LINES ON THE HORIZON

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HADAMARD AND FRÉCHET, READERS OF VOLTERRA

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ABSTRACT. This chapter provides a detailed examination of the
manner in which elements drawn from a reading of Volterra's work
on the generalization of the concept of function and differential cal-
culus became decisive for the research programs of first Hadamard
and then Fréchet, and how this passing of the baton to a new
generation marked a turning point in the evolution of studies on
partial differential equations, and more broadly, for all of twentieth-
century functional analysis.

INTRODUCTION

This present chapter concerns a very specific aspect of the relation-
ship between French and Italian mathematicians at the turn of the
19th and 20th centuries, and is centred on the work of Vito Volterra
(1860-1940). This emblematic figure of the Italian scientific milieu has
already been extensively studied, including two major biographies ([43]
and [45]). The omnipresence of this mathematician in the historical
works that examine the scientific, and even more generally intellectual,
life in Italy in the decades following Italian unification shows the fun-
damental role played by Volterra. Born at the very moment that the
Italian peninsula achieved political unity, Vito Volterra may properly
be considered as embodying the aspirations and transformations of the
new nation, at a time when she wished to claim her place in the world

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of European science. Meteoric progress had been made in this direction
since the Risorgimento; in the years following the French defeat in 1870
and the rise of Germany, when a concerned Darboux wrote to Houël
that if things continued in that way, the Italians would also surpass
the French on the mathematical scene (see [42]), the prediction did
not seem unrealistic. Under the leadership of eminent mathematicians
such as Enrico Betti (1823-1892) and Ulisse Dini (1845-1918), various
institutions rose in prestige in the country, such as the Scuola Normale
Superiore in Pisa, where the young Volterra was trained and where he
was able to learn the latest developments in mathematics and physics
from all horizons, Germany in the first place. Dini had worked a great
deal following works of Dirichlet, Riemann and their successors on the
properties of functions of a real variable. His masterful treatise [24]
inaugurated a new phase in the trend which tended to focus increas-
ingly on functions as the object of studies in their own right. Thanks
to the works of Dini, certain families of functions were characterized by
properties related to the existence of limits, to their Riemannian inte-
grability, to different interpretations of differentiability, and so forth.

Volterra, reprising his master Dini, thus had at his disposal a suf-
ficiently stable definition of the notion of function, and a rough idea
of classes grouping these objects according to affinity. These classes,
which prefigured certain algebraic structures, allowed him to consider
functions as variable elements on which to work particularly with in-
finitesimal methods. These ideas were useful to the young Volterra for
rethinking issues of mathematical physics and mechanics, which natu-
rally appear as outgrowths of the infinity of parameters composing a
function. In the first part of this chapter, we outline the main stages
of the story. In particular, we will describe how the young scientist
played along with a mathematical construction that followed its own
developmental process, and was more concerned with the issues of ap-
plications that it inspired. Within a few months Volterra published a
number articles introducing two concepts, functions depending on func-
tions, and a sort of geometric twin of that, the functions of lines. He
also developed a variety of techniques for manipulating these objects
and formulated equations to which they were the solutions. The birth
of functional calculus, as we will refer to it hereinafter, was a decisive
step in the transferral to more sophisticated situations of infinitesimal
methods that had appeared at first to be restricted to the original case
of numerical functions of a real variable.

What primarily concerns us here is how two French mathematicians,
Jacques Hadamard (1865-1963) and Maurice Fréchet (1878-1973), came
into contact with the work of Volterra, took it on themselves to develop
it, and, in the case of Fréchet, to criticize and to some extent go be-
yond it on their own terms. The two figures belong to two successive
generations, the older one to Volterra's generation. Their trajectories

are interesting to us, among others reasons, because of the uniqueness of the tandem they formed with their Italian colleague. Hadamard and Fréchet, as we will show, were in a relationship of master and disciple; this meant that each declined their reading of Volterra's work in a different way. Hadamard's reading of Volterra is the subject of the second part of the chapter. He took part in the extensive research program on partial differential equations (referred to as PDEs in what follows) that Hadamard began in 1890 on the occasion of his arrival in Bordeaux, especially after his meeting with Pierre Duhem (1861-1916).

This is by far not the first time that the Volterra-Hadamard duo has been studied to stress its importance in the study of PDEs that overtook physics in the second half of the nineteenth century, especially following Riemann's work on shock waves. We can cite in particular the voluminous scientific biography by Maz'ya and Shaposhnikova [74] and above all the chapter by Jeremy Gray [44], which presents a detailed study of the roles of Poincaré, Volterra and Hadamard in this history. Our objective is much more modest than these panoramic overviews, and focuses on the passing of the baton from Volterra to Hadamard and Fréchet. Expressing it lightly, Gray says that in reprising Volterra's work on line functions, *his friend Hadamard* ([44], p.127) introduced the term *functional*. However, this word *friend* is really too static to describe the growing, evolving relationship between the two men; this deserves greater attention, because it is highly correlated with Hadamard's research program in mathematical physics. This seems a convincing illustration of the hypothesis that Italian progress in mathematics was so valuable that the French mathematicians had to watch more closely the work that was being carried out on the other side of the Alps. We will therefore provide a fairly detailed account of the construction of Hadamard's program and how its progressive development put him in touch with Volterra. In fact, as we shall see, it was more broadly speaking the work of many Italian mathematicians that would at that time nourish French research on these topics. Volterra's considerations on line functions and differential calculus were only recognised in a later moment by Hadamard as possibly providing tools to solve problems of PDEs where the initial conditions undergo deformation over time, as in some wave situations. In the second section of the paper, we will examine the original circumstances of the encounter between the two mathematicians, who came to know and gradually appreciate each other between one international conference held in Zurich in 1897 and another in Heidelberg in 1904, when their personal relationship proper began.

In the case of Fréchet, the situation is very different. When he came into contact with Volterra, he was a young student fresh out of the École Normale Supérieure. Under the advisement of his mentor Hadamard and supported by Émile Borel (1871-1956), a great friend of Volterra's

since the Congress of Zurich, Fréchet sought a thesis topic with the Italian mathematician. The correspondence that began at that time shows that for a few months the young mathematician tried to follow in the footsteps of the older one. We will show how the young man quickly and markedly deviated from this initial orientation to follow his own path of radical originality, a path whose modernity somewhat baffled many of his contemporaries. Exploding the framework imposed by Volterra on classes of functions, Fréchet introduced a topological view of abstract spaces and general differential calculus that became the natural framework for further developments in functional analysis, relegating Volterra to the rank of pioneer, although not without some clashes, as we shall see.

Reflections on the birth of functional analysis and Volterra's role are certainly not new too. In the 1980s Reinhardt Siegmund-Schultze's comprehensive thesis [84] provided a thorough study of the subject. More recently, in another direction, Tom Archibald and Rossana Tazzioli [7] published an important contribution on the relationship between functional analysis, equations integrals and nuclei introduced by Fredholm operators as they were studied in France and Italy at the period we are concerned with, and Volterra also plays a central role here. Our research, however, makes a number of additions to these studies, emphasizing the reading of Volterra by his two French colleagues, but also highlighting a network of Italian and French contributions to the debate that the perspective chosen by other historians has not necessarily shed light on. For example, it is interesting to note that the story described in [7] has close affinities with the topics we will discuss, but also includes aspects that do intersect them at all (and vice versa), revealing how the bubbling research on functional spaces at the beginning of the twentieth century may have resulted in surprisingly independent lines of exploration. The lines we describe here appear to us to be those where the direct influence of Volterra's work on French mathematicians was the strongest and most decisive.

1. VITO VOLTERRA: THE TOOLS OF ANALYSIS AT THE SERVICE OF MATHEMATICAL PHYSICS

1.1. *Mathematical physics and rigorous analysis*

Vito Volterra was just over twenty-five years old when he published a note [100] —presented by Betti— in the *Rendiconti dell'Accademia dei Lincei* that represents the beginning of his research in functional analysis and introduces a concept that is central to his work: the notion of *a function that depends on other functions*. This publication can rightly be thought of as marking a singular moment in the history of nineteenth-century mathematics. It is also often identified today, and was even by the protagonists of the period, as the founding act of a new

science. Volterra wrote his degree thesis on hydrodynamics under the advisement of Betti at the Scuola Normale Superiore in Pisa. It was this period that witnessed the large movement to transfer the rigour of German mathematical analysis into Italian mathematics, particularly through the translation of the works of Riemann, under the impetus of Betti and Dini. Betti mainly focused on issues related to complex variables, while Dini primarily worked in real analysis. During the two years that he spent at the Scuola Normale, the young Volterra was seduced by Dini's lectures —Dini had published in 1878 his fundamental book on the theory of the functions of a real variable [22]. Volterra's first publications clearly show Dini's influence. In 1881, when he was still a student at the Scuola Normale, Volterra published two articles ([94], [95]) with evocative titles —*Alcune osservazioni sulle funzioni punteggiate discontinue* (Some observations on pointwise discontinuous functions) and *Sui principi del calcolo integrale* (On the principles of integral calculus)— in the *Giornale di Matematica.*

He described in detail what is meant by the smallness of a set and gives an example of a subset of $\mathbb{R}$ which is nowhere dense yet has a non-null measure (in today's words). He further exhibited differentiable functions whose derivative, albeit limited, is not Riemann integrable, demonstrating that in general integration and differentiation are not inverse operations of each other. As we can see, the brilliant student Volterra showed that he had grasped the nature of various central issues of nineteenth-century analysis, modernised by Dirichlet, Riemann and their heirs beginning from their studies on trigonometric series.[1]

However after these notes of 1881, Volterra did not again intervene in these questions concerning the basis of real analysis, preferring instead to turn to the study of functions of a complex variable and differential equations. In the years following his degree thesis, Volterra published many articles in mathematical physics that reflect the presence and influence of Betti in his choice of research topics. The detailed study of the theory of elasticity by Gabriel Lamé (1795-1870) [64], which will be discussed in the next section, gave Volterra the opportunity to publish his best-known product of this era ([96]). Lamé treated the propagation of light in birefringent media assuming that the incident beam is split into two polarized rays that vibrate in perpendicular planes. Twenty years later, Sofia Kovalevskaya (1850-1891) took this up, criticising Lamé's method in an article published in *Acta Mathematica* in 1886 ([66]). Shortly after Kovalevskaya's death, Volterra noted an error in her approach, in that she had never taken into account the discontinuity

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[1]For the history of this very important chapter in the development of integration, see the fine work by Hawkins [61].

of one of the parameters. He corrected the error and provided in his article the righteous differential system for light waves ([22], p.173 ff).[2]

1.2. *Differential equations and functions of lines*

The two years 1887 and 1888 were particularly fruitful for Volterra. They actually saw the publication of several sets of work, each comprising a series of articles, which show the originality of his thinking and his growing independence from the ideas of his teachers, Betti and Dini. Especially noteworthy are the first three articles [97], [98] and [99] that mark the development of a proper theory of linear differential equations. Here we should note Volterra's clear orientation towards mathematical physics. The subjects of these three articles are no longer based on the examination of concrete physical situations, but on a strictly mathematical analysis. More to the point for our purposes, in a single year, 1887, Volterra published three notes [100], [101] and [102] in which he introduced the concept of a function that depends on other functions in order to study the quantities that all depend on the values that one or more functions of one variable can assume in a given interval. Volterra uses the expression *function that depends on other functions*, clearly stating in his text that he is not dealing here with functions of functions, that is, a function obtained by composition of functions in the sense of Dirichlet. Volterra insists on the essential fact that the concept of function that depends on other functions attributes two different roles to the elements involved. Such a function has functions as variables, the latter maintaining a sort of general and indeterminate character proper to the nature of being a variable. For simplicity's sake, we will use the term *functional* (which will be introduced by Hadamard, as we shall see in the next section) to refer to the concept of Volterra. Volterra's insistence on indicating the variable x in his expressions involving the variable function φ —thus systematically written $\varphi(x)$— is a good illustration of the fact that this concept of variable function was still in its infancy for him. This also creates difficulties for the reader accustomed to modern presentation. These fluctuations in language and notation remind us that at that time functional analysis was not yet born as a field of research, and show how the abstract form it received in the twentieth century helped to synthesise the concepts introduced by Volterra and others; the third part of our article, devoted to Fréchet, also concerns the first steps in this spectacular synthesis. For Volterra, the domain of functionals, that is to say, the set of elements on which the operations is carried

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[2]This publication provided the occasion for Volterra to begin an important correspondence with Gösta Mittag-Leffler (1846-1927), who had founded the journal *Acta Mathematica* in 1882 and directed it energetically for almost half a century. A study of this correspondences (see [75]), which was most frequent in the years 1888-1892, offers a very vivid picture of the young Volterra's boundless activity.

out, is not a general set but is systematically made up of the class of functions of one variable which are all continuous in an interval $[A, B]$. The notion of uniform metric on an abstract space is not yet clarified (it will be later, again by Fréchet) and the distance between two functions is exclusively given by a property of upper boundary property inherited from Cauchy and Weierstrass. Thus, a functional y is said to be continuous if[3]:

> making vary $\varphi(x)$ of a variation $\psi(x)$ such that the absolute value $\psi(x)$ is always less than ε, the corresponding variation in y can be made smaller than an arbitrarily small σ[4] [100, p.296]

The main objective of the three notes of 1887 on functions of functions is to extend to functionals the concept of derivative as well as that of differential. Specifically, the central section of the first note [100] is based on the notion of variation of a function that depends on another function and designates with the symbol σ the first-order term of the variation Δ of y for a small variation in the variable (which here is a function). It is significant that this approach takes place within the general development of analysis at a time when mathematicians were attempting to define a differential calculus for functions of several real variables that has the greatest possible analogy with what was known at the time for functions defined on $\mathbb{R}$. In keeping with the work being done in these last decades of the nineteenth century, Volterra did not seek to specifically form a notion of a differential or of a differentiable function, but to generalize the classic formulas $df(x) = f'(x)dx$ and $df(x_1, x_2, \ldots, x_n) = \sum f'_{x_i} dx_i$ that were known for the functions of one or several variables.

One of his main results is that of having shown that, under reasonable conditions on the functional y defined on the space of continuous functions in the interval $[A, B]$, its variation can be expressed as an integral of the type[5]

$$\delta y[\varphi(x)] = \int_A^B y'[\varphi(x), t]\delta\varphi(x)dt$$

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[3]We must emphasize here the scope of this statement, which does not produce a generality that will only be arrived later. Volterra's terms are intended to define the continuity of the functional in the case of continuous functions in [ab]. In other words, the set of continuous functions should not be thought of here as an archetype of a more general abstract space.

[4]si, faisant varier $\varphi(x)$ d'une variation $\psi(x)$ telle qu'en valeur absolue $\psi(x)$ soit toujours inférieure à ϵ, la variation correspondante de y peut être rendue inférieure à σ arbitrairement petit.

[5]In 1887 Volterra introduced the notation $y|[\varphi(x), t]|_B^A$, which we simplify as $y[\varphi(x), t]$ in our commentary.

in which can be specifically discerned the analogy with the corresponding formulas in real analysis and how to generalize them. The operation of integration generalizes the sum used in the expression of the differential referred to above, δy represents the variation of the functional y engendered by the variation $\delta\varphi$ of the independent variable, and y' denotes what Volterra called the *functional derivative* of y. The *ad hoc* conditions proposed by Volterra to permit the proof of his integral formula are quite clearly inspired by the analytical-geometric methods of the calculus of variations, consisting in the creation of a localised perturbation. Let us consider, Volterra says, a subinterval $[m, n]$ of $[A, B]$ and a variation θ of the variable function φ such that $0 \le \theta(x) \le \varepsilon$ for all $x \in [m, n]$. We set $\int_m^n \theta(x)dx = \sigma$, which thus represents the area comprised within the graph of φ and that of $\varphi + \theta$. Finally, δy represents the variation $y(\varphi + \theta) - y(\varphi)$. Volterra then formulates four hypotheses ([100, p.296]):

(1) $\frac{\delta y}{\epsilon h}$ is always less than a constant M.
(2) If ϵ and h tend to 0, such that the interval $[m, n]$ always contains the point t, the ratio $\frac{\delta y}{\sigma}$ tends to a finite limit, denoted $y'[\varphi(x), t]$, and called *functional derivative* of y.
(3) This limit is uniform for the possible choices of φ and t.
(4) $\varphi \mapsto y'[\varphi(x), t]$ and $t \mapsto y'[\varphi(x), t]$ are continuous.

