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A Gaussian martingale which is the sum of two independent Gaussian non-semimartingales

Marc Yor

Abstract

In this paper two examples of two independent centered Gaussian processes are given such that at least one of them is not a semimartingale but their sum is a martingale.

Keywords: Martingales; semimartingales; Gaussian processes; Brownian bridges.

AMS MSC 2010: 60G15; 60G44.

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1 Certain mixed Fractional Brownian motions are semimartingales

In his thesis, P. Cheridito [1, 2] obtained the following remarkable result: if $(B_t, t \geq 0)$ and $(B_t^{(H)}, t \geq 0)$ denote two independent Gaussian processes, the first one being a Brownian motion, and the second one a fractional Brownian motion with Hurst parameter $H \in [3/4, 1]$, i.e.,

$$E \left[ B_t^{(H)} \right] = 0 \quad \text{and} \quad E \left[ (B_t^{(H)} - B_s^{(H)})^2 \right] = |t - s|^{2H}, \quad s, t \geq 0,$$

then, for every $\alpha \in \mathbb{R}$, the sum:

$$\Sigma_t^{(H)} = B_t + \alpha B_t^{(H)}, \quad t \geq 0,$$

is a semimartingale with respect to its own natural filtration.

Notice that, for $H = 1$, one has: $B_t^{(1)} = t\xi$, where $\xi$ is a standard Gaussian variable, and consequently, $(\Sigma_t^{(1)}, t \geq 0)$ is a semimartingale in the filtration $B_t^{(\xi)} := \sigma \{ B_s, s \leq t; \xi \}$, made right continuous, hence, a fortiori, with respect to its own filtration. However, for $H \in [3/4, 1]$, $(B_t^{(H)}, t \geq 0)$ has zero quadratic variation, but infinite variation on any time interval, hence it is not a semimartingale with respect to its own filtration, which makes Cheridito’s result remarkable.

Note: Throughout the rest of this paper, when we say that a process $(\Pi_t, t \geq 0)$ is a semimartingale with no further qualification, we mean: semimartingale with respect to its own filtration made right continuous and $P$-complete.

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2 Some related questions

In the light of Cheridito’s result, one may ask the following question:

(*) to give a “simpler” example of a pair of independent centered Gaussian processes, 
\( (X_t, t \geq 0) \) and \( (Y_t, t \geq 0) \), one of which at least is not a semimartingale, but such that 
the sum is a semimartingale.

In Section 3, we shall give an example where \( (X_t, t \geq 0) \) is constructed from a Brownian 
bridge, and is not a semimartingale whereas \( (Y_t, t \geq 0) \) has bounded variation. In Section 
4, pushing the construction of Section 3 one step further, we shall give another example 
of (*), where neither \( (X_t) \) nor \( (Y_t) \) is a semimartingale. For the moment, we simply note that, 
in order to obtain some positive answer to (*), at least one of the Gaussian 
processes \( (X_t) \) or \( (Y_t) \) must have some non-zero quadratic variation, i.e., \( \sum_{t} (\Delta X_t)^2 \) 
does not converge to 0, where \( \tau_n = \{0 = t_0 < t_1 < \cdots < t_{p_n} = 1\} \), \( \Delta X_{t_i} = X_{t_i} - X_{t_{i-1}} \), 
and \( \sup_{\tau_n} (t_i - t_{i-1}) \to 0 \). This assertion follows from the

Lemma 2.1.

(i) Assume that \( X \) and \( Y \) are two independent centered Gaussian processes, and \( \tau \) is 
a subdivision of \([0, 1]\). Then

\[
\max \left( E \left[ \sum_{\tau} |\Delta X_{t_i}| \right] ; E \left[ \sum_{\tau} |\Delta Y_{t_i}| \right] \right) 
\leq E \left[ \sum_{\tau} |\Delta (X + Y)_{t_i}| \right] 
\leq E \left[ \sum_{\tau} |\Delta X_{t_i}| + \sum_{\tau} |\Delta Y_{t_i}| \right].
\]

(ii) If both, \( X \) and \( Y \), have zero quadratic variation and at least one of them has 
infinite variation on a set of positive probability, then \( X + Y \) also enjoys these two 
properties.

