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## An optimization method for elastic shape matching

### *Une méthode d'optimisation pour l'appariement de formes élastiques*

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#### ABSTRACT

This note addresses the following shape matching problem: given a 'template' shape, numerically described by means of a computational mesh, and a 'target' shape, known only via a signed distance function to its boundary, we aim at deforming iteratively the mesh of the template shape into a computational mesh of the target shape. To achieve this goal, we rely on techniques from shape optimization. Under the sole assumption that both shapes share the same topology, the desired transformation is realized as a sequence of elastic displacements, which are obtained by minimizing an energy functional based on the distance between the two shapes. The proposed method has been implemented in a finite elements setting and numerical examples in two and three dimensions are presented to illustrate its efficiency.

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#### RÉSUMÉ

Dans cette note, nous nous intéressons au problème d'appariement de formes suivant : étant donné une forme de référence, représentée numériquement par un maillage de calcul, et une forme cible, connue seulement par l'intermédiaire de la fonction de distance signée à celle-ci, notre objectif consiste à déformer itérativement le maillage de la forme de référence en un maillage de la forme cible. Pour ce faire, nous nous appuyons sur des techniques d'optimisation de formes. Sous l'hypothèse que les deux formes ont la même topologie, la transformation cherchée s'obtient comme une suite de déplacements élastiques, solutions d'un problème de minimisation d'une énergie basée sur la distance entre les formes. La méthode a été implémentée en deux et trois dimensions d'espace et nous présentons des exemples numériques permettant d'apprécier son efficacité.

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**1. Introduction**

Shape morphing or matching arises in a wide variety of situations in areas from biomedical engineering to computer graphics and scientific computing. Beyond the specific stakes to each particular application, the general issue is to find one transformation from a given ‘template’ shape  $\Omega_0$  into a ‘target’  $\Omega_T$ . Such a transformation may be used as a means to appraise how much  $\Omega_0$  and  $\Omega_T$  differ from one another – for instance in shape retrieval, classification or recognition – or to achieve physically the transformation from  $\Omega_0$  to  $\Omega_T$  (in shape registration, reconstruction, or shape simplification). See for instance [15] and references therein for an overview of several related applications.

Understandably enough, a great deal of work has been devoted to shape matching, and we limit ourselves to mentioning a few approaches. In [5], the authors start by distributing sample points on the contour of both shapes, which will be matched according to their ‘shape context’. They eventually infer a global transformation from this point-to-point correspondence. In the field of computational anatomy, a series of articles (see, e.g., [4,9,11]) have suggested to describe a sought diffeomorphism between  $\Omega_0$  and  $\Omega_T$  as the flow of a velocity field  $v$ , and to cast the search for  $v$  as an optimal control problem. The resulting mapping is used to study features of organs, detect anomalies, etc. More recently, in the field of Computer Graphics, the optimal transport point of view has been used to displace an input tetrahedral mesh onto a given object [13].

This note addresses the following problem: given a ‘template’ shape  $\Omega_0$ , numerically described by means of a (conforming) computational mesh, and a ‘target’ shape  $\Omega_T$ , known only via the signed distance function to its boundary, we aim at deforming (iteratively) the mesh of  $\Omega_0$  into a computational mesh of  $\Omega_T$ . Such a technique could be applied, for instance, to the reconstruction of a computational mesh  $\Omega_T$  from invalid data, to transport quantities of interest from  $\Omega_0$  to  $\Omega_T$ , etc. The precise range of applications we have in mind will be described in a forthcoming, longer article.

To achieve our purpose, we rely on a method that has much in common with that of [2], borrowing techniques from shape optimization, and more generally optimal control. Under the assumption that  $\Omega_0$  and  $\Omega_T$  share the same topology, the desired transformation from  $\Omega_0$  to  $\Omega_T$  is realized as a sequence of elastic displacements, which are obtained by minimizing an energy functional based on the distance between  $\Omega_0$  and  $\Omega_T$ . In doing so, it is expected that the deformation will be easier to achieve in numerical practice, and in particular by limiting the troubles due to mesh tangling.

