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PAPER

Quantum–like nonseparable structures in optical beams

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Abstract

When two or more degrees of freedom become coupled in a physical system, a number of observables of the latter cannot be represented by mathematical expressions separable with respect to the different degrees of freedom. In recent years it appeared clear that these expressions may display the same mathematical structures exhibited by multiparty entangled states in quantum mechanics. In this work, we investigate the occurrence of such structures in optical beams, a phenomenon that is often referred to as 'classical entanglement'. We present a unified theory for different kinds of light beams exhibiting classical entanglement and we indicate several possible extensions of the concept. Our results clarify and shed new light upon the physics underlying this intriguing aspect of classical optics.

1. Introduction

A composite physical system, namely one made of at least two identifiable parts, say A and B, which are denoted subsystems, can be prepared in such a way that the latter are not independent. In the realm of classical physics this means, for example, that the probability $P(a \in A,\ b \in B)$ for the events $a,\ b$ associated to subsystems $A,\ B$, respectively, cannot be factored as $P(a \in A,\ b \in B) = P(a \in A)P(b \in B)$ [1]. Conversely, for a composite quantum system, statistical dependence of the subsystems $A,\ B$ means that the state vector $|\Psi\rangle$ describing a physical state of the whole system, cannot be decomposed in the tensor product

$$|\Psi\rangle = \left|\psi_A\right\rangle \otimes \left|\psi_B\right\rangle, \tag{1}$$

where $|\psi_A\rangle$ represents the state of the subsystem A and $|\psi_B\rangle$ represents the state of the subsystem B. Here, we are not interested in the deep conceptual implications of equation (1) but follow, rather, the 'die-hard pragmatist's' approach [2] and denote as entangled any state vector that does not factorizes as in equation (1); namely,

$$\text{entangled} = \text{non-separable}. \tag{2}$$

Traditionally, entanglement has been regarded either as a peculiar feature of quantum mechanics or, instead, as a powerful resource especially for quantum information science [3]. In this paper we adhere to the latter view and aim at showing how some potentially useful characteristics of *quantum* entanglement can be replicated in *classical* systems. In fact, our ultimate goal is *not* to replace or simulate entangled quantum systems with classical ones in some actual operations. Instead, the aim is to study how to make quantum entanglement potentialities accessible to classical physics applications as recently demonstrated, e.g., in classical polarization metrology [4].

Thus, the main purpose of this paper is to revisit the concept of the so-called 'classical entanglement' in optics [5, 6], and to present a brief but comprehensive overview of it. We would like to stress that 'classical entanglement' is *not* substitutive of bona fide quantum entanglement, but is a feature exhibited by some classical systems. In a sense, which will become more clear later, the name *classical entanglement* denotes the occurrence

of some mathematical and physical aspects of quantum entanglement in classical beams of light. In this sense, classical entanglement should not be confused with 'entanglement simulations in classical optics', namely the use of classical fields to reproduce non-classical correlations between distinct measurement apparatuses [7, 8]. In any case, classical entanglement does not belong to the rich field of studies denoted by the name 'quantum–classical analogies' [9–11]. A precise definition of what is usually meant with 'classical entanglement', will be given in section 2.

As a final important remark, the term 'classical' in the name *classical entanglement*, indicates the non-quantum nature of the excitation of the electromagnetic field. In this paper, typically, we deal with bright beams of light as, e.g., laser beams. However, whether the beam is very intense or very weak, is a factor that has not influence upon classical entanglement, as it will be shown in section 2. Yet, it should be noticed that single-photon excitations permit only the quantum mechanical representation as Fock states and, therefore, will not be considered here. However, it has been recently demonstrated that single photons can be prepared in a quantum state entangled with the vacuum [12–16]. Single-photon-vacuum entanglement resembles classical entanglement in that there is only one individual physical system, a single-photon in the quantum case and a single bright beam in the classical one, and two (or more) entangled modes of the electromagnetic field [17–19]. This concept will be further discussed in the next section.

2. Two types of quantum entanglement

Consider a quantum system S made of two parts, denoted with S_1 and S_2, which are dubbed 'subsystems'. For example, *two* particles of mass m constrained to move along a line with coordinates $x_1,\ x_2$, respectively, tied to the equilibrium point by two equal springs of elastic constant $k = m\omega^2$, constitute a composite (bipartite) system whose dynamics is governed by the Hamiltonian $\hat{H} = \hat{H}_1 + \hat{H}_2$, where

$$\hat{H}_\alpha = \frac{1}{2\,m}\hat{p}_\alpha^2 + \frac{1}{2}m\omega^2\hat{x}_\alpha^2, \qquad (\alpha = 1,\ 2). \tag{3}$$

In this case the two subsystems $S_1,\ S_2$ are naturally identified with the two particles[5].

As a second example, consider now a *single* particle of mass m moving upon the plane $(x_1,\ x_2)$ and tied to the equilibrium point $x_1 = 0 = x_2$ by a spring of elastic constant $k = m\omega^2$. This is a two-dimensional harmonic oscillator with Hamiltonian

$$\begin{aligned}\hat{H} &= \frac{1}{2\,m}\left(\hat{p}_1^{\,2} + \hat{p}_2^{\,2}\right) + \frac{1}{2}m\omega^2\left(\hat{x}_1^{\,2} + \hat{x}_2^{\,2}\right)\\ &= \hat{H}_1 + \hat{H}_2,\end{aligned} \tag{4}$$

where $\hat{H}_\alpha$ is again given by the expression in equation (3). In this case the two subsystems $S_1,\ S_2$ are clearly identified with the two Cartesian coordinates of the single particle. Not surprisingly, the Hamiltonian $\hat{H}$ is the same in both cases and the generic state vector $|\Psi\rangle$ satisfying the Schrödinger equation $i\hbar\partial|\Psi\rangle/\partial\, t = \hat{H}\ |\Psi\rangle$, belongs to a Hilbert space $\mathscr{H}$ made as the tensor product of spaces associated to each subsystem: $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$.