The publication of article [100] is undoubtedly a milestone in the development of functional analysis. Beginning with a new mathematical concept, Volterra constructed for the first time a new differential calculus that allowed him to envision higher order derivatives and arrive at a Taylor formula in this general framework. Volterra revisited his system of assumptions again and again, thus the framework presented in [100] was still provisional. Aside from the ambiguities in the choice of symbolism and terminology that we have underlined, we see that it is necessary that the increase $\theta(x)$, to which the function φ is submitted, always have a constant sign, a condition that will disappear during the course of Volterra's successive publications. Similarly, the fourth hypothesis will evolve into the strongest condition, but naturally verified in some typical situations, that the derivative y' is uniformly continuous, making it possible to give a simpler proof of the result. This property will also become the hypothesis canonically proposed to guarantee the existence of the first derivative of a functional. In their 1936 work that became a point of reference ([123]), Pérès[6] and Volterra also remarked that they prove the result of representation under conditions that are obviously not as general as possible. Another observation

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[6]On Joseph Pérés (1890-1962), see [73].

that should be noted is that Volterra is not concerned about the independence of the first derivative of the functional with respect to the particular choice of θ. As mentioned, the main purpose in [100] is not to study the formal properties of differentials or to specify the class of differentiable functions but rather to seek a representation theorem of the variation and it is on this occasion that a notion of differentiability is introduced. This will be greatly extended in subsequent work by Fréchet, which we examine below.[7]

We mentioned that the representation theorem was valid under certain assumptions. Volterra's second note on the functions that depend on other functions ([101]) seeks to relax these constraints and focuses particularly on functionals for which the conditions of the first article are not all met at exceptional points t. Thus Volterra notes that there may be some cases where around certain points of the segment $[A, B]$, hypothesis 1 of the previous note, which required the variation of the functional to be an infinitesimal of an order higher or equal to εh , cannot be verified. The three cases examined by Volterra, where this assumption is not satisfied, make it possible to confirm that the representation formula for the differential established from the beginning remains valid, with the possible addition of terms that depend on exceptional points. In the third section we will return to this point, which gave rise to a dispute between Volterra and Fréchet regarding the generality of the theory of functionals.

The third note [102] focuses on particular issues, where the functional or its derivative has a specific form which is defined by its dependence on the relation to the variable φ or its derivatives at given points. Volterra especially examines the case where there is a function F such that $y'|[\varphi(x), t]| = F(\varphi(t))$ or where $y|[\varphi(x)]| = \int_A^B \int_A^B F(\varphi(t), \varphi(t_1))dt_1$. He also studies the case in which an ordinary differential equation

$$f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}, \varphi(x), \varphi'(x), \ldots, \varphi^{(m)}(x)\right) = 0$$

is given, where the function φ is also given, as are the initial conditions

$$y(A), y'(A), \ldots, y^{(n-1)}(A).$$

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[7]These representational issues also arise in connection with specific works in the context of functional calculus, as in the case of the studies by René Gateaux (1889-1914), as well as those by Paul Lévy (1886-1971) in potential theory, introducing a notion of differential engendered by a variation of the independent variable in a given direction ψ. On this exciting chapter linking functional calculus and probability theory, see [71] and [9]. We see in [9] that when Levy began a correspondence with Fréchet after World War I, he never misses a chance to voice strong criticism of the narrow framework proposed by Volterra for the derivation of higher-order functionals. Younger by almost thirty years, Levy was probably not in position to fully comprehend how radically novel the older mathematician's approach was at the time it appeared.

One thus considers the functional Y which to φ associates the value in B of the solution associated to φ. As Volterra himself observes, that value depends on that of φ in the entire interval $[A, B]$. This then raises the question of the derivative of that functional, which Volterra obtains by means of the solution of an auxiliary equation. The consideration of particular functionals in the form of integrals given as solutions to differential equations provides an initial glimpse of the reasons that led Volterra to introduce the concept of functional. His initial motivations were in fact those of a mathematical physicist. However, behind the new mathematical notion that he introduced and the calculus that it allowed him to develop, he discovered tools essential for posing other problems of analysis. Volterra also seemed surprised to see how the idea that is the source of the concept of functional made it possible to revisit some classic chapters of mathematics and was already present in some elementary observations and experiments of physics in which one seeks identify the dependence of certain continuous parameters. His first note therefore opens with an optimistic vision, with Volterra saying that he will introduce

> some considerations that will serve to illuminate concepts
> that I think are necessary to introduce for an extension
> of Riemann's theory on functions of a complex variable,
> and which I think lend themselves to uses in various other
> research areas.[8] [100, p.294]

A few lines further, we read:

> in fact in many questions of physics and mechanics, as
> well as in the integration of partial differential equations,
> there may be a need to consider quantities that depend
> on all the values that one or more functions of one varia-
> ble assume in given intervals ... For example, the tem-
> perature at a point of a conductive blade depends on all
> the values that the temperature takes on the edge ...[9]

The prediction of possible applications of the notion of functional, which Volterra presents in his first note [100], effectively materialises in some of the research he carried out in the years that followed. In general there is no direct reference to functions that depend on other functions, but to their geometric versions —the *functions of lines*—

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[8]alcune considerazioni le quali servono a chiarire dei concetti che credo necessari introdurre per una estensione della teoria di Riemann sulle funzioni di variabili complesse, e che penso possano tornar giovevoli anche in varie altre ricerche.

[9]infatti in molte questioni di Fisica e di Meccanica, e nella integrazione di equazioni differenziali alle derivate parziali, capita di dover considerare delle quantità che dipendono da tutti i valori che una o più funzioni di una variabile prendono in dati intervalli ... Così per esempio la temperatura in un punto di una lamina conduttrice dipende da tutti i valori che la temperatura ha al contorno ... [100, p.294]

that Volterra examined in two notes published in that same year, 1887, in the *Rendiconti dell'Accademia dei Lincei* ([103],[104]). Here the motivation behind the study of functions of lines is similar to that which generally tends to associate with each concept of analysis a geometric representation.[10] As Volterra wrote:

> The usefulness of geometric representation in the domain of variability of a function is well known... . One can obtain a geometric image of the same kind for the functions that depend on another function. ([103, p.315]

The objectives are first of all identical to those set forth earlier in the study of the functions that depend on other functions:

> Such an idea is familiar to physicists: it presents itself spontaneously when one thinks of certain electrical phenomena ... For certain studies that I hope to be able to communicate in a very short time, it is beneficial to consider functions of lines of a three-dimensional field.[11]
> [103, p.315]

The study of functions of lines in fact permitted giving the representation theory of a form even more visible than what was known for the differential of a function of several variables. For the functional $\varphi[L]$ —here φ indicates the functional and no longer the function on which it operates!— Volterra provides the following notation of its variation:

$$\delta\varphi = \int_L (X\delta x + Y\delta y + Z\delta z)ds$$

where X denotes the derivative of φ with respect to x, defined as the limit of the ratio $\frac{\Delta_x\varphi}{\epsilon h}$ when ε and h tend to 0, ε and h having the sense given in [100], while $\Delta_x\varphi$ is the variation of the functional that corresponds to the variation of the line L with respect to the x axis. The quantities Y and Z are defined in an analogous way. The three partial derivatives of φ corresponding to x, y, z are not independent of each other but are linked by relationships described in the form of equalities. These ideas are then extended to the second derivatives and developed more particularly in the second note on the functions of lines to study *simple functions*, which verify the equality $\varphi[L_1 + L_2] = \varphi[L_1] + \varphi[L_2]$

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[10]The notion of *line* covers several aspects that will be evoked via Volterra's texts in the what follows. Although at the time *line* could be interpreted as a function with values in the plane, with varying properties and regularity, Volterra makes very little appeal to this parametric version. For him the line is primarily manipulated as a geometric, almost physical object, for which he will define an operation of addition (the piecing together of two lines) and conceive a notion of neighborhood (word belonging to the vocabulary of an observer) that does not rely on parameterisation.

[11]una tale idea è familiare ai fisici; essa si presenta spontaneamente quando si pensa a certi fenomeni elettrici (...) Per alcuni studi, che spero di poter comunicare quanto prima giova considerare le funzioni delle linee di un campo a tre dimensioni.

in which L_1 and L_2 are two paths with the restrictions described in the first note: the lines are closed or terminate at the boundary of the area in which they are considered, there are a finite number of singular points and elsewhere there is a tangent, and finally, there are no knots. The sum is defined, with a suitable parameterization, as the line obtained by removing the common part of L_1 and L_2. These properties of linearity drawn from problems of functionals emerged in various forms devised by many mathematicians in that period. We can observe a strong resemblance to the notion of distributive function introduced by Peano at almost the same time ([78]).[12] Moreover, we will see in the next section that Volterra's *simple functions* fall within the scope of Hadamard's *additive functions*. We can also recall the work of Emmanuel Carvallo (1856-1945), who grasped early on the concept of linear operation (which he calls *operator*) to study systems of functional equations (see [20]).

1.3. *The first results of functional analysis*

While in the twentieth century the functional became one of the mathematical objects most used, in the last decade of the nineteenth century the concept was still in its infancy. In the introduction to his article [114], Volterra reported that functions of lines arise in several questions of physics and may also be related to questions of analysis. His goal in the article was to show how they could be used in the theory of functions of complex variables. In an earlier article published in 1887[13], Volterra had already mentioned the use of functions of lines to generalise Riemann's definition of complex function. In effect, he wrote that Riemann's considerations in reference to a two-dimensional space can be extended to three-dimensional spaces provided that instead of functions defined on such a space, one begins with functions that depend on lines on this space. In [114], Volterra describes his notion in detail:

> In the theory of functions of a complex variable, we suppose, in a way, that the values of imaginary variables are extended on a surface, with the condition that the differential relations of the variables only depend on the points of the surface... Is it possible to generalise this theory by referring to a three-dimensional space? This is the problem I proposed. We can solve the question, but to approach it, it is necessary to use what I have called the functions of a line. ... How will the generalisation which I have just mentioned be linked to that

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[12]The difficult relationship between Peano and Volterra forms an explosive chapter of Italian mathematics in that period. For more on this, see [45] p.36-42.

[13]This is the first of a series of three notes [105, 106, 107] on this subject published between 1887 and 1888.

known theory? It is easy to show that it is related to the
theory of functions of several complex variables. Nearly
a year ago, Mr Poincaré, in generalising the theorem of
Cauchy, proved that the integral of a uniform function of
two complex variables taken on a closed surface is zero, if
one can distort and reduce the surface to a point without
encountering any singularities. From this it can be de-
duced that if the surface of integration is not closed, the
integral depends on the lines that form the boundary of
the surface. So we see that the integration of functions
of two variables leads to functions of lines.[14] [114, p.364-
365]

Two years later, in a new article [117], Volterra showed that it was
possible to use functions of lines to extend to double integrals Jacobi-
Hamilton's theory of the calculus of variations:

The Jacobi-Hamilton procedure is based on the simple
integral (of which the variation is to be made null) con-
sidered as a function of its limits and of the arbitrary
values assigned to the unknown functions in the limits
themselves... If one goes from simple integrals to the ca-
se of double integrals, instead of the two limits of the
integral, we have one or more lines that form the boun-
dary of the area of integration.[15] [117, p.464]

It is in this type of context that functions of lines come into play, per-
mitting the construction of an element analogous to the characteristic
function set forth in the Hamilton-Jacobi theory, and extending the

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[14]Dans la théorie des fonctions d'une variable imaginaire, on suppose, en quelque
sorte, que les valeurs des variables imaginaires sont étendues sur une surface, avec
la condition que les rapports différentiels des variables ne dépendent que des points
de la surface ... Est-ce qu'on peut généraliser cette théorie en se rapportant à un
espace à trois dimensions ? Voilà le problème que je me suis proposé. On peut
résoudre la question, mais pour l'aborder il faut recourir à ce que je viens d'appeler
les fonctions d'une ligne ... A quelle théorie connue va se rattacher la généralisation
dont je viens de parler ? Il est bien aisé de montrer qu'elle se rattache à la théorie
des fonctions de plusieurs variables imaginaires. Il y a presque une année, M.
Poincaré, en généralisant le théorème de Cauchy, à démontré que l'intégrale d'une
fonction uniforme de deux variables imaginaires prise sur une surface fermée est
nulle, si l'on peut déformer et réduire la surface à un point sans rencontrer de
singularités. On peut déduire de là que, si la surface d'intégration n'est pas fermée,
l'intégrale dépend des lignes qui forment le contour de la surface. Donc on voit que
l'intégration des fonctions de deux variables conduit aux fonctions des lignes.

[15]Il procedimento Jacobi-Hamilton si fonda sull'esame dell'integrale semplice (di
cui si vuole annullare la variazione) considerato come funzione dei suoi limiti e
dei valori assegnati ad arbitrio alle funzioni incognite nei limiti stessi (...). Se
si passa dagli integrali semplici al caso degli integrali doppi, invece dei due limi-
ti dell'integrale, abbiamo una o più linee che formano il contorno del campo di
integrazione.

concept of multiple integrals. In a subsequent article ([120]), Volterra
explains how functions of lines make it possible to develop a general
vision for the addition problem for elliptic functions and the connec-
tion with partial differential equations. Consider the following sum of
multiple integrals

$$(1) \quad J = \int\int_{\alpha_1} \frac{dydz}{\sqrt{\frac{\beta_2}{\lambda}z^2 - \frac{\beta_3}{\lambda}y^2 + \beta_1}}$$

$$+ \int\int_{\alpha_2} \frac{dzdx}{\sqrt{\frac{\beta_3}{\mu}x^2 - \frac{\beta_1}{\mu}z^2 + \beta_2}} + \int\int_{\alpha_3} \frac{dydz}{\sqrt{\frac{\beta_1}{\nu}y^2 - \frac{\beta_2}{\nu}x^2 + \beta_3}}$$

with the condition $\lambda + \mu + \nu = 0$. The main theorem proved by Volterra
in [120] says that this sum is constant when the integration domains
are limited by the projections $\alpha_1, \alpha_2, \alpha_3$ on the three coordinate planes
of a curve drawn on the algebraic surface with equation

$$(2) \quad \lambda x \sqrt{\frac{\beta_2}{\lambda}z^2 - \frac{\beta_3}{\lambda}y^2 + \beta_1}$$

$$+ \mu y\sqrt{\frac{\beta_3}{\mu}x^2 - \frac{\beta_1}{\mu}z^2 + \beta_2} + \nu z\sqrt{\frac{\beta_1}{\nu}y^2 - \frac{\beta_2}{\nu}x^2 + \beta_3} = C.$$

Volterra then makes the following observation : the existence of a so-
lution to the partial differential equation

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0$$

can be interpreted as the condition of existence of a first-order function
of line $F[(L)]$ such that its surface derivatives $\dfrac{dF}{d(y,z)}, \dfrac{dF}{d(y,z)}, \dfrac{dF}{d(y,z)}$
(defined in [104]) are respectively given by X, Y and Z. Now, such a
function of line F is constant on any line L of the surface Σ defined by
the equation $f = $ constant where f is solution of the partial differential
equation

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0.$$

Hence a possible link between the additive relation involving multi-
ple integrals and some properties of some functions of line. Volterra
remains however rather vague about the aforementioned link.

It is also interesting to note that in [120] Volterra cites a long para-
graph from a similar work by Picard and extracts from letters that show
that the two mathematicians were in contact regarding this question:

> With regard to this I am pleased to present to the Acade-
> my an extract from two letters that our illustrious corre-
> sponding member Mr Picard sent me following the com-
> munication I had sent to him of previous proposals and

that he authorised me to publish: '... I believe I reco-
gnise in the question that you show me something that
appears to relate to a study that I began but which I
have never gone into thoroughly and that I have never
published.'[16] [120, p.333]

2. Jacques Hadamard: clairvoyant catalyst of French-Italian mathematical relations

In July 1904, a few weeks before the opening of the International
Congress of Mathematicians in Heidelberg, Jacques Hadamard, who
knew he was working on topics similar to those of Volterra's interests
and specialties, thought it prudent to ask his Italian colleague what
he would speak about at the congress. This exchange marks the real
beginning of their personal relationship. The themes that occupied
Hadamard in those years appertained to the broad field of mathemat-
ical physics. He was specifically interested in issues surrounding the
study of partial differential equations and their solutions. Such equa-
tions model the basic situations of mechanics and, more specifically,
of wave problems. In fact, during the Heidelberg congress, Hamadard
and Volterra co-chaired the session of 10 August 1904 devoted to these
problems, during which they each also gave a talk, Volterra on wave
theory and Hadamard on limit conditions in partial differential equa-
tions in physics; Arnold Sommerfeld and Robert William Genesis were
the two other speakers in that session. Moreover, two days later, on 12
August, Hadamard gave a new talk in another session that this time
he co-chaired with Tullio Levi-Civita, in which he presented a paper
on the fundamental solutions of linear partial differential equations,
followed by a discussion in which Volterra was the main speaker.

