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Discrete harmonic functions on an orthant in $\mathbb{Z}^d$

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Abstract

We give a positive answer to a conjecture on the uniqueness of harmonic functions stated in [20]. More precisely we prove the existence and uniqueness of a positive discrete harmonic function for a random walk satisfying finite range, centering and ellipticity conditions, killed at the boundary of an orthant in $\mathbb{Z}^d$.

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1 Introduction

An explicit description of the Martin compactification for random walks is usually a non-trivial problem and the most of the existing results are obtained for so-called homogeneous random walks, when the transition probabilities of the process are invariant with respect to the translations over the state space E (see [5], [18], [22]). In [4] Doney describes the harmonic functions and the Martin boundary of a random walk $(Z(n))$ on $\mathbb{Z}$ killed on the negative half-line $\{z : z < 0\}$. Alili and Doney [2] extend this result for the corresponding space-time random walk $S(n) = (Z(n), n)$. Kurkova and Malyshev [13] describe the Martin boundary for nearest neighbor random walks on $\mathbb{Z} \times \mathbb{N}$ and on $\mathbb{Z}_+^2$ with reflected conditions on the boundary. The recent results of Raschel [20] and Kurkova and Raschel [14] identify the Martin compactification for random walks in $\mathbb{Z}_+^2$ with jumps at distance at most 1 and absorbing boundary. All these results use methods that seem to be unlikely to apply in a general situation. The methods of Doney [4] and Alili and Doney [2] rely in an essential way on the one-dimensional structure of the process $(Z(n))$. Kurkova and Malyshev [13], Raschel [20] and Kurkova and Raschel [14] use an analytical method where the geometrical structure of the elliptic curve defined by the jump generating function of the random walk plays a crucial role. For small steps walks the method uses the fact that the corresponding elliptic curve is homeomorphic to the torus. In a very recent paper Fayolle and Raschel [6] show how to extend the method for random walks with arbitrary big jumps (in this case the algebraic curve is no more homeomorphic to the torus).

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The aim of our paper is to offer a general approach allowing to study positive harmonic functions for random walks killed at the boundary of an orthant in $\mathbb{Z}^d$ without assuming invariance with respect the translations over the state space.

More precisely let $\Gamma = -\Gamma \subset \mathbb{Z}^d$, be a symmetric finite subset of $\mathbb{Z}^d$ containing all unit vectors in $\mathbb{Z}^d$, i.e. all the vectors $e_k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^d$, where the 1 is the k-th component; and $\pi : \mathbb{Z}^d \times \Gamma \to [0, 1]$ be such that

$$\sum_{e \in \Gamma} \pi(x, e) = 1, \quad x \in \mathbb{Z}^d.$$

Then, we let $(S_j)_{j \in \mathbb{N}}$, be the Markov chain defined by

$$\mathbb{P}[S_{j+1} = x + e / S_j = x] = \pi(x, e); \quad e \in \Gamma, x \in \mathbb{Z}^d, j = 0, 1, \cdots$$

We shall impose the following three conditions on the step set Γ and the probabilities $\pi(x, e)$:

(Finite range) $|\Gamma| < \infty$,

(Centering) $\displaystyle\sum_{e \in \Gamma} \pi(x, e) e = 0$,

(Ellipticity) there exists $\alpha > 0$ such that $\pi(x, \pm e) \geq \alpha; \quad x \in \mathbb{Z}^d, \ |e| = 1$.

We shall denote by:

$$\mathcal{D} = \{(x_1, ..., x_d) \in \mathbb{Z}^d; \quad x_1 > 0, \cdots, x_d > 0\}$$

and we shall be interested in characterizing the positive harmonic functions for the random walk killed at the boundary of the orthant $\mathcal{D}$, i.e. in functions $u : \overline{\mathcal{D}} \to \mathbb{R}$ such that:

(i) For any $x \in \mathcal{D}, \ \ u(x) = \displaystyle\sum_{e \in \Gamma} \pi(x, e) u(x + e)$,

(ii) if $\quad x \in \mathcal{D}, \quad u(x) > 0$,

(iii) if $\quad x \in \partial\mathcal{D}, \quad u(x) = 0$,

where we denote by

$$\partial\mathcal{D} = \{x \in \mathcal{D}^c / x = z + e, \quad \text{for some } z \in \mathcal{D}, \text{ and } e \in \Gamma\}; \quad \text{and } \overline{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}.$$

Theorem 1.1. *Let $(S_j)_{j \in \mathbb{N}}$ be a spatially inhomogeneous random walk satisfying the above centering, finite range and ellipticity conditions. Then, up to a multiplicative constant, there exists a unique positive harmonic function for the random walk killed at the boundary.*

Note that general results on harmonic functions for killed random walks in half spaces $\mathbb{Z}^+ \times \mathbb{Z}^{d-1}$ and orthants were obtained in the non-zero drift case in [8, 9, 14]. In this case, the Martin boundary is composed of infinitely many harmonic functions. Regarding random walks with zero drift, important results were obtained recently by Raschel in [20] in the quarter plane. In a sense our results can be seen as complementing those of [20] because they provide an answer to the conjecture stated in [20] in the case of random walks with finite support (cf. [20, Conjecture 1, §4.1]). Our methods, using ideas introduced in [16, 17], allow on the other hand to generalize from the quarter plane to orthants in higher dimensions and to treat the spatially inhomogeneous walks.

