This suggests to introduce the space $L^2(d\mu)$ where the measure $d\mu(\mathbf{v})$ is defined on the phase space by $d\mu := d\mu(\mathbf{v})/F(\mathbf{v})$, $\mathbf{v}\in D_{\mathbf{v}}$. Then with the help of (A.4) we led to

$$\frac{f_i^*}{F_i^*} = \frac{f_i}{F_i}\,d\mu(\mathbf{v})\otimes d\mu(\mathbf{v}^*) \qquad \text{a.e.}$$

from which we deduce the result about the null space of $V[f_i]$.

The solvability conditions imply that $\displaystyle\int_{D_{\mathbf{v}}} h_i\,d\mathbf{v}=0$ is a necessary condition for the solvability of $V_i[f_i]=h_i$. The Fredholm alternative follows from a direct application of the Lax-Milgram theorem applied to the variational formula $\displaystyle\int_{D_{\mathbf{v}}} V_i[f_i]\,g\,d\mathbf{v} = \int_{D_{\mathbf{v}}} h\,g\,d\mathbf{v}$ on the closed subspace $\left\{f_i\in L^1(D_{\mathbf{v}}) : \int_{D_{\mathbf{v}}} f_i\,d\mathbf{v}=0\right\}$. Next, by integration with respect to $\mathbf{v}^*$ of the following identity

$$f_iF_i^* - f_i^*F_i = \left(\frac{f_i}{F_i}-\frac{f_i^*}{F_i^*}\right)F_iF_i^*,$$

we get

$$f_i\int_{D_{\mathbf{v}}} F_i^*d\mathbf{v}^* - \left(\int_{D_{\mathbf{v}}} f_i^*d\mathbf{v}^*\right)F_i = \int_{D_{\mathbf{v}}}\left(\frac{f_i}{F_i}-\frac{f_i^*}{F_i^*}\right)F_iF_i^*d\mathbf{v}^*. \quad \text{(A.5)}$$

The right-hand side of the (A.5) is estimated, with the help of the Cauchy-Schwarz inequality, as

$$|f_i-\langle f_i\rangle F_i|^2 \leqslant \left(\int_{D_{\mathbf{v}}}\left(\frac{f_i}{F_i}-\frac{f_i^*}{F_i^*}\right)T(\mathbf{v},\mathbf{v}^*)F^*d\mathbf{v}^*\right)\left(\int_{D_{\mathbf{v}}}\frac{F_i^2}{T}F^*d\mathbf{v}^*\right).$$

Therefore

$$\begin{aligned}\int_{D_{\mathbf{v}}}\frac{f_i-\langle f_i\rangle F_i}{F_i}\,\nu_i\,d\mathbf{v} &\leqslant \left(\sup_{\mathbf{v}\in D_{\mathbf{v}}}\nu_i\int_{D_{\mathbf{v}}}\frac{F_i}{T(\mathbf{v},\mathbf{v}^*)}F^*d\mathbf{v}^*\right)\\ &\quad\times\left(\int_{D_{\mathbf{v}}}\left(\frac{f_i}{F_i}-\frac{f_i^*}{F_i^*}\right)^2 T(\mathbf{v},\mathbf{v}^*)F_i^*d\mathbf{v}d\mathbf{v}^*\right)\\ &\leqslant \kappa\int_{D_{\mathbf{v}}} V_i[f_i]\,\frac{f_i}{F_i}\,d\mathbf{v}. \end{aligned} \quad \text{(A.6)}$$

By using (A.4) and (A.6) we obtain the estimation (3.11). The proof is now completed. ■

A.2 Computations of the turning term

Let $T(\mathbf{v},\mathbf{v}^*)$ be the turning kernel defined by Eq. (4.32). Therefore the corresponding turning operator reads:

$$\begin{aligned} V[f] &= \frac{1}{\omega}\int_{D_{\mathbf{v}}}\left[\left(1+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbf{v}^*\right)f(\mathbf{v}^*)-\left(1+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbf{v}^*\right)f(\mathbf{v})\right]d\mathbf{v}^* \\ &= \frac{1}{\omega}\left[\int_{D_{\mathbf{v}}} f(\mathbf{v}^*)\,d\mathbf{v}^*+\frac{a}{s^2}\int_{D_{\mathbf{v}}}(\mathbf{v}\cdot\mathbf{v}^*)\,f(\mathbf{v}^*)\,d\mathbf{v}^*\right. \\ &\qquad \left.-\int_{D_{\mathbf{v}}} f(\mathbf{v})\,d\mathbf{v}^*-\frac{a}{s^2}\int_{D_{\mathbf{v}}}(\mathbf{v}\cdot\mathbf{v}^*)\,f(\mathbf{v})\,d\mathbf{v}^*\right] \end{aligned}$$