The fundamental difference between the two cases considered above is that in the first case the two subsystems are identified with two distinct physical objects, the two particles, which can be spatially separated. Conversely, in the second case there are not two individual physical objects to set apart but only two orthogonal coordinates attached to a single physical object: the sole particle. This simple fact has serious consequences when the state vector $|\Psi\rangle$ is entangled, namely when $|\Psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$. In the words of Spreeuw [20]:

> "(there is) a profound difference between two types of entanglement: (i) true, multiparticle entanglement and (ii) a weaker form of entanglement between different degrees of freedom of a single particle. Although these two types look deceptively similar in many respects, only type (i) can yield non-local correlations. Only the type (ii) entanglement has a classical analogy."

In this paper, borrowing from the jargon of the theory of optical coherence functions [21], we denote entanglement of type (i) and (ii) as *inter*system and *intra*system entanglement, respectively. As remarked by Spreeuw, intersystem entanglement can occur only in quantum systems and may lead to the so-called quantum non-locality [22, 23], a fundamental aspect of quantum mechanics that should not be confused with quantum entanglement [24]. Conversely, intrasystem entanglement may appear in both quantum and classical systems

[5] In this qualitative discussion we are intentionally oversimplifying the situation in order to keep simple the language. More properly, one should identify the two subsystems with the two harmonic oscillators.

and has a local nature by definition because the two or more entangled degrees of freedom are localized within the same physical object.

In the last two decades it became clear that intrasystem entanglement also occurs frequently in classical optics. In this case intrasystem entanglement is usually dubbed *classical entanglement* [6, 25–28] [6]. A typical example thereof is given by a collimated optical beam with *non-uniform* polarization pattern. The electric field of a generic paraxial beam of light can be written as $\mathbf{E}(\boldsymbol{\rho}, z, t) = 2\,\mathrm{Re}\,\{\mathbf{U}(\boldsymbol{\rho}, z)\exp[\mathrm{i}k(z - c\,t)]\}$, where $\mathbf{U}(\boldsymbol{\rho}, z)$ is the complex amplitude of the field (technically called: analytic signal [29]), $\boldsymbol{\rho} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y$ denotes the transverse position vector, k is the wave-number and the axis z is taken along the direction of propagation of the beam. Then, the analytic signal of a non-uniformly polarized paraxial beam can be represented by a *non-separable* vector function of the form

$$\mathbf{U}(\boldsymbol{\rho}, z) = \boldsymbol{a}_1 b_1(\boldsymbol{\rho}, z) + \boldsymbol{a}_2 b_2(\boldsymbol{\rho}, z), \tag{5}$$

where $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ are two constant vectors perpendicular to the propagation axis z, the functions $b_1(\boldsymbol{\rho}, z)$, $b_2(\boldsymbol{\rho}, z)$ denotes two distinct solutions of the paraxial equation. In this instance the polarization vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and the spatial mode functions $b_1(\boldsymbol{\rho}, z)$, $b_2(\boldsymbol{\rho}, z)$ describe two independent degrees of freedom, which play the role of the two subsystems in quantum mechanics. The degrees of freedom are independent in the sense that it is possible to assign arbitrary values to the polarization of a paraxial beam of light irrespective of its spatial mode function and vice versa.

An expression of the form (5) is clearly non-separable, namely it is not possible to rewrite it as the simple product between one constant polarization vector $\boldsymbol{a}$ and one mode function $b(\boldsymbol{\rho}, z)$: $\mathbf{U}(\boldsymbol{\rho}, z) \neq \boldsymbol{a}\, b(\boldsymbol{\rho}, z)$. In this sense, equation (5) has the same mathematical structure (isomorphism) of a two-qubit entangled state vector $|\Psi\rangle$ belonging to a bipartite Hilbert space $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ of dimension 4 [3]. It is well known that such state $|\Psi\rangle$ can always be written in terms of a Schmidt decomposition of the form [30, 31]

$$|\Psi\rangle = \sqrt{\lambda_1}\ |u_1\rangle|v_1\rangle + \sqrt{\lambda_2}\ |u_2\rangle|v_2\rangle, \tag{6}$$

where $\{|u_1\rangle, |u_2\rangle\}$ and $\{|v_1\rangle, |v_2\rangle\}$ are orthonormal bases for $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and $\lambda_1 \geqslant \lambda_2 \geqslant 0$ are real non-negative coefficients. If the state is normalized to 1, then $\lambda_1 + \lambda_2 = 1$. If either $\lambda_1 = 0$ or $\lambda_2 = 0$ the state is factorable and the two subsystems are independent. Vice versa, if $\lambda_1, \lambda_2 \neq 0$, the state vector $|\Psi\rangle$ is entangled. The amount of entanglement can be quantified by the Schmidt number (or participation ratio) K defined as:

$$K = \frac{\left(\lambda_1 + \lambda_2\right)^2}{\lambda_1^2 + \lambda_2^2}, \tag{7}$$

with $1 \leqslant K \leqslant 2$ [32, 33]. $K = 1$ characterizes factorable state vectors, while $K = 2$ denotes maximal entanglement occurring whenever $\lambda_1 = \lambda_2$. In a similar manner, it is not difficult to show that a non-separable vector function of the form (5) can always be rewritten as

$$\mathbf{U}(\boldsymbol{\rho}, z) = \sqrt{\lambda_1}\ \hat{\boldsymbol{u}}_1 v_1(\boldsymbol{\rho}, z) + \sqrt{\lambda_2}\ \hat{\boldsymbol{u}}_2 v_2(\boldsymbol{\rho}, z), \tag{8}$$

where

$$\int \mathbf{U}^{\star}(\boldsymbol{\rho}, z) \cdot \mathbf{U}(\boldsymbol{\rho}, z)\ \mathrm{d}^2\rho = \lambda_1 + \lambda_2, \tag{9}$$

denotes the total intensity of the beam and the integration extended upon the whole xy plane with $\mathrm{d}^2\rho = \mathrm{d}x\mathrm{d}y$. Here we use the notation $(\hat{\boldsymbol{u}}_\alpha, \hat{\boldsymbol{u}}_\beta)_P = \hat{\boldsymbol{u}}_\alpha^{\star} \cdot \hat{\boldsymbol{u}}_\beta = \delta_{\alpha\beta}$, with $\alpha, \beta \in \{1, 2\}$, and

$$(v_\alpha, v_\beta)_S = \int v_\alpha^{\star}(\boldsymbol{\rho}, z) v_\beta(\boldsymbol{\rho}, z)\ \mathrm{d}^2\rho = \delta_{\alpha\beta}, \tag{10}$$

where $(\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2)_P$ and $(v_1, v_2)_S$ symbolize the scalar product in the polarization (subscript P) and in the spatial (subscript S) Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Given the decomposition (8), one can again formally quantify the amount of 'classical entanglement' via the Schmidt number K given in equation (7) which holds irrespective of the normalization of the state. Therefore, the total intensity of the beam $\lambda_1 + \lambda_2$ does not affect classical entanglement.

3. Three kinds of classical entanglement

Nowadays, three methods to prepare optical beams exhibiting intrasystem entanglement are quite popular. In all the three cases the goal is to prepare beams of light possessing some properties of entangled states of two qubits. This is achieved by manipulating two relevant *binary* degrees of freedom of the electromagnetic field, each qubit

[6] The names 'nonquantum entanglement' [26] and 'structural inseparability' [27, 28] are also sometimes used.

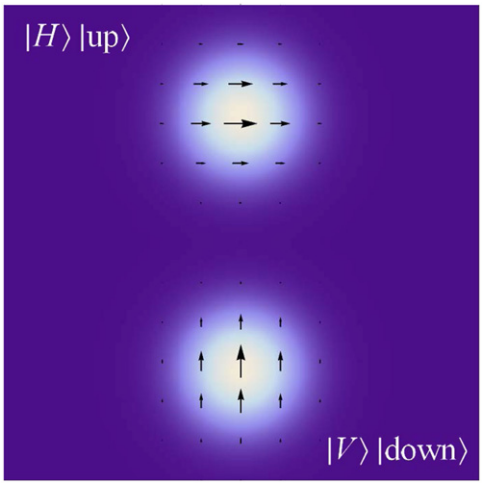

Figure 1. Polarization and intensity patterns in the transverse plane $z = 0$, of the twofold optical beam represented by equation (13). The black arrows denotes the polarization of light.

being encoded in one degree of freedom. According to what pair of binary degrees of freedom are chosen, one can have (1) polarization-position entanglement, (2) position–position entanglement and (3) polarization-spatial entanglement. In the remainder of this section we shall illustrate and compare these three kinds of classical entanglement within the framework of paraxial optics that allows for a unified description of these cases.

3.1. Polarization-position entanglement

The first example of intrasystem entanglement in optical beams was given by Peres [31], Spreeuw [5] and Cerf *et al* [34]. Consider an unpolarized beam of light passing through a Calcite crystal. Crossing the crystal, the beam splits in two beams traveling along two different paths (say 'up' and 'down'), with orthogonal linear polarization (say, '*H*orizontal' and '*V*ertical'), as shown in figure 1. Therefore, the beam can be described in terms of two significant binary degrees of freedom: the polarization and the position of the path. As shown in section 2, the paraxial twofold beam exiting the crystal can be represented by the non-separable vector field

$$\mathbf{U}(\boldsymbol{\rho}, z) = \hat{\boldsymbol{e}}_H\, U_{\rm up}(\boldsymbol{\rho}, z) + \hat{\boldsymbol{e}}_V\, U_{\rm down}(\boldsymbol{\rho}, z), \tag{11}$$

where the polarization vectors are orthogonal by definition $(\hat{\boldsymbol{e}}_H, \hat{\boldsymbol{e}}_V)_P = 0$, and $(U_{\rm up}, U_{\rm down})_S = 0$ when the two paths are non-overlapping and therefore fully distinguishable. Thus, equation (11) has a Schmidt form analogous to equation (8), with $\lambda_1 = \lambda_2 = 1$ and represents a classical optics analogue of a maximally entangled state of two qubits of the form

$$|H\rangle|\mathrm{up}\rangle + |\mathrm{V}\rangle|\mathrm{down}\rangle. \tag{12}$$