How did the two mathematicians come to find themselves together
in Heidelberg as the two indisputed specialists in mathematical physics
and PDEs? Although almost exactly the same age (Volterra was the el-
der by five years), the mathematical careers of Hadamard and Volterra
had followed quite distinct trajectories. From the beginning of his pro-
fessional life Volterra was fascinated, as we have seen, by questions of
mathematical physics. Starting with his first articles published in 1881,
as we discussed in the first section, Volterra's work included studies on
the distribution of heat or electrical energy in materials, issues related
to potential theory, which he made the subject of his habilitation thesis
at the Scuola Normale of Pisa in 1883. On the other hand, we have

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[16]A questo riguardo sono ben lieto di poter presentare all'Accademia l'estratto
di due lettere che il nostro illustre corrispondente signor PICARD mi ha dirette in
seguito alla comunicazione fattagli delle precedenti proposizioni e che egli mi auto-
rizza a pubblicare : ' ... je crois reconnaître dans la question que vous m'indiquez
quelque chose qui doit avoir un rapport avec une étude que j'avais commencée mais
que je n'ai pas approfondie et que je n'ai pas non plus publiée.'

amply shown that Volterra also dealt with more specifically mathematical considerations regarding questions of integration, or extensions of the concept of function (with the concept of *functions of lines*) and the development of an appropriate calculus that can permit modelling of new physical situations such as phenomena of heredity. For his French colleague, as we shall see, the situation was quite different. In this present section we will examine how Hadamard began to take an interest in this type of problem. Such research is an outgrowth of the long and rich history of the development of the study of PDEs in the nineteenth century, which we will briefly review here.

Hadamard's work on PDEs is abundant and fundamental. One of its original aspects, very valuable for the historian, is that Hadamard consistently provided a very honest historical, or at least chronological, account to explain how his work is inscribed in the process of construction the theory, as for example at the conference he gave to 1928 International Congress of Mathematicians in Bologna ([59]). A complete history of Hadamard's contributions to the theory of PDEs is beyond the scope of this present article; as already mentioned in the introduction, the reader would do well to consult the extensive biographical work of Vladimir Maz'ya and Tatyana Shaposhnikova devoted to the mathematician [74], especially chapters 14 and 15, as well as panoramic overview by Jeremy Gray [44]. For our part we will focus on a specific aspect which, although present in the works just cited, may be discussed in greater detail in the light of our own study, whose focus is more restricted.

The 1890s saw the development of intense correspondence between French and Italian mathematicians regarding PDEs, exchanges in which Volterra quickly takes centre stage. Therefore, the two questions we attempt to answer in this section, focussed on our two protagonists, are as follows. How did the Italian works, including those of Volterra, come to be known to Hadamard? How was his relationship with Volterra established in the few years between their first meeting in 1897 and the congress of 1904, where they became close friends? These questions lead us to look more closely at the role played by the previous international conferences (1897 in Zürich and 1900 in Paris) and, going beyond the example of Hadamard, to shed light on how works by Italian mathematicians during the decade between 1880 and 1890 were reviewed and extended in France at the turn of the century.

2.1. *Elasticity, waves and PDEs*

2.1.1. *Lamé, Riemann and du Bois-Reymond*

Although it is difficult to identify an indisputable starting point for the systematic study of PDEs in the late nineteenth century, it seems

legitimate to underline, as does Jeremy Gray ([44]), the central importance of the study of wave propagation in elastic medium. This theory, which especially involves the study of the characteristics that describe the deformation of materials, effectively forms a bridge between different fields of mechanics: those of continuous media and fluids in particular, but also with regard to thermodynamics, by addressing the question of the compression of gasses. It had found its mathematical expression in the research of Lamé, of which here we will mention just a few brief aspects (for details, see [89], [90] and [41]). Lamé's interest in elasticity stemmed from a number of mechanical studies undertaken during his long stay in Russia in the company of Benoît Clapeyron (1799-1864), notably his studies on suspension bridges. However, beyond these questions of engineering, Lamé pursued a broader goal. His conception of science and the role played by mathematics was not without echoes, and we will see it adopted later by Duhem and Hadamard. Joseph Bertrand described them thus:

> In the eyes of Lamé, science was a single entity, and the relationships, even among the individual formulas, among theories still distinct, were sure indicators of a more general doctrine that must one day embrace them all. The distinction between pure mathematics and applied mathematics was, in his eyes, dangerous and false.[17]
> [13, p.19]

Lamé's aim was to show that a theory of elasticity could be used to support a unified wave theory where elasticity of the ether served to explain both the phenomena of heat and the propagation of light. Returning to France after the consolidation of diplomatic relations with Russia that followed the July Revolution of 1830[18], and appointed professor at the École Polytechnique in 1831, Lamé settled on a program of applying modern geometry and analysis to develop a theory of elasticity. In 1833 Clapeyron and Lamé published a note [21] presenting the equations describing the internal equilibrium of homogeneous solid bodies: they show that the equations of elastic equilibrium (involving the directional stress introduced by Cauchy) are identical to Navier's equations for molecular forces. In 1852 appeared the *Leçons sur la théorie mathématique de l'élasticité des corps* [67]. In the ninth lesson, Lamé studied small motions of an elastic membrane in the form

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[17]Aux yeux de Lamé, la science était une, et les rapprochements, même dans les seules formules, entre des théories encore distinctes étaient l'indice certain d'une doctrine plus générale qui doit un jour les embrasser toutes. La distinction entre les mathématiques pures et les mathématiques appliquées était, à ses yeux, dangereuse et fausse.

[18]Clapeyron, who had been close to the movement leading to the Decembrist Revolt in 1825, had been, like Lamé, forced to flee from the repression of the Saint-Simonians, and returned at about the same time.

of linear hyperbolic PDEs. His works were reprised and extended by Riemann (see [18]).

In 1864 appeared the great treatise [27] by Paul du Bois-Reymond (1818-1896) on PDEs, in which the German mathematician introduced the now classic classification for second-order linear equations (hyperbolic, elliptic and parabolic). This was one of the first books that sought to impose a semblance of unity on the landscape of PDEs, which was still dominated by a catalogue of specific methods for particular cases. Du Bois-Reymond also made known the works of Riemann for the case of the equation of the propagation of sound [82], with the first draft of a method of characteristics that consisted in finding the curves along which the solution of the equation is obtained through ordinary differential equations. Described by du Bois-Reymond and Gaston Darboux (1842-1917) in various works of the 1870s and 1880s, this method makes it possible to obtain the value of the solution for certain hyperbolic PDEs in the plane at a point situated within a quadrilateral formed by the characteristic curves of the equation in function of the values assumed on its sides. Luigi Bianchi (1856-1928), one of the first Italians to work on these questions, extended certain of these works to the elliptic case; in 1889 in [11] he described the role played by the work of du Bois-Reymond and Darboux:

> In the second volume of the fine lessons on the general theory by Mr Darboux and in the last memoir by du Bois-Reymond, published in the 104th volume of Crelle's Journal, are contained results of great importance for the theory of second-order partial differential linear equations with two independent variables x, y. In particular, for hyperbolic equations, that is, those in which the two systems of characteristic lines are real and distinct, is proven the fundamental theorem, according to which in every quadrilateral bounded on the plane xy by four characteristics, when the values assumed by the integral along two adjacent sides of the quadrilateral are determined, the values of the integral itself in the entire interior region of the quadrilateral are identified [Darboux, §364; Du Bois Reymond §15]. This theorem is correlated, for the equations of the elliptic type (with imaginary characteristics), to another by which the values assumed by the integral within a connected area are generally identified by the values that the integral receives on the boundaries of the area. Several simple observations contained in the present note make it possible to establish this result with great generality. The process itself is readily extensible, as will be seen, to the case of any number of independent variables. But there

is no treatment at all of the much more difficult question of whether these boundary values can effectively be arbitrarily given, a question that, with the exception of a few special cases, does not appear possible to solve at present.[19] [11, pp.35-36]

2.1.2. *Kirchhoff*

In 1882, the physicist Gustav Kirchhoff (1824-1887) intervened unexpectedly in the theory of PDEs. In [65], he proposed in the spherical case of the wave equation (for a space of dimension 3) a new approach with the aid of Green's formulas, in expressing solutions in integral form, extending Poisson's formula of the cylindrical case (for a space of dimension 2). Kirchhoff had come across the idea of using Green's theorem in Helmholtz ([62]), who used it to obtain the equation of vibrations of air in a tube: on these pioneering application of Green's formula, one may consult [5], [6] and [91]. The details regarding the expressions obtained by Kirchhoff can be found in [29] (p.140 ff), and here we will only mention that, unlike Poisson's formula, for which the integral acts over the whole disk of radius at centered on the point under consideration —a being the speed of wave, t the time— Kirchhoff's formula in dimension 3 is an integral acting over the surface of the sphere of radius at. Kirchhoff then interpreted this result as the analytical expression of Huygens's principle explaining the propagation of a light wave by considering the points step by step as small localized sources of emission of spherical waves, a principle which had never in fact received a satisfactory mathematical formulation and had been considered a simple explanatory artifice; Kirchhoff's discovery finally gave it its rightful place in the arsenal of mathematical physics. The

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[19]Nel tomo 2° delle belle lezioni sulla teoria generale del sig. Darboux e nell'ultima Memoria del Du Bois Reymond, inserita nel 104o volume del Giornale di Crelle, sono contenuti risultati di grande importanza per la teoria delle equazioni lineari a derivate parziali del 2o ordine con due variabili indipendenti x, y. In particolare per le equazioni del tipo iperbolico, nelle quali cioè i due sistemi di linee caratteristiche sono reali e distinti, viene dimostrato il teorema fondamentale, secondo il quale in ogni quadrilatero racchiuso sul piano xy da quattro caratteristiche, fissati i valori che l'integrale assume lungo due lati adiacenti del quadrilatero, risultano individuati i valori dell'integrale stesso in tutta la regione interna del quadrilatero [Darboux, §364; Du Bois Reymond §15]. A questo teorema fa riscontro, per le equazioni del tipo ellittico (a caretteristiche immaginarie), l'altro che i valori assunti dall'integrale nell'interno di un campo connesso sono generalmente individuati dai valori che l'integrale riceve sul contorno del campo. Alcune semplici osservazioni contenute nella presente Nota permettono appunto di stabilire con molta generalità questo risultato. Il processo stesso è immediatamente estendibile, come si vedrà, al caso di un numero qualunque di variabili independenti. Però non viene qui affatto trattata la questione molto più difficile se tali valori al contorno possano darsi effettivamente ad arbitrio, questione che, salvo pochi casi particolari, non sembra per ora prossima a risolversi.

1882 article was immediately widely circulated, despite some inaccuracies in the mathematical treatment identified by Gian Antonio Maggi (1856-1937), an Italian mathematician from Messina who had been a student of Kirchhoff in Berlin; in 1888 Maggi proposed a slightly different proof [70]. Eugenio Beltrami (1835-1900), thanks to a slightly more sophisticated formulation of Green's theorem, arrived in 1889 in [12] at a correction of Kirchhoff's original proof, while retaining his original idea of seeking the solutions dependent only on the norm $r = \sqrt{x^2 + y^2 + z^2}$.

2.1.3. *Pierre Duhem*

There then appeared a central figure in our history, the physicist Pierre Duhem. A complex and brilliant personality, Duhem taught in Lille from 1887 to 1891, and a year in Reims, before settling down as a professor of theoretical physics at Bordeaux in 1894, where he remained until the end of his life. His thesis on thermodynamic potential had strong repercussions: it was rejected by the jury because of the ferocious opposition of Marcelin Berthelot (1827-1907), because Duhem had questioned —rightly— the principle of maximum work.[20] Berthelot became an implacable enemy, and did all that was humanly possible to hinder the career of his young colleague, blocking his access to all chairs in Paris. It may be added that this head-on opposition was also fuelled by Duhem's position in the political spectrum, there being no mystery about his strong sympathies for the far right political movement *Action Française*, his hostility to the Republican regime and his uncompromising commitment with the rigid, Gallican fringe of the Roman Catholic Church.[21]

Duhem's monumental output comprises both technical studies and texts on philosophy of science, as well as many educational treatises of the highest calibre.

In his course at the Faculty of Science at Lille, published in 1891 [28], Duhem expounded many aspects of physics in which PDEs played a central role, particularly the theory of elasticity. Taking stock of the results that had been achieved up to that point for hyperbolic PDEs, he showed ([28], Tome II, Livre III, Chapitre VIII) that Kirchhoff's method discussed above can be established for dimensions 1 or 3.

Duhem was an effective promoter of his own course, sending copies out himself to many people, including Eugenio Beltrami, whom he asked to transmit it to Italian colleagues involved in mathematical

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[20]The principle of maximum work had been stated by Berthelot in 1879 in his *Essai de Mécanique chimique fondé sur la thermochimie* [10]: 'Tout changement chimique accompli sans l'intervention d'une énergie étrangère tend vers la production du corps ou du système de corps qui dégage le plus de chaleur.'

[21]In particular, see [63] which gives a very vivid picture of Duhem's philosophical-political position.

physics. Beltrami did so, sending it to Ernesto Padova (1845-1896), then professor of theoretical mechanics at the University of Padua, perhaps because Padova had just published a paper in the journal *Nuovo Cimento* ([77])[22] proposing a unitary mechanical theory of electrical, magnetic and luminous phenomena.

In a letter that he sent to Duhem in January 1892[23] to thank him for his book, Padova also insisted that Duhem express his opinion on these problems, but noted that *his friend Volterra* (who was then still in Pisa for some time to come, where he was in particular assistant to Dini) had also presented, during the same session of the Lincei, a note on electrodynamics, in which he arrived in part to the same results by another route.

The case of Ernesto Padova is a good illustration of the difficulty experienced by mathematicians nourished in the mid-nineteenth century by the milk of Lagrange and Laplace regarding the question of the universality of the mechanistic approach to physical phenomena. In March 1892, after reading Duhem's book in detail, Padova wrote him a long letter containing an impassioned appeal in support of mechanical theories, regarding which, in his opinion, Duhem was *too severe* and Poincaré too reckless:

> We cannot, because a mechanical theory adequately explains certain facts, say: things are going thus! this is how matter is constituted! that is perfectly right and you are right to fight this human vanity or presumption, but that does not prevent mechanical theories from occupying a much more important place than purely physical theories in the discovery of natural phenomena ... There is in many analysts today a tendency to reject these theories as a whole and I fear it will eventually create a gulf between them and physicists. ... In saying: if a phenomenon has a complete mechanical explanation, it will include an infinity of others that render an equally good account of all the particulars revealed by experience, doesn't Poincaré, while stating a true

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[22]Developed further in [76].

[23]Duhem's scientific correspondence is found in the Archives of the Paris Académie des Sciences.

fact, cast discredit on all mechanical interpretations?[24]

[E. Padova à P. Duhem, mars 1892]

In the same letter, Padova remarks that Volterra, to whom he suggested that Duhem send his works, is himself in quite the same vein as Poincaré with regard to mechanical theories.

In April 1892 Duhem sent his course to Volterra. In thanks[25], Volterra sent his own article [116] on Hertz's equations, and told him about the forthcoming publication of his article on light vibrations in birefringent media in *Acta Mathematica* [118].

In July 1892, Volterra, who had read Duhem's course, took pen in hand to send his observations to the author. Having developed Duhem's fine discussion of Huygens's principle in light of Kirchhoff, he mentions having found what appears to him to be an extension of those results:

> You have shown that only two cases in the general equation of elastic vibrations
>
> $$(1) \qquad \frac{\partial^2 V}{\partial t^2} = A^2 \sum_1^m \frac{\partial^2 V}{\partial x_i^2}$$
>
> have integrals of the form
>
> $$(2) \qquad V = \psi F(r - At)$$
>
> where F is an arbitrary function and ψ is a function of r only. These are the cases where $m = 1, m = 3$. The first corresponds to the problem of vibrating strings. The second, to the question of spherical waves in 3-dimensional space. Since the Kirchhoff formula is based on the existence of the integral (2) in the case $m = 3$ we must conclude that we cannot proceed in the same way to find an analogous formula in the case of cylindrical waves or elastic membranes, and for generalising the same formulas for the vibration in an m-dimensional

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[24]Qu'on ne puisse, parce qu'une théorie mécanique explique suffisamment bien certains faits, dire : les choses se passent ainsi ! voilà comment est constituée la matière ! c'est parfaitement juste et vous avez bien raison de combattre cette vanité ou présomption humaines, mais cela n'empêche pas qu'aux théories mécaniques n'appartienne une place bien plus importante qu'aux théories purement physiques dans la découverte des phénomènes naturels ... Il y a dans beaucoup d'analystes aujourd'hui une tendance à rejeter en bloc ces théories et je crains qu'elle ne finisse par creuser un abîme entre eux et les physiciens. ... En disant : si un phénomène comporte une explication mécanique complète, il en comportera une infinité d'autres qui rendront également bien compte de toutes les particularités révélées par l'expérience, M. Poincaré, tout en énonçant un fait vrai ne jette-t-il pas de discrédit sur toutes les interprétations mécaniques ?