The paper is organized as follows. Section 2 gives in its first part the main technical ingredients in the proof of Theorem 1.1. A lower estimate of the harmonic measure

providing a Hölder continuity property at the boundary is established. Specifically, Proposition 1 gives a control of the growth of positive harmonic functions vanishing on the boundary. A Carleson-type estimate in then derived. In section 2.2 we establish the existence of a positive harmonic function on $\mathcal{D}$. Section 3 is devoted to prove the uniqueness of such positive harmonic function on $\mathcal{D}$.

2 Proof of Theorem 1.1

2.1 Key technical ingredients

The proof of Theorem 1.1 relies on a systematic use of two fundamental principles of potential theory, maximum principle and Harnack principle.

Let $A \subset \mathbb{Z}^d$ denote a bounded domain (i.e. a finite connected set of vertices of $\mathbb{Z}^d$). We let

$$\partial A = \{x \in A^c / x = z + e, \quad \text{for some } z \in A, \text{ and } e \in \Gamma\}; \quad \text{and } \bar{A} = A \cup \partial A.$$

We say that a function $u : \bar{A} \to \mathbb{R}$ is harmonic on A if

$$u(x) = \sum_{e \in \Gamma} \pi(x, e) u(x + e), \quad x \in A.$$

The following maximum principle is immediate

Theorem 2.1. ***(Maximum principle)*** *Let $A \subset \mathbb{Z}^d$ be a bounded domain in $\mathbb{Z}^d$ and $u : \bar{A} \to \mathbb{R}$ a harmonic function on A. Assume that $u \geq 0$ on ∂A. Then $u \geq 0$ in A.*

The following theorem is a centered version of the Harnack inequality established by Lawler [15] for spatially inhomogeneous random walk with bounded symmetric increments.

Theorem 2.2. ***(Harnack principle)*** *(Cf. [11]) Let $u : B_{2R}(y) \to \mathbb{R}$ a nonnegative harmonic function on a discrete Euclidean ball centered at y and with radius $2R$ ($R = 1, 2 \cdots$ and $y \in \mathbb{Z}^d$). Then*

$$\max_{x \in B_R(y)} u(x) \leq C \min_{x \in B_R(y)} u(x),$$

where $C = C(d, \alpha, \Gamma)$ is independent of $y \in \mathbb{Z}^d$, $R \geq 1$ and u.

In the proof of Theorem 1.1, together with Theorems 2.1 and 2.2, we need their parabolic versions.

Let $B = A \times \{a \leq k \leq b\} \subset \mathbb{Z}^d \times \mathbb{Z}$ where $A \subset \mathbb{Z}^d$ is a bounded domain of $\mathbb{Z}^d$ and $a < b \in \mathbb{Z}$. We let:

$$\begin{aligned} \partial_\ell B &= \cup_{a<k<b} \partial A \times \{k\} \\ \partial_p B &= \partial_\ell B \cup \left(\bar{A} \times \{a\}\right) \\ \bar{B} &= B \cup \partial_p B \end{aligned}$$

$\partial_p B$ is the parabolic boundary of B and $\partial_\ell B$ its lateral boundary. We say that $u : \bar{B} \to \mathbb{R}$ is caloric in B if

$$u(x, k+1) = \sum_{e \in \Gamma} \pi(x, e) u(x + e, k), \quad (x, k) \in A \times \{a \leq k < b\}.$$

Caloric functions share with harmonic functions the fundamental property that there exist quantitative relations between their values at different points. Maximum principle extends so that we have a control of the values of a caloric function inside the cylinder B by its values on the parabolic boundary $\partial_p B$. Harnack principle generalizes for

nonnegative caloric functions on two conditions: being at distance R – in the parabolic sense – from the parabolic boundary and observing a waiting period of time $\approx R^2$. This requires to be at a distance R^2 from the bottom and R from the lateral boundary in addition to the waiting time. We then obtain a control of the values of the function in the lower sub-cylinder $B_R(y) \times \{s - 3R^2 \leq k \leq s - 2R^2\}$ by those in the upper sub-cylinder $B_R(y) \times \{s - R^2 \leq k \leq s\}$.

Theorem 2.3. ***(Parabolic maximum principle)*** *Let $B = A \times \{a \leq k \leq b\} \subset \mathbb{Z}^d \times \mathbb{Z}$, where $a < b \in \mathbb{Z}$ and $A \subset \mathbb{Z}^d$ is a bounded domain in $\mathbb{Z}^d$, and $u : \bar{B} \to \mathbb{R}$ a caloric function in B. If $u \geq 0$ on $\partial_p B$ then $u \geq 0$ on B.*

Theorem 2.4. *(Cf. [12]) Let u be a nonnegative caloric function on $B_{2R}(y) \times \{s - 4R^2 \leq k \leq s\}$, where $(y, s) \in \mathbb{Z}^d \times \mathbb{Z}$ and $R \geq 1$. Then*

$$\max\left\{u(x,k); \quad x \in B_R(y), \quad s - 3R^2 < k < s - 2R^2\right\}$$

$$\leq C \min\left\{u(x,k); \quad x \in B_R(y), \quad s - R^2 < k < s\right\},$$

where $C = C(d, \alpha, \Gamma)$ is independent of $(y, s) \in \mathbb{Z}^d \times \mathbb{Z}$, $R \geq 1$ and u.

2.2 A lower estimate of the harmonic measure

For a finite set $A \subset \mathbb{Z}^d$ we denote by τ_A the exit time from A. We define the harmonic measure $h_A^x(E)$ $(E \subset \partial A)$ of $(S_j)_{j \in \mathbb{N}}$ in A at the point $x \in A$, by

$$h_A^x(E) = \mathbb{P}_x\left[S_{\tau_A} \in E\right].$$

Lemma 2.5. *Let $y \in \partial\mathcal{D}$ and $R \geq 1$ denote a large integer. Then*

$$h^x_{B_{2R}(y)\cap\mathcal{D}}\left[\partial\mathcal{D} \cap B_{2R}(y)\right] \geq \theta, \quad x \in B_R(y) \cap \mathcal{D}, \tag{2.1}$$

where $\theta = \theta(d, \alpha, \Gamma) > 0$ is independent of y and R.