For the sake of definiteness, let us choose $U_{\rm up}(\boldsymbol{\rho}, z) = U(x, y - a, z)$ and $U_{\rm down}(\boldsymbol{\rho}, z) = U(x, y + a, z)$, where $2a > 0$ quantifies the distances between the two beams and $U(\boldsymbol{\rho}, z)$ denotes any solution of the paraxial equation. By definition, $U(x, y \mp a, z)$ represents a beam displaced up and down by $\pm a$ along the (vertical) y-axis. Thus, equation (11) can be rewritten as

$$\mathbf{U}(\boldsymbol{\rho}, z) = \hat{\boldsymbol{e}}_H U(x, y - a, z) + \hat{\boldsymbol{e}}_V\, U(x, y + a, z). \tag{13}$$

The orthogonality requirement $(U_{\rm up}, U_{\rm down})_S = 0$ now becomes $I(a) = 0$, where $I(a)$ is the overlap integral

$$I(a) = \int U^{\star}(x, y - a, z)\, U(x, y + a, z)\, \mathrm{d}x\mathrm{d}y. \tag{14}$$

This condition is trivially satisfied when the two beams are non-overlapping, namely when the functions $U(x, y - a, z)$ and $U(x, y + a, z)$ have spatially disjoint supports[7] and, therefore, $U_{\rm up}^{\star} U_{\rm down} = 0$, namely:

$$U^{\star}(x, y - a, z)\, U(x, y + a, z) = 0. \tag{15}$$

[7] Let us briefly remind the concept of support of a function. Let $f\colon X \to \mathbb{C}$ be a complex-valued function defined in the arbitrary domain X. The support of f, written $\mathrm{supp}(f)$, is the set of points in X where f is non-zero: $\mathrm{supp}(f) = \{x \in X \mid f(x) \neq 0\}$.

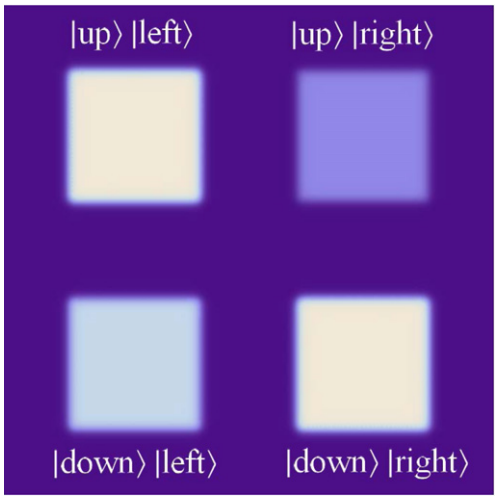

Figure 2. Illustration of a fourfold optical beam in the transverse plane $z = 0$ described by equation (18). The two yellow square spots along the diagonal constitute a twofold beam described by equation (21) and represent the entangled vector state (20).

For example, for a fundamental Gaussian beam of waist (spot size) w_0 and normalized amplitude

$$U(\boldsymbol{\rho}, z) = \sqrt{\frac{kL}{\pi}} \frac{1}{z - \mathrm{i}L} \exp\left(\frac{\mathrm{i}k}{2} \frac{|\boldsymbol{\rho}|^2}{z - \mathrm{i}L}\right), \tag{16}$$

where $L = kw_0^2/2$ is the Rayleigh range of the beam, it is not difficult to show that

$$I(a) = \exp\left(-\frac{a^2}{w_0^2/2}\right). \tag{17}$$

As expected, one obtains $I(a) \simeq 0$ only when the separation between the two beams is much bigger than the beam waist: $a \gg w_0$. Conversely, when $I(a) \neq 0$ the spatial mode functions $U(x, y - a, z)$ and $U(x, y + a, z)$ are not reciprocally orthogonal and, therefore, equation (13) is no longer in a Schmidt form. In this case a new Schmidt decomposition must be performed to bring $\mathbf{U}(\boldsymbol{\rho}, z)$ to the form (8).

3.2. Position–position entanglement

A second way to encode two qubits in optical beams was proposed by Puentes *et al* [35] (a similar method to process optical beams was previously proposed by Caulfield and Shamir [36] and by Spreeuw and coworkers [37]) and found numerous applications in recent years [38–42]. The key idea is to encode two qubits in the transverse positions of four non-overlapping beams of light propagating along a common axis, say z. In the xy-plane of equation $z = 0$, these beams form an array of four bright spots with the same polarization, say $\boldsymbol{e}$, but different phase and intensity, as shown in figure 2. The first qubit is encoded in the vertical position ('up' and 'down') of such fourfold beam, and the second qubit in the horizontal position ('left' and 'right'). Then, the fourfold beam at $z = 0$ can be described by the analytic signal $\mathbf{U}(\boldsymbol{\rho}, 0) = \boldsymbol{e}U(x, y)$, where [41]

$$\begin{aligned} U(x, y) = {} & A_{00}U(x + a, y - a) + A_{01}U(x - a, y - a) \\ & + A_{10}U(x + a, y + a) + A_{11}U(x - a, y + a), \end{aligned} \tag{18}$$

with $A_{ij} \in \mathbb{C}$, $(i, j = 0, 1)$ being numerical constants settling the intensity and the phase of each of the four beams, and the four functions

$$U\left(x + (-1)^j a, y - (-1)^i a\right) = \mathrm{rect}\left(\frac{y - (-1)^i a}{b}\right) \mathrm{rect}\left(\frac{x + (-1)^j a}{b}\right), \tag{19}$$

fix the position and the (square) spatial profiles of the beams where, again, $i, j = 0, 1$. In equation (19), $0 < b < 2a$ is the horizontal and vertical width of each of the four beams, and the rectangle function $\mathrm{rect}(\xi)$ is equal to 1 for $|\xi| < 1/2$, to 1/2 for $|\xi| = 1/2$ and to 0 for $|\xi| > 1/2$ [43].