[25]Volterra and Duhem's correspondence is found in the archives of the Académie des Sciences de Paris and in the archives of the Accademia dei Lincei.

space. For some time I have tried to obtain formulas
that, not having the same form as that of Kirchhoff,
could substitute it in the case of cylindrical waves and
have the same meaning and extend the results to the
general case. In pursuing this objective, I have seen that
if one does not pose the condition that ψ is a function of
r only, we can find the integrals of equation (1) having
the form (2), but I have shown that (apart from the cases
$m = 1, m = 3$) the function ψ must have singularities
(polydromes, etc.) such that starting from these inte-
grals and using Kirchhoff's method, we find results that
differ substantially from those of Kirchhoff. Due to this
one cannot attain the goal. That is why I tried another
route.[26] [V. Volterra à P. Duhem, 24 juillet 1892]

Volterra then set forth in this long letter the essence of what would
constitute his note to the Accademia dei Lincei [119]. For a hyper-
bolic system in the cylindrical case (spatial dimension $n = 2$), Volterra
showed that it is possible to obtain the formulas of integral representa-
tion extending the formulas of Poisson and Kirchhoff, by means of an
extension of the method of characteristics (formulas A, B, D, E in [119],

---

[26]Vous avez montré qu'il n'y a que deux cas à l'équation générale des vibrations
élastiques

$$(1) \qquad \frac{\partial^2 V}{\partial t^2} = A^2 \sum_1^m \frac{\partial^2 V}{\partial x_i^2}$$

possède des intégrales de la forme

$$(2) \qquad V = \psi F(r - At)$$

où F est une fonction arbitraire et ψ est une fonction de r seulement. Ce sont les
cas où $m = 1, m = 3$. Le premier correspond au problème des cordes vibrantes. Le
second à la question des ondes sphériques dans l'espace à 3 dimensions. Puisque la
formule de Kirchhoff est fondée sur l'existence de l'intégrale (2) dans le cas $m = 3$
on doit conclure qu'on ne peut pas procéder de la même façon pour trouver une
formule analogue dans le cas des ondes cylindriques ou des membranes élastiques,
et pour généraliser les mêmes formules pour les vibrations dans un espace à m
dimensions. Depuis quelque temps, j'ai tâché d'obtenir des formules qui, n'ayant
pas la même forme que celle de Kirchhoff pouvaient la substituer dans le cas des
ondes cylindriques et en avoir la même signification et étendre le résultat au cas
général. En poursuivant ce but, j'ai vu que si l'on ne pose pas la condition que ψ
soit une fonction de r seulement, on peut trouver des intégrales de l'équation (1)
ayant la forme (2), mais j'ai démontré que (en dehors des cas $m = 1, m = 3$) la
fonction ψ doit avoir des singularités (polydromie etc.) telles qu'en partant de
ces intégrales et en employant la méthode de Kirchhoff, on trouve des résultats qui
diffèrent substantiellement de ceux de Kirchhoff. Par là on ne peut donc atteindre
le but. C'est pourquoi j'ai essayé un autre chemin.

p.166-168).[27] For Volterra, this was a satisfactory expression of Huygens's principle, describing the wave in the elastic medium beginning with localised waves.

Duhem replied immediately and enthusiastically:

> I know of nothing analogous to the work of which you speak; I would add that it seems to me extremely interesting; the exceptional character that I recognised in the two cases $n = 1, n = 3$ struck me vividly, and I much desired greater knowledge of the general case; but as I am not a mathematician, I could not conceive of clarifying this difficult issue myself. I had recently urged my friend Paul Painlevé, whose name is certainly not unknown to you, to deal with this question. He had not yet had time to think about it. All of this will combine to show you how much pleasure and interest your memoir will bring me when you have published it.[28] [P. Duhem à V. Volterra, 30 juillet 1892]

A few months later, after reading Volterra's article, Duhem once again wrote him an enthusiastic letter :

> Kirchhoff's formula with the transformation that it must be subjected to when one wishes to extend it either to the equation $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = a^2 \frac{\partial^2 V}{\partial t^2}$ or to Lamé's equations, appears to me to be one of the finest achievements that

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[27]Volterra's results on this type of PDE were later extended to the case of any dimension by Orazio Tedone (1870-1922) in several publications between 1893 and 1898 (see in particular [92]). Volterra's approach was taken in the case of non-constant coefficients by Jean-Marie Le Roux (1863-1949) in his thesis of 1895 [68] and a newcomer from Bordeaux, Joseph Coulon (dates unknown), whose article [23] was published in 1898, generalizing Tedone's results for the heat equation to equation

$$\frac{\partial^2 U}{\partial x_1^2} + \cdots + \frac{\partial^2 U}{\partial x_p^2} - \frac{\partial^2 U}{\partial y_1^2} - \cdots - \frac{\partial^2 U}{\partial y_q^2} = 0.$$

These works were in turn later extended by the research of Robert d'Adhémar (1874-1941) in [1], [2] et [3]. On these subjects, see [40]. As for Coulon, an ecclesiastic, he discussed his thesis (on the integration of second-order partial differential equations by the method of characteristics) in Paris in 1902, then went to Fribourg in Switzerland to direct the French section of the Collège Saint-Michel.

[28]Je ne connais rien d'analogue au travail dont vous me parlez; j'ajouterai qu'il me paraît extrêmement intéressant; le caractère exceptionnel que j'avais reconnu aux deux cas $n = 1, n = 3$ m'avait vivement frappé, et je désirais beaucoup une connaissance plus approfondie du cas général; mais comme je ne suis nullement mathématicien, je ne pouvais songer à élucider moi-même cette difficile question. J'avais récemment poussé mon ami Paul Painlevé, dont le nom ne vous est certainement pas inconnu, à s'occuper de cette question. Il n'avait pas encore eu le temps d'y songer. Tout cela vous marque suffisamment combien de plaisir et d'intérêt me causera votre mémoire lorsque vous l'aurez publié.

has been made in a long time in the field of second-order partial differential equations. While Kirchhoff was a great innovator, you can, in my opinion, claim a great part of these new conquests. So accept my very sincere congratulations.[29]

2.1.4. *A fine Bordeaux vintage*

In 1893 Jacques Hadamard was appointed to the University of Bordeaux. This was his first academic position, after three years teaching at the Lycée Buffon in Paris, years which, if he spent them without pleasure —Hadamard's talents for secondary education do not seem to have been highly developed— had nevertheless allowed him to finish his thesis in peace, discussed in 1892 at the Sorbonne ([46]). They also provided him with the opportunity to find a rare pearl in the person of his young pupil Maurice Fréchet (1878-1973), whose mathematics education and future career he followed from that moment on with tireless zeal.[30] In Bordeaux, Hadamard was first assigned temporarily to teach the course of mechanics and astronomy before being named, in 1896, to the chair of astronomy and theoretical mechanics.

The scientific conditions in Bordeaux proved particularly suitable for the young Hadamard, due to the presence in town of a learned society, the *Société des Sciences Physiques et Naturelles* (SSPN), founded in 1855. The mathematician Jules Houël, who became librarian of the society in 1865, put his polyglot talents at the service of the SSPN. This made it possible for him to interact with mathematicians from around the world, to translate many works, to make the library one of the richest scientific libraries in France, and to raise the Société's publications to a prestigious level on the international mathematical scene.[31]

In 1894, Pierre Duhem, whom Hadamard had known personally for a short time when he began his education at the École Normale Supérieure just as that of Duhem ended, arrived in Bordeaux to take over the chair of theoretical physics.

In his 1927 article on Duhem [58], Hadamard never misses a chance to pay tribute not only to the extraordinary scientific curiosity of the elder scholar, but also to his conception of a mathematical physicist, which had seduced him. In all likelihood it was this shared passion

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[29]La formule de Kirchhoff avec les transformations qu'il faut lui faire subir lorsqu'on veut l'étendre soit à l'équation $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = a^2 \frac{\partial^2 V}{\partial t^2}$ soit aux équations de Lamé, me semblent être une des plus belles conquêtes qui aient été faites depuis longtemps dans le domaine des équations aux dérivées partielles du second ordre. Si Kirchhoff a été un grand novateur, vous pouvez cependant, ce me semble, réclamer une belle part de ces nouvelles conquêtes. Recevez donc mes bien sincères félicitations. [P. Duhem à V. Volterra, 28 novembre 1892]

[30]See the next section.

[31]On Jules Houël in Bordeaux, see [93] and [81].

for scientific activity in all its forms that allowed the two colleagues in Bordeaux to spend an enormous amount of time together without their strong political differences disturbing their relationship. Since his days at the École Normale, Hadamard had perceived Duhem's enthusiasm for new theories in physics, such as Hugoniot's notion of the propagation of shock waves in fluids, which Duhem had learned of through his course at the University of Lille. That enthusiasm was visibly contagious.

Hadamard himself wrote:

> For my part our reunion at the Faculty of Sciences of Bordeaux brought me the rare good fortune of completing the reading [of Duhem's course] with a valuable and continuous exchange of views. To this reading, I owe the greater part of all my subsequent work devoted to the calculus of variations, Hugoniot's theory, to hyperbolic partial derivative equations, to Huygens's principle.
>
> Duhem himself returned to almost all these questions in the course of his immense work, and most of the theories he had so happily and so luminously expounded were suggested to him here by a particular remark, and there by additions of fundamental importance.[32] [58, p.644-645]

2.2. *From congress to congress*

In 1897, Hadamard, appointed non-tenured professor of analytical and celestial mechanics at the Collège de France, returned to Paris. It was also during this year that he became aware of the Italian work on PDEs, and in particular during the first International Congress of Mathematicians in Zürich. Only a small number of French mathematicians attended, perhaps because they saw little use of going abroad when, in their opinion, Paris was the cardinal point of the mathematical world, the place where all that is important ended up happening; maybe they were also reluctant to go to a place that was considered to be too much under German influence. This point probably deserves to be studied more thoroughly. Borel, who was part of the French delegation, seized every opportunity to criticize this attitude in the long

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[32]Pour ma part notre réunion à la Faculté des Sciences de Bordeaux me procura la rare fortune [de] compléter la lecture [du cours de Duhem] par de précieux et continuels échanges de vues. A cette lecture, je dois la plus grande partie de mes travaux ultérieurs tous consacrés au Calcul des variations, à la théorie d'Hugoniot, aux équations aux dérivées partielles hyperboliques, au principe de Huygens.

Duhem lui-même revenait sur presque toutes ces questions, dans la suite de son immense labeur, et la plupart des théories qu'il avait si heureusement et si lumineusement exposées lui suggérèrent ici des remarques de détail, là des compléments d'une importance fondamentale.

and savoury account that he wrote [16]. He also emphasized the importance, in his opinion, of face-to-face encounters with his colleagues, allowing the *living word* to take precedence over *cold print*. The Italian delegation was itself an important one, and Volterra appeared among the most enthusiastic supporters of the formula. It is thus remarkable that on this occasion the two men became acquainted and struck up a friendship that lasted until Volterra's death in 1940. The fiery beginnings of their relationship are discussed elsewhere [72] , and here we will cite only the first letter that Borel sent to Volterra, which illustrates the optimistic climate that followed the Congress:

> If personal relationships that have been forged between us in Zürich were to end after three years or more, the greatest and most enjoyable benefits of the Congress would be lost. But I quite intend for that not to happen for us.[33]

Hadamard was also a member of the French delegation, and he also met Volterra for the first time on this occasion, but unlike Borel, their meeting seems to have remained without any immediate effects on a personal level. Having begun various works on mathematical physics during his stay in Bordeaux, Hadamard came to Zürich to speak about PDEs, but from a prospective that was somewhat particular. In a short talk intriguingly entitled *Sur certaines applications possibles de la théorie des ensembles* [48], Hadamard explained the interest that led him to study certain sets of functions in order to solve problems of extrema, and to study the properties of these sets. Inspired by the calculus of variations, it was especially in view of applications to PDEs whose solution could be solution to such a problem of extremum, namely, a function that maximizes a certain functional (Hadamard did not use the term *functional*, which would appear in his vocabulary a few years later). Hadamard especially evoked the study of the set of continuous real functions on the interval [0,1] endowed with the uniform norm and the cardinality of a covering by balls with a given radius.[34] Hadamard's talk provoked reactions. Borel pointed out [15] that he himself had considered some sets of functions in his studies of series, as the set of the functions which are the coefficients of the decomposition in power series of the solution of a PDE. But it was the short contribution by Salvatore Pincherle (1853-1936), who was also a member of the Italian delegation, which is much more significant for

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[33]Si les relations personnelles qui se sont nouées à Zürich devaient s'éteindre pendant trois ans ou plus, le plus grand et le plus agréable des avantages des Congrès serait perdu. Mais je compte bien qu'il n'en sera pas ainsi pour nous. (Borel à Volterra, 14 novembre 1897)

[34]We obviously use modern terminology here. One can recognize an origin of the questions of precompacity which Fréchet will consider some years later, as we shall see in the next part.

our purposes. Pincherle reported that several Italian mathematicians, such as Ascoli, Volterra, Arzelà and he himself, had for some years been examining these questions regarding sets of functions, envisioning functions as points of a set and even as a continuum.

We have no knowledge of what Hadamard's reaction might have been; Pincherle's remark seems not to have had any effect at first. It must be said that Hadamard was then absorbed in work of another kind, although this also involved PDEs, which were by that time central to his research, and in particular in the preparation of his course at the Collège de France of the years 1898-1900, which was published shortly after with title *Leçons sur la propagation des ondes et les équations de l'hydrodynamique* [54]
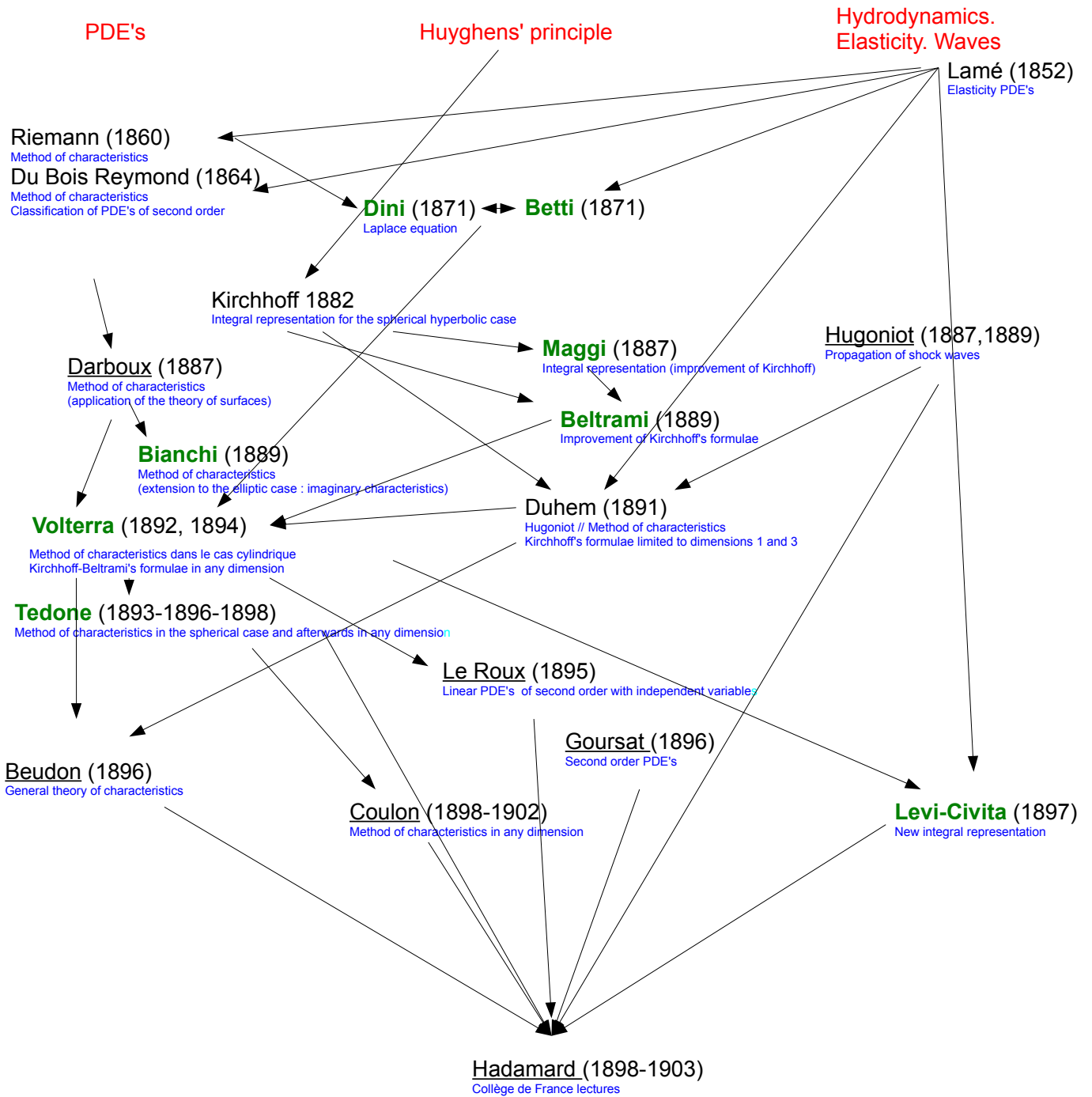
Fig. 1. The sources for the course on the PDEs that Hadamard
taught at the Collège de France (1898-1900). Names of the Italians
are in bold, those of the French are underlined; the dates are those of
the principal publications on PDEs

There, he repeated and extended Duhem's course of 1891, assembling for the first time the results obtained through solutions to hyperbolic PDEs, notably a general theory of characteristics inspired by the thesis of Jules Beudon (1869-1900) [14].

on the theory of systems of PDEs. Hadamard defined the characteristic surfaces associated with an equation of arbitrary order as the location of discontinuity of certain derivatives and showed that this approach permitted a good mathematical translation of the analytical expression of the Hugoniot's considerations on shock waves (see [74], 14.2).

