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Existence, uniqueness and global behavior of the solutions to some nonlinear vector equations in a finite dimensional Hilbert space

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Abstract

The initial value problem and global properties of solutions are studied for the vector equation: $\left(\|u'\|^l u'\right)' + \|A^{\frac{1}{2}}u\|^\beta Au + g(u') = 0$ in a finite dimensional Hilbert space under suitable assumptions on g.

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1 Introduction

Let H be a finite dimensional real Hilbert space, with norm denoted by $\|.\|$. We consider the following nonlinear equation

$$\left(\|u'\|^l u'\right)' + \|A^{\frac{1}{2}}u\|^\beta Au + g(u') = 0, \tag{1.1}$$

where l and β are positive constants, and A is a positive and symmetric linear operator on H. We denote by $(\cdot,\cdot)$ the inner product in H. The operator A is coercive, which means :

$$\exists \lambda > 0, \ \forall u \in D(A), \quad (Au, u) \geq \lambda \|u\|^2.$$

We also define

$$\forall u \in H, \ \|A^{\frac{1}{2}}u\| := \|u\|_{D(A^{\frac{1}{2}})},$$

a norm equivalent to the norm in H. We assume that $g : H \to H$ is locally Lipschitz continuous.

When $l = 0$ and $g(u') = c|u'|^\alpha u'$ Haraux **[4]** studied the rate of decay of the energy of non-trivial solutions to the scalar second order ODE. In addition, he showed that if $\alpha > \frac{\beta}{\beta+2}$ all non-trivial solutions are oscillatory and if $\alpha < \frac{\beta}{\beta+2}$ they are non-oscillatory. In the oscillatory case he established that all non-trivial solutions have the same decay rates, while in the non-oscillatory case he showed the coexistence of exactly two different decay rates, calling slow solutions those which have the lowest decay rate, and fast solutions the others.

Abdelli and Haraux **[1]** studied the scalar second order ODE where $g(u') = c|u'|^\alpha u'$, they proved the existence and uniqueness of a global solution with initial data $(u_0, u_1) \in \mathbb{R}^2$. They used some modified energy function to estimate the rate of decay and they used the method introduced by Haraux **[4]** to study the oscillatory or non-oscillatory of non-trivial solutions. If $\alpha > \frac{\beta(l+1)+l}{\beta+2}$ all non-trivial solutions are oscillatory and if $\alpha < \frac{\beta(l+1)+l}{\beta+2}$ they are non-oscillatory. In the non-oscillatory rate, as in the case $l = 0$, the coexistence of exactly two different decay rates was established.

In this article, we use some techniques from Abdelli and Haraux **[1]** to establish a global existence and uniqueness result of the solutions, and under some additional conditions on g (typically $g(s) \sim c\|s\|^\alpha s$), we study the asymptotic behavior as $t \to \infty$. A basic role will be played by the total energy of the solution u given by the formula

$$E(t) = \frac{l+1}{l+2}\|u'(t)\|^{l+2} + \frac{1}{\beta+2}\|A^{\frac{1}{2}}u(t)\|^{\beta+2}. \tag{1.2}$$

The plan of this paper is as follows: In Section 2 we establish some basic preliminary inequalities, and in Section 3 we prove the existence of a solution $u \in \mathcal{C}^1(\mathbb{R}^+, H)$ with $\|u'\|^l u' \in \mathcal{C}^1(\mathbb{R}^+, H)$ for any initial data $(u_0, u_1) \in H \times H$ under relevant conditions on g and the conservation of total energy for each such solution. In Section 4 we establish the uniqueness of the solution in the same regularity class under additional conditions on g. In Section 5 we prove convergence of all solutions to 0 under more specific conditions on g and we estimate the decay rate of the energy. Finally, in Section 6, we discuss the optimality of these estimates when $g(s) = c\|s\|^{\alpha}s$ and $l < \alpha < \frac{\beta(1+l)+l}{\beta+2}$; in particular, by relying on a technique introduced by Ghisi, Gobbino and Haraux **[3]**, we prove the existence of a open set of initial states giving rise to slow decaying solutions.

2 Some basic inequalities

In this section, we establish some easy but powerful lemmas which generalize Lemma 2.2 and Lemma 2.3 from **[2]** and will be essential for the existence and uniqueness proofs of the next section. Troughout this section, H denotes an arbitrary (not necessary finite dimensional) real Hilbert space with norm denoted by $\|.\|$.

Proposition 2.1. *Let $(u, v) \in H \times H$ and $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then the condition*

$$(\alpha - \beta)(\|u\| - \|v\|) \geq 0,$$

implies

$$(\alpha u - \beta v)(u - v) \geq \frac{1}{2}(\alpha + \beta)\|u - v\|^2.$$

Proof. An easy calculation gives the identity

$$(\alpha u - \beta v)(u - v) = \frac{1}{2}(\alpha - \beta)(\|u\|^2 - \|v\|^2) + \frac{1}{2}(\alpha + \beta)\|u - v\|^2.$$

Since

$$(\alpha - \beta)(\|u\|^2 - \|v\|^2) = (\|u\| + \|v\|)(\alpha - \beta)(\|u\| - \|v\|),$$

the result follows immediately. □

For the next results, we consider a number $R > 0$ and we set

$$J_R = [0, R]; \quad B_R := \{u \in H, \ \|u\| \leq R\}$$

Proposition 2.2. *Let $f : J_R \longrightarrow \mathbb{R}^+$ be non increasing and such that for some positive numbers p, c:*

$$\forall s \in J_R, \quad f(s) \geq cs^p.$$

Then we have

$$\forall (u, v) \in B_R \times B_R, \quad (f(\|u\|)u - f(\|v\|)v)(u - v) \geq \frac{c}{2}(\|u\|^p + \|v\|^p)\|u - v\|^2.$$

Proof. Applying the previous result with $\alpha = f(\|u\|)$ and $\beta = f(\|v\|)$ the result follows immediately. □

Proposition 2.3. *Let $f : J_R \longrightarrow \mathbb{R}^+$ be non increasing and such that for some positive numbers p, c:*

$$\forall s \in J_R, \quad f(s) \geq cs^p.$$

Then we have

$$\forall (u, v) \in B_R \times B_R, \quad (f(\|u\|)u - f(\|v\|)v)(u - v) \geq \delta\|u - v\|^{p+2},$$

with $\delta = \frac{c}{2^{\max\{p,1\}}}$.

Proof. It is sufficient to apply the previous result combined with the inequality

$$\|u-v\|^p \leq 2^{\max\{p-1,0\}}(\|u\|^p + \|v\|^p).$$

$\square$

Corollary 2.4. *Let $f : J_R \longrightarrow \mathbb{R}^+$ be non increasing and such that for some positive numbers p, c:*

$$\forall s \in J_R, \quad f(s) \geq cs^p.$$

Then we have for some constant $C = C(c,p) > 0$,

$$\forall (u,v) \in B_R \times B_R, \quad \|u-v\| \leq C\|f(\|u\|)u - f(\|v\|)v)\|^{\frac{1}{p+1}}.$$

Proof. This inequality follows immediately from Cauchy-Schwarz inequality combined with the conclusion of the previous proposition. $\square$

Lemma 2.5. *Assume that A is a positive, symmetric, bounded operator on H. for some constant $D > 0$ we have $\forall (w,v) \in H \times H$*

$$\forall (w,v) \in H \times H, \ \|\|A^{\frac{1}{2}}w\|^\beta Aw - \|A^{\frac{1}{2}}v\|^\beta Av\| \leq D \max(\|A^{\frac{1}{2}}v\|, \|A^{\frac{1}{2}}w\|)^\beta \|w-v\|.$$