namely

$$\begin{aligned} V[f] &= \frac{1}{\omega}\left[\int_{D_{\mathbf{v}}} f(\mathbf{v}^*)\,d\mathbf{v}^*+\frac{a}{s^2}\left(\mathbf{v}\cdot\int_{D_{\mathbf{v}}}\mathbf{v}^*\,f(\mathbf{v}^*)\,d\mathbf{v}^*\right)\right. \\ &\qquad \left.-f(\mathbf{v})\int_{D_{\mathbf{v}}} d\mathbf{v}^*-\frac{a}{s^2}\left(\mathbf{v}f(\mathbf{v})\cdot\int_{D_{\mathbf{v}}}\mathbf{v}^*\,d\mathbf{v}^*\right)\right]. \end{aligned}$$

Since $\displaystyle\int_{D_{\mathbf{v}}}\mathbf{v}\,d\mathbf{v}=0$ we have

$$V[f]=\frac{1}{\omega}\left[\varrho+\varrho\frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}-\omega f(\mathbf{v})\right]=\frac{\varrho}{\omega}\left(1+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}\right)-f(\mathbf{v}).$$

From the identity given by Eq. (4.33) one has

$$F_{\varrho_{i*},\mathbb{U}_{i*}}(\mathbf{v})=\frac{\varrho_{i*}}{\omega}\left(1+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_{i*}\right)$$

and then

$$\begin{aligned} \int_{D_{\mathbf{v}}} F_{\varrho_{i*},\mathbb{U}_{i*}}(\mathbf{v})\,F_{\varrho_{j^*},\mathbb{U}_{j^*}}(\mathbf{v})\,d\mathbf{v} &= \frac{\varrho_{i*}\varrho_{j*}}{\omega^2}\int_{D_{\mathbf{v}}}\left(1+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_{i*}\right)\left(1+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_{j*}\right)d\mathbf{v} \\ &= \frac{\varrho_{i*}\varrho_{j*}}{\omega^2}\int_{D_{\mathbf{v}}}\left(1+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_{i*}+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_{j*}+\frac{a^2}{s^4}\,(\mathbf{v}\cdot\mathbb{U}_{i*})(\mathbf{v}\cdot\mathbb{U}_{j*})\right)d\mathbf{v} \end{aligned}$$

Since

$$\int_{D_{\mathbf{v}}} v_h\,v_k\,d\mathbf{v}=\frac{1}{d}\,s^2\,s^{d-1}\,|\mathbb{S}^{d-1}|\,\delta_{hk},=\frac{\omega}{d}\,s^2\,\delta_{hk},$$

one obtains

$$\begin{aligned} \int_{D_{\mathbf{v}}} F_{\varrho_{i*},\mathbb{U}_{i*}}(\mathbf{v})\,F_{\varrho_j^*,\mathbb{U}_j^*}(\mathbf{v})\,d\mathbf{v} &= \frac{\varrho_{i*}\varrho_{j*}}{\omega}\left(1+\frac{s^2}{d}\,\frac{a^2}{s^4}\,\mathbb{U}_{i*}\cdot\mathbb{U}_{j*}\right) \\ &= \frac{\varrho_{i*}\varrho_{j*}}{\omega}\left(1+\frac{a^2}{ds^2}\,\mathbb{U}_{i*}\cdot\mathbb{U}_{j*}\right). \end{aligned}$$

Finally, since $\int_{D_{\mathbf{v}}} v_i v_j v_k \, d\mathbf{v} = 0$, one obtains