In 1900, Hadamard relied on Huygens's principle. In his article of that year [49], he arrived at the conclusion that Volterra's interpretation was questionable from the point of view of physics.[35]

That same year the International Congress of Mathematics took place, this time in Paris. Volterra and Hadamard met once again, and gave each a talk on PDEs in the same session of 10 August 1900. Volterra's talk, soberly entitled *Sur les équations aux dérivées partielles* [121] reprised the integral formulas that he set out in his article of 1894. That of Hadamard, entitled *Sur les équations aux dérivées partielles à caractéristiques réelles* [50], showed that the clear separation between the elliptic case (imaginary characteristics) for which Cauchy's problem generally has no solutions, and the hyperbolic case where it in fact has some, is in fact only valid in the simple case where the equation is given in the space as a whole. In cases where the boundary conditions affect a portion of the space —such as Dirichlet or Neuman boundary conditions— the situation is more complicated, and the two types of equations have common properties.

There is no trace of discussions between the two mathematicians. We know that a few weeks after the Congress Hadamard sent his first letter to Volterra, who had just been appointed to the University of Rome, but its nature was of the most practical kind[36]:

> I have written to you in Turin, not knowing whether you
> have already moved to Rome, and I hope that my letter
> reaches you just the same. Its purpose is to beg you to
> send me, or better yet, send to Mr Duporcq, Secretary

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[35]Hadamard's criticism was based on the fact that in its original version, the principle enunciated by Huygens in fact implies that every point is at rest after the passage of the wave, which is a specific property of the equations in an odd-numbered dimension of space (such as the spherical case considered by Kirchhoff), since the integral expression of the solution is related to a sphere of radius at. In the case of even-numbered dimensions, to the contrary, the integral formula that extends to the entire interior of the ball prevents the point from ever returning to a state of rest and therefore makes it subject to a residual motion which gives the article its title.

[36]The correspondence between Hadamard and Volterra is conserved in the Archives of the Accademia dei Lincei, Rome.

of the Congress, 162 Boulevard Péreire in Paris, the text
of the talk that you gave on equations with real charac-
teristics, which we would like to have for the proceedings
of the congress.

I am pleased to have this opportunity to remember
myself to you, and ask you accept, with all my best
wishes to Mrs Volterra, the assurance of my kindest re-
gards.[37] [J. Hadamard à V. Volterra, août ou septembre
1900]

Thus we see that Volterra and Hadamard, while being on cordial
terms, do not appear to have been particularly intimate. In light of
what we have shown, we can conclude that while Hadamard esteemed
his Italian colleague, attributing to him a place of merit among those
investigating the methods for solving hyperbolic PDEs, he expected
nothing especially grandiose of Volterra's work. That situation would
change in the years between then and the Heidelberg Congress of 1904.

2.3. *The discovery of functions of lines*

In 1902, Hadamard wrote the article [52], his first publication using
a formalism where functions are taken as variable elements, five years
after his talk at the Zürich conference and Pincherle's observations,
mentioned above. In his 1945 book about the psychology of a mathe-
matician during the process of research [60], he also testifies to his own
surprise over how that concept intervened in his work:

Much more surprising is the fate of the extension given
to that initial conception [of calculus of variations] in
the last part of the nineteenth century, chiefly under the
powerful impulse of Volterra. Why was the great Ital-
ian geometer led to operate on functions as infinitesimal
calculus had operated on numbers, that is, to consider
a function as a continuously variable element? Only
because he realized that this was a harmonious way of
completing the architecture of the mathematical build-
ing, just as the architect sees that the building will be
better poised by the addition of a new wing. One could
already imagine that such a harmonious creation could

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[37]Je vous ai écrit à Turin, ne sachant si vous êtes déjà installé à Rome et j'espère
que ma lettre vous joindra quand même. Elle a pour but de vous prier de vouloir
bien m'envoyer ou, mieux encore, envoyer à M. Duporcq, secrétaire du Congrès,
162 boulevard Péreire à Paris, le texte de la communication que vous avez faite sur
les équations à caractéristiques réelles, et que nous voudrions bien avoir pour les
comptes-rendus du congrès.

Je suis heureux d'avoir cette occasion de me rappeler à votre souvenir et vous
prie de recevoir, avec tous mes respects pour Madame Volterra, l'assurance de mes
sympathiques sentiments.

be of help for solving problems concerning functions considered in the previous fashion; but that "functionals", as we called the new conception, might be directly related to reality could not be thought as otherwise than mere absurdity. Functionals seemed to be an essentially and completely abstract creation of mathematicians. Now, precisely the absurd has occurred. Hardly intelligible and conceivable as it seems, in the ideas of contemporary physicists (in the recent theory of "wave mechanics"), the new notion, the treatment of which is accessible only to students already familiar with very advanced calculus, is absolutely necessary for the mathematical representation of any physical phenomenon. Any observable element, such as a pressure, a speed, etc., which one used to define a number, can no longer be considered as such but is mathematically represented by a functional! [60, p.129-130]

Hadamard embellished the situation somewhat by attributing an ethereal aesthetic vision to Volterra's invention of functions of lines. As we have seen, in his 1887 article [108], the Italian mathematician had underlined how in numerous problems of mechanics and physics there are many naturally occuring quantities that depend on all the values given by one function or another, a situation that he illustrated again and again with the example of the temperature at a point of a conductive blade, which depends on all the temperatures on the edge. In his article [52] of 1902, Hadamard dealt with the notion of derivative of a line function such as Volterra had formulated. Under a certain number of assumptions of regularity, he had given an integral expression to it. However, Hadamard remarked that the derivative in question satisfies the properties of a *fonction linéaire* (linear function), a notion that he himself had introduced in 1901 in his slender, surprisingly little known book on Taylor series [51]. In chapter VII, Hadamard observes that in a number of situations of extension of analytic functions, it is necessary to pay attention to functional transformations, such as Borel's method, who to $f(x) = \sum a_m x^m$ made correspond the *fonction associée* (associated function) $F(x) = \sum a_m x^m / m!$. Setting $\tilde{f}(x) = \int_0^\infty e^{-t} F(tx) dt$, one thus defines a function that in good cases extends f beyond the circle of convergence. In chapter VIII, Hadamard then systematically introduces, following Bourlet (and Pincherle), functions that associate a function to another function, which he names *transmutations*, an example of which is the operator derivative. A transmutation is defined on a class of functions that Hadamard called a *functional field*. He then introduced a particular case of a linear transmutation A verifying

$A(f_1 + f_2) = A(f_1) + A(f_2)$. The same situation was described as *distributive* by Pincherle and *additive* by Bourlet, and these are the only two authors who appear in the chapter's bibliography.

The notion of linear transformation thus makes it possible to envision quite general forms for the derivatives of a line function. Focusing specifically on how it is possible to define the second derivative of a functional, Hadamard showed that it requires a more subtle treatment and that it cannot be expressed in general in the form of an integral by means of the second partial derivative as a naive extension of the form given by Volterra to the first derivative might suggest.[38]

However, it is especially in a paper published in 1903 [55] that Hadamard appears to have begun to see the advantages deriving from the theory of functions of lines, since he himself said that it seemed most practical for studying certain situations by directly manipulating a function as a variable, without having to be limited to analytic functions to go back to sequences of scalar coefficients. Extending by analogy Volterra's formalism to surface functions, Hadamard obtains different representations for linear functionals in the sense that he had introduced previously. A key aspect was the presentation of a fundamental example in which this type of formalism could be used: the first variation of Green's function g_A^B relative to two points A and B within a surface S that is deformed.[39]

Hadamard now became aware of having found terrain on which to stand with Volterra, or even compete with him. There is a touch of anxiety in the letter he sent him a few weeks before the conference in Heidelberg:

> I have always forgotten to ask you what you intend to speak about at the Congress of Heidelberg (in the section of applied mathematics). May I ask you to tell me about this, so I will not go over ground you have already broken?

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[38]It is important to place this discussion in its proper context, to recall that here we are still several years before Fréchet established the notion of differential precisely to deal with this type of situation.

[39]The Green's function g associated with the Laplacian Δ in a domain D is defined by means of the impulse responses defined on the domain. Taking as an argument a pair of points (A, B) of the domain D, $g(A, B)$ (in Hadamard's notation g_A^B) denotes the value in B of the harmonic function in the domain deprived of point A, null on the boundary and infinite in A. It was introduced in 1830 by George Green to obtain by convolution with a function f the solution of the equation $\Delta u = f$. In the modern approach to linear PDEs in the framework of the theory of distributions, it is generally preferred to use the concept of fundamental solution, which is closely connected to it (see [88], p.241ff).

> Please accept my apologies for the bother, and the as-
> surance of my devoted very sentiments.[40] [J. Hadamard
> to V. Volterra, no date]

Volterra's generous reply on the eve of the congress, of which a note
is conserved, probably reassured Hadamard.

> Thank you very much for your kind letter. I intend to
> say a few words at the Heidelberg Congress on wave
> theory. The subject is so vast that I am sure that if
> you also wish to talk about it there will be no inter-
> ference between the two talks. Here are more details
> about my program. I begin by looking at whether the
> limits of what is called wave theory are well marked
> and I will attempt to treat this aspect by considering a
> few questions. I will then attempt to show some lem-
> mas that I believe I will find in the analytical theory of
> dual distribution and I will try to compare Kirchhoff's
> method with Mrs Kovalevsky's method. In relation to
> the vibrations of membranes, which I taught this year
> in my course, one can use the method of images when
> the membrane is rectangular and in other cases where it
> is bounded by straight lines (you have already touched
> on this subject in your paper). I wish to note that this
> method gives much simpler results in analogous prob-
> lems of heat and electricity because it does not reduce
> to series. This usually occurs for hyperbolic equations.
> I will not fail in this regard to cite your paper[41] of the
> French Mathematical Society of 1903, in which when
> you use the wave method. If I have time I would like
> to touch on the relationship between vibration of mem-
> branes and wave theory. It might be that my plan is too
> vast. If you would like to make some remarks about it
> for me, I will be much obliged. I look forward to see-
> ing you in Heidelberg. Mrs. Volterra will accompany

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[40]J'ai toujours oublié de vous demander ce que vous avez l'intention de traiter
au Congrès de Heidelberg (Section des mathématiques appliquées). Puis-je vous
demander de me renseigner sur ce point, afin que je n'aille point sur vos brisées?

Recevez avec mes excuses pour le dérangement que je vous cause l'assurance de
mes sentiments bien dévoués.

[41]Volterra refers to [55].

me and she will be pleased to meet Mrs. Hadamard.[42]
[Volterra to Hadamard, 27 July 1904]

This new encounter in Heidelberg (their third Congress together!) between Volterra and Hadamard was a good one, and their relationship, like that between Borel and Volterra, was now one of esteem and mutual affection.[43]

In several subsequent articles, Hadamard pursued his variational approach to the physics of vibrations. He sought, for example, the law of variation of functions which are solutions to equations of the type $\Delta\Delta V = kV$ when one varies the shape of the boundary of the domain, a type of formulation that occurs naturally in the theory of elasticity when one considers the equilibrium conditions of fixed elastic plates. Consequently, Hadamard's functional equations shed light on the role of evolution equations that one might attempt to study to obtain the form of the various physical parameters. Taking stock of all these questions, Hadamard finally composed the lengthy paper [56], which was awarded the Prix Vaillant. We find here the origin of the works by Paul Lévy (1886-1971), who was present at Hadamard's lectures at the Collège de France in 1909, to whom he suggested the idea of a thesis

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[42]Je vous remercie beaucoup de votre aimable lettre. J'ai l'intention de dire quelques mots au congrès de Heidelberg sur la théorie des ondes. Le sujet est si vaste que je suis sûr que si vous voulez parler aussi du même sujet il n'y aura pas que des interférences entre les deux communications.

Voilà à peu près quel est mon programme. Je commence par chercher si les limites de ce qu'on appelle théorie des ondes sont bien marquées et je pense le [traiter en ?] quelques questions. Je tâche après de montrer quelques lemmes que je crois trouver dans la théorie analytique de la double distribution et je tâche de comparer la méthode de Kirchhoff avec la méthode de Mme Kovalevski.

Par rapport aux vibrations des membranes j'ai enseigné cette année dans mon cours qu'on peut employer la méthode des images lorsque la membrane est rectangulaire et que d'autres cas où elle est limitée par des lignes droites (vous avez déjà touché ce sujet dans votre note). Je désire faire la remarque que cette méthode donne des résultats beaucoup plus simples dans les problèmes analogues de la chaleur et de l'électricité car on ne tombe pas sur des séries. Cela arrive en général pour les équations de type hyperbolique.

Je ne manquerai pas de noter à ce propos votre note de la Société math. de France de 1903 où vous employez la méthode des ondes. Si j'aurai le temps je voudrais toucher à une relation entre les vibrations des membranes et la théorie des ondes. Peut être le programme est trop vaste. Si vous voulez bien me faire quelques remarques je vous en serai fort obligé.

Je serai heureux de vous voir à Heidelberg. Mme Volterra m'accompagne et elle sera heureuse d'y rencontrer Mme Hadamard.

[43]An overwhelming expression of friendship is shown during the tragic moments that Borel and Hadamard went through during World War II (see [73]). Further, when Volterra was in the grips of a struggle with the Fascist regime in 1930, Borel and Hadamard did everything they could to aid him.

on the systematic study of functional equations.[44] Hadamard always
maintained the highest regard for the work carried out by Lévy in his
thesis[45].

3. Maurice Fréchet: In search of a General Functional Analysis

In the previous section we mentioned the fateful encounter at the
Lycée Buffon in Paris between Hadamard and his young pupil Fréchet,
whose progress he followed untiringly. When Fréchet, some sixteen
years after the older mathematician, returned to the École Normale
Supérieure, he benefited from all the relationships that had been estab-
lished between Hadamard and Volterra, and the new interest of French
mathematicians in the work of Italian analysts, especially Pincherle
and Volterra.

In 1902 Hadamard published his article [52] and Fréchet rapidly
worked his way through Volterra's articles of 1887 on line functions.
By 1904, Fréchet had published in his turn an article of his own on the
subject ([30]).

The ideas that influenced and nourished Fréchet's work on functional
analysis have been the subject of several historical studies that have fo-
cused specifically on the essential concept underlying his vision: the no-
tion of 'abstract space' in analysis. In particular, the interested reader
may consult the pioneering thesis by Reinhard Siegmund-Schultze [84]
and the very lengthy, three-part work by Angus E. Taylor [85, 86, 87].