The proof of (2.1) we give below is the only place where caloric functions are used. It may seem strange that an issue regarding harmonic functions requires going through caloric functions. Indeed, it would have been more natural to estimate the left hand side of (2.1) by $h^x_{B_{2R}(y)}(\partial B_{2R}(y) \cap \mathcal{D}^c)$ (using the maximum principle) and then establish the desired lower estimate for $h^x_{B_{2R}(y)}(\partial B_{2R}(y) \cap \mathcal{D}^c)$. Such an approach may lead to the conclusion provided we have good estimates of the Green kernel as in [7, p. 511]; or by using a barrier argument as in [3, p. 157], but this is difficult to implement for inhomogeneous walks. A way to get round the difficulty is to apply Harnack principle to a harmonic extension of $h^x_{B_{2R}(y)}(\partial B_{2R}(y) \cap \mathcal{D}^c)$, but harmonic functions are difficult to extend; hence the idea of going through a caloric function defined on a cylinder outside of $\mathcal{D} \times \mathbb{Z}$ and much easier to extend so that it is possible to apply parabolic Harnack principle. The idea of such a construction is inspired by the proof of lemma 4.1 in [21].

Proof. Let $z = y - \dfrac{R}{2} e_i$ $(i = 1, \cdots, d)$ (the direction e_i to be chosen in an appropriate way depending on the piece of the boundary to which y belongs) and u_1, u_2 and u_3 the three caloric functions defined by:

- $u_1 : (B_{2R}(y) \cap \mathcal{D}) \times \{-4R^2 \leq k \leq 0\} \to \mathbb{R}$,

$$u_1(x,k) = h^x_{B_{2R}(y)\cap\mathcal{D}}\left(\partial\mathcal{D} \cap B_{2R}(y)\right).$$

- $u_2 : \Omega = B_{2R}(y) \times \{-4R^2 \le k \le 0\} \to \mathbb{R}$, the solution of the parabolic boundary problem

$$\begin{cases} u_2(x,k+1) = \sum_{e\in\Gamma} \pi(x,e) u_2(x+e,k), \quad \text{if} \quad (x,k) \in B_{2R}(y) \times \{-4R^2 \le k < 0\} \\ u_2(x,-4R^2) = \mathbf{1}_{B_{R/8}(z)}(x) \\ u_2|_{\partial_p \Omega \setminus B_{\frac{R}{8}}(z)} = 0 \end{cases}$$

- $u_3 : \Omega' = B_{R/8}(z) \times \{-4R^2 \le k \le 0\} \to \mathbb{R}$, the solution of the parabolic boundary problem

$$\begin{cases} u_3(x,k+1) = \sum_{e\in\Gamma} \pi(x,e) u_3(x+e,k), \quad \text{if} \quad (x,k) \in B_{R/8}(z) \times \{-4R^2 \le k < 0\} \\ u_3(x,-4R^2) = \mathbf{1}_{B_{R/8}(z)}(x) \\ u_3|_{\partial_l \Omega'} = 0 \end{cases}$$

The parabolic maximum principle implies:

$$\begin{aligned} u_1(x,k) &\ge u_2(x,k), \quad (x,k) \in (\mathcal{D} \cap B_{2R}(y)) \times \{-4R^2 \le k \le 0\} \\ u_2(x,k) &\ge u_3(x,k), \qquad (x,k) \in B_{R/8}(z) \times \{-4R^2 \le k \le 0\} \end{aligned}$$

Combining with parabolic Harnack principle gives that for $x \in B_R(y) \cap \mathcal{D}$:

$$h^x_{B_{2R}(y)\cap\mathcal{D}}\left[\partial\mathcal{D} \cap B_{2R}(y)\right] \ge c\, u_3(z,-R^2) \ge \theta > 0,$$

simply because the caloric function u_3 can be extended to a larger domain (e.g. $B_{\frac{R}{8}}(z) \times \{-10R^2 \le k \le 0\}$) so that $u_3 = 1$ on $B_{\frac{R}{8}}(z) \times \{-10R^2 \le k \le -4R^2\}$. □

2.3 The Hölder continuity at the boundary

An important consequence of the lower bound of Lemma 2.5 is that:

Proposition 2.6. *Let $y \in \partial\mathcal{D}$ and $R \ge 1$ denote a large integer. Let u be a nonnegative harmonic function in $\mathcal{D} \cap B_{2R}(y)$ which vanishes on $\partial\mathcal{D} \cap B_{2R}(y)$. Then*

$$\max_{x\in\mathcal{D}\cap B_R(y)} u(x) \le \rho \max_{x\in\overline{\mathcal{D}\cap B_{2R}(y)}} u(x) \tag{2.2}$$

with $0 < \rho = \rho(d,\alpha,\Gamma) < 1$.

Proof. Let $x \in \mathcal{D} \cap B_R(y)$. Let τ denote the exit time from $\mathcal{D} \cap B_{2R}(y)$.