By selecting only two spots along the diagonal $x + y = 0$, one achieves the position–position optical beam representation of the two-qubit entangled state

$$|\mathrm{up}\rangle|\mathrm{left}\rangle + |\mathrm{down}\rangle|\mathrm{right}\rangle, \tag{20}$$

where the spatial separation $a > b/2$ between the beams guarantees that $\langle \text{up}|\text{down}\rangle = 0 = \langle \text{left}|\text{right}\rangle$. The analytic signal of such twofold beam can be written as

$$\begin{aligned} U(x, y) = & \; U(x + a, y - a) + U(x - a, y + a) \\ = & \; \text{rect}\left(\frac{y - a}{b}\right)\text{rect}\left(\frac{x + a}{b}\right) + \; \text{rect}\left(\frac{y + a}{b}\right)\text{rect}\left(\frac{x - a}{b}\right), \end{aligned} \tag{21}$$

where $\text{rect}[(y - a)/b] \sim |\text{up}\rangle$, $\text{rect}[(\text{y} + \text{a})/\text{b}] \sim |\text{down}\rangle$ and $\text{rect}[(x + a)/b] \sim |\text{left}\rangle$, $\text{rect}[(\text{x} - \text{a})/\text{b}] \sim |\text{right}\rangle$.

The functions $U(x + a, y - a)$ and $U(x - a, y + a)$ in equation (21) are non-overlapping only if $b < 2a$. In this case the vertical and horizontal positional degrees of freedom are binary and the equation (21) is automatically in a Schmidt form and displays maximum entanglement.

3.3. Polarization-spatial entanglement

The third method to achieve intrasystem entanglement in optical beams, exploits polarization and the so-called first-order spatial modes [44] of the electromagnetic field, as binary degrees of freedom. It is a well established result of polarization optics that the polarization vector space can be represented by the polarization Poincaré sphere [45]. It is also known that the vector space formed by the first-order spatial mode can be mapped into a Poincaré sphere [46]. The direct product of polarization and spatial vector spaces contains a subspace spanned by the so-called cylindrically polarized beams of light and can be represented as the direct sum of two 'hybrid' Poincaré spheres [47, 48]. In recent years, many fundamental and applied researches upon polarization-spatial entanglement in optical beams have been carried out [4, 6, 25–28, 47, 49–56].

The analytic signal of the more general paraxial beam in this polarization-spatial space takes the form

$$\mathbf{U}(\boldsymbol{\rho}, z) = A_{00}\boldsymbol{e}_H U_{10}(\boldsymbol{\rho}, z) + A_{01}\boldsymbol{e}_H U_{01}(\boldsymbol{\rho}, z) + A_{10}\boldsymbol{e}_V U_{10}(\boldsymbol{\rho}, z) + A_{11}\boldsymbol{e}_V U_{01}(\boldsymbol{\rho}, z), \tag{22}$$

where $A_{ij} \in \mathbb{C}$, $(i, j \in \{0, 1\})$ are numerical constants, and $U_{nm}(\boldsymbol{\rho}, z)$ denotes the Hermite–Gauss solution of the paraxial wave equation of order $N = n + m$ with $N = 1$. These solutions are also known as transverse electromagnetic (TEM_{nm}) modes and are orthogonal with respect to the spatial scalar product: $(U_{nm}, U_{n'm'})_S = \delta_{nn'}\delta_{mm'}$, with $n, n', m, m' \in \{0, 1, 2, \ldots\}$ [57].

Choosing $A_{00} = 1 = A_{11}$ and $A_{10} = 0 = A_{01}$ in equation (22), one obtains a representation of the so-called radially polarized beam of light

$$\mathbf{U}(\boldsymbol{\rho}, z) = \boldsymbol{e}_H U_{10}(\boldsymbol{\rho}, z) + \boldsymbol{e}_V U_{01}(\boldsymbol{\rho}, z), \tag{23}$$

illustrated in figure 3. The beam in equation (23) is automatically in a Schmidt form and furnishes the polarization-spatial optical representation of the two-qubit entangled state

$$|H\rangle|\text{TEM}_{10}\rangle + |V\rangle|\text{TEM}_{01}\rangle. \tag{24}$$

Non-uniformly polarized beams of light exhibition classical entanglement have recently found practical applications in quantum information [28] and polarization metrology [4, 26].

3.4. Comparison

The three kinds of classically entangled optical beams considered in this section have a quite different nature. First of all, both polarization-position and polarization-spatial entanglement are consequences of a natural partition between *different* degrees of freedom, namely *polarization* and *position/spatial*. Conversely, position–position entanglement occurs because of an arbitrarily chosen partition of the $\mathbb{R}^2$ plane, being the two binary positional degrees of freedom of the same type. This means, for example, that it is possible to represent the vector field (21) in a separable form by simply choosing a 45°-rotated Cartesian reference frame.