Proof. We can write

$$\begin{aligned}\|\|A^{\frac{1}{2}}w\|^\beta Aw - \|A^{\frac{1}{2}}v\|^\beta Av\| &= \|A^{\frac{1}{2}}(\|A^{\frac{1}{2}}w\|^\beta A^{\frac{1}{2}}w - \|A^{\frac{1}{2}}v\|^\beta A^{\frac{1}{2}}v)\| \\ &\leq \|A^{\frac{1}{2}}\|\|\|A^{\frac{1}{2}}w\|^\beta A^{\frac{1}{2}}w - \|A^{\frac{1}{2}}v\|^\beta A^{\frac{1}{2}}v\|.\end{aligned}$$

Direct calculations (see also **[2]**, Lemma 2.2) yield

$$\begin{aligned}\|\|A^{\frac{1}{2}}w\|^\beta A^{\frac{1}{2}}w - \|A^{\frac{1}{2}}v\|^\beta A^{\frac{1}{2}}v\| &\leq C_1 \max(\|A^{\frac{1}{2}}v\|, \|A^{\frac{1}{2}}w\|)^\beta \|A^{\frac{1}{2}}w - A^{\frac{1}{2}}v\| \\ &\leq C_2 \max(\|A^{\frac{1}{2}}v\|, \|A^{\frac{1}{2}}w\|)^\beta \|w-v\|,\end{aligned}$$

and the result is proved. $\square$

3 Global existence and energy conservation for equation (1.1)

In this section, we study the existence of a solution for the initial value problem associated to equation (1.1) where $g : H \to H$ is a locally Lipschitz continuous function which satisfies the following hypothesis:

$$\exists k_1 > 0, \quad k_2 > 0, \ \forall v, \ (g(v), v) \geq -k_1 - k_2\|v\|^{l+2}. \tag{3.1}$$

Theorem 3.1. *Let $(u_0, u_1) \in H \times H$. The problem (1.1) has a global solution satisfying*

$$u \in \mathcal{C}^1(\mathbb{R}^+, H), \quad \|u'\|^l u' \in \mathcal{C}^1(\mathbb{R}^+, H) \quad \textit{and} \ \ u(0) = u_0, \ \ u'(0) = u_1.$$

Proof. To show the existence of the solution for (1.1), we consider the auxiliary problem

$$\begin{cases} (\varepsilon+\|u_\varepsilon'\|^2)^{l/2}u_\varepsilon''+l(u_\varepsilon',u_\varepsilon'')(\varepsilon+\|u_\varepsilon'\|^2)^{l/2-1}u_\varepsilon'+\|A^{\frac12}u_\varepsilon\|^\beta Au_\varepsilon+g(u_\varepsilon')=0,\\ u_\varepsilon(0)=u_0,\quad u_\varepsilon'(0)=u_1. \end{cases} \tag{3.2}$$

Here, $\varepsilon>0$ is a small parameter, devoted to tend to zero. For simplicity in the sequel we shall write

$$l/2:=m>0.$$

Assuming the existence of such a solution u_ε, multiplying (3.2) by u_ε', we find

$$[(\varepsilon+\|u_\varepsilon'\|^2)^m+2m(\varepsilon+\|u_\varepsilon'\|^2)^{m-1}\|u_\varepsilon'\|^2](u_\varepsilon',u_\varepsilon'')+\|A^{\frac12}u_\varepsilon\|^\beta(Au_\varepsilon,u_\varepsilon')+(g(u_\varepsilon'),u_\varepsilon')=0,$$

then

$$(u_\varepsilon',u_\varepsilon'')=-\frac{\|A^{\frac12}u_\varepsilon\|^\beta(Au_\varepsilon,u_\varepsilon')+(g(u_\varepsilon'),u_\varepsilon')}{(\varepsilon+\|u_\varepsilon'\|^2)^{m-1}(\varepsilon+(l+1)\|u_\varepsilon'\|^2)}. \tag{3.3}$$

From (3.2), we obtain that u_ε is a solution of

$$\begin{cases} (\varepsilon+\|u_\varepsilon'\|^2)^m u_\varepsilon''-l\dfrac{\|A^{\frac12}u_\varepsilon\|^\beta(Au_\varepsilon,u_\varepsilon')+(g(u_\varepsilon'),u_\varepsilon')}{\varepsilon+(l+1)\|u_\varepsilon'\|^2}u_\varepsilon'+\|A^{\frac12}u_\varepsilon\|^\beta Au_\varepsilon+g(u_\varepsilon')=0,\\ u_\varepsilon(0)=u_0,\quad u_\varepsilon'(0)=u_1. \end{cases} \tag{3.4}$$

Conversely, (3.4) implies (3.3). Then, replacing (3.3) in (3.4), we obtain (3.2). Therefore (3.2) is equivalent to (3.4).

i) A priori estimates:

Next, we introduce

$$F_\varepsilon(u_\varepsilon,u_\varepsilon') = l\frac{\|A^{\frac12}u_\varepsilon\|^\beta(Au_\varepsilon,u_\varepsilon')+(g(u_\varepsilon'),u_\varepsilon')}{(\varepsilon+\|u_\varepsilon'\|^2)^m(\varepsilon+(l+1)\|u_\varepsilon'\|^2)}u_\varepsilon'-\frac{\|A^{\frac12}u_\varepsilon\|^\beta Au_\varepsilon+g(u_\varepsilon')}{(\varepsilon+\|u_\varepsilon'\|^2)^m}.$$

Then (3.4), can be rewritten as

$$\begin{cases} u_\varepsilon''=F_\varepsilon(u_\varepsilon,u_\varepsilon'),\\ u_\varepsilon(0)=u_0,\quad u_\varepsilon'(0)=u_1. \end{cases} \tag{3.5}$$

Since the vector field F_ε is locally Lipschitz continuous, the existence and uniqueness of u_ε in the class $\mathcal{C}^2([0,T),H)$, for some $T>0$ is classical. Multiplying (3.2) by u_ε', we obtain by a simple calculation the following energy identity:

$$\frac{d}{dt}E_\varepsilon(t)+(g(u_\varepsilon'(t)),u_\varepsilon'(t))=0,$$

where

$$E_\varepsilon(t)=\frac{l+1}{l+2}(\varepsilon+\|u_\varepsilon'\|^2)^{m+1}-\varepsilon(\varepsilon+\|u_\varepsilon'\|^2)^m+\frac{1}{\beta+2}\|A^{\frac12}u_\varepsilon(t)\|^{\beta+2}.$$

Indeed, for any function $v\in\mathcal{C}^1([0,T),H)$ we have the following sequence of identities

$$(((\varepsilon+\|v\|^2)^m v)',v)=2m(\varepsilon+\|v\|^2)^{m-1}\|v\|^2(v,v')+(\varepsilon+\|v\|^2)^m(v,v')$$

$$= 2m(\varepsilon+\|v\|^2)^{m-1}(\varepsilon+\|v\|^2)(v,v') - 2m\varepsilon(\varepsilon+\|v\|^2)^{m-1}(v,v') + (\varepsilon+\|v\|^2)^m(v,v')$$
$$= (2m+1)(\varepsilon+\|v\|^2)^m(v,v') - 2m\varepsilon(\varepsilon+\|v\|^2)^{m-1}(v,v')$$
$$= \frac{d}{dt}\left[\frac{l+1}{l+2}(\varepsilon+\|v\|^2)^{m+1} - \varepsilon(\varepsilon+\|v\|^2)^m\right].$$

Moreover for some constant $C>0$ independent of ε, we have

$$E_\varepsilon(t) + C \geq \frac{l+1}{l+2}(\varepsilon+\|u_\varepsilon'\|^2)^{m+1} - \varepsilon(\varepsilon+\|u_\varepsilon'\|^2)^m + C \geq \frac{1}{4}\|u_\varepsilon'(t)\|^{l+2}$$

and then as a consequence of (3.1)

$$\frac{d}{dt}E_\varepsilon(t) = -(g(u_\varepsilon'(t)), u_\varepsilon'(t)) \leq k_1 + k_2\|u_\varepsilon'(t)\|^{l+2}$$
$$\leq k_3 + k_4 E_\varepsilon(t).$$