We have

$$\begin{aligned} u(x) &= \sum_{z\in\partial(\mathcal{D}\cap B_{2R}(y))} u(z)\mathbb{P}_x\left[S_\tau = z\right] \\ &= \sum_{z\in\partial(\mathcal{D}\cap B_{2R}(y))\setminus(\partial\mathcal{D}\cap B_{2R}(y))} u(z)\mathbb{P}_x\left[S_\tau = z\right]. \end{aligned}$$

Hence

$$\begin{aligned} u(x) &\le \mathbb{P}_x\left[S_\tau \in \partial\left(\mathcal{D} \cap B_{2R}(y)\right) \setminus \left(\partial\mathcal{D} \cap B_{2R}(y)\right)\right] \max_{x\in\overline{\mathcal{D}\cap B_{2R}(y)}} u(x) \\ &= \left(1 - \mathbb{P}_x\left[S_\tau \in \partial\mathcal{D} \cap B_{2R}(y)\right]\right) \max_{x\in\overline{\mathcal{D}\cap B_{2R}(y)}} u(x). \end{aligned}$$

Using Lemma 2.5 and taking supremum on $x \in \mathcal{D} \cap B_R(y)$ we deduce that:

$$\max_{x \in \mathcal{D} \cap B_R(y)} u(x) \leq (1-\theta) \max_{x \in \overline{\mathcal{D} \cap B_{2R}(y)}} u(x),$$

which implies (2.2) with $\rho = 1 - \theta$. □

2.4 The Carleson principle

This principle provides control of the bound of nonnegative harmonic functions vanishing on a portion of the boundary.

Theorem 2.7. *Let $R \geq 1$ denote a large integer. Assume that u is a nonnegative harmonic function in $\mathcal{D}$. Assume that $u = 0$ on $\partial\mathcal{D} \cap B_{2R}$ (where B_R is the discrete Euclidean ball centered on 0 of radius R). Then*

$$\sup \{u(x), x \in \mathcal{D} \cap B_R\} \leq C\, u(Re), \tag{2.3}$$

where $e = (1, ..., 1)$ and $C = C(d, \alpha, \Gamma) > 0$.

Proof. To prove (2.3) we first observe that the ellipticity assumption implies that $u(x) \leq Ce^{C|x-y|}u(y)$, $x, y \in \mathcal{D}$; where $C = C(d, \alpha, \Gamma) > 0$. This "local" Harnack principle allows us to assume that the distance of $x = (x_1, ..., x_d) \in \mathcal{D} \cap B_{2R}$ to the boundary $\partial\mathcal{D}$ is sufficiently large. Without loss of generality we may assume that $\min_{1 \leq i \leq d} x_i = x_1 \geq C$. On the other hand, the Harnack principle implies (by the classical chain argument) that

$$u(x) \leq C \left(\frac{R}{x_1}\right)^{\gamma} u(Re); \quad x \in B_{2R} \cap \mathcal{D}, \tag{2.4}$$

where γ and C are positive constants depending only on d, α and Γ. Let $x_0 = (0, x_2, ..., x_d) \in \partial\mathcal{D}$. By Proposition 2.6 applied to the positive harmonic function in $B_{2x_1}(x_0) \cap \mathcal{D}$, we have

$$u(x) \leq \rho u(z),$$

where $z \in \overline{\mathcal{D} \cap B_{2x_1}(x_0)}$ satisfies

$$u(z) = \max\{u(y), \;\; y \in \overline{\mathcal{D} \cap B_{2x_1}(x_0)}\}.$$

We distinguish two cases: (i) where $x_1 \leq \dfrac{1}{2^N}(2R - \mid x \mid)$ and (ii) where $x_1 > \frac{1}{2^N}(2R - \mid x \mid)$. The constant N that appears in the definition of these conditions is appropriately large and will be chosen later.

Case (i). In this case we observe that $z \in \mathcal{D} \cap B_{2R}$. This is because:

$$\begin{aligned} \mid z \mid &\leq \mid z - x \mid + \mid x \mid \leq 8x_1 + \mid x \mid \\ &\leq \frac{1}{2^{N-3}}(2R - \mid x \mid) + \mid x \mid \\ &\leq \frac{R}{2^{N-4}} + (1 - \frac{1}{2^{N-3}})|x| \\ &= \frac{R}{2^{N-4}} + (1 - \frac{1}{2^{N-3}})2R \leq 2R. \end{aligned}$$

We have

$$2R - \mid z \mid \geq 2R - \mid x \mid - 8x_1 \geq (1 - \frac{1}{2^{N-3}})(2R - \mid x \mid).$$

From which it follows that

$$\begin{aligned}(2R-\mid x\mid)^{\gamma}u(x) &\leq (1-\frac{1}{2^{N-3}})^{-\gamma}(2R-\mid z\mid)^{\gamma}\rho\, u(z)\\ &\leq (1-\frac{1}{2^{N-3}})^{-\gamma}\rho \max_{x\in\mathcal{D}\cap B_{2R}}(2R-\mid x\mid)^{\gamma}u(x)\end{aligned}$$

and we choose N large enough such that $(1-\frac{1}{2^{N-3}})^{-\gamma}\rho=\rho'<1$.

Case (ii). In this case we observe that:

$$(2R-\mid x\mid)^{\gamma}u(x)\leq 2^{\gamma N}x_1^{\gamma}u(x)\leq C\,2^{\gamma N}R^{\gamma}u(Re)$$

where the last inequality follows from (2.4).