In his study, Taylor describes as 'indirect' Volterra's influence on the
concept of abstract spaces:

> Fréchet's short paper of 1904 [on Weierstrass's theorem]
> broke absolutely fresh ground. Although Vito Volterra's
> work certainly had some influence on Fréchet's work taken
> as a whole, I think a good deal of it was exerted indirectly,
> through Hadamard. I see little or no reason for think-
> ing that Volterra contributed directly to the shaping of
> Fréchet's ideas on L-classes, V-classes, or E-classes [85,
> pp.286-287].

The paper of 1904 to which Taylor refers to here ([31]) introduced
for the first time in Fréchet's work a very general vision that allows
him to conceive 'functional operations' on 'sets' constituted of certain
'categories of arbitrary elements' ('numbers, surfaces, etc.') in bring-
ing into play elements of topology. Naturally Fréchet did not use the

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[44]'Puisque vous avez mis la main sur le sujet, je vous l'abandonne' (since you
have put your hand to the subject, I leave it to you); this is how Lévy described
Hadamards's reaction in his autobiography [69] (p.42).

[45]See [72] and also what was said by Hadamard in his talk at the Bologna
congress in 1928 [15, p.152]. For more details on Lévy's work, see also [9], Section
6.

word 'topology', but what matters, for the quite special purposes that we describe in what follows, are the ideas that we now would classify in the field of 'topology', such as compactness, or the properties of one or another particular type of convergence of series. According to Taylor, that paper marks the starting point of Fréchet's progression to abstract spaces, and he in some way relegates the studies carried out at the same time on line functions to a place among his youthful works. However, we believe there is legitimate reason to reconsider the importance of Fréchet's reading of Volterra's work for his notion of abstract spaces and, more generally, for his work in functional analysis. In particular, we will show that it is essentially the search for a general means of understanding the 'new' functions set forth by Volterra that will, starting with the publications of 1904, give rise to two types of developments which would only come together little by little into a general framework.

On the one hand, by formulating the idea of function that Volterra put forward in a simple manner, Fréchet developed a theory in line with the definition of Dirichlet (cf. [25]) and Weierstrass, who would take a very general form obtained by process of abstraction characteristic of the approach of the young French mathematician. The notion of abstract space appears here as a true preliminary to a general theory of functions or, better, 'operations', a term that Fréchet used in the form of 'operation' or 'functional operation' to designate a function in a wider sense. At the beginning of his thesis he specifies the terms: if E is a set formed of arbitrary elements (numbers, points, features, lines, surfaces, etc.), a 'functional operation' in E is a mathematical object that to any element A of this set makes correspond a specific number, $U(A)$. The study of these operations is the subject of calcul fonctionnel (cf. [35]). In his dissertation Fréchet uses the term 'function' to describe what becomes a special case of 'operation', namely, a classic function of one or more real variables.

On the other hand, following the path opened by Volterra, Fréchet will elaborate tools suitable for dealing with problems of variations arising in mathematical physics. We will analyze below the process used by Fréchet in seeking to bring the problems in this area together within a single conceptual framework, particularly by means of the notion of 'linear operation', and the search for the representation theorems of these functions. Our aim is to show that these two channels were developed in parallel, and took root in a rereading of Volterra's ideas, before they gradually came together again in a general theory of linear operations defined on topological vector spaces. Abstract spaces will be at the centre of much of the subsequent research and are still today of paramount importance in functional analysis.

3.1. *The first contact with line functions*

Shortly after finishing his training at the École Normale in autumn 1903, Fréchet wrote a long letter to Volterra, on the advice of both Borel and Hadamard, who had suggested that he would probably find material in the Italian geometer's work that could supply a subject for a thesis. In his letter Fréchet stated that, starting from Volterra's articles, he had already begun working on line functions, which he had learned about through Hadamard's course, mentioned in the previous section. He then goes into a technical discussion of these line functions in order that Volterra might help him decide upon an interesting subject for his thesis. This might lead us to think that, although Fréchet makes only a passing reference in the first letter, he also knew of Pincherle's approach through having read his articles, as well as through the 1901 treatise that Pincherle wrote with Ugo Amaldi ([4]).

Although the letter was not in the form of an article, it offers a very detailed exposition and shows that Fréchet had already given much thought to Volterra's conceptions. In fact, in essence it contains the results that would be published soon after in [30]. Fréchet defines the notion of extended function in terms similar to those of Volterra:

> Mathematical physics has led to the study of functions
> that is much more general than the functions depending
> on the value of one or more variables (functions that I
> will call *ordinary*). I wish to speak of expressions that
> are determined only by the knowledge of all the values
> of one or more *ordinary functions*.[46] [30, p.557]

However, as soon as this general approach is outlined, Fréchet limits its exploration to the most 'simple' cases, and specifies the framework that his study covers:

> We restrict ourselves to the case of the function U_L whose
> value varies only with the *shape* of a line L planar or
> curved, continuous, closed, and thus whose tangent varies
> continuously, except at a finite number of isolated points.[47]
> [30, p.557]

Although he does not say so explicitly, Fréchet envisions only U_L functions of real or complex values. The point of view adopted for the variable is that of the parameterized family of curves that make it

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[46]La physique mathématique conduit à l'étude de fonctions beaucoup plus générales que les fonctions dépendant de la valeur d'une ou de plusieurs variables (fonctions que j'appellerai *ordinaires*). Je veux parler des expressions qui ne sont déterminées que par la connaissance de toutes les valeurs d'une ou de plusieurs *fonctions ordinaires*.

[47]Nous nous bornerons au cas des fonctions U_L dont la valeur varie seulement avec la *forme* d'une ligne L plane ou gauche, continue, *fermée* et dont la tangente varie d'une manière continue, sauf en des points isolés en nombre fini.

possible to define the (uniform) limits and the variation of the function U_L:

> Consider a family G of these lines, dependent on a parameter α such that if α tends to α_0, L tends uniformly at all points to the corresponding points of L_0. For these lines L, U_L will be a function of α and we add the assumption that it is a function that is continuous and differentiable in α. Under these conditions, we can speak, as in the calculus of variations, of the first variation of U_L: δU_L[48], which depends, of course, on the family G under consideration.[49] [30, p.557]

In the study Fréchet makes direct reference to Volterra's works and in particular [115], where the notion of function lines made it possible to formulate a framework for generalizing the study of functions, extending the concept of variable that varies from point to the curve, and then to the surface etc. Yet the approach of Fréchet does not resume directly to their account this view. He questions the best overall design as possible to deal with problems that have been identified as functional problems, that is to say, whose questioning relates to functions.[51] The presentation of Fréchet suggests that he is seeking from that article in a way to see the functions of general lines and as a tool to read through a single abstract concept all the situations encountered.

As we have seen, in 1887 Volterra had attempted to adapt the classic strategy of studying the functions of the real variable: a dependence, the 'function', which he viewed in a broad sense, hence a variation of the variable that engenders a variation the value of the function that in turn provides the notion of derivative. Finally, in this sequence one obtains an expression of the variation of which the first-order terms give the differential. The analysis in the case of lines is done in the

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[48]The reader can consult the course of Joseph Alfred Serret (1819-1885) [83], published the first time in 1868 and reprinted many times thereafter, for an overview of notations and definitions of the terms 'derived', 'augmented,' differential', etc. in use at the time for ordinary functions. The texts of Volterra, Hadamard or Fréchet never redefine these concepts, which they use in new, different contexts. We must precisely distinguish and analyse them. In particular here δU_L denotes the differential relative to the parameter α (see Ch XII of [83]).

[49]Considérons une famille G de ces lignes, dépendant d'un paramètre α de façon que, si α tend vers α_0, L tende *uniformément* en tous ses points vers les points correspondants de L_0. Pour ces lignes L, U_L sera une fonction de α et nous ajoutons l'hypothèse que ce soit une fonction continue et dérivable en α. Dans ces conditions, nous pourrons parler, comme dans le calcul des variations, de la variation première de U_L : δU_L[50], laquelle dépend, bien entendu, de la famille G considérée.

[51]For an overview of the issues at the core of the research discussed here, one can see Hadamard's 1898-1899 course at the Collège de France, published in 1903 [51]. It may be interesting to read in parallel the Leçons sur le calcul des variation [57] published later, in 1910, which gives an broader panorama of the questions embraced by this term at the beginning of the twentieth century.

same way by Volterra, starting with the deformation of a line L and observing the variation produced for the function U_L that will give the form of the differential.

Fréchet's work proposes an approach that reveals a genuine difference in perspective that he justifies by noting that Volterra's presentation does not make it possible to treat the classic functionals arising from the calculus of variations. This different point of view appears even in the choice of the notations used for line function: while Volterra used the notation $\varphi|[L]|$ to denote the line function, Fréchet goes back to a notation that had been used earlier by Hadamard, U_L, where the connection with the concept of (line) function is not completely transparent. Here the symbol U_L is used instead by Fréchet to denote certain elements of the calculus of variations, elements he eventually identifies as line functions and whose properties he seeks to make evident.

The strategy for generalisation adopted by Fréchet is that of considering in the first place the simplest functionals encountered in the calculus of variation. Thus Fréchet begins with an integral of the type $I_L = \int_L P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$ the variation of which can be written $\delta I_L = \int_L (I'_x \delta x + I'_y \delta y + I'_z \delta z)ds$. Hence, one way to generalise consists in taking that property of a particular functional as a general definition. It is nevertheless necessary to ensure that the new definition covers new cases, which Fréchet does by providing specific examples. Line functions are thus defined beginning with the variation that Fréchet names after Volterra:

> We will thus give the name 'Volterra's function', or 'function $(\mathcal{V})$, to all closed line functions U_L, satisfying the condistion that we posed in no. 1, and such that one has
>
> $$\delta U_L = \int_L (U'_x \delta x + U'_y \delta y + U'_z \delta z)ds,$$
>
> U'_x, U'_y, U'_z having quantities determined in each point M on the entire closed line L.[52] [30, p.560]

Further, this definition permits defining several other classes of functions: 'functions $(\mathcal{V})$ of the first degree, that is, those of the form I_L, which were used to forge the extended definition, and 'simple functions $(\mathcal{V})$', which are presented as ordinary functions of a single function $(\mathcal{V})$ of the first degree.

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[52]Nous appellerons donc *fonction de Volterra*, ou *fonction* $(\mathcal{V})$, toute fonction de ligne fermée, U_L, satisfaisant aux conditions que nous avons posées au n. 1 et telle que l'on ait

$$\delta U_L = \int_L (U'_x \delta x + U'_y \delta y + U'_z \delta z)ds,$$

U'_x, U'_y, U'_z étant des quantités déterminées en chaque point M de toute ligne fermée L.

This approach leads Fréchet to find the characteristic properties of different classes of functions $(\mathcal{V})$. In particular, here we find, in a second moment, the notion of linearity that Volterra had also put forward:

> Let us recall, in order to generalise it, a theorem of Mr Volterra's. We denote, as he did, by $L + L'$ a closed contour (which may include several closed curves) and constituted by the contours L and L', where the common parts (if any exist) have been removed, travelled in the opposite direction (for U_L depends in general on the direction of travel of L).
>
> Mr Volterra has shown that the functions of first degree are the only functions $(\mathcal{V})$ that verify the functional equation
> $$U_{L+L'} = U_L + U_{L'}.$$
> More generally, we will now show that the simple functions are the only functions $(\mathcal{V})$ that verify the functional equation
> $$U_{L+L'} = \varphi(U_L, U_{L'}),$$
> φ being an ordinary function (continuous and derivable) of U_L and $U_{L'}$.[53] [30, p.562]

These objects will become the focus of a series of studies carried out by Fréchet in the years to follow, and we will show in a later section that this development is not without a connection to the ideas that made it possible to forge his abstract spaces.

Finally, we believe it is important to emphasise how Fréchet positioned himself and the strategy he adopted in the search for a general framework of study, and to compare these elements with the ideas chosen by Volterra before him.

Volterra conceived the notion of function that depends on other functions in a very general way in order to conceive all the situations that arise in problems of variations. Let us recall, as we have seen in the first section, that it was only in a later moment that he restricted his point of view to particular cases for the study of problems of variations,

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[53]Rappelons, pour le généraliser, un théorème de M. Volterra. Désignons avec lui par $L+L'$ un contour fermé (pouvant comprendre plusieurs courbes fermées) et constitué par les contours L et L', où l'on a supprimé les parties communes (s'il en existe) parcourues en sens contraire (car U_L dépend en général du sens de parcours de L). M. Volterra a démontré que les fonctions du premier degré sont les seules fonctions $(\mathcal{V})$ qui vérifient l'équation fonctionnelle
$$U_{L+L'} = U_L + U_{L'}.$$
Plus généralement, nous allons montrer que les fonctions simples sont les seules fonctions $(\mathcal{V})$ qui vérifient l'équation fonctionnelle
$$U_{L+L'} = \varphi(U_L, U_{L'}),$$
φ étant une fonction *ordinaire* (continue et dérivable) de U_L et $U_{L'}$.

by means of the notion of line functions and the property of linearity. Moreover, in order to transpose the usual properties of ordinary functions, such as continuity or differentiability, he had conceived a notion of neighbourhood or of perturbation, which he declined in function of the situations, from the most general to the most particular. For functions that depend on other functions, one cannot properly speak of neighbourhood and a perturbation is a function θ whose values are everywhere small. In the case of line functions, Volterra develops a much more geometric version of neighbourhood, which he calls the domain of line L ([115, pp.237–238]), consisting in a tube around the line L.

Fréchet's approach is significantly different. He uses a strategy of abstraction that permits him to take the classic situations into account in order to extract an effective general point of view. This strategy will be applied to the reading of Volterra's papers and will result in two types of developments.

The first arises from the notion of function which is based on Volterra, and appears to be a major factor in what might be called the process of generalisation in analysis (from finite to infinite, as the protagonists themselves say). Fréchet will therefore rethink the notion of function and attempt to extract the essential elements that make it possible to develop a general theory, doing away with the necessity of seeing the variable as a real number, hence embarking all the topological tools then known on $\mathbb{R}$. This first approach will be developed in [30] and will constitute the heart of Fréchet's doctoral thesis.

On the other hand, in discussing Volterra's ideas, Fréchet proposes a second point of view: he decides this time to begin with the notion of line function, which is also the idea Hadamard had had (see [52]). In his case he begins with a simple case where the parallel with the results already established in some particular case of the theory of variations is clear and from which he extracts the elements that could make it possible to conceive a general framework. Thus, as we have shown earlier, he decided to define first of all a type of line function according to the form of variation. It is thus by beginning with the analysis of this simple case that Fréchet, by a process of abstraction, puts forward the notion of linear functional. He thus deals with two related problems: on the one hand, that of the representation of linear operations, and on the other hand, integration, that is to say, how to find the function when its differential is known, which is none other than a line function. These problems will essentially be developed in three papers entitled 'Sur les Opérations linéaires' published between 1904 and 1907 ([32], [33], [34]).

3.2. *The extension of the study of functions*

3.2.1. *The generalisation of a theorem by Weierstrass*

In [31], Fréchet proposes a first direction for the general study of functions adapted to variational problems. The introduction positions the article along the lines of Hadamard's work, not only for the notation U of a functional, but above all for the questions of minima that he raised in 1897 (see [48]), as we saw earlier in our second section:

> One knows how important it would be, in a large number of problems, to know whether a quantity U depending on certain elements (points, functions, etc.) had effectively reached a minimum in the field under consideration. Dirichlet's principle offers one of the most striking justifications of this remark ...[54] [31, p.848]

Fréchet immediately recalls the result he has in mind and would like to retain in the more general framework:

> The problem is solved in the particular case where U is a simple function of x (or of several independent variables). Weierstrass has in effect shown that any continuous function in a bounded interval attains its maximum at least once.[55] [31, p.848]

This property depends on many of the ingredients that Volterra himself had relied on in his approach, namely the central use of a notion (extended) of function and continuity. However, it appears to Fréchet that in their approaches analysts relied on the intervention of the nature and certain of the elements that serve as object or variable to obtain the modern results of the usual theory of functions. He thus conceived a general framework by taking these aspects into account, and the definition obtained appears to be an extension of that of Dirichlet:

> We assume given a certain class[56] C of arbitrary elements (numbers, surfaces, etc.), in which it is possible to discern the distinct elements. We can say that U_A is a uniform function (or functional operation) in a set E of elements

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[54]On sait l'importance qu'il y aurait, dans un grand nombre de problèmes, à savoir si une quantité U dépendant de certains éléments (points, fonctions, etc.) atteint effectivement un minimum dans le champ considéré. Le principe de Dirichlet offre une des justifications les plus frappantes de cette remarque...