By Gronwall's inequality, this implies

$$\forall t \in [0,T), \quad \|u_\varepsilon(t)\| \leq M_1, \ \|u_\varepsilon'(t)\| \leq M_2, \tag{3.6}$$

for some constants M_1, M_2 independent of ε. Hence, u_ε and u_ε' are uniformly bounded and u_ε is a global solution, in particular $T>0$ can be taken arbitrarily large.

ii) Passage to the limit:

In order to pass to the limit as $\varepsilon \to 0$ we need to know that u_ε and u_ε' are uniformly equicontinous on $(0,T)$ for any $T>0$. For u_ε it is clear, since $\|u_\varepsilon'\|$ is bounded.
Moreover we have

$$(\varepsilon+\|u_\varepsilon'\|^2)^{l/2}u_\varepsilon'' + l(u_\varepsilon', u_\varepsilon'')(\varepsilon+\|u_\varepsilon'\|^2)^{l/2-1}u_\varepsilon' = ((\varepsilon+\|u_\varepsilon'\|^2)^{l/2}u_\varepsilon')',$$

hence by the equation

$$\|((\varepsilon+\|u_\varepsilon'\|^2)^{l/2}u_\varepsilon')'\| \leq \|A^{\frac{1}{2}}u_\varepsilon(t)\|^\beta\|Au_\varepsilon(t)\| + \|g(u_\varepsilon'(t))\|,$$

and since g is locally Lipschitz continuous, hence bounded on bounded sets, we obtain

$$\|((\varepsilon+\|u_\varepsilon'\|^2)^{l/2}u_\varepsilon')'\| \leq M_3.$$

Therefore the functions $(\varepsilon+\|u_\varepsilon'\|^2)^m u_\varepsilon'$ are uniformly Lipschitz continuous on $\mathbb{R}+$. We claim that u_ε' is a uniformly (with respect to ε) locally Hölder continuous function of t. Indeed by applying Corollary 2.4 with $f(s) = (\varepsilon+s^2)^m$ and $p = 2m = l$ we find for some $C>0$

$$\|u_\varepsilon'(t_1) - u_\varepsilon'(t_2)\| \leq C\|(\varepsilon+\|u_\varepsilon'\|^2)^m u_\varepsilon'(t_1) - (\varepsilon+\|u_\varepsilon'\|^2)^m u_\varepsilon'(t_2)\|^{\frac{1}{l+1}} \leq C'\|t_1 - t_2\|^{\frac{1}{l+1}}.$$

As a consequence of Ascoli's theorem and a priori estimate (3.6) combined with (3.5), we may extract a subsequence which is still denoted for simplicity by (u_ε) such that for every $T>0$

$$u_\varepsilon \to u \quad \text{in } \mathcal{C}^1((0,T),H),$$

as ε tends to 0. Integrating (3.2) over $(0,t)$, we get

$$\begin{aligned}(\varepsilon+\|u_\varepsilon'\|^2)^m u_\varepsilon'(t) \quad &- \quad (\varepsilon+\|u_\varepsilon'\|^2)^m u_\varepsilon'(t)(0) \\ &= \quad -\int_0^t \|A^{\frac{1}{2}}u_\varepsilon(s)\|^\beta A u_\varepsilon(s)\,ds - \int_0^t g(u_\varepsilon'(s))\,ds.\end{aligned} \tag{3.7}$$

From (3.7), we then have, as ε tends to 0

$$(\varepsilon+\|u_\varepsilon'\|^2)^m u_\varepsilon'(t) \to -\int_0^t \|A^{\frac{1}{2}}u(s)\|^\beta A u(s)\,ds - \int_0^t g(u'(s))\,ds + \|u'(0)\|^l u'(0) \quad \text{in } \mathcal{C}^0((0,T),H).$$

Hence

$$\|u'\|^l u' = -\int_0^t \|A^{\frac{1}{2}}u(s)\|^\beta A u(s)\,ds - \int_0^t g(u'(s))\,ds + \|u'(0)\|^l u'(0), \tag{3.8}$$

and $\|u'\|^l u' \in \mathcal{C}^1((0,T),H)$. Finally by differentiating (3.8) we conclude that u is a solution of (1.1).

□

Remark 3.2. *It is not difficult to see that the solution u constructed in the existence theorem satisfies the energy identity $\frac{d}{dt}E(t) = -(g(u'(t)),u'(t))$. The following stronger result shows that this identity is true for any solution even if uniqueness is not known. For infinite dimensional equations such as the Kirchhoff equation, both uniqueness and the energy identity for general weak solutions are old open problems.*

Theorem 3.3. *Let $(u_0,u_1) \in H \times H$. Then any solution u of (1.1) such that*

$$u \in \mathcal{C}^1(\mathbb{R}^+,H), \quad \|u'\|^l u' \in \mathcal{C}^1(\mathbb{R}^+,H) \quad \textit{and} \quad u(0)=u_0, \quad u'(0)=u_1,$$

satisfies the formula

$$\frac{d}{dt}E(t) = -(g(u'(t)),u'(t)). \tag{3.9}$$

The proof of this new result relies on the following simple lemma

Lemma 3.4. *Let J be any interval, assume that $v \in \mathcal{C}(J,H)$ and $\|v\|^l v \in \mathcal{C}^1(J,H)$ then*

$$\|v(t)\|^{l+2} \in \mathcal{C}^1(J,H),$$

and

$$((\|v(t)\|^l v(t))', v(t)) = \frac{d}{dt}\Big(\frac{l+1}{l+2}\|v(t)\|^{l+2}\Big), \quad \forall t \in J.$$

Proof. **Case 1.** For any $t_0 \in J$, if $v(t_0) \neq 0$ then $v(t) \neq 0$ in the neighborhood of t_0 and in this neighborhood we have $v \in \mathcal{C}^1(J,H)$ with

$$\begin{aligned}\frac{d}{dt}\left(\|v(t)\|^{l+2}\right) &= \frac{d}{dt}\left(\|v(t)\|^2\right)^{l/2+1} \\ &= \left(\frac{l}{2}+1\right)(\|v(t)\|^2)^{l/2}\frac{d}{dt}\left(\|v(t)\|^2\right) \\ &= 2\left(\frac{l}{2}+1\right)\|v(t)\|^l(v(t),v'(t)).\end{aligned} \tag{3.10}$$

On the other hand, we find

$$\begin{aligned}((\|v(t)\|^l v(t))', v(t)) &= (\|v(t)\|^l v'(t), v(t)) + (\|v(t)\|^l)'(v(t), v(t)) \\ &= \|v(t)\|^l (v'(t), v(t)) + l\|v(t)\|^l (v'(t), v(t)) \\ &= (l+1)\|v(t)\|^l (v'(t), v(t)),\end{aligned} \tag{3.11}$$

and from (3.10)-(3.11), we obtain

$$((\|v(t)\|^l v(t))', v(t)) = \frac{d}{dt}\Big[\frac{l+1}{l+2}\|v(t)\|^{l+2}\Big], \quad \text{for} \quad t = t_0.$$

Case 2. If $v(t_0) = 0$ then since $\|v(t)\|^l v(t) \in \mathcal{C}^1(J, H)$, we have clearly

$$((\|v\|^l v)'(t_0), v(t_0)) = 0.$$

Moreover in the neighborhood of t_0 we have

$$\|v(t)\|^l v(t) = 0(|t - t_0|),$$

in other terms

$$\|v(t)\|^{l+1} \leq C|t - t_0|,$$

therefore

$$\|v(t)\|^{l+2} \leq C^{\frac{l+2}{l+1}}|t - t_0|^{1+\frac{1}{l+1}},$$

then

$$(\|v(t)\|^{l+2})' = 0, \quad \text{for} \quad t = t_0,$$

and finally

$$((\|v(t)\|^l v(t))', v(t)) = \frac{d}{dt}\Big[\frac{l+1}{l+2}\|v(t)\|^{l+2}\Big] = 0 \quad \text{for} \quad t = t_0.$$

□

We now give the proof of Theorem 3.3.