Combining case (i) and case (ii) we deduce that for all $x\in\mathcal{D}\cap B_{2R}$

$$(2R-\mid x\mid)^{\gamma}u(x)\leq \rho' \max_{x\in\mathcal{D}\cap B_{2R}}(2R-\mid x\mid)^{\gamma}u(x)+C2^{\gamma N}R^{\gamma}u(Re).$$

Using the fact that $\rho'<1$ and $(2R-\mid x\mid)\approx R$ for $x\in\mathcal{D}\cap B_R$ we deduce the estimate (2.3). □

2.5 Existence of a positive harmonic function on $\mathcal{D}$

In order to establish the existence of a positive harmonic function on $\mathcal{D}$ vanishing on $\partial\mathcal{D}$, let:

$$u_l(x)=\alpha_l(G_{l+1}(x,e)-G_l(x,e)),\quad x\in B_{2^l}\cap\mathcal{D},$$

where $G_l(x,e)$; $l=1,2,\cdots$ is the Green function of $(S_n)_{n\in\mathbb{N}}$ in the discrete ball B_{2^l}, with pole at $e=(1,\cdots,1)$ and where the α_l are chosen so that $u_l(e)=1$; $l=1,2,\cdots$. Harnack principle combined with Carleson estimate (2.3) implies that the u_l satisfy

$$u_l(x)\leq C,\quad x\in B_{2^k}\cap\mathcal{D};\ l>k,$$

with a constant $C=C(k)$ depending only on k. The diagonal process then allows us to deduce the existence of a positive harmonic function defined globally in $\mathcal{D}$ and vanishing on $\partial\mathcal{D}$. More precisely, for each $k=1,2,...$ there exists an extraction φ_k so that $(u_{\varphi_k(n)})_{n\geq 1}$ converges on $B_{2^k}\cap\mathcal{D}$. An easy induction on k gives a sequence of harmonic functions $\left(u_{\varphi_1\circ\varphi_2\circ\cdots\circ\varphi_k(n)}\right)_{n\geq 1}$ converging on $B_{2^k}\cap\mathcal{D}$. Taking $\psi(n)=\varphi_1\circ\cdots\circ\varphi_n(n)$ and setting $u(y)=\lim_{n\to\infty}u_{\psi(n)}(y)$ gives the required harmonic function. A similar construction of the harmonic function u can be found in [10, p. 104].

3 Uniqueness of the positive harmonic function

3.1 Boundary Harnack principle

The previous section provides us a positive harmonic function in $\mathcal{D}$ vanishing on $\partial\mathcal{D}$. It remains to show that up to a multiplicative constant, this function is unique. One way that this can be seen is by using boundary Harnack principle.

Theorem 3.1. *Assume that u and v are two positive harmonic functions in $\mathcal{D}$. Assume that $u=0$ on $\partial\mathcal{D}$. Then*

$$\sup_{x\in\mathcal{D}\cap B_{R/2}}\frac{u(x)}{v(x)}\leq C\,\frac{u(Re)}{v(Re)},\quad R\geq C, \tag{3.1}$$

where $C=C(d,\alpha,\Gamma)>0$.

Proof. Using (2.3) we see that

$$u(x) \leq Cu(Re), \quad x \in \partial B_R \cap \mathcal{D}.$$

Hence the maximum principle implies that

$$u(x) \leq Cu(Re) h^x_{B_R \cap \mathcal{D}} [\partial B_R \cap \mathcal{D}] \tag{3.2}$$

for all $x \in B_R \cap \mathcal{D}$. For $x \in (\partial B_R \cap \mathcal{D}) \cap \{x_i \geq \epsilon R, \ \ i = 1, ..., d\}$ (for an appropriate small $\epsilon > 0$), Harnack principle gives

$$v(x) \geq cv(Re),$$

and so the maximum principle implies that

$$v(x) \geq cv(Re) h^x_{B_R \cap \mathcal{D}} [(\partial B_R \cap \mathcal{D}) \cap \{x_i \geq \epsilon R, \ \ i = 1, ..., d\}] \tag{3.3}$$

for all $x \in B_R \cap \mathcal{D}$. The proof will follow from the following claim.

Claim. Let $x = (x_1, ..., x_d) \in B_{R/2} \cap \mathcal{D}$ with $0 < x_i < \epsilon R$ for some $i = 1, ..., d$. Then

$$h^x_{B_R \cap \mathcal{D}} [\partial B_R \cap \mathcal{D}] \leq C \ h^x_{B_R \cap \mathcal{D}} [(\partial B_R \cap \mathcal{D}) \cap \{x_i \geq \epsilon R, \ \ i = 1, ..., d\}] \tag{3.4}$$

provided that ϵ is small enough.

Let us take the claim for granted and show how to finish the proof. We can assume, in the proof of (3.1), that $\text{dist}(x, \partial \mathcal{D}) \leq \epsilon R$ (otherwise (3.1) is an immediate consequence of Harnack principle). Combining (3.2) and (3.4) we deduce that

$$\frac{u(x)}{u(Re)} \leq C h^x_{B_R \cap \mathcal{D}} [(\partial B_R \cap \mathcal{D}) \cap \{x_i \geq \epsilon R, \ \ i = 1, ..., d\}] \, .$$

Since

$$h^x_{B_R \cap \mathcal{D}} [(\partial B_R \cap \mathcal{D}) \cap \{x_i \geq \epsilon R, \ \ i = 1, ..., d\}] \leq C \frac{v(x)}{v(Re)},$$

by (3.3), our inequality is verified. □

Proof of the claim. Let $x = (x_1, ..., x_d) \in B_{R/2} \cap \mathcal{D}$. Without loss of generality we can assume that $x_1 = \text{dist}(x, \partial \mathcal{D}) < \epsilon R$. Using, as in §2.4, a sequence of balls, connecting x to $\frac{R}{2} e$ and observing that

$$h^{\frac{R}{2} e}_{B_R \cap \mathcal{D}} [\partial B_R \cap \mathcal{D}] \geq c$$