**2. Presentation of the method**

Let  $\Omega_0, \Omega_T \subset \mathbb{R}^d$ ,  $d = 2, 3$  be respectively ‘template’ and ‘target’ shapes, i.e. bounded Lipschitz domains. We assume that they share the same topology, but are not necessarily close to one another. Our purpose is to map  $\Omega_0$  onto  $\Omega_T$ , which we achieve numerically by deforming a mesh  $\mathcal{T}_0$  of  $\Omega_0$  into one for  $\Omega_T$ . For this purpose, we rely on the shape optimization setting.

*2.1. Shape matching as a shape optimization problem*

The discrepancy between a reference shape  $\Omega$  and a target shape  $\Omega_T$  is measured by the following functional  $J(\Omega)$  of the domain:

$$J(\Omega) = \int_{\Omega} d_{\Omega_T}(x) dx, \tag{1}$$

which involves the Euclidean *signed distance function*  $d_{\Omega_T}$  to  $\Omega_T$ , defined as:

$$\forall x \in \mathbb{R}^d, d_{\Omega_T}(x) = \begin{cases} -d(x, \partial\Omega_T) & \text{if } x \in \Omega_T, \\ 0 & \text{if } x \in \partial\Omega_T, \\ d(x, \partial\Omega_T) & \text{if } x \in \mathbb{R}^d \setminus \bar{\Omega}_T. \end{cases}$$

In the above formula,  $d(\cdot, \partial\Omega_T)$  denotes the usual Euclidean distance function to  $\partial\Omega_T$ .

In order to decrease the value of  $J(\Omega)$ , the domain  $\Omega$  must expand in the regions of the ambient space  $\mathbb{R}^d$ , where  $d_{\Omega_T}$  is negative (that is, in the regions comprised in  $\Omega_T$ ), and to retract in those where it is positive. Note that the functional  $J(\Omega)$  has a unique, global minimizer  $\Omega = \Omega_T$ , and no extra local minimum point provided  $\Omega_T$  is connected. It is then expected that an iterative (e.g., gradient-based) algorithm for minimizing  $J(\Omega)$ , starting from  $\Omega_0$ , will lead to an interesting way to transform  $\Omega_0$  into  $\Omega_T$ .

*2.2. Shape derivative of the functional  $J(\Omega)$*

Several notions of differentiation with respect to the domain are available in the literature. One which is very-well tailored for our purpose is Hadamard’s boundary variation method (see, e.g., [1,6,12,14]), whereby variations of a bounded, Lipschitz domain  $\Omega \subset \mathbb{R}^d$  are considered under the form (see Fig. 1, left):

$$\Omega_{\theta} = (I + \theta)(\Omega), \quad \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d).$$

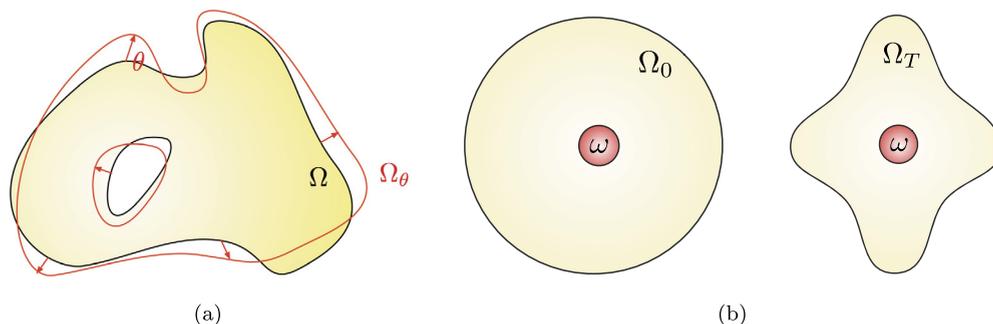


Fig. 1. (a) Variation  $\Omega_\theta$  of a shape  $\Omega$  according to Hadamard's method; (b) Target and template shapes sharing a common fixed subset  $\omega$ .