Proof. Setting $v = u'$, from Lemma 3.4, we deduce

$$((\|u'(t)\|^l u'(t))', u'(t)) = \frac{d}{dt}\Big[\frac{l+1}{l+2}\|u'(t)\|^{l+2}\Big], \quad \forall t \in J.$$

By multiplying equation (1.1) by u', we obtain easily

$$\frac{d}{dt}E(t) = -(g(u'(t)), u'(t)).$$

□

4 Uniqueness of solution for (u_0, u_1) given

In this section we suppose that

$$\forall R \in \mathrm{I\!R}^+, \forall (u,v) \in B_R \times B_R, \quad \|g(u) - g(v)\| \leq k_3(R)\|u - v\|, \tag{4.1}$$

for some positive constant $k_3(R)$.

Theorem 4.1. *Let $(u_0, u_1) \in H \times H$, J an interval of $\mathbb{R}$ and $t_0 \in J$. Then (1.1) has at most one solution*

$$u \in \mathcal{C}^1(J,H), \quad \|u'\|^l u' \in \mathcal{C}^1(J,H) \quad \textit{and} \;\; u(t_0) = u_0, \;\; u'(t_0) = u_1.$$

Remark 4.2. *The uniqueness of solutions of (1.1) will be proved under conditions on the initial data (u_0, u_1). The next proposition concerns the uniqueness result for $u_1 \neq 0$.*

Proposition 4.3. *Let $\tau \in \mathbb{R}^+$ and $J = (\tau, T)$, $T > \tau$. Then there is at most one solution of (1.1) with $u(\tau) = u_0$ and $u'(\tau) = u_1$ for $T - \tau$ small enough such that*

$$u \in \mathcal{C}^1(J,H), \quad \textit{and} \quad \|u'\|^l u' \in \mathcal{C}^1(J,H).$$

Proof. Since $u'(\tau) \neq 0$, the second derivative $u''(\tau)$ exists and $u''(t)$ also exists for $\tau \leq t < \tau + \varepsilon$ with ε small enough.
On $(T, T+\varepsilon)$, (1.1) reduces to

$$u'' = l\|u'\|^{2(\frac{l}{2}-1)} u' \frac{\|A^{\frac{1}{2}}u\|^\beta (Au, u') + (g(u'), u')}{(l+1)\|u'\|^l \|u'\|^l} - \frac{\|A^{\frac{1}{2}}u\|^\beta Au + g(u')}{\|u'\|^l},$$

the existence and uniqueness of u in the class $\mathcal{C}^2(J,H)$ for this equation is classical. $\square$

Proposition 4.4. *Let $a \neq 0$. Then for J an interval containing 0 and such that $|J|$ is small enough, equation (1.1) has at most one solution satisfying*

$$u \in \mathcal{C}^1(J,H), \quad \|u'\|^l u' \in \mathcal{C}^1(J,H) \quad \textit{and} \;\; u_0 = a, \;\; u_1 = 0.$$

Proof. From (1.1), we obtain

$$(\|u'\|^l u')'(0) = -\|A^{\frac{1}{2}}a\|^\beta Aa = \xi,$$

where $\|\xi\| \neq 0$.
We set $\|u'\|^l u' = \psi(t)$, then $\psi(0) = 0$. For any $t \neq 0$ we have

$$\frac{\psi(t)}{t} = \frac{1}{t}\int_0^t \psi'(s)\, ds.$$

It follows that $\dfrac{\psi(t)}{t} \to \xi$ as $t \to 0$, therefore for $|t|$ small enough we have,

$$\|\psi(t)\| \geq \frac{t}{2}\|\xi\|.$$

Hence, $\|u'\|^l \geq \eta t^{\frac{l}{l+1}}$ for $|t|$ small enough and some $\eta > 0$.

Integrating (1.1) over $(0,t)$, we have since $u'(0) = 0$

$$\|u'(t)\|^l u'(t) = -\int_0^t \|A^{\frac{1}{2}} u(\tau)\|^\beta A u(\tau)\, d\tau - \int_0^t g(u'(\tau))\, d\tau.$$

Let $u(t)$ and $v(t)$ be two solutions, then $w(t) = u(t) - v(t)$ satisfies

$$\begin{aligned} \|u'(t)\|^l u'(t) - \|v'(t)\|^l v'(t) = &-\int_0^t (\|A^{\frac{1}{2}} u(\tau)\|^\beta A u(\tau) - \|A^{\frac{1}{2}} v(\tau)\|^\beta A v(\tau))\, d\tau \\ &- \int_0^t (g(u'(\tau)) - g(v'(\tau)))\, d\tau. \end{aligned} \tag{4.2}$$

Applying Corollary 2.4 with $f(s) = s^l$ and $p = l$, we get

$$\|\|u'(t)\|^l u'(t) - \|v'(t)\|^l v'(t)\| \geq \eta t^{\frac{l}{l+1}} \|u'(t) - v'(t)\|,$$

and applying Lemma 2.5, (4.1) and from (4.2), we now deduce

$$\begin{aligned} \|w'(t)\| &\leq \frac{C}{t^{\frac{l}{l+1}}} \int_0^t \int_0^\tau \|w'(s)\|\, ds\, d\tau + \frac{C}{t^{\frac{l}{l+1}}} \int_0^t \|w'(\tau)\|\, d\tau \\ &\leq C(T) t^{-\frac{l}{l+1}} \int_0^t \|w'(\tau)\|\, d\tau. \end{aligned} \tag{4.3}$$

Setting $\phi(t) = \displaystyle\int_0^t \|w'(\tau)\|\, d\tau$, by solving (4.3) on $[\delta, t]$ we obtain

$$\phi(t) \leq \phi(\delta) e^{C(T) \int_\delta^t s^{-l/(l+1)}\, ds},$$

and by letting $\delta \to 0$ we conclude that $w(t) = 0$. A similar argument gives the uniqueness for t negative with $|t|$ small enough. □

Proposition 4.5. *For any interval J and any $t_0 \in J$ if a solution u of (1.1) satisfies*

$$u \in \mathcal{C}^1(J, H), \quad \|u'\|^l u' \in \mathcal{C}^1(J, H) \quad \text{and} \quad u(t_0) = u'(t_0) = 0,$$

then $u \equiv 0$.

Proof. From theorem 2.3 we know that

$$\frac{d}{dt} E(t) = -(g(u'(t)), u'(t)).$$

Using (4.1), we have

$$|\frac{d}{dt} E(t)| = |(g(u'(t)), u'(t))| \leq k_3 \|u'\|^{l+2} \leq k_3 E(t).$$

Now, let $t_0 \in J$ such that $E(t_0) = 0$. By integration we get

$$|E(t)| \leq |E(t_0)| e^{k_3 |t - t_0|} = 0.$$

□

5 Energy estimates for equation (1.1)

In this section, we suppose that

$$\exists \eta_1 > 0, \quad \forall v, \ \ \|g(v)\| \le \eta_1 \|v\|^{\alpha+1}, \tag{5.1}$$

and

$$\exists \eta_2 > 0, \quad \forall v, \ \ (g(v), v) \ge \eta_2 \|v\|^{\alpha+2}, \tag{5.2}$$

for some $\alpha > 0$.