(with a similar proof as in §2.2), we deduce by Harnack principle that

$$1 \leq C_1 \left(\frac{R}{x_1} \right)^{\gamma} h^x_{B_R \cap \mathcal{D}} [\partial B_R \cap \mathcal{D}] \, .$$

Hence

$$h^x_{B_R \cap \mathcal{D}} [\partial B_R \cap \mathcal{D}] \geq \frac{1}{C_1} \left(\frac{x_1}{R} \right)^{\gamma} .$$

On the other hand, the same considerations, as in §2.3, show that

$$h^x_{B_R \cap \mathcal{D}} [(\partial B_R \cap \mathcal{D}) \cap \{x_i \leq \epsilon R, \ \ i = 1, ..., d\}] \leq C_2 \, \rho^{\frac{R}{x_1}} \, ,$$

where $0 < \rho < 1$. Setting $\rho = e^{-r}$, for an appropriate $r > 0$ and choosing ϵ sufficiently small and satisfying

$$C_2 \epsilon^{-\gamma} e^{-\frac{r}{\epsilon}} < \frac{1}{C_1}$$

gives the inequality (3.4).

3.2 The denouement

Let us consider u_1 and u_2, two positive harmonic functions on $\mathcal{D}$ vanishing on $\partial\mathcal{D}$. By the boundary Harnack principle, we have

$$\frac{1}{C}\frac{u_1(x)}{u_1(Re)} \leq \frac{u_2(x)}{u_2(Re)} \leq C\,\frac{u_1(x)}{u_1(Re)}, \quad x \in B_{R/2} \cap \mathcal{D}, \quad R \geq C,$$

where $C > 1$ is an appropriate large positive constant. Assume that u_1 and u_2 are normalized so that $u_1(e) = u_2(e) = 1$. It follows then that

$$\frac{1}{C}u_1(Re) \leq u_2(Re) \leq Cu_1(Re), \quad R \geq C.$$

Then by letting $R \to +\infty$ we deduce that:

$$\frac{1}{C^2} \leq \frac{u_2(x)}{u_1(x)} \leq C^2, \quad x \in \mathcal{D}.$$

Define

$$u_3(x) = u_2(x) + \frac{1}{C^2-1}\left(u_2(x) - u_1(x)\right), \quad x \in \mathcal{D}.$$

We have

$$u_3(x) = \frac{C^2}{C^2-1}\left(u_2(x) - \frac{1}{C^2}u_1(x)\right) \geq 0, \quad x \in \mathcal{D},$$

and clearly u_3 is harmonic function, vanishing on the boundary $\partial\mathcal{D}$ and satisfies the normalization condition $u_3(e) = 1$. We iterate and define (for $p \geq 4$)

$$u_p(x) = u_{p-1}(x) + \frac{1}{C^2-1}\left(u_{p-1}(x) - u_1(x)\right).$$

An easy computation shows that u_p is a non-negative harmonic function on $\mathcal{D}$, vanishing on the boundary $\partial\mathcal{D}$, satisfying the normalization condition $u_p(e) = 1$ and such that

$$\frac{1}{C^2} \leq \frac{u_p(x)}{u_1(x)} \leq C^2, \quad x \in \mathcal{D}.$$

On the other hand, combining

$$\begin{aligned} u_p(x) &= \frac{C^2}{C^2-1}u_{p-1}(x) - \frac{1}{C^2-1}u_1(x) \\ u_1(x) &= \frac{C^2}{C^2-1}u_1(x) - \frac{1}{C^2-1}u_1(x). \end{aligned}$$

We deduce that

$$u_p(x) - u_1(x) = \frac{C^2}{C^2-1}\left(u_{p-1}(x) - u_1(x)\right), \quad x \in \mathcal{D},\ p \geq 3.$$

From which it follows that

$$u_p(x) = \left(\frac{C^2}{C^2-1}\right)^{p-2}\left(u_2(x) - u_1(x)\right) + u_1(x).$$

Dividing by $u_1(x)$ we deduce that:

$$\frac{1}{C^2} \leq \left(\frac{C^2}{C^2-1}\right)^{p-2}\left(-1 + \frac{u_2(x)}{u_1(x)}\right) + 1 \leq C^2, \quad x \in \mathcal{D}, p \geq 4.$$

Letting $p \to +\infty$ we deduce that we necessarily have

$$u_1(x) = u_2(x), \quad x \in \mathcal{D}.$$

The previous construction of the sequence (u_p) is inspired by the proof of Lemma 6.2 of [1].

3.3 Examples

Here we illustrate our main result by two examples of a spatially inhomogeneous random walk in the quarter plane $\mathbb{Z}_+^2$.

Example 1. We consider a random walk with alternating transition probabilities defined as follows. On the lines $\mathcal{L}_k = \{(x, x+2k);\ x \in \mathbb{Z}\}$ $(k \in \mathbb{Z})$ the transition probabilities are those of the simple random walk, see 1.

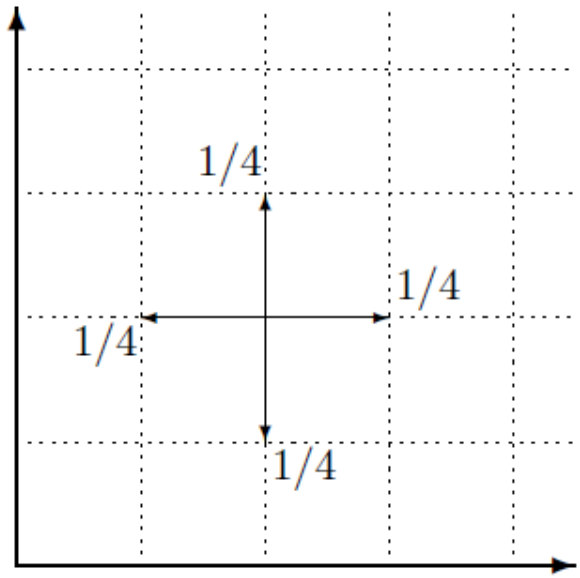

Figure 1: The simple random walk in the quarter plane.