Accordingly, a function  $F(\Omega)$  of the domain is said to be *shape differentiable* at  $\Omega$  if the mapping  $\theta \mapsto F(\Omega_\theta)$ , from  $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$  into  $\mathbb{R}$ , is Fréchet differentiable at  $\theta = 0$ . The associated Fréchet differential is denoted as  $\theta \mapsto F'(\Omega)(\theta)$  and called the shape derivative of  $F$ ; the following expansion then holds:

$$F(\Omega_\theta) = F(\Omega) + F'(\Omega)(\theta) + o(\theta), \text{ where } \frac{|o(\theta)|}{\|\theta\|_{W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)}} \xrightarrow{\theta \rightarrow 0} 0.$$

By a classical calculation, the shape derivative of the function  $J(\Omega)$  defined in (1) reads:

$$\forall \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \quad J'(\Omega)(\theta) = \int_{\partial\Omega} d_{\Omega_T} \theta \cdot n \, ds.$$

This paves the way for an iterative algorithm, producing a sequence  $(\Omega_k)_{k=0,\dots}$  of shapes (and corresponding meshes  $\mathcal{T}_k$ ), which are ‘closer and closer’ to  $\Omega_T$ : at each step,  $\Omega_k$  is updated according to

$$\Omega_{k+1} = (I + \theta_k)(\Omega_k), \text{ where } \theta_k \text{ is (an extension to } \Omega_k \text{ of) } -d_{\Omega_T} n_{\Omega_k}, \tag{2}$$

and  $n_{\Omega_k}$  stands for the unit normal vector to  $\partial\Omega_k$ , pointing outward  $\Omega_k$ . The mesh  $\mathcal{T}_k$  is updated as:

$$\forall x \text{ vertex of } \mathcal{T}_k, \quad x \mapsto x + \theta_k(x). \tag{3}$$

### 2.3. Parameterization by elastic displacements

The formal procedure summarized in (2) boils down to deforming a shape  $\Omega$  in the negative direction of the  $L^2(\partial\Omega)^d$  gradient of the differential  $\theta \mapsto J'(\Omega)(\theta)$ . Unfortunately, this reveals unsuited when it comes to deforming a mesh  $\mathcal{T}$  of  $\Omega$  by moving its vertices. Indeed, the vector field  $\theta = -d_{\Omega_T} n$  featured in (2) is defined only on the boundary of  $\Omega$ ; it has therefore to be extended to  $\Omega$  as a whole so that it can be a guide for displacing the vertices of  $\mathcal{T}$ . Moreover, if no particular attention is paid to this extension, the extended displacement field may impose an important stretching in  $\Omega$ , making the motion of the vertices of  $\mathcal{T}$  via (3) impossible to achieve without invalidating the mesh.

These difficulties can be alleviated by using the gradient of  $\theta \mapsto J'(\Omega)(\theta)$  associated with another inner product. This *velocity extension-regularization* issue is quite classical in shape optimization (see [10] and references therein), and can be thought of as an efficient preconditioning of the naive procedure (2).

In the present context, imagine that all the considered shapes  $\Omega$  are filled with a linear elastic material. Also, assume that any such shape  $\Omega$  contains a given subset  $\omega \subseteq \Omega$  on which it is clamped. We now obtain a descent direction for  $J(\Omega)$  as the unique solution  $u_\Omega$  belonging to  $H^1_\omega(\Omega)^d := \{v \in H^1(\Omega)^d, v = 0 \text{ in } \omega\}$  of the linearized elasticity system (hereafter written in variational form)