Theorem 5.1. *Assuming $\alpha > l$, there exists a positive constant η such that if u is any solution of (1.1) with $E(0) \neq 0$*

$$\liminf_{t \to +\infty} t^{\frac{l+2}{\alpha-l}} E(t) \ge \eta. \tag{5.3}$$

(i) If $\alpha \ge \frac{\beta(1+l)+l}{\beta+2}$, then there is a constant $C(E(0))$ depending on $E(0)$ such that

$$\forall t \ge 1, \quad E(t) \le C(E(0)) t^{-\frac{l+2}{\alpha-l}}.$$

(ii) If $\alpha < \frac{\beta(1+l)+l}{\beta+2}$, then there is a constant $C(E(0))$ depending on $E(0)$ such that

$$\forall t \ge 1, \quad E(t) \le C(E(0)) t^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}.$$

Proof. From the definition of $E(t)$ we have

$$\|u'(t)\|^{\alpha+2} \le C(l, \alpha) E(t)^{\frac{\alpha+2}{l+2}},$$

where $C(l, \alpha)$ is a positive constant, hence from (3.9) and (5.1) we deduce

$$\frac{d}{dt} E(t) \ge -C(l, \alpha, \eta_1) E(t)^{\frac{\alpha+2}{l+2}}.$$

Assuming $\alpha > l$ we derive

$$\begin{aligned} \frac{d}{dt} E(t)^{-\frac{\alpha-l}{l+2}} &= -\frac{\alpha-l}{l+2} E'(t) E(t)^{-\frac{\alpha+2}{l+2}} \\ &\le \frac{\alpha-l}{l+2} C(l, \alpha, \eta_1) = C_1. \end{aligned}$$

By integrating, we get

$$E(t) \ge (E(0)^{-\frac{\alpha-l}{l+2}} + C_1 t)^{-\frac{l+2}{\alpha-l}},$$

implying

$$\liminf_{t \to +\infty} t^{\frac{l+2}{\alpha-l}} E(t) \ge \eta = C_1^{-\frac{l+2}{\alpha-l}}.$$

Hence (5.3) is proved. Now, we show (i) and (ii), we consider the perturbed energy function

$$E_\varepsilon(t) = E(t) + \varepsilon(\|u\|^{2\gamma} u, \|u'\|^l u'), \tag{5.4}$$

where $l > 0$, $\gamma > 0$ and $\varepsilon > 0$.
By Young's inequality, with the conjugate exponents $l+2$ and $\dfrac{l+2}{l+1}$, we get

$$|(\|u\|^{2\gamma}u, \|u'\|^{l}u')| \leq C_1\|A^{\frac{1}{2}}u\|^{(2\gamma+1)(l+2)} + c_2\|u'\|^{l+2}.$$

We choose γ so that $(2\gamma+1)(l+2) \geq \beta+2$, which reduces to

$$\gamma \geq \frac{\beta - l}{2(l+2)}. \tag{5.5}$$

Then, for some $C_1 > 0, M > 0$, we have

$$\begin{aligned} |(\|u\|^{2\gamma}u, \|u'\|^{l}u')| &\leq C_1\|A^{\frac{1}{2}}u\|^{\beta+2} + c_2\|u'\|^{l+2} \\ &\leq ME(t). \end{aligned} \tag{5.6}$$

By using (5.6), we obtain from (5.4)

$$(1 - M\varepsilon)E(t) \leq E_\varepsilon(t) \leq (1 + M\varepsilon)E(t).$$

Taking $\varepsilon \leq \dfrac{1}{2M}$, we deduce

$$\forall t \geq 0, \ \ \frac{1}{2}E(t) \leq E_\varepsilon(t) \leq 2E(t). \tag{5.7}$$

On the other hand, we have

$$E'_\varepsilon(t) = -(g(u'), u') + \varepsilon((\|u\|^{2\gamma}u)', \|u'\|^{l}u') + \varepsilon(\|u\|^{2\gamma}u, (\|u'\|^{l}u')').$$

We observe that

$$\begin{aligned} (\|u\|^{2\gamma}u)' &= ((\|u\|^2)^{\gamma})'u + \|u\|^{2\gamma}u' \\ &= 2\gamma\|u\|^{2(\gamma-1)}(u, u')u + \|u\|^{2\gamma}u', \end{aligned}$$

and we have

$$\begin{aligned} ((\|u\|^{2\gamma}u)', \|u'\|^{l}u') &= 2\gamma\|u\|^{2(\gamma-1)}\|u'\|^{l}(u, u)(u', u') + (\|u\|^{2\gamma}u', \|u'\|^{l}u') \\ &= 2\gamma\|u\|^{2\gamma}\|u'\|^{l+2} + (\|u\|^{2\gamma}u', \|u'\|^{l}u'). \end{aligned}$$

Then, we can deduce that

$$\begin{aligned} E'_\varepsilon(t) &= -(g(u'), u') + \varepsilon 2\gamma\|u\|^{2\gamma}\|u'\|^{l+2} + \varepsilon(\|u\|^{2\gamma}u', \|u'\|^{l}u') \\ &\quad - \varepsilon(\|u\|^{2\gamma}u, \|A^{\frac{1}{2}}u\|^{\beta}Au) - \varepsilon(\|u\|^{2\gamma}u, g(u')). \end{aligned} \tag{5.8}$$

We now estimate the right side of (5.8),
The fourth term:

$$\begin{aligned} -\varepsilon(\|u\|^{2\gamma}u, \|A^{\frac{1}{2}}u\|^{\beta}Au) &= -\varepsilon\|u\|^{2\gamma}\|A^{\frac{1}{2}}u\|^{\beta}(u, Au) \\ &= -\varepsilon\|u\|^{2\gamma}\|A^{\frac{1}{2}}u\|^{\beta+2} \\ &\leq -c\varepsilon\|A^{\frac{1}{2}}u\|^{2\gamma+\beta+2}. \end{aligned} \tag{5.9}$$

The second term:

$$\|u\|^{2\gamma}\|u'\|^{l+2} \leq C_2\|A^{\frac{1}{2}}u\|^{2\gamma}\|u'\|^{l+2}.$$

The third term:

$$|(\|u\|^{2\gamma}u', \|u'\|^l u')| = \|u\|^{2\gamma}\|u'\|^l(u', u') \leq C_3\|A^{\frac{1}{2}}u\|^{2\gamma}\|u'\|^{l+2}.$$

Applying Young's inequality, with the conjugate exponents $\frac{\alpha+2}{\alpha-l}$ and $\frac{\alpha+2}{l+2}$, we have

$$\|A^{\frac{1}{2}}u\|^{2\gamma}\|u'\|^{l+2} \leq \delta\|A^{\frac{1}{2}}u\|^{2\gamma\frac{\alpha+2}{\alpha-l}} + C(\delta)\|u'\|^{(l+2)\frac{\alpha+2}{l+2}}.$$

We assume

$$\frac{(\alpha+2)2\gamma}{\alpha-l} \geq 2\gamma+\beta+2,$$

which reduces to the condition

$$\gamma \geq \frac{(\beta+2)(\alpha-l)}{2(l+2)}, \tag{5.10}$$

and taking δ small enough, we have for some $P > 0$, therefore the second and the third terms becomes

$$\begin{aligned} \varepsilon 2\gamma\|u\|^{2\gamma}\|u'\|^{l+2} + \varepsilon(\|u\|^{2\gamma}u', \|u'\|^l u') &\leq \varepsilon C(2\gamma+1)\|A^{\frac{1}{2}}u\|^{2\gamma}\|u'\|^{l+2} \\ &\leq \frac{\varepsilon}{4}\|A^{\frac{1}{2}}u\|^{2\gamma+\beta+2} + \varepsilon P\|u'\|^{\alpha+2}. \end{aligned} \tag{5.11}$$

Using (5.2), (5.8), (5.9) and (5.11), we have

$$E'_\varepsilon(t) \leq (-\eta_2+\varepsilon P)\|u'\|^{\alpha+2} - \varepsilon\|A^{\frac{1}{2}}u\|^{2\gamma+\beta+2} + \frac{\varepsilon}{4}\|A^{\frac{1}{2}}u\|^{2\gamma+\beta+2} - \varepsilon(\|u\|^{2\gamma}u, g(u')). \tag{5.12}$$

Applying Young's inequality, with the conjugate exponents $\alpha+2$ and $\frac{\alpha+2}{\alpha+1}$ and using (5.1), we have

$$\begin{aligned} -(\|u\|^{2\gamma}u, g(u')) &\leq \delta(\|u\|^{(2\gamma+1)(\alpha+2)} + C'(\delta)\|g(u')\|^{\frac{\alpha+2}{\alpha+1}} \\ &\leq C_4\delta\|A^{\frac{1}{2}}u\|^{(2\gamma+1)(\alpha+2)} + C'(\delta)\|u'\|^{\alpha+2}. \end{aligned}$$