[55]La question est résolue dans le cas particulier où U est une simple fonction de x (ou de plusieurs variables indépendantes). Weierstrass a en effet démontré que toute fonction continue dans un intervalle limité y atteint au moins une fois son maximum.

[56]Care must be taken not to attribute a modern meaning to the term "class" (*catégorie*). Here 'class' is used to form a certain typology. It follows that in this statement a set is formed of elements of a certain determined type ('numbers', or 'surfaces', etc.).

of C, if to any element A of E there corresponds a well-determined number U_A.[57] [31, p.849]

We can remark a somewhat similar approach in [30], in the introduction to which Fréchet had already described the set of lines on which the functional acted, as we have mentioned earlier.

In [30], as did Volterra before him, Fréchet defines continuity in a particular way for each situation. In contrast, in [31] Fréchet takes a much more general approach in which a notion of sequential convergence is given a priori:

> To arrive at the notion of continuity of such a function, we presume having arrived at a definition that gives a precise meaning to this phrase: *the infinite set $A_1, A_2, \cdots, A_n, \cdots$ of the elements of C has a limit B.* This definition will suffice for us, in any case, when the following two conditions are satisfied : 1st, if the sequence $A_1, A_2, \cdots, A_n, \cdots$ has a limit, then any sequence $A_{p_1}, A_{p_2}, \cdots, A_{p_n}, \cdots$ formed of elements of the first sequence with increasing indices also has the same limit; 2nd, if none of the elements $A_1, A_2, \cdots, A_n, \cdots$ of an arbitrary sequence is different from A, that sequence has a limit which is A.[58] [31, p.849]

The abstract nature of this notion of convergence will then make it possible to define the notions of closed set, continuity and compact set in a way that is likewise independent of the nature of the elements of C. For example, the notion 'compact set' is defined by Fréchet as follows:

> We will call *compact set* any set E such that there is at least one element common to an arbitrary infinite sequence of sets $E_1, E_2, \ldots, E_n, \ldots$ contained in E, where these (possessing at least one element each) are closed and each is contained in the previous.[59] [31, p.849]

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[57]Nous supposons donnée une certaine catégorie C d'éléments quelconques (nombres, surfaces, etc.), dans laquelle on sache discerner les éléments distincts. Nous pourrons dire que U_A est une fonction (ou opération fonctionnelle) uniforme dans un ensemble E d'éléments de C, si à tout élément A de E correspond un nombre bien déterminé U_A.

[58]Pour arriver à la notion de continuité d'une telle fonction, nous supposerons acquise une définition qui donne un sens précis à cette phrase : *la suite infinie $A_1, A_2, \cdots, A_n, \cdots$ d'éléments de C a une limite B.* Il nous suffira que cette définition, d'ailleurs quelconque, satisfasse aux deux conditions suivantes : 1° si la suite $A_1, A_2, \cdots, A_n, \cdots$ a une limite, toute suite $A_{p_1}, A_{p_2}, \cdots, A_{p_n}, \cdots$ formée d'éléments d'indices croissants de la première suite a aussi une limite qui est la même : 2° si aucun des éléments $A_1, A_2, \cdots, A_n, \cdots$ d'une suite quelconque n'est distinct de A, cette suite a une limite qui est A.

[59]Nous appellerons *ensemble compact* tout ensemble E tel qu'il existe toujours au moins un élément commun à une suite infinie quelconque d'ensembles

In this framework, Weierstrass's theorem receives a formulation whose generality guarantees its applicability in many situations:

> THEOREM. Any functional operation U_A uniform and continuous in a compact and closed set E: 1st, is bounded in E; 2nd, attains at least once its upper limit.[60] [31, p.849]

This idea will be developed and specified in Fréchet's thesis, which we will discuss below. However, it is also his strategy for seeking a general framework for treating the problems that will interest us in the rest of in our study. It illustrates in effect the spectacular way that Fréchet steered the concepts of his Italian predecessor into new territory.

3.2.2. *Fréchet's doctoral thesis*

Fréchet's doctoral thesis was published in 1906 under the title 'Sur quelques points du calcul fonctionnel' ([35]). We will not give an extensive presentation, but will limit ourselves to highlighting a few elements that extend the ideas we have just mentioned.

Fréchet's ambition is to construct a general framework for modern analysis that would encompass a number of classes of functions with specific properties. After recalling that in the previous decade several mathematicians (Le Roux, Volterra, Arzelà, Hadamard) had generalized the notion of function by considering increasingly extensive cases, Fréchet engages in a strategy that consists in conceiving the broadest and most indeterminate possible for the variable.

> We say that a functional operation U is defined in a set E of arbitrary elements (numbers, curves, points, etc.) when to any element A of E there corresponds a determined numeric value of U: $U(A)$. The search for the properties of such operations constitute the object of functional calculus.[61] [30, p.1]

From the beginning, Fréchet remarks that nothing seems to play naturally and in a uniform way for all situations the role of intervals, a notion that occupied a determinant role in analysis for the functions of a real variable. In a second step, like Volterra, he thought it necessary to extend the notion of continuity.

$E_1, E_2, \ldots, E_n, \ldots$ contenus dans E, lorsque ceux-ci (possédant au moins un élément chacun) sont fermés et chacun contenu dans le précédent.

[60]THÉORÈME. Toute opération fonctionnelle U_A uniforme et continue dans un ensemble compact et fermé E : 1° est bornée dans E; 2° y atteint au moins une fois sa limite supérieure.

[61]Nous dirons qu'une opération fonctionnelle U est définie dans un ensemble E d'éléments de nature quelconque (nombres, courbes, points, etc.) lorsqu'à tout élément A de E correspond une valeur numérique déterminée de U : $U(A)$. La recherche des propriétés de ces opérations constitue l'objet du Calcul Fonctionnel.

We say that a functional operation V uniform in a set E of elements of a class (L) is continuous in E if, whatever element A in E is the limit of a sequence of elements $A_1, A_2, \ldots, A_n, \ldots$ of E, we always have:

$$V(A) = \lim_{n=\infty} V(A_n).$$

[62] [35, p.7]

In passing, Fréchet notes that this concept is too general to permit defining uniform continuity in arbitrary sets. He will then, by means of the special cases that he deems to be most significant, progressively consider new conditions that make it possible to develop a complete theory of functions. The examination of particular fields and of uniform continuity (which will play a key role in his approach) spur him to define a notion of neighborhood and to isolate the sets where this notion of neighborhood is defined, which he calls V classes.

Likewise, to use topological concepts in sequential form, Fréchet places himself in a framework of completeness. There again, the generalisation works by conserving a property obtained in the classic case as a result of a theorem, this property now characterising a general class:

We say then that a class (Y) admits a generalisation of Cauchy's theorem if any sequence of elements of that class, which satisfies Cauchy's conditions, has a (necessarily unique) limit element.[63] [35, p.23]

Finally, Fréchet introduces one final, more restrictive class denoted by (E) thanks to a particular neighborhood he calls "écart" and corresponds to our notion of distance. Nevertheless, at this point in time Fréchet still thinks that the proper level of generality is represented by his broader notion of class (V).

In the majority of proofs of known theorems, the property b) of the distance [écart] intervenes in the reasoning. However, the theory developed in this chapter shows that it is not indispensable, and that it suffices to make use of neighborhood without the need for it to significantly complicate the reasoning.[64] [35, p.30]

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[62]Nous dirons qu'une opération fonctionnelle V uniforme dans un ensemble E d'éléments d'une classe (L) est continue dans E, si, quel que soit l'élément A de E limite d'une suite d'éléments $A_1, A_2, \ldots, A_n, \ldots$ de E, on a toujours :

$$V(A) = \lim_{n=\infty} V(A_n).$$

[63]Nous dirons alors qu'une classe (Y) admet une généralisation du théorème de CAUCHY si toute suite d'éléments de cette classe, qui satisfait aux conditions de CAUCHY, a un élément limite (nécessairement unique).

[64]Dans la plupart des démonstrations des théorèmes connus, la propriété b) de l'écart intervient dans les raisonnements. Cependant la théorie développée dans

In the remainder of his thesis, Fréchet examines the consequences that can be drawn from his new concepts, notably by revisiting a number of classic cases. We will not dwell on this here. We will, however, complete this section by showing how Fréchet, beginning with the concept of generalised function set forth by Volterra, developed his ideas of abstract space and classes (L), (V), (E), etc. in order to outline the general framework that seemed essential to relieve analysis of unnecessary encumbrances. The result of this choice of a very general framework is that it provides the opportunity for Fréchet to give his theorems definitive postulates that prove exceptionally close to those utilised today. Hadamard did not exaggerate when he wrote in a report on Fréchet's work on the occasion on his candidacy for membership in the Académie des Sciences,[65]

> Mr Fréchet taught us to reason about sets that are completely abstract, that is to say, composed of elements about which one makes, at least in the beginning, no hypothesis. He goes in a stroke to extreme generality, a generality which, by definition, can never be exceeded.[66]

3.3. *Linear operations*

We will now examine the second development that grew out of Fréchet's article [30] ("Fonctions de lignes fermées") and thus directly from Volterra's legacy.

In his three papers on linear operations [32], [33] and [34], Fréchet does not adopt the general framework for studying Volterra functions. These articles directly concern a problem that Fréchet had mentioned in [30]. We recall that Fréchet had chosen, following his reading of Volterra, to elaborate functionals defined by their variation, especially when that variation is itself a linear functional. This then translated into an equality of the type:

$$\delta U_L = \int_L (U'_x \delta x + U'_y \delta y + U'_z \delta z) ds.$$

Besides its focus on the linear aspect of the functional, this equation poses the problem of the integration of this formula. The integration consists in determining the functional U_L solely by starting with its derivatives U'_x, U'_y, U'_z and the relationship we have referred to above.

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ce Chapitre montre qu'elle n'est pas indispensable et qu'il suffit de se servir du voisinage sans avoir besoin pour cela de compliquer notablement le raisonnement.

[65]The quote is taken from that report (1934), which is conserved in the Archives of the Académie des Sciences in Paris.

[66]M. Fréchet nous a appris à raisonner sur des ensembles entièrement abstraits, c'est-à-dire composés d'éléments sur lesquels on ne fait, tout au moins en commençant, aucune hypothèse. Il va d'un coup à l'extrême généralité, une généralité qui, par définition, ne pourra jamais être dépassée.

Fréchet had devoted the last section of his 1904 article on line functions to this question, and arrived at a necessary and sufficient condition that a function $(\mathcal{V})$ is of the first degree (up to an additive constant), stipulating that U'_x, U'_y, U'_z depend only on the point M and the tangent to L at this point ([30, p.570]). These questions then find in Fréchet a natural extension in the general problem of the representation of linear functionals, in the form of an integral, or in the form of a Taylor series, as Volterra had already suggested.

It is this sequence of ideas that Fréchet intends to pursue and develop in his three papers [32], [33] and [34], beginning with the works by Hadamard, which he first seeks to extend.[67]

At the beginning of [32] Fréchet explains the perspective in which he regards operations by making reference to Hadamard:

> We say that an operation is defined if one makes a determined and finite real number U_f correspond to any function $f(x)$ that is real and continuous between two fixed numbers a and b. We call with Mr Hadamard a linear operation any operation that enjoys the following two properties:
>
> (1) it is distributive, that is, if f_1 and f_2 are two continuous functions between a and b, one always has
>
> $$U_{f_1+f_2} = U_{f_1} + U_{f_2}$$
>
> (2) it is continuous, that is, that U_{f_1} tends to U_{f_2} when the function f_1 tends uniformly to the function f_2 between a and b.
>
> [68] [32, p.493]

The views we see expressed here differ from what was proposed in the thesis. In this present context, an operation has as an argument an ordinary continuous function and the continuity of the operation is given by the uniform convergence naturally available for these functions. These elements are consistent with the ideas exchanged between Volterra and Fréchet, the latter, as we have seen, nourished by Hadamard's subtle

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[67]These notes are of utmost importance for understanding the evolution of ideas in the birth of functional analysis. For a precise analysis of the elements involved in this perspective, see [64].

[68]Nous dirons qu'une *opération* est définie si l'on fait correspondre un nombre réel déterminé et fini U_f à toute fonction $f(x)$ réelle et continue entre deux nombres fixes a et b. Nous appellerons avec M. Hadamard *opération linéaire* toute opération qui jouit de deux propriétés suivantes :

(1) elle est *distributive*, c'est-à-dire que si f_1 et f_2 sont deux fonctions continues entre a et b, on a toujours

$$U_{f_1+f_2} = U_{f_1} + U_{f_2}$$

(2) elle est *continue*, c'est-à-dire que U_{f_1} tend vers U_{f_2} lorsque la fonction f_1 tend *uniformément* vers la fonction f_2 entre a et b.

remarks. The ideas are those used to reread known situations in the light of functionals.

The first paper [32] then reprises the result of representation of such an operation given by Hadamard in the form of limit. The essential aim of the article consists in finding a development similar to Taylor series for such operations. Here Fréchet reprises Hadamard's ideas, and particularly his approach centred on the decomposition of functions in Taylor series ([47, 51, 53]).

Fréchet then proposes a series of generalisations essentially based on the linearity of U, its continuity for uniform convergence and the possibility to decompose the functions in a series that converges uniformly. In particular, he uses the uniform convergence of Césaro means proven by Fejer to study the convergence of the Fourier series associated with a continuous periodic function ([61, p.166-167]). The idea is notably not to appeal to a hypothesis of analyticity, which we have long seen that Fréchet considered too restrictive to deal with problems of mathematical physics.

Finally, thanks to the development of functions into series that are increasingly more general, Fréchet rediscovers Hadamard's theorem in all its generality:

$$U_f = \lim_{n=\infty} \int_0^{2\pi} f(y)H_n(y)dy.$$

Fréchet does not stop there; he goes on to analyse the formula obtained. He thus asks what are the functions H_n for which this expression defines a linear operation, and what are the functions for which the limit itself admits an integral representation with the aid of a continuous function H. By showing that the 'pathological' functions H_n nevertheless make it possible to give sense of such an integral, Fréchet will again consider expressions of the type $\int_0^\pi f(y)H(y)dy$, an expression in which it is not even supposed that H is Lebesgue integrable.

Again we find a process dear to Fréchet: isolating a form that was itself obtained as a consequence of a traditional approach (here when one has recourse to a theorem of uniform convergence), in order to construct a generalised vision.

Fréchet's second paper [33] will focus, on the one hand, on refining these considerations on the integral representation but will also finish by setting off in a new direction and reflecting on the nature of the functions f that can be considered as the object of the operation. This is an important step to which Fréchet devotes a separate section in his article, entitled 'Importance du champ fonctionnel dans lequel on définit une opération linéaire'. At the end of the paper, Fréchet shows in particular that it is possible to define functionals for non-continuous functions.

This new direction will bring Fréchet to write a third paper [34] in which the field of all the functions f become the central focus. The two developments that came out of his reading of Volterra that we have followed are both found here, each extensively modified or inclined towards the search for a general analysis. What in fact occurs here is a complete reorganization of the discourse. This article, in the image of the doctoral thesis, reprises a presentation that begins with the domain of definition of the functional —presuming by now that it is a completely arbitrary field of functions— and supposing the existence of a property of the type $U_{cf} = cU_f$ for any real c that ensues from the fact of the continuity expressed in a particular way for each functional field. In contrast to the topological aspects seen in the thesis, the emphasis here is on the structural elements of vector space. Nevertheless, this very much involves the conditions necessary for the expression of linearity that Fréchet wishes to maintain here.