Now, let us compare polarization-position (PP) and polarization-spatial (PS) entanglement. To begin with, it is clear that equation (23) has the same form of equation (13). Both expressions are written as a Schmidt sum. Each term in the sums is given by the product of a polarization vector times a scalar function. The two scalar functions in equation (13) are orthogonal to each other and the same applies to the two scalar functions in equation (23). However, and here is the profound difference, the functions in equation (13) are 'trivially' orthogonal simply because they are non-zero in different spatially disjoint regions, as shown by equation (15), i.e., the beams are non-overlapping. Conversely, the functions U_{10} and U_{01} in equation (23) have the same support(see footnote 7) and $\left(U_{10}, U_{01}\right)_S = 0$, although

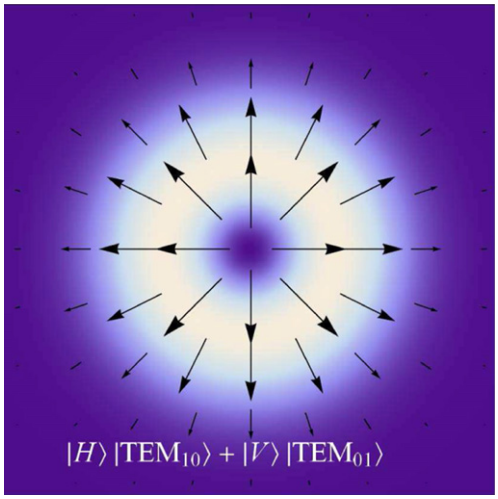

Figure 3. Polarization (black arrows) and intensity pattern of a radially polarized optical beam in the transverse plane $z = 0$, as described by equation (23).

$$U_{10}^{\star}(\boldsymbol{\rho}, z)\, U_{01}(\boldsymbol{\rho}, z) \neq 0. \tag{25}$$

Therefore, in the polarization-spatial entanglement there is a *single* beam of light, the radially polarized one, encoding both qubits. This is very different from the polarization-position case where one needs *two* spatially separated (therefore, fully distinguishable) beams to encode two entangled qubits.

This concept may be further clarified noticing that equation (23) represents a *coherent* superposition of beams with orthogonal polarization, while equation (13) represents, de facto, an *incoherent* superposition of orthogonally polarized beams. To be more quantitative, we may calculate the covariance matrix [29] of both PP and PS beams defined as

$$J^{\mathrm{PX}} = \int \mathcal{J}^{\mathrm{PX}}(\boldsymbol{\rho}, z)\ \mathrm{d}^2\rho, \tag{26}$$

with $X \in \{P, S\}$, and

$$\mathcal{J}^{\mathrm{PP}}(\boldsymbol{\rho}, z) = \begin{bmatrix} |U_{\mathrm{up}}|^2 & U_{\mathrm{up}} U_{\mathrm{down}}^{\star} \\ U_{\mathrm{up}}^{\star} U_{\mathrm{down}} & |U_{\mathrm{down}}|^2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} |U_{\mathrm{up}}|^2 & 0 \\ 0 & |U_{\mathrm{down}}|^2 \end{bmatrix}, \tag{27}$$

where the last line is a straightforward consequence of equation (15), and

$$\mathcal{J}^{\mathrm{PS}}(\boldsymbol{\rho}, z) = \begin{bmatrix} |U_{10}|^2 & U_{10} U_{01}^{\star} \\ U_{10}^{\star} U_{01} & |U_{01}|^2 \end{bmatrix}. \tag{28}$$

Spatial integration in equation (26) has the physical meaning of disregarding the position/spatial degrees of freedom. It is the analogous of the 'trace' operation in quantum mechanics, with respect to the unobserved subsystem. A straightforward calculation shows that in both cases one obtains $J^{\mathrm{PX}} = I_2$, where I_2 denotes the 2×2 identity matrix. This is expected because both beams represents maximally entangled states [58, 59] and the corresponding covariance matrix must describe completely unpolarized light. However, the diagonal form of equation (27) reveals that equation (13) is in some sense more similar to an incoherent superposition already before integration. In this respect, polarization-spatial entanglement is the 'closest' one, amongst the three types of entanglement considered here, to bona fide quantum entanglement.

4. Outlook: from 2 qubits to 3 qubits entanglement and more

In the case of polarization-spatial entanglement, we have considered each Hermite–Gauss mode $U_{nm}(\boldsymbol{\rho}, z)$ as a single function. However, from the case of position–position entanglement we have learned that the Cartesian coordinates x and y may be also considered as independent degrees of freedom. In this section, we combine these two concepts to build optical beam representations of tripartite states of qubits, each party being associate to a specific degree of freedom.