This term will be dominated by the negative terms assuming

$$(2\gamma+1)(\alpha+2) \geq 2\gamma+\beta+2 \Leftrightarrow (\alpha+1)(2\gamma+1) \geq \beta+1.$$

This is equivalent to the condition

$$\gamma \geq \frac{\beta-\alpha}{2(\alpha+1)}, \tag{5.13}$$

and taking δ small enough, we have

$$-\varepsilon(\|u\|^{2\gamma}u, g(u')) \leq \frac{\varepsilon}{4}\|A^{\frac{1}{2}}u\|^{2\gamma+\beta+2} + P'\varepsilon\|u'\|^{\alpha+2}.$$

By replacing in (5.12), we have

$$E'_\varepsilon(t) \leq (-\eta_2 + Q\varepsilon)\|u'\|^{\alpha+2} - \frac{\varepsilon}{2}\|A^{\frac{1}{2}}u\|^{2\gamma+\beta+2},$$

where $Q = P + P'$. By choosing ε small, we get

$$\begin{aligned} E'_\varepsilon(t) &\leq -\frac{\varepsilon}{2}\Big(\|u'\|^{\alpha+2} + \|A^{\frac{1}{2}}u\|^{2\gamma+\beta+2}\Big) \\ &\leq -\frac{\varepsilon}{2}\Big((\|u'\|^{l+2})^{\frac{\alpha+2}{l+2}} + (\|A^{\frac{1}{2}}u\|^{\beta+2})^{\frac{2\gamma+\beta+2}{\beta+2}}\Big). \end{aligned} \tag{5.14}$$

This inequality will be satisfied under the assumptions (5.5), (5.10) and (5.13) which lead to the sufficient condition

$$\gamma \geq \gamma_0 = \max\Big\{\frac{\beta-l}{2(l+2)}, \frac{(\beta+2)(\alpha-l)}{2(l+2)}, \frac{\beta-\alpha}{2(\alpha+1)}\Big\}. \tag{5.15}$$

We now distinguish 2 cases.

(i) If $\alpha \geq \dfrac{\beta(1+l)+l}{\beta+2}$, then clearly $\dfrac{(\beta+2)(\alpha-l)}{2(l+2)} \geq \dfrac{\beta-l}{2(l+2)}$.

Moreover

$$\frac{\beta-\alpha}{2(\alpha+1)} = \frac{1}{2}\left(\frac{\beta+1}{\alpha+1} - 1\right) \leq \frac{1}{2}\left(\frac{\beta+1}{\frac{\beta(1+l)+l}{\beta+2}+1} - 1\right) = \frac{\beta-l}{2(l+2)}.$$

In this case $\gamma_0 = \dfrac{(\beta+2)(\alpha-l)}{2(l+2)}$ and choosing $\gamma = \gamma_0$, we find

$$2\gamma + \beta + 2 = \frac{\alpha+2}{l+2}(\beta+2),$$

since $\dfrac{2\gamma+\beta+2}{\beta+2} = 1 + \dfrac{\alpha-l}{l+2}$, replacing in (5.14), we obtain for some $\rho > 0$

$$E'_\varepsilon(t) \leq -\rho E(t)^{1+\frac{\alpha-l}{l+2}} \leq -\rho' E_\varepsilon(t)^{1+\frac{\alpha-l}{l+2}},$$

where ρ and ρ' are positive constants.

(ii) If $\alpha < \dfrac{\beta(l+1)+l}{\beta+2}$, then $\dfrac{(\beta+2)(\alpha-l)}{l+2} < \dfrac{\beta-l}{l+2}$.

Moreover

$$\begin{aligned} \frac{\beta-\alpha}{2(\alpha+1)} - \frac{\beta-l}{2(l+2)} &= \frac{(\beta-\alpha)(l+2)-(\beta-l)(\alpha+1)}{2(\alpha+1)(l+2)} \\ &= \frac{\beta(l+1)+l-\alpha(\beta+2)}{2(\alpha+1)(l+2)} > 0. \end{aligned}$$

In this case $\gamma_0 = \dfrac{\beta-\alpha}{2(\alpha+1)}$ and choosing $\gamma = \gamma_0$, we find

$$2\gamma + \beta + 2 = (\beta+2)\left(1 + \frac{2\gamma}{\beta+2}\right) = (\beta+2)\left(1 + \frac{\beta-\alpha}{(\alpha+1)(\beta+2)}\right), \tag{5.16}$$

since $\gamma > \dfrac{(\beta+2)(\alpha-l)}{l+2}$, we have

$$\frac{2\gamma+\beta+2}{\beta+2} = 1 + \frac{2\gamma}{\beta+2} > 1 + \frac{\alpha-l}{l+2} = \frac{\alpha+2}{l+2},$$

replacing in (5.14), we obtain

$$E'_\varepsilon(t) \leq -\delta\varepsilon(\|u'\|^{l+2} + \|A^{\frac{1}{2}}u\|^{\beta+2})^{\frac{2\gamma+\beta+2}{\beta+2}},$$

for some $\delta > 0$. Using (5.16), we have

$$E'_\varepsilon(t) \leq -\rho E^{(1+\frac{\beta-\alpha}{(\alpha+1)(\beta+2)})} \leq -\rho' E_\varepsilon(t)^{(1+\frac{\beta-\alpha}{(\alpha+1)(\beta+2)})},$$

where ρ and ρ' are positive constants.

□

6 Existence of slow and fast solutions for equation (1.1)

In the case where $\alpha < \frac{\beta(1+l)+l}{\beta+2}$, Theorem 5.1 gives two different decay rates for the lower and the upper estimates of the energy. In the scalar case, this fact was explained in [1] by the existence of two (and only two) different decay rates of the solutions, corresponding precisely to the lower and upper estimates. The solutions behaving as the lower estimate were called "fast solutions" and those behaving as the upper estimate were called "slow solutions." Moreover in the scalar case, it was shown that the set of initial data giving rise to "slow solutions" has non-empty interior in the phase space $\mathbb{R}^2$.

In the general case, by reducing the problem to a related scalar equation, it is rather immediate to show the coexistence of slow and fast solutions in the special case of power nonlinearities. More precisely we have

Proposition 6.1. *Let $\alpha < \frac{\beta(1+l)+l}{\beta+2}$ and $c > 0$. Then the equation*

$$(\|u'\|^l u')' + \|A^{\frac{1}{2}}u\|^\beta Au + c\|u'\|^\alpha u' = 0$$

has an infinity of "fast solutions" with energy comparable to $t^{-\frac{l+2}{\alpha-l}}$ and an infinity of "slow solutions" with energy comparable to $t^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}$ as t tends to infinity.

Proof. Let $\lambda > 0$ be any eigenvalue of A and $A\varphi = \lambda\varphi$ with $\|\varphi\| = 1$. Let $(v_0, v_1) \in \mathbb{R}^2$ and v be the solution of

$$(|v'|^l v')' + C_1|v|^\beta v + C_2|v'|^\alpha v' = 0,$$

where C_1, C_2 are positive constants to be chosen later. Then $u(t) = v(t)\varphi$ satisfies

$$(\|u'\|^l u')' + \|A^{\frac{1}{2}}u\|^\beta Au + c\|u'\|^\alpha u' = (|v'|^l v')'\|\varphi\|^l\varphi + |v|^\beta v\|A^{\frac{1}{2}}\varphi\|^\beta A\varphi + c|v'|^\alpha v'\|\varphi\|^\alpha\varphi$$

$$= [(|v'|^l v')' + \lambda^{\frac{\beta}{2}+1}|v|^\beta v + c|v'|^\alpha v']\varphi.$$

Choosing $C_1 = \lambda^{\frac{\beta}{2}+1}$ and $C_2 = c$, $u(t) = v(t)\varphi$ becomes a solution of the vector equation. The existence of an infinity of "fast solutions" and an infinity of "slow solutions" are then an immediate consequence of the same result for the scalar equation proven in **[1]**. $\square$

Remark 6.2. In the special case $A = \lambda I$, we can take for φ any vector of the sphere $\|\varphi\| = 1$. We obtain in this way an open set of slow solutions in $\mathbb{R}^{N+1}$ corresponding to the initial data of the form $(v_0\varphi, v_1\varphi)$. The dimension of the set of slow solutions obtained by this procedure is $N - 1 + 2 = N + 1$ and that of the set of fast solutions is N. It is natural to conjecture that the set of all slow solutions has dimension $2N$ and the set of all fast solutions has dimension $2N - 1$. The conjecture on fast solutions seems delicate to prove as well as the alternative (existence of only two decay rates) remains to prove even for $A = I$. On the other hand, by generalizing a modified energy method introduced in [3] , we can prove the existence of an open set of slow solutions. This is the object of the last result of this paper.