On the lines $\mathcal{L}'_k = \{(x, x+2k+1);\ x \in \mathbb{Z}\}$ the walk is allowed to visit its eight nearest neighbors with probabilities $p_{i,j} = \frac{1}{8}$, $-1 \le i, j \le 1$, see 2.

The resulting random walk with killing at $\partial\mathbb{Z}_+^2$ is represented on 3.

Easy computation shows that the function

$$u(x, y) = xy \tag{3.5}$$

satisfy, as in the case of the simple random walk, conditions i), ii) and iii) of §1 with respect to our random walk and is then the unique harmonic function found in §2.

The previous example can be generalized as follows. Let $(S_n)_{n\in\mathbb{N}}$ denote a random walk as in 2, but where the transition probabilities $p_{ij} = p_{ij}(x, y)$ depend on the position (x, y). Assume that

$$p_{1,0};\quad p_{-1,0};\quad p_{0,-1};\quad p_{-1,-1} \ge \alpha > 0$$

(i.e the ellipticity condition is satisfied) and that $(S_n)_{n\in\mathbb{N}}$ is centered:

$$\sum_{-1\le i,j\le 1} ip_{ij} = \sum_{-1\le i,j\le 1} jp_{ij} = 0. \tag{3.6}$$

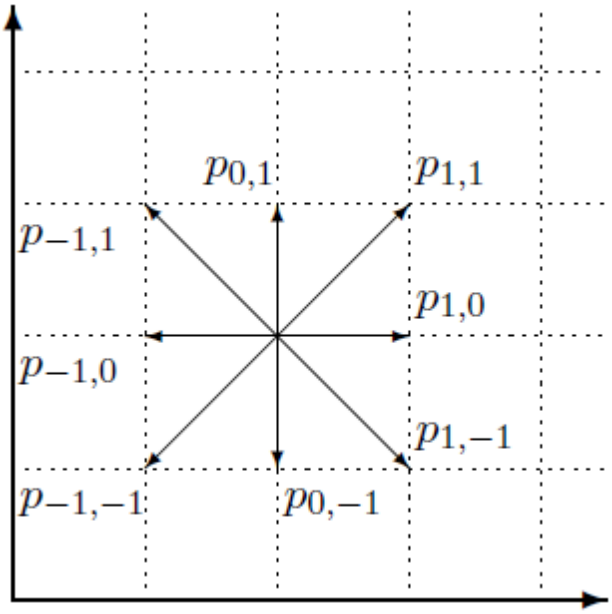

Figure 2: Walks with small steps in the quarter plane.

Then u as defined by (3.5) is harmonic for the random walk $(S_n)_{n\in\mathbb{N}}$, if it satisfies the finite difference equation

$$\sum_{-1\le i,j\le 1} p_{ij}(x,y)u(x+i,y+j) = u(x,y). \tag{3.7}$$

We observe that the right hand side of (3.7) is given by

$$\begin{aligned}\sum_{-1\le i,j\le 1} p_{ij}(x,y)(x+i)(y+j) &= \left(\sum_{-1\le i,j\le 1} p_{ij}(x,y)\right) xy \\ &+ \left(\sum_{-1\le i,j\le 1} jp_{ij}(x,y)\right) x + \left(\sum_{-1\le i,j\le 1} ip_{ij}(x,y)\right) y \\ &+ \sum_{-1\le i,j\le 1} ijp_{ij}(x,y).\end{aligned}$$

Using (3.6) and the fact that $\sum_{-1\le i,j\le 1} p_{ij} = 1$, we see that equation (3.7) is satisfied if and only if

$$\sum_{-1\le i,j\le 1} ijp_{ij}(x,y) = 0.$$

This condition is obviously satisfied by the random walk represented on 3.

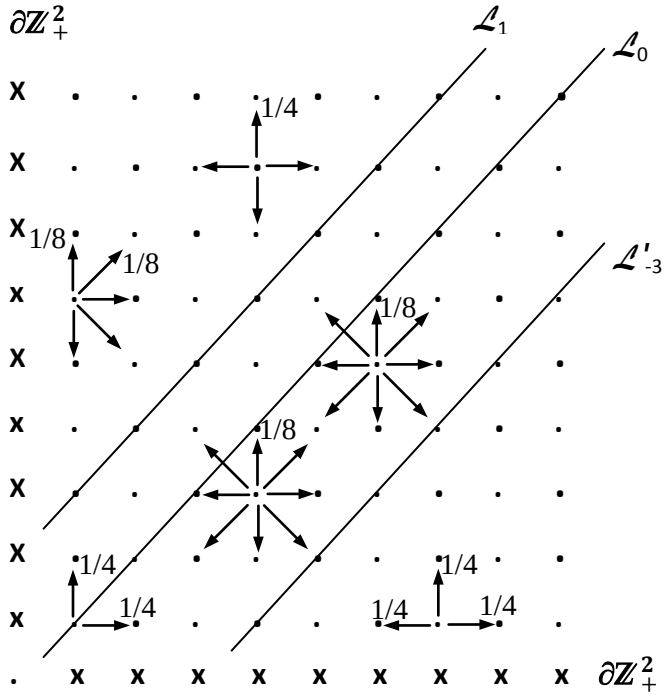

Figure 3: The walk has alternating transition probabilities inside the quarter plane and is killed on the boundary of $\mathbb{Z}_+^2$ which consists of $\{(x,\,0)\,;\; x \in \mathbb{N}^*\} \cup \{(0,\,y)\,;\; y \in \mathbb{N}^*\}$.