Proof. (i) Only the LHS inequality needs to be proven; but this follows from

\[
E \| \Delta (X + Y)_{t_i} \| = \sqrt{\frac{2}{\pi}} \| \Delta X_{t_i} + \Delta Y_{t_i} \|_2 \geq \sqrt{\frac{2}{\pi}} \| \Delta X_{t_i} \|_2 = E \| \Delta X_{t_i} \|.
\]

(ii) It is clear that \( X + Y \) has zero quadratic variation. On the other hand, it follows from 
(i) and our hypothesis in (ii) that

\[
E \left[ \int_0^1 |d(X_s + Y_s)| \right] = \infty.
\]

Now it follows from Fernique’s integrability result for the norms of Gaussian vectors 
that \( \int_0^1 |d(X_s + Y_s)| \) cannot be finite a.s.

3 Brownian bridges and a first solution to (*)

Let \( u > 0 \), and denote by \((\eta_u(t), t \leq u)\) a Brownian bridge of length \( u \), i.e., \( (B_t, t \leq u) \) 
conditioned to be equal to 0 at time \( u \). Recall that it can be realized as \( \eta_u(t) = B_t - \frac{t}{u} B_u \), 
\( \eta_u \) is independent of \( B_u \), and its canonical decomposition is

\[
\eta_u(t) = \beta_t - \int_0^t ds \frac{\eta_u(s)}{u - s}, \quad t \leq u,
\]

where \((\beta_t, t \leq u)\) is a Brownian motion in the filtration \((\mathcal{F}_t^{(u)}, t \leq u)\) of \( \eta_u \). Furthermore, 
there is the following
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**Proposition 3.1.** Let $f \in L^2([0,u])$. Then

(i) The process

$$\int_0^t f(s)d\eta_u(s) = \int_0^t f(s)d\beta_s - \int_0^t ds f(s)\frac{\eta_u(s)}{u-s}$$

is well defined for any $t \leq u$ with

$$\int_0^u f(s)d\eta_u(s) = (L^2 \text{ and a.s.}) \lim_{t \uparrow u} \int_0^t f(s)d\eta_u(s).$$

(ii) $\left(\int_0^t f(s)d\eta_u(s), t \leq u \right)$ is a semimartingale with respect to $(\mathcal{P}^{(u)}_t, t \leq u)$ if and only if

$$\int_0^u ds |f(s)| \frac{1}{\sqrt{u-s}} < \infty.$$ 

**Proof.** (i) The $L^2$ and a.s. convergence results are easily obtained from the representations of $\eta_u$ as $\eta_u(t) = B_t - \frac{t}{u} B_u$.

(ii) The semimartingale property of $\left(\int_0^t f(s)d\eta_u(s), t \leq u \right)$ is clearly equivalent to

$$\int_0^u ds |f(s)| \frac{\eta_u(s)}{u-s} < \infty.$$ 

The arguments developed in the proof of Theorem 3 in Jeulin and Yor [3] show that this is equivalent to

$$\int_0^u ds |f(s)| \frac{1}{\sqrt{u-s}} < \infty.$$ 

In order to give explicit examples for $(\ast)$ in the sequel of this paper, let us point out that for $u \in [0,1]$ and $\alpha \in ]1/2,1]$, the function

$$\psi(s) = \frac{1}{\sqrt{u-s}} |\log(u-s)|^{-\alpha} 1_{(u/2<s<u)}$$

satisfies

$$\int_0^u ds \psi^2(s) < \infty \quad \text{but} \quad \int_0^u ds \psi(s) \frac{1}{\sqrt{u-s}} = \infty.$$ 

To obtain a solution to $(\ast)$, we decompose a Brownian motion $(B_t, t \leq u)$ as

$$B_t = \eta_u(t) + \frac{t}{u} B_u, \quad t \leq u,$$

and we consider $f_* \in L^2([0,u])$ such that

$$\int_0^u ds |f_*(s)| \frac{1}{\sqrt{u-s}} = \infty \quad \text{and} \quad f_*(s) \neq 0 \text{ for every } s.$$ 

Then, taking

$$X_t = \int_0^t f_*(s)d\eta_u(s) \quad \text{and} \quad Y_t = \frac{B_u}{u} \int_0^t f_*(s)ds,$$

we obtain a solution to $(\ast)$ since $X$ and $Y$ are independent and $X_t + Y_t = \int_0^t f_*(s)dB_s$ is a martingale.
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4 A “full” solution to $(\ast)$