Theorem 6.3 (Existence of slow solutions)**.** *Assume that g satisfies (5.2) and $l < \alpha < \frac{\beta(1+l)+l}{\beta+2}$. Then, there exist a nonempty open set $\mathcal{S} \subset H \times H$ and a constant M such that, for every $(u_0, u_1) \in \mathcal{S}$, the unique global solution of equation (1.1) with initial data (u_0, u_1) satisfies*

$$\|u(t)\| \geq \frac{M}{(1+t)^{\frac{\alpha+1}{\beta-\alpha}}} \quad \forall t \geq 0. \tag{6.1}$$

Proof. Assuming $(u_0, u_1) \in H \times H$ and $u_0 \neq 0$, we consider the following constants

$$\sigma_0 := \left(\frac{(\beta+2)(l+1)}{l+2}\|u_1\|^{l+2} + \|A^{\frac{1}{2}}u_0\|^{\beta+2}\right)^{\frac{1}{\beta+2}},$$

and

$$\sigma_1 := \frac{\|u_1\|^2}{\|A^{\frac{1}{2}}u_0\|^{2(\frac{\beta-\alpha}{\alpha+1}+1)}}.$$

For any $\varepsilon_0 > 0$, $\varepsilon_1 > 0$, the set $\mathcal{S} \subset H \times H$ of initial data such that $\sigma_0 < \varepsilon_0$, $\sigma_1 < \varepsilon_1$ is clearly a nonempty open set which contains at least all pairs (u_0, u_1) with $u_1 = 0$ and $u_0 \neq 0$ and $\|u_0\|$ small enough. We claim that if ε_0 and ε_1 are small enough, for any $(u_0, u_1) \in \mathcal{S}$, the global solution of (1.1) satisfies (6.1).

First of all, from (3.9) and (5.2), we see that

$$\frac{d}{dt}E(t) \leq -\eta_2 \left\|u'(t)\right\|^{\alpha+2} < 0,$$

hence $E(t) \leq E(0)$ for every $t \geq 0$, and from (1.2), we deduce that

$$\|A^{\frac{1}{2}}u(t)\| \leq [(\beta+2)E(0)]^{\frac{1}{\beta+2}} = \sigma_0 \quad \forall t \geq 0. \tag{6.2}$$

Then, we shall establish that if ε_0 and ε_1 are small enough, we have

$$u(t) \neq 0 \quad \forall t \geq 0, \tag{6.3}$$

and for some $C > 0$

$$\frac{\|u'(t)\|^2}{\|A^{\frac{1}{2}}u(t)\|^{2(\frac{\beta-\alpha}{\alpha+1}+1)}} \leq C \quad \forall t \geq 0. \tag{6.4}$$

Assuming this inequality, it follows immediately that u is a slow solution. Indeed let us set $y(t) := \|u(t)\|^2$. We observe that

$$\begin{aligned} \left|y'(t)\right| &= 2\left|\left(u'(t), u(t)\right)\right| \leq 2\left\|u'(t)\right\| \cdot \|u(t)\| \\ &\leq 2\frac{\|u'(t)\|}{\|A^{\frac{1}{2}}u(t)\|^{\frac{\beta-\alpha}{\alpha+1}+1}} \cdot \|A^{\frac{1}{2}}u(t)\|^{\frac{\beta-\alpha}{\alpha+1}+1}\|u(t)\| \\ &\leq 2\sqrt{C}\|u(t)\|^{\frac{\beta-\alpha}{\alpha+1}+2} \leq 2\sqrt{C}\,|y(t)|^{\frac{\beta-\alpha}{2(\alpha+1)}+1}, \end{aligned}$$

and in particular

$$y'(t) \geq -2\sqrt{C}\,|y(t)|^{\frac{\beta-\alpha}{2(\alpha+1)}+1} \quad \forall t \geq 0.$$

Taking into account that

$$\left(y(t)^{-\frac{\beta-\alpha}{2(\alpha+1)}}\right)' = -\frac{\beta-\alpha}{2(\alpha+1)}y(t)^{-\frac{\beta-\alpha}{2(\alpha+1)}-1}y'(t),$$

we have

$$\left(y(t)^{-\frac{\beta-\alpha}{2(\alpha+1)}}\right)' \leq \sqrt{C}\frac{\beta-\alpha}{\alpha+1}.$$

Integrating between 0 and t and since $y(0) > 0$, we deduce that there exists a constant M_1 such that

$$y(t)^{-\frac{\beta-\alpha}{2(\alpha+1)}} \leq M_1(t+1) \quad \forall t \geq 0.$$

This inequality concludes the proof. So we are left to prove (6.3) and (6.4).

Let us set

$$T := \sup\{t \geq 0 : \forall \tau \in [0,t], \quad u(\tau) \neq 0\}.$$

Since $u(0) \neq 0$, we have that $T > 0$ and if $T < +\infty$, then $u(T) = 0$.

Let us consider, for all $t \in [0,T)$, the energy

$$H(t) := \frac{\|u'(t)\|^2}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}}, \tag{6.5}$$

where $\gamma := \dfrac{\beta-\alpha}{\alpha+1} + 1$.

We differentiate H,

$$H'(t) = \frac{\frac{d}{dt}\left(\left\|u'(t)\right\|^{2+l}\right)}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}\left\|u'(t)\right\|^{l}} - \frac{\frac{d}{dt}\left(\|A^{\frac{1}{2}}u(t)\|^{2\gamma}\left\|u'(t)\right\|^{l}\right)}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}\left\|u'(t)\right\|^{l}}H(t).$$

Taking into account (1.2) and (3.9), we deduce

$$\begin{aligned} H'(t) &= -\rho\frac{(g(u'(t)), u'(t))}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}\left\|u'(t)\right\|^{l}} - \rho\frac{\|A^{\frac{1}{2}}u(t)\|^{\beta-2\gamma}\left(u'(t), Au(t)\right)}{\left\|u'(t)\right\|^{l}} \\ &\quad -\frac{\frac{d}{dt}\left(\|A^{\frac{1}{2}}u(t)\|^{2\gamma}\left\|u'(t)\right\|^{l}\right)}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}\left\|u'(t)\right\|^{l}}H(t) =: H_1 + H_2 + H_3, \end{aligned} \tag{6.6}$$

where $\rho := \dfrac{l+2}{l+1}$.