We begin with the simple observation that an Hermite–Gauss mode can be always factorized as

$$U_{nm}(\boldsymbol{\rho}, z) = u_n(x, z)\, u_m(y, z), \tag{29}$$

where $n,\ m = 0,\ 1,\ 2,\ \ldots$. This means, for example, that the radially polarized beam (23), can be rewritten in the form

$$\mathbf{U}(\boldsymbol{\rho}, z) = \boldsymbol{e}_H u_1(x, z)\, u_0(y, z) + \boldsymbol{e}_V\ u_0(x, z)\, u_1(y, z), \tag{30}$$

which is isomorphic to the three-qubit state vector

$$|0\rangle_p |1\rangle_x |0\rangle_y + |1\rangle_p |0\rangle_x |1\rangle_y, \tag{31}$$

where $\boldsymbol{e}_H \sim |0\rangle_p$, $\boldsymbol{e}_V \sim |1\rangle_p$, $u_1(x, z) \sim |1\rangle_x$, $u_0(y, z) \sim |0\rangle_y$ and $u_0(x, z) \sim |0\rangle_x$, $u_1(y, z) \sim |1\rangle_y$, and the label '$p$' stands for 'polarization'. The state vector (31) lives in a Hilbert space with $2 \times 2 \times 2$ dimensions, namely it represents a tripartite system. Of course, a state of the form (31) can be easily generalized to a vector state acting in a $2 \times N \times M$ Hilbert space. The idea of considering entanglement between x and y Cartesian coordinates in paraxial beams has been also recently exploited by Agarwal and coworkers [60]. A tripartite representation of an optical beam like the one in equation (30) can be used, for example, to implement a quantum-like teleportation scheme. Moreover, polarization/spatial coupled modes have recently found applications in decoherence-free subspaces studies [61], quantum cryptography [62], quantum computing [63] and quantum logic gates [64]. Further examples of possible applications are briefly illustrated below. However, before proceeding, we remark that key to the use of classically entangled beams, is the experimenter's capability to create and to manipulate (especially in unitary manner) optical modes[8]. Such operations can be performed by means of different schemes as, e.g., the ones presented in [56, 65] and references therein.

4.1. GHZ state

The state vector given in equation (31) as a form similar to the so-called GHZ state [66]:

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{32}$$

In the language of classical optics, the lowest-order implementation of the state (32) is given by the beam

$$\mathbf{U}_{\mathrm{GHZ}}(\boldsymbol{\rho}, z) = \frac{1}{\sqrt{2}}\Big[\boldsymbol{e}_H U_{00}(\boldsymbol{\rho}, z) + \boldsymbol{e}_V U_{11}(\boldsymbol{\rho}, z)\Big]. \tag{33}$$

The beam represented by equation (33) clearly has a non-uniform polarization pattern, as shown in figure 4.

4.2. *W* state

Another famous tripartite quantum state is the so-called W state [67]:

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \tag{34}$$

which has the same mathematical structure of the following classical beam:

$$\mathbf{U}_W(\boldsymbol{\rho}, z) = \frac{1}{\sqrt{3}}\Big[\boldsymbol{e}_H U_{01}(\boldsymbol{\rho}, z) + \boldsymbol{e}_H U_{10}(\boldsymbol{\rho}, z) + \boldsymbol{e}_V\ U_{00}(\boldsymbol{\rho}, z)\Big]. \tag{35}$$

Also $\mathbf{U}_W(\boldsymbol{\rho}, z)$ describes a beam with a non-uniform polarization pattern, as illustrated in figure 5 below

4.3. NOON states

Another class of entangled quantum states that can be encoded in paraxial beams of light, are the so-called generalized NOON states [68]:

$$|\mathrm{NOON}\rangle = \frac{1}{\sqrt{2}}\Big(|N, 0\rangle + \mathrm{e}^{\mathrm{i}N\theta}\ |0, N\rangle\Big). \tag{36}$$

The classical optics representation of (36) is a purely scalar superposition of HG beams:

$$U_{\mathrm{NOON}}(\boldsymbol{\rho}, z) = \frac{1}{\sqrt{2}}\Big[U_{N0}(\boldsymbol{\rho}, z) + \mathrm{e}^{\mathrm{i}N\theta} U_{0N}(\boldsymbol{\rho}, z)\Big]. \tag{37}$$

The real and imaginary parts of the analytic signal given in equation (37) are shown in figure 6 below for $N = 4$ and $\theta = \pi/3$: super-resolution and super-sensitivity (see, e.g., [69] for a proper definition of the two terms), are intriguing properties of the NOON states that could be investigated using classical beams of light of the form (37). While it is known that super-resolution can be achieved with classical light [69, 70], the question whether

[8] We thank an anonymous Referee for pointing out this important issue.

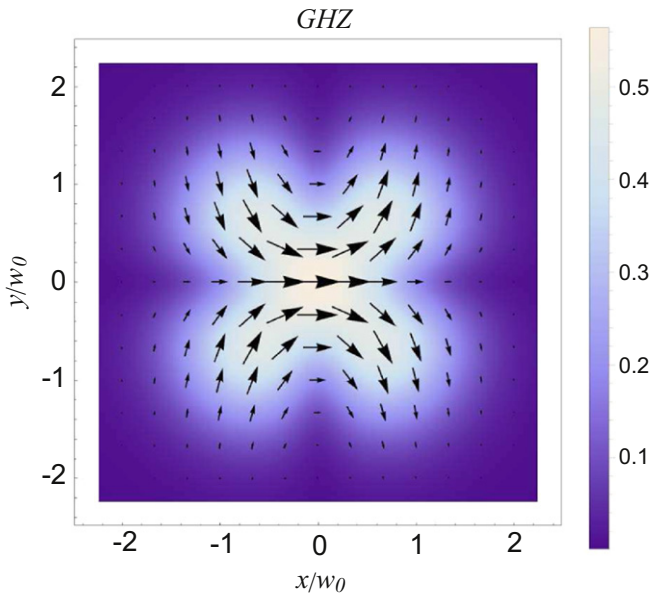

Figure 4. Polarization (black arrows) and intensity pattern of the non-uniformly polarized beam described by equation (33) analogous to the quantum GHZ vector state (32).