Let $u \in [0, 1]$. We shall use the same idea as in Section 3, but twice instead of once, by decomposing first $(B_t, t \leq u)$ into $\eta_u(t) + \frac{t}{1-u}B_u$, and then

$$ (\hat{B}_t \equiv B_{t+} - B_u, \ t \leq 1-u) \quad \text{into} \quad \tilde{\eta}_{1-u}(t) + \frac{t}{1-u}\hat{B}_{1-u}. $$

(4.1)

Next, for $f \in L^2([0, 1])$, we write

$$
\begin{align*}
\int_0^t f(s)dB_s &= \int_0^t f(s)1_{[s \leq u]}dB_s + 1_{(u \lessdot t)}\int_u^t f(s)dB_s \\
&= \int_0^t f(s)1_{[s \leq u]}d\eta_u(s) + \frac{B_u}{u} \int_0^t f(s)1_{[s \leq u]}ds \\
&\quad + 1_{(u \lessdot t)}\int_u^t f(s)d\tilde{\eta}_{1-u}(s-u) + 1_{(u \lessdot t)}\frac{B_1 - B_u}{1-u}\int_u^t f(s)ds.
\end{align*}
$$

We then choose $f_u \in L^2([0, 1])$ such that

$$
\int_0^u |f_u(s)| \frac{ds}{\sqrt{1-s}} = \infty, \quad \int_u^1 |f_u(s)| \frac{ds}{\sqrt{1-s}} = \infty \quad \text{and} \quad f_u(s) \neq 0 \text{ for all } s < 1.
$$

Then

$$
X_t = \int_0^t f_u(s)1_{[s \leq u]}d\eta_u(s) + 1_{(u \lessdot t)}\frac{B_1 - B_u}{1-u}\int_u^t f_u(s)ds
$$

and

$$
Y_t = 1_{(u \lessdot t)}\int_u^t f_u(s)d\tilde{\eta}_{1-u}(s-u) + \frac{B_u}{u}\int_u^t f_u(s)1_{[s \leq u]}ds
$$

are two independent Gaussian processes such that $X_t + Y_t = \int_0^t f_u(s)dB_s$ is a martingale.

Using the semimartingale characterization in part (ii) of Proposition 3.1, it is easily shown that neither $X$ nor $Y$ is a semimartingale. However, we give a few details:

Concerning $(X_t)$, we see that $X_t = \tilde{X}_t$ for $t \leq u$, where $\tilde{X}_t = \int_0^t f_u(s)1_{[s \leq u]}d\eta_u(s)$. Hence the non-semimartingale property of $X$ follows from that of $\tilde{X}$ as discussed in Section 3.

Concerning $(Y_t)$, we have

$$
Y_u = \frac{B_u}{u}\int_0^u f_u(s)ds \quad \text{and} \quad Y_t - Y_u = \int_u^t f_u(s)d\tilde{\eta}_{1-u}(s-u), \quad t \in [u, 1].
$$

Now $Y$, being a Gaussian process, could only be a semimartingale if it were a quasi-semimartingale; see, e.g., Stricker [4]. If

$$
\mathcal{Y}_{u+t} = \sigma\{B_u, \tilde{\eta}_{1-u}(s), s \leq t\}
$$

and $(\bar{\mathcal{T}}^{1-u}_t)$ is the filtration of $\tilde{\eta}_{1-u}$, it follows from the independence of $B_u$ and $\tilde{\eta}_{1-u}$ that for $s < t$:

$$
E[Y_{u+t} - Y_{u+s} | \mathcal{Y}_{u+s}] = E[Y_{u+t} - Y_{u+s} | \bar{\mathcal{T}}^{1-u}_s].
$$

From Section 3 we know that $(Y_t - Y_u)$ is not a $\mathcal{Y}^{1-u}$-quasi-martingale. So it is not a $\bar{\mathcal{T}}^{1-u}$-quasi-martingale. It follows that $(Y_t)$ is not a $\mathcal{Y}$-quasi-martingale and therefore, also not a $\mathcal{Y}$-semimartingale.
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References

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