> Definition of the field. Consider a field of functions of variable x defined in the interval $(0, 2\pi)$. I will assume that if two functions belong to the field, it is the same of their sum. To any function of the field, $f(x)$, we can make correspond a well determined number U_f. We thus define an operation in this field functions. We say that this operation is distributive if given any functions $f_1(x)$, $f_2(x)$ of the field, one has:
>
> $$U_{f_1} + U_{f_2} = U_{f_1 + f_2}$$
>
> One can conclude in particular that one has for any distributive operation:
>
> $$cU_f = U_{cf}$$
>
> for any rational constant c. For this relationship to occur even for an irrational c, it is sufficient that U_f satisfies a certain complementary condition. We will state below this complementary condition, but we will state it in a particular way for each field of functions that we will then examine.[69] [34, p.433]

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[69]Définition du champ. Considérons un champ de fonctions de la variable x définies dans l'intervalle $(0, 2\pi)$. Je supposerai que si deux fonctions appartiennent au champ, il en est de même de leur somme. A toute fonction du champ, $f(x)$, nous pourrons faire correspondre un nombre bien déterminé U_f. Nous définirons ainsi une opération dans ce champ de fonctions. Nous dirons que cette opération est *distributive*, si quelles que soient les fonctions $f_1(x)$, $f_2(x)$ du champ, on a :

$$U_{f_1} + U_{f_2} = U_{f_1 + f_2}$$

On en conclut en particulier que l'on a pour toute opération distributive :

$$cU_f = U_{cf}$$

As we have seen, Fréchet makes no reference to any notion of linear space, although this is already found in the works of Peano and Pincherle. In effect, at the end of Peano's 1888 book [79] there is a chapter entitled 'Trasformazioni di sistemi lineari' (Transformations of linear systems). Peano first of all defines the notion 'linear system' in terms very close to those used today to define vector spaces. Finally, in a second moment, he defines the 'operations', which make an element of one system correspond to an element of another system (or possibly of the same system). Immediately the notions of linearity or distributivity (both terms are used by Peano) are set forth. In this optic, Peano evokes, by means of the structure of 'linear system', the minimum elements required to develop a theory of 'linear transformations' in which all the properties necessary for establishing the proofs are explicitly set forth in the definitions introduced at the beginning. A few years later, in 1901, Pincherle and Amaldi published a book entitled 'Le operazioni distributive e le loro applicazioni all'analisi' [80], which opens with a construction similar to that of Peano, explicating first of all 'general sets of linear systems' (insieme / sistema lineare generale), that is, sets that possess a number of properties that define what we call vector spaces (the wording differs only slightly from that of Peano). It is within this framework that is defined, in the second chapter, the notion of 'operation' in the form of 'correspondence' (corrispondenza) between the elements of two linear systems.

We can thus see that Fréchet positions himself very differently, and does not reprise in his account the ideas of his predecessors: although the previous quote begins with the words 'Definition of field', Fréchet does not select any particular property common to fields of functions that he will use. The collections of functions he mentions, besides being stable by addition in order to define distributive operations, are considered with all of their properties without discrimination. We do not know at this point what properties of the sets of functions will be useful in establishing the proofs of the theorems. It is the notion of function field, which is not defined abstractly here, that integrates all the properties necessary to develop a general vision, that is, a vision that embraces all the particular cases of function fields that Fréchet envisions at the end of the article. Finally, the definition of 'distributive operation' is introduced explicitly and synthetically in this context.

The end of [34] allows Fréchet to specify in different functional fields the particular nature of the concepts of continuity and of convergence.

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quelle que soit la constante rationnelle c. Pour que cette relation ait lieu même pour c irrationnel, il suffit que U_f satisfasse à une certaine condition complémentaire. Nous allons énoncer plus loin cette condition complémentaire, mais nous l'énoncerons d'une manière particulière pour chacun des champs de fonctions que nous allons examiner.

The initial motivations that led Fréchet to conceive this organisation are significantly different from those that motivated the ideas expounded in his thesis. Nevertheless, in both we see a common goal underlying the mathematician's thinking: Fréchet's aim is to propose a general and abstract vision of analysis, and more particularly at this stage for functional analysis, which he conceives beginning with diverse situations identified as belonging to a single domain. Moreover —and this links him strongly to Volterra— the notion of generalized function, which little by little becomes functional operation and shortly after operator, remains the key, determinant element in this general vision.

CONCLUSION

We have aimed at showing two outcomes of French mathematicians' readings of the work of Volterra. Hadamard was probably attracted by Volterra's progress in a mathematical vision that harmonised with his own, and a way of interpreting problems that was immediately compatible with his own work. He saw in the work on line functions a program that was ambitious and suitable to problems of PDEs arising from mathematical physics that occupied him in the early twentieth century. The acceptance of the ideas developed by his Italian colleague was total and within a few years the relationship between the two men turned into a genuinely personal as well as scientific friendship. Later Hadamard and Volterra would meet often, especially at international congresses, and their works on functional analysis often contain laudatory references to each other. In 1909 Hadamard was appointed professor at the Collège de France and in 1912 was elected to the Paris Académie des Sciences, two strategic positions that allowed him to help disseminate the ideas of his Italian colleague. The joint efforts of Hadamard and Borel culminated in 1912 with the invitation to Volterra to give a series of lectures at the Sorbonne on line functions, published shortly after in a book edited by Joseph Pérès [122].

We have seen how, in a sort of benevolent gesture,[70] Hadamard had encouraged Fréchet to take an interest in Volterra's work, and how Borel willingly facilitated contacts between the young mathematician and his friend Volterra. A remarkable result of these strong personal ties between Borel, Hadamard and Volterra was the establishment, in the early 1910s, when several new modes of funding became available in France, of scholarships for French students to go to Rome to study with Volterra. Pérès was the first student to make the journey in 1912, followed the next year by Gateaux.[71] This trend was quickly disrupted by the outbreak of World War I. The war and its attendant tragedies also played a large role in the evolving relationship between Volterra

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[70]On this subject, see [19].

[71]For more on this, see [72].

and Hadamard (and more generally between Volterra and all his French colleagues).[72]

We have observed that the young Fréchet was rereading the works on line functions at the very time when he was forging his own research program and inaugurating his conception of functional analysis in a general and abstract sense. It seems that Fréchet very quickly decided to construct a new mathematical edifice that he named 'general analysis', and which he could not conceive of as the culmination of the program proposed by Volterra. A veritable divergence of points of view was in fact born, a divergence that led to a certain degree of tension between the two mathematicians. We find the traces of this friction in an exchange of letters between Volterra and Fréchet in spring 1913. On the occasion of the publication of [122], Fréchet wrote a letter to Volterra to say that he himself was going to publish on the subject:

> My dear Sir and colleague,
> As you have published a book on line functions, I wanted to send you two articles that I have written on the subject. Unfortunately one is printed, but has not yet appeared, the other is not yet printed. The first in any case is the development two notes published in Comptes Rendus Ac. Sc. in Paris (1911, vol. 152, pp.854[73] and 1050), or rather the parts of these two papers regarding to functional calculus. It may be of interest to you for me to summarise the essential ideas. If by chance you are interested in these reflections, I could send you the manuscript of my paper [presented] to the Congress of Sociétés Savantes in 1912, which as I mentioned is in press but has not yet been issued. It is 19 pages long.[74]
> [M. Fréchet to V. Volterra, 26 May 1913]

What ensued was a tense discussion between the two mathematicians and an exchange of letters that show how each wished to make his point of view known. The publications that Fréchet mentioned in

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[72]For this, we refer to [73].

[73]One in fact finds this on p.845, and this is probably a slip of the pen in Fréchet's letter.

[74]Monsieur et cher collègue

Comme vous publiez un livre sur les fonctions de lignes, j'aurais voulu pouvoir vous envoyer deux articles que j'ai écrit à ce sujet. Malheureusement l'un sous presse tarde à paraitre, l'autre n'est pas encore imprimé. Le premier est d'ailleurs le développement de deux notes parues dans les Comptes Rendus de l'Ac. des Sc. de Paris (1911, tome 152, pp 854 et 1050) ou plutôt des parties de ces deux notes relatives au Calcul Fonctionnel.

Peut-être vous intéresserait-il que j'en résume l'essentiel. Si par hasard ces réflexions vous intéressaient, je pourrais vous envoyer le manuscrit de ma note au Congrès des Soc. Savantes de 1912 qui déjà imprimée comme je le disais plus haut n'est pas encore parue. Elle a 19 pages.

the previous letter ([36], [37]) are both entitled 'Sur la notion de differentielle'. They certainly do not make any reference to Volterra's work on functional calculus; they are two very brief papers that do not go into detail, but which set forth a general way of conceiving the differential. Paper [38] is more detailed and focuses over fourteen pages on elements that might bring Volterra to mind. Volterra's works are cited in passing, and this probably caused a change of humour in the Italian mathematician.

> To arrive at a generalisation of the theorems of differential calculus, one must first of all generalise the notion of derivative where differential. One might base oneself, to carry out this extension, on the method used in the calculus of variations, which is but a chapter of functional calculus.
>
> This is the path followed by Mr Volterra, who had the merit of developing the first coherent theory of differential functional calculus. It consists in operating with the variation of the functional in the sense of the term as used in the calculus of variations. U_f is the functional defined in the field of continuous functions within a given interval I. Mr Volterra considers the case in which the quantity $U_{f(x)+\epsilon\varphi(x)}$ has a differential with respect to ϵ for $\epsilon = 0$: this differential will by definition the variation of U_f for the argument $f(x)$. He remarks that, under certain simple conditions, this variation is of the form
>
> $$\epsilon \int_a^b \varphi(y)k(y)dy$$
>
> $k(y)$ being an independent function of $\varphi(y)$.
>
> However, Mr Hadamard observed that the calculus of variations already offers us the example of a very simple functional whose variation cannot be put in this form. He therefore reduces Volterra's condition to: the variation of U_f must simply be a linear functional ... with respect to the increase $\epsilon\varphi(x)$ of the point $f(x)$.
>
> This is the essential indication which will form my point of departure. Nevertheless, it appears to me necessary to make the existence of the variation derive from that of the differential and in consequence define first of all the differential of a functional.[75] [38, p.47]

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[75]Pour arriver à généraliser les théorèmes du calcul différentiel, il faut généraliser d'abord la notion de dérivée où de différentielle. On pourrait se baser, pour effectuer cette extension, sur la méthode employée dans le Calcul des Variations, qui n'est qu'un chapitre du Calcul Fonctionnel.

This quote allows us to see the issues of this dispute, which will play out over several months without ever really being resolved. In the first place, Fréchet fails to cite all of the works of Volterra, who had himself considered singular points and forms of variations taking into account these particular cases. Volterra sent a written request to Fréchet asking him kindly to rectify this omission. In a letter dated 17 November 1913 he writes, 'I also hope that you will correct what you have said in relation to singular points. I attach great importance to this point, which I have emphasised since my first memoirs of 1887'.[76]

Fréchet had quickly, from the very first letters of Volterra and with much deference, promised to rectify his articles by mentioning the exact extent of Volterra's results. He was in fact able to do so in [39], which appeared in 1915. Yet even in this article citing Volterra, Fréchet introduced nuances and clearly marks how he differs:

> Historical Overview
>
> The first attempt to apply to functionals the procedures of differential calculus appears to be due to Mr Volterra. ...
>
> Mr Volterra did not fail to remark that such a definition was not entirely satisfactory, since it leaves out many of the expressions that intervene in the calculus of variations, namely those variations of definite integrals where the limits are not fixed. Such variations have in effect, besides a definite integral of the form (1) $[\delta U_L = \int_L U_{L,x} \delta y \, dx]$, finite terms at the limits. He

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C'est la voie suivie par M. Volterra, qui a eu le mérite de développer le premier une théorie cohérente du Calcul Différentiel Fonctionnel. Elle consiste à opérer avec la variation de la fonctionnelle au sens où on entend ce mot dans le Calcul des Variations. Soit la fonctionnelle U_f définie dans le champ des fonctions continues dans un intervalle donné I. M. Volterra considère le cas où la quantité $U_{f(x)+\epsilon\varphi(x)}$ a une différentielle par rapport à ϵ pour $\epsilon = o$: cette différentielle sera par définition la *variation* de U_f pour l'argument $f(x)$. Il remarque que, sous certaines conditions simples, cette variation est de la forme

$$\epsilon \int_a^b \varphi(y)k(y)dy$$

$k(y)$ étant une fonction indépendante de $\varphi(y)$.

Mais M. Hadamard a fait observer que déjà le Calcul des Variations nous offre l'exemple de fonctionnelles très simples dont la variation ne peut se mettre sous cette forme. Il réduit donc la condition de Volterra à celle-ci : la variation de U_f doit être simplement une fonctionnelle linéaire [...] par rapport à l'accroissement $\epsilon\varphi(x)$ de l'argument $f(x)$.

C'est là l'indication essentielle qui formera mon point de départ. Cependant il me paraît nécessaire de faire découler l'existence de la variation de celle de la différentielle et par conséquent de définir d'abord la différentielle d'une fonctionnelle.

[76]J'espère aussi que vous aurez corrigé ce que vous avez dit par rapport aux points singuliers. Je tiens beaucoup à ce point que j'ai mis en évidence depuis mes premiers mémoires de 1887.

> therefore agreed to add to the second member of (1) the
> terms that depend, in a special way, as he phrased it, on
> certain exceptional points. By adopting the definition
> of Mr Volterra, one is inspired by the first applications
> that have presented themselves and that Mr Volterra has
> treated with a success that practically justifies his defini-
> tion. However, it was desirable from the point of view of
> logic and to ensure the future development of the theory
> to deduce the definition of a unique and general prin-
> ciple. Mr Hadamard thus proposed to 'consider as a
> functional to which one can extend the methods of infin-
> itesimal calculus, all functionals U_y whose variation is a
> linear function of the variation of y'.[77] ([39, p.136])

The expression 'appears to be due to Mr Volterra' enabled Fréchet
to imply that the subject did not belong to anyone, and shows that
by now he felt prepared to take it on in a way that he believed to be
innovative with respect to what Volterra and Hadamard had proposed.
The dispute between the two mathematicians was actually of short
duration, even though Fréchet adopted a somewhat ambiguous line of
conduct. In his publications, he sometimes failed to cite Volterra as
the initiator of functional analysis, while writing him letters filled with
of tributes and acknowledgments:

> Dear Sir, I should be sorry if you might believe that I do
> not appreciate the true value of your essential contribu-
> tion to functional calculus. First of all, I see as much as
> you do no 'difference between your definition of function
> that depends on all the values of another function, and
> functional, as well as between their calculus'. . . .

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[77]Aperçu historique

Le premier essai pour appliquer aux fonctionnelles les procédés du Calcul Différen-
tiel semble être dû à M. Volterra. [...]

M. Volterra ne manqua pas de remarquer qu'une telle définition n'était pas
entièrement satisfaisante, puisqu'elle laisse de côté une grande partie des expressions
qui interviennent dans le Calcul des Variations, à savoir celles des variations des
intégrales définies où les limites ne sont pas fixes. De telles variations comportent
en effet, outre une intégrale définie de la forme (1) $[\delta U_L = \int_L U_{L,x} \delta y dx]$, des termes
finis aux limites. Il convint donc d'ajouter au second membre de (1) des termes
qui dépendent, d'une manière spéciale, selon son expression, de certains points
exceptionnels. En adoptant la définition de M. Volterra, on s'inspire des premières
applications qui se sont présentées et que M. Volterra a traitées avec un succès qui
justifie pratiquement sa définition. Mais il était souhaitable au point de vue logique
et pour assurer le développement futur de la théorie de déduire la définition d'un
principe unique et général. M. Hadamard proposa donc de "considérer comme
fonctionnelles auxquelles on peut étendre les méthodes du Calcul Infinitésimal,
toutes les fonctionnelles U_y dont la variation est une fonctionnelle linéaire de la
variation de y."

If my current ideas may be of some interest, in any case, I myself recognise it as a secondary refinement of the beautiful theory that you have constructed.

It is precisely because I consider your theory to have remained well enough known that I have not believed it necessary to insist on its importance and have directly proposed some refinements of the details.[78] [M. Fréchet à V. Volterra, 21 novembre 1913]

The short-lived dispute aside, the exchanges between the two mathematicians reveal the significant difference in vision that we have highlighted in our study, which marks, we might say, a change of generation. While the idea of function allowed the two mathematicians to develop in each case a general vision and conceive a rereading of problems of functionals in an abstract setting, Fréchet had an early desire to build a genuinely general theory which naturally embraced the problems of mathematical physics dear to his masters. It is in this quest for a general analysis that he felt the need to break away from Volterra's vision. This shows no lack of respect or recognition, but a bifurcation that was necessary to carry these ideas forward. An inevitable consequence was that the centre of gravity for subsequent studies in functional analysis moved far away from the sunny skies of Rome.

Translated from the French by Kim Williams.

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[78]Cher Monsieur,

Je serais désolé si vous pouviez croire que je n'apprécie pas à sa juste valeur votre contribution essentielle au Calcul Fonctionnel. Tout d'abord pas plus que vous je ne vois "de différence entre votre définition de fonction qui dépend de toutes les valeurs d'une autre fonction et les fonctionnelles ainsi que dans leurs calculs". [...]

Si mes idées actuelles peuvent avoir quelque intérêt, ce n'est en tout cas, je le reconnais moi même, que comme perfectionnement secondaire de la belle théorie que vous avez édifiée.

C'est précisément parce que je considère que votre théorie est maintenant suffisamment connue que je n'avais pas cru inutile d'insister sur son importance et que j'avais proposé directement quelques perfectionnement de détails.

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