$$\begin{aligned}
\int_{D_{\mathbf{v}}} \mathbf{v} F_{\varrho_{i*},\mathbb{U}_{i*}}(\mathbf{v})\, F_{\varrho_j^*,\mathbb{U}_j^*}(\mathbf{v})\, d\mathbf{v} &= \frac{\varrho_{i*}\varrho_{j*}}{\omega^2} \int_{D_{\mathbf{v}}} \left(1 + \frac{a}{s^2}\, \mathbf{v}\cdot\mathbb{U}_{i*}\right)\left(1 + \frac{a}{s^2}\, \mathbf{v}\cdot\mathbb{U}_{j*}\right) \mathbf{v}\, d\mathbf{v} \\
&= \frac{\varrho_{i*}\varrho_{j*}}{\omega^2}\, \frac{a^2}{s^4} \int_{D_{\mathbf{v}}} (\mathbb{U}_{i*} + \mathbb{U}_{j*})\, v_i\, v_k\, d\mathbf{v} \\
&= \frac{\varrho_{i*}\varrho_{j*}}{\omega^2}\, \frac{a^2}{s^4}\, (\mathbb{U}_{i*} + \mathbb{U}_{j*})\, \frac{\omega}{d}\, s^2 = \frac{\varrho_{i*}\varrho_{j*}}{\omega}\, \frac{a^2}{ds^2}\, (\mathbb{U}_{i*} + \mathbb{U}_{j*}).
\end{aligned}$$

A.3 Proof of Lemma 4.2

Going back to Lemma 4.1 yields:

$$\begin{aligned}
\int_{D_{\mathbf{v}}} J_i[\mathbf{f}]\, d\mathbf{v} &= \sum_{j=1}^{n} \int_{D_u\times D_u} \mathcal{A}_{ij}(u_*, u^*, u)\, \frac{\varrho_{i*}\varrho_{j^*}}{|D_{\mathbf{v}}|} \left(1 + \frac{a^2}{ds^2}\, \mathbb{U}_{i*}\cdot\mathbb{U}_{j^*}\right) du_*\, du^* \\
&- \sum_{j=1}^{n} \int_{D_u} \frac{\varrho_i\, \varrho_{j^*}}{|D_{\mathbf{v}}|} \left(1 + \frac{a^2}{ds^2}\, \mathbb{U}_i\cdot\mathbb{U}_{j^*}\right) du^*,
\end{aligned}$$

and then

$$\begin{aligned}
\int_{D_{\mathbf{v}}} J_i[\mathbf{f}] dv &= \frac{1}{|D_{\mathbf{v}}|} \sum_{j=1}^{n} \int_{D_u\times D_u} \mathcal{A}_{ij}(u_*, u^*, u)\, \varrho_{i*}\, \varrho_{j^*}\, du_*\, du^* \\
&+ \frac{a^2}{ds^2|D_{\mathbf{v}}|} \sum_{j=1}^{n} \int_{D_u\times D_u} \mathcal{A}_{ij}(u_*, u^*, u)\, (\varrho_{i*}\mathbb{U}_{i*}\cdot\varrho_{j^*}\mathbb{U}_{j^*})\, du_*\, du^* \\
&- \frac{\varrho_i}{|D_{\mathbf{v}}|} \sum_{j=1}^{n} \int_{D_u} \varrho_j^* du^* - \frac{a^2}{ds^2}\, \varrho_i\mathbb{U}_i \cdot \sum_{j=1}^{n} \int_{D_u} \varrho_{j^*}\mathbb{U}_{j^*}\, du^*, \\
&= \frac{1}{|D_{\mathbf{v}}|} \left(J_i[\boldsymbol{\varrho}] + \frac{a^2}{ds^2}\, J_i[\boldsymbol{\varrho}\mathbf{U}]\right). \qquad \text{(A.7)}
\end{aligned}$$

Therefore the first equality is proved. For the first order moment of J_i, simple computations give:

$$\begin{aligned}
\int_{D_{\mathbf{v}}} \mathbf{v} J_i[\mathbf{f}]\, d\mathbf{v} &= \frac{a^2}{ds^2} \sum_{j=1}^{n} \int_{D_u\times D_u} \mathcal{A}_{ij}(u_*, u^*, u)\, \frac{\varrho_{i*}\varrho_{j^*}}{|D_{\mathbf{v}}|}\, (\mathbb{U}_{i*} + \mathbb{U}_{j^*})\, du_*\, du^* \\
&- \frac{a^2}{ds^2} \sum_{j=1}^{n} \int_{D_u} \frac{\varrho_i\, \varrho_{j^*}}{|D_{\mathbf{v}}|} \left(\mathbb{U}_i + \mathbb{U}_{j^*}\right) du^*.
\end{aligned}$$