Let us estimate H_1, H_2 and H_3. Using (5.2) and (6.5), we observe that

$$\begin{aligned} H_1 &\leq -\rho\,\eta_2\frac{\|u'(t)\|^{\alpha+2-l}}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}} = -\rho\,\eta_2\left(\frac{\|u'(t)\|^{2}}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}}\right)^{\frac{\alpha+2-l}{2}}\frac{\|A^{\frac{1}{2}}u(t)\|^{\gamma(\alpha+2-l)}}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}} \\ &= -\rho\,\eta_2 H^{\frac{\alpha+2-l}{2}}(t)\|A^{\frac{1}{2}}u(t)\|^{\gamma(\alpha-l)}. \end{aligned} \tag{6.7}$$

In order to estimate H_2, we use Young's inequality applied with the conjugate exponents $\frac{\alpha+2-l}{1-l}$ and $\frac{\alpha+2-l}{\alpha+1}$,

$$\begin{aligned} |H_2| &\leq \rho\left\|u'(t)\right\|^{1-l}\cdot\frac{\|A^{\frac{1}{2}}u(t)\|^{\beta+1}}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}} = \rho\frac{\left\|u'(t)\right\|^{1-l}}{\|A^{\frac{1}{2}}u(t)\|^{\frac{2\gamma(1-l)}{\alpha+2-l}}}\cdot\frac{\|A^{\frac{1}{2}}u(t)\|^{\frac{2\gamma(1-l)}{\alpha+2-l}+\beta+1}}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}} \\ &\leq \delta_1\left(\frac{\|u'(t)\|^{1-l}}{\|A^{\frac{1}{2}}u(t)\|^{\frac{2\gamma(1-l)}{\alpha+2-l}}}\right)^{\frac{\alpha+2-l}{1-l}} + \delta_2\left(\|A^{\frac{1}{2}}u(t)\|^{\frac{2\gamma(1-l)}{\alpha+2-l}+\beta+1-2\gamma}\right)^{\frac{\alpha+2-l}{\alpha+1}}, \end{aligned}$$

where $\delta_1 := \rho\,\eta_2\left(\dfrac{1-l}{\alpha+2-l}\right)$ and $\delta_2 := \dfrac{\rho}{\eta_2}\left(\dfrac{\alpha+1}{\alpha+2-l}\right)$, and taking into account that $\gamma = \frac{\beta+1}{\alpha+1}$, we deduce

$$\begin{aligned} |H_2| &\leq \delta_1\left(\frac{\|u'(t)\|^{2}}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}}\right)^{\frac{\alpha+2-l}{2}}\frac{\|A^{\frac{1}{2}}u(t)\|^{\gamma(\alpha+2-l)}}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}} + \delta_2\|A^{\frac{1}{2}}u(t)\|^{\gamma(\alpha-l)} \\ &= \delta_1 H^{\frac{\alpha+2-l}{2}}(t)\|A^{\frac{1}{2}}u(t)\|^{\gamma(\alpha-l)} + \delta_2\|A^{\frac{1}{2}}u(t)\|^{\gamma(\alpha-l)}. \end{aligned} \tag{6.8}$$

For simplicity in the sequel we shall write

$$l/2 := m > 0.$$

In order to estimate H_3, we use (6.5) and we have that

$$
\begin{aligned}
H_3 &= -\frac{\frac{d}{dt}\left(\|A^{\frac{1}{2}}u(t)\|^{2\gamma(1+m)}H(t)^m\right)}{\|A^{\frac{1}{2}}u(t)\|^{2\gamma}\|u'(t)\|^l}H(t) = -mH(t)^m H'(t)\frac{\|A^{\frac{1}{2}}u(t)\|^{2\gamma m}}{\|u'(t)\|^l} \\
&\quad -2\gamma(1+m)H(t)^{m+1}\frac{\|A^{\frac{1}{2}}u(t)\|^{2(\gamma m-1)}}{\|u'(t)\|^l}\big(u'(t), Au(t)\big) =: H_4 + H_5.
\end{aligned}
$$

We observe that taking into account (6.5), we have

$$
H_4 = -mH'(t), \tag{6.9}
$$

and

$$
|H_5| \le 2\gamma(1+m)H(t)^{m+1}\frac{\|A^{\frac{1}{2}}u(t)\|^{2\gamma m}}{\|u'(t)\|^l}\cdot\frac{\|u'(t)\|}{\|A^{\frac{1}{2}}u(t)\|} = 2\gamma(1+m)H(t)^{\frac{3}{2}}\|A^{\frac{1}{2}}u(t)\|^{\gamma-1}.
$$

We observe

$$
\gamma - 1 > \gamma(\alpha - l) \longleftrightarrow \beta - \alpha > (\beta+1)(\alpha - l),
$$

which reduces to

$$
\alpha < \frac{\beta(1+l)+l}{\beta+2},
$$

and taking into account (6.2), we have

$$
|H_5| \le 2\gamma(1+m)\sigma_0^{\gamma-1-\gamma(\alpha-l)}H(t)^{\frac{3}{2}}\|A^{\frac{1}{2}}u(t)\|^{\gamma(\alpha-l)}. \tag{6.10}
$$

Then, taking into account (6.7)-(6.10) in (6.6), we deduce that

$$
H'(t) \;\le\; \|A^{\frac{1}{2}}u(t)\|^{\gamma(\alpha-l)}\left[-\widehat{\rho}\,\eta_2 H^{\frac{\alpha+2-l}{2}}(t) + \frac{\widehat{\rho}}{\eta_2} + 2\gamma\sigma_0^{\gamma-1-\gamma(\alpha-l)}H(t)^{\frac{3}{2}}\right], \tag{6.11}
$$

where $\widehat{\rho} := \frac{\rho(\alpha+1)}{(1+m)(\alpha+2-l)}$.

Now, let

$$
h(s,\Gamma) = -\widehat{\rho}\,\eta_2 s^{\frac{\alpha+2-l}{2}} + \frac{\widehat{\rho}}{\eta_2} + 2\gamma\Gamma s^{\frac{3}{2}},
$$

where $\Gamma := \sigma_0^{\gamma-1-\gamma(\alpha-l)}$.

We observe that $h((2/\eta_2^2)^{2/(\alpha+2-l)},\Gamma) = -\dfrac{\widehat{\rho}}{\eta_2} + 2\gamma\Gamma\left(2/\eta_2^2\right)^{\frac{3}{\alpha+2-l}}$. In particular

$$
h\left((2/\eta_2^2)^{2/(\alpha+2-l)},\Gamma\right) \to -\frac{\widehat{\rho}}{\eta_2} < 0, \quad \text{if } \Gamma \to 0.
$$

Let us therefore assume that σ_0 is sufficiently small to achieve

$$
h((2/\eta_2^2)^{2/(\alpha+2-l)}, \sigma_0^{\gamma-1-\gamma(\alpha-l)}) < 0.
$$

We claim that if $H(0) = \sigma_1 < \varepsilon_1 := (2/\eta_2^2)^{2/(\alpha+2-l)}$, then $H(t)$ is bounded for all $t \in (0,T)$. Indeed if H is not bounded for all $t \in (0,T)$, then there exists $\bar{T} \in (0,T)$ such that $H(\bar{T}) = \varepsilon_1$ and $H(t) < \varepsilon_1$ for all $t \in (0,\bar{T})$. By (6.11) and taking into account that $u \in \mathcal{C}^1(\mathbb{R}^+, H)$ and $\|u'\|^l u' \in \mathcal{C}^1(\mathbb{R}^+, H)$, we have

$$H'(\bar{T}) \leq \|A^{\frac{1}{2}}u(\bar{T})\|^{\gamma(\alpha-l)} h(\varepsilon_1, \sigma_0^{\gamma-1-\gamma(\alpha-l)}) \leq \left\|u(\bar{T})\right\|^{\gamma(\alpha-l)} h(\varepsilon_1, \sigma_0^{\gamma-1-\gamma(\alpha-l)}) < 0,$$

then since $\bar{T} \in (0,T)$ implies $\left\|u(\bar{T})\right\| > 0$, H decreases near $\bar{T}$. This contradicts the definition of $\bar{T}$ and we can claim that H is bounded for all $t \in (0,T)$.

Finally, we claim that $T = +\infty$. Let us assume by contradiction that this is not the case. Then, taking into account that H is bounded for all $t \in [0,T)$, we observe that

$$\left\|u'(t)\right\|^2 \leq C\|A^{\frac{1}{2}}u(t)\|^{2\gamma} \leq C\left\|u(t)\right\|^{2\gamma} \quad \forall t \in [0,T).$$

Therefore, from the continuity of the vector (u', u) with values in $H \times H$ it now follows that $u(T) = 0$ implies $u'(T) = 0$, hence $u \equiv 0$ by backward uniqueness, a contradiction. Hence, $T = +\infty$, we obtain (6.3) and (6.4), and the conclusion follows.

□

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