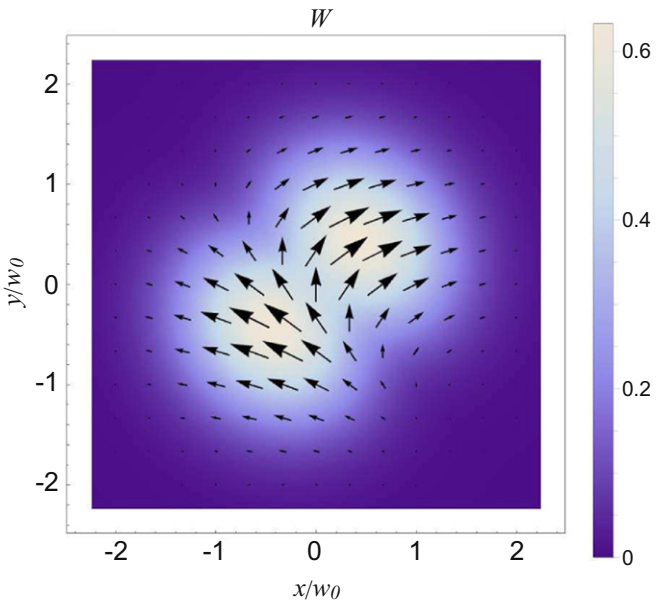

Figure 5. Polarization (black arrows) and intensity pattern of the non-uniformly polarized beam described by equation (35) corresponding to the quantum *W* vector state (34). Differently from the pattern shown in figure 4, this beam does not exhibit Cartesian symmetry.

super-sensitivity could be obtained by classical beams of the form (37), is perfectly open. Moreover, these optical beams could furnish a good 'laboratory' to study decoherence in Schrödinger cat states under easily controllable conditions.

5. Summary

The seemingly oxymoronic name 'classical entanglement' actually denotes the occurrence of some typical quantum mechanical features in classical systems and it should not be regarded as a substitute for quantum entanglement. In this article we have studied classical entanglement exhibited by optical beams prepared in three different manners. Within the context of paraxial optics, we have been able to provide a theory unifying the

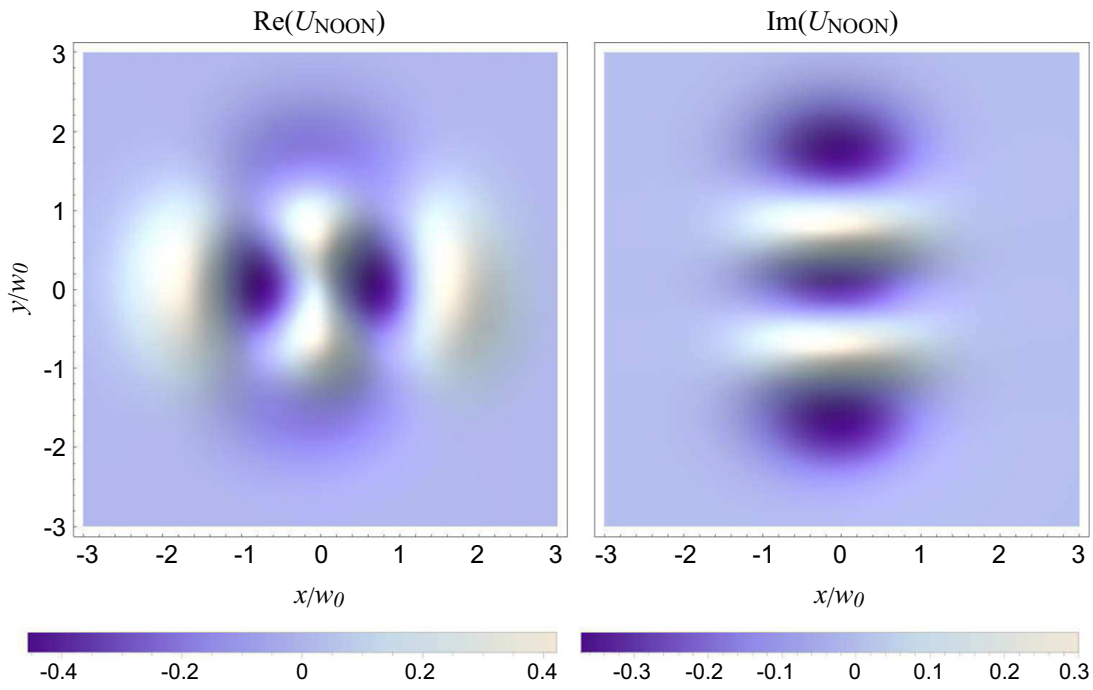

Figure 6. Real and imaginary parts of the 'NOON' beam field described by equation (37) evaluated for $N = 4$ and $\theta = \pi/3$.

representation of these three kinds of beams. We also have classified the latter according to what pair of binary degrees of freedom of the light is chosen to encode the 'entangled qubits'. Moreover, we have demonstrated that despite of the formal similarity between the mathematical expressions for the beams in all the three cases, the physical characteristics of the light (coherent or incoherent) may be very different. Finally, we have suggested a few ideas about how to enlarge the already rich phenomenology of classical entanglement.

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