A second example. We now give a second example, more sophisticated than the previous one and whose harmonic function is not of the product form xy. We start from the random walk in the Weyl chamber of the the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ (cf. [20, §3.5]) and we reflect this walk with respect to the diagonal $x = y$. The obtained walk is assigned to the even lines $\mathcal{L}_k$ where it is allowed to visit its six neighbors with equal probabilities

$$p_{1,0} = p_{-1,0} = p_{0,1} = p_{0,-1} = p_{1,-1} = p_{-1,1} = 1/6.$$

On the odd lines $\mathcal{L}'_k$ we modify these probabilities as follows:

$$p_{1,0} = p_{-1,1} = p_{0,-1} = \frac{2-\alpha}{6}, \quad p_{0,1} = p_{-1,0} = p_{1,-1} = \frac{\alpha}{6},$$

where $0 < \alpha < 1$ is fixed. Easy computations show that

$$u(x,y) = \frac{xy(x+y)}{2},$$

(the same function as in [20, §3.5]) is harmonic for our inhomogeneous walk. Finally, we observe that we can allow α to take different values on the odd lines as it varies within a range ensuring ellipticity.

References

[1] A. Ancona: Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine Lipschitzien. *Ann. Inst. Fourier* **37**, (2001), no. 3, 313–338.

[2] L. Alili and R. A. Doney: Martin boundaries associated with a killed random walk. *Ann. Inst. H. Poincaré Probab. Statist* **37**, (2001), no. 3, 313–338. MR-1831986

[3] P. Bauman: Positive solutions of elliptic equations in nondivergence form and their adjoints. *Arkiv Math* **22**, (1984), 153–173. MR-0765409

[4] R. A. Doney: The Martin boundary and ratio limit theorems for killed random walks. *J. London Math. Soc.* **2**, (1998), no. 58, 761–768. MR-1678162

[5] J. L. Doob: Discrete potential theory and boundaries. *J. Math. and Mech.* **8**, (1959), 433–458. MR-0107098

[6] G. Fayolle and K. Raschel: About a possible analytic approach for walks in the quarter plane with arbitrary big jumps. *C. R. Math. Acad. Sci. Paris* **353**, (2015), no. 2, 89–94. MR-3300936

[7] R. A. Hunt and R. L. Wheeden: Positive harmonic functions on Lipschitz domains. *Trans. Amer. Math. Soc.* **132**, (1970), 507–527. MR-0274787

[8] I. Ignatiouk-Robert: Martin boundary of a reflected random walk on a half space. *Probab. Theory Related Fields* **148**, (2010), no. 1-2, 197–294. MR-2653227

[9] I. Ignatiouk-Robert and Ch. Loree: Martin boundary of a killed random walk on a quadrant. *The Annals of Probability* **38**, (2010), no. 3, 1106–1142. MR-2674995

[10] D. Jerison and C. Kenig: Boundary behavior of harmonic functions in non-tangentially accessible domains. *Advances in Math.* **146**, (1982), 1–68. MR-0716504

[11] H. J. Kuo and N. S. Trudinger: Positive difference operators on general meshes. *Duke Math. J.* **83**, (1996), no. 2, 415–433. MR-1390653

[12] H. J. Kuo and N.S. Trudinger: Evolving monotone difference operators on general space-time meshes. *Duke Math.J.* **91**, (1998), 587–607. MR-1604175

[13] I.A. Kurkova and V.A. Malyshev: Martin boundary and elliptic curves. *Markov Processes Related Fields* **4**, (1998), 203–272. MR-1641546

[14] I. Kurkova and K. Raschel: Random walks in $\mathbb{Z}_+^2$ with non-zero drift absorbed at the axes. *Bulletin de la Société Mathématique de France* **139**, (2011), 341–387. MR-2869310

[15] G. F. Lawler: Estimates for differences and Harnack inequality for difference operators coming from random walks with symmetric, spatially inhomogeneous, increments. *Proc. London Math. Soc.(3)* **63**, (1991), no. 3, 552–568. MR-1127149

[16] S. Mustapha: Gaussian estimates for spatially inhomogeneous random walks on $\mathbb{Z}^d$. *The Annals of Probability* **34**, (2006), no. 1, 264–283. MR-2206348

[17] S. Mustapha: Gambler's ruin estimates for random walks with symmetric spatially inhomogeneous increments. *Bernoulli* **13**, (2007), no. 1, 131–147. MR-2307398

[18] M. Picardello, and W. Woess: Martin boundaries of Cartesian products of Markov chains. *Nagoya Math. J.* **128**, (1992) 153–169. MR-1197035

[19] K. Raschel: Greenfunctions and Martin compactification for killed random walks related to SU(3). *Electronic Communications in Probability* **15**, (2010) 176–190. MR-2651549

[20] K. Raschel: Random walks in the quarter plane, discrete harmonic functions and conformal mappings, with an appendix by Sandro Franceschi. *Stochastic Processes and their Applications* **124**, (2014) 3147–3178. MR-3231615

[21] M. V. Safonov and Y. Yuan: Doubling properties for second order parabolic equations. *Annals of Mathematics* **150**, (1999), 313–327. MR-1715327

[22] W. Woess: Random walks on infinite graphs and groups. *Cambridge University Press*, Cambridge, 2000. MR-1743100

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