That is

$$\begin{aligned}\int_{D_{\mathbf{v}}} \mathbf{v}\; J_i[\mathbf{f}]\, d\mathbf{v} &= \frac{a^2}{ds^2|D_{\mathbf{v}}|}\sum_{j=1}^{n}\int_{D_u\times D_u}\mathcal{A}_{ij}(u_*,u^*,u)(\varrho_{i*}\mathbb{U}_{i*}\varrho_{j^*}+\varrho_{j^*}\mathbb{U}_{j^*}\varrho_{i*})\,du_*du^*\\ &- \frac{a^2}{ds^2|D_{\mathbf{v}}|}\sum_{j=1}^{n}\int_{D_u\times D_u}\mathcal{A}_{ij}(u_*,u^*,u)(\varrho_i\mathbb{U}_i n_{j^*}+\varrho_{j^*}\mathbb{U}_{j^*}\varrho_i)\,du_*du^*\\ &= \frac{a^2}{ds^2|D_{\mathbf{v}}|}\left(A_i[\boldsymbol{\varrho},\boldsymbol{\varrho}\mathbf{U}]+B_i[\boldsymbol{\varrho},\boldsymbol{\varrho}\mathbf{U}]\right) \qquad (A.8)\end{aligned}$$

so that (4.40) is also proved. □

A.4 Pressure computation

The pressure term is defined as follows:

$$\begin{aligned}\mathbb{P}_i(t,\mathbf{x},\mathbf{v}) &= \int_{D_{\mathbf{v}}}(\mathbf{v}-\mathbb{U}_i)\otimes(\mathbf{v}-\mathbb{U}_i)\,f_i(t,\mathbf{x},\mathbf{v},u)\,d\mathbf{v}\\ &= \int_{D_{\mathbf{v}}}(\mathbf{v}-\mathbb{U}_i(t,\mathbf{x},u)\otimes(\mathbf{v}-\mathbb{U}_i(t,\mathbf{x},u)\,F_{\varrho_i\mathbb{U}_i}(\mathbf{v})\,d\mathbf{v}.\end{aligned}$$

Straightforward computations give

$$\begin{aligned}\mathbb{P}_i(t,\mathbf{x},\mathbf{v}) &= \int_{D_{\mathbf{v}}}(\mathbf{v}-\mathbb{U}_i(t,\mathbf{x},u)\otimes(\mathbf{v}-\mathbb{U}_i(t,\mathbf{x},u)F_{\varrho_i\mathbb{U}_i}d\mathbf{v}\\ &= \int_{D_{\mathbf{v}}}(\mathbf{v}\otimes\mathbf{v})\,F_{\varrho_i\mathbb{U}_i}\,d\mathbf{v}-2\int_{D_{\mathbf{v}}}\mathbf{v}\otimes\mathbb{U}_i\,F_{\varrho_i\mathbb{U}_i}\,d\mathbf{v}+\int_{D_{\mathbf{v}}}\mathbb{U}_i\otimes\mathbb{U}_i\,F_{\varrho_i\mathbb{U}_i}\,d\mathbf{v},\end{aligned}$$

that in the Example 1 case, it reads:

$$\begin{aligned}\mathbb{P}_i &= \frac{\varrho_i}{\omega}\int_{D_{\mathbf{v}}}(\mathbf{v}\otimes\mathbf{v})\left(1+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_i\right)d\mathbf{v}-\frac{2\varrho_i}{\omega}\int_{D_{\mathbf{v}}}\mathbf{v}\left(1+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_i\right)d\mathbf{v}\otimes\mathbb{U}_i\\ &+\frac{\varrho_i}{\omega}\,\mathbb{U}_i\otimes\mathbb{U}_i\int_{D_{\mathbf{v}}}\left(1+\frac{a}{s^2}\,\mathbf{v}\cdot\mathbb{U}_i\right)d\mathbf{v}.\end{aligned}$$

Taking into account the following relations

$$\int_{D_{\mathbf{v}}}d\mathbf{v}=\omega\quad\int_{D_{\mathbf{v}}}\mathbf{v}\,d\mathbf{v}=0,\quad\int_{D_{\mathbf{v}}}v_iv_k\,d\mathbf{v}=\frac{s^2}{d}\,\omega\,\delta_{ik},\quad\int_{D_{\mathbf{v}}}v_iv_jv_k\,d\mathbf{v}=0,$$

one obtains

$$\mathbb{P}_i = \frac{s^2}{d}\,\varrho_i\,\mathbb{I}-\frac{2a}{d}\,\varrho_i\mathbb{U}_i\otimes\mathbb{U}_i+\varrho_i\mathbb{U}_i\otimes\mathbb{U}_i.$$

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