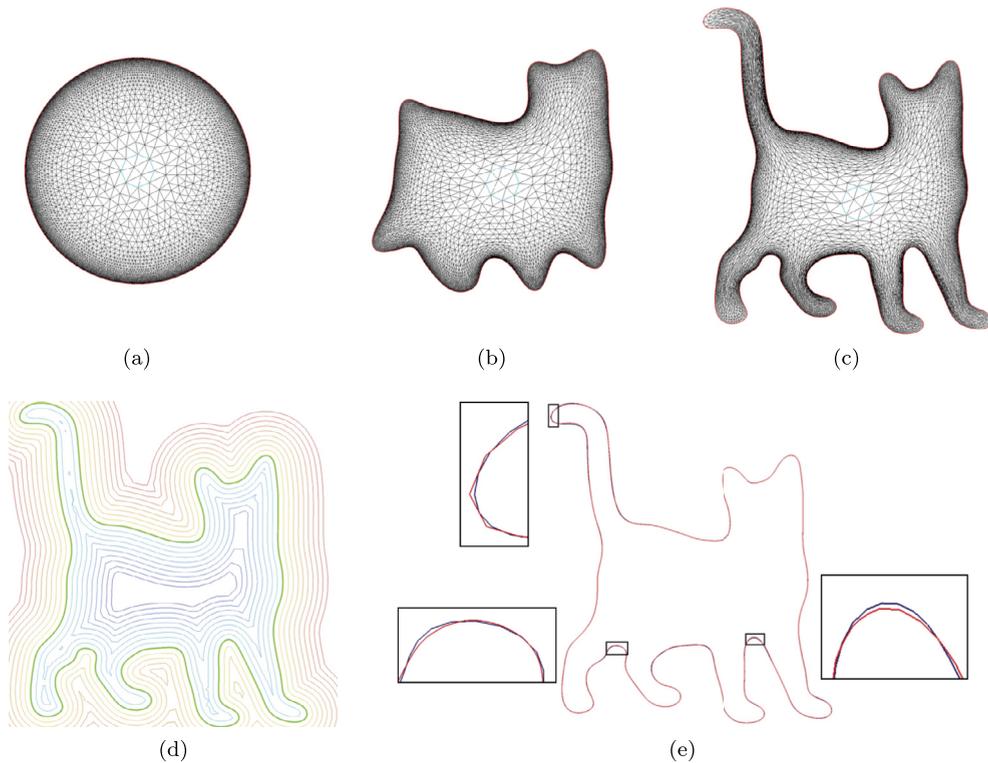
$$\forall v \in H^1_\omega(\Omega)^d, \quad \int_\Omega \sigma(u_\Omega) : \varepsilon(v) \, dx = -J'(\Omega)(v) = - \int_{\partial\Omega} d_{\Omega_T} v \cdot n \, ds, \tag{4}$$

where  $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$  is the linearized strain tensor and  $\sigma$  is the associated stress tensor via the Hooke's law with Lamé coefficients  $\lambda, \mu$ :

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda \operatorname{tr}(\varepsilon(u))I.$$

This vector field  $u_\Omega$  is naturally a descent direction for  $J(\Omega)$  since  $J'(\Omega)(u_\Omega) \leq 0$  thanks to (4), and its advantages over the ‘natural’ deformation field  $\theta$  defined in (2) are twofold:

- (i)  $u_\Omega$  is defined on the whole shape  $\Omega$ ; owing to the regularizing effect of elliptic equations, it is intrinsically smoother than  $\theta = -d_{\Omega_T} n$  (see for instance [7]),



**Fig. 2.** An example in 2D. (a) Template shape  $\Omega_0$ . (b) Deformed shape  $\Omega_k$  for  $k = 90$ . (c) Deformed shape  $\Omega_k$  for  $k = 2100$ . (d) Isovalues of the signed distance function to the target shape  $\Omega_T$  defined on the fixed mesh  $\mathcal{T}_D$ . (e) Discrepancy between  $\Omega_T$  and  $\Omega_{2100}$ .

- (ii) owing to the mechanical features of elastic displacements (notably their ‘rigidity’), it is expected that  $u_\Omega$  will be more amenable to the displacement of the mesh  $\mathcal{T}$  into a valid mesh via (3); see, e.g., [3] for an example of use of elastic displacements in the context of mesh displacement.

**Remark 1.** From the numerical point of view, the choice of a subset  $\omega$  corresponds to a global alignment of shapes (cf. Fig. 1, right). This restriction we used to guarantee the well-posedness of Problem (4) could be replaced by adding a 0th-order term.

### 3. Numerical issues

As far as the numerical setting is concerned, the template shape  $\Omega_0$  is discretized as a simplicial mesh (i.e. a triangulation), and the target shape  $\Omega_T$  is supplied through its signed distance function, e.g., as a  $\mathbb{P}^1$  piecewise affine function on the fixed mesh  $\mathcal{T}_D$  of a large computational domain  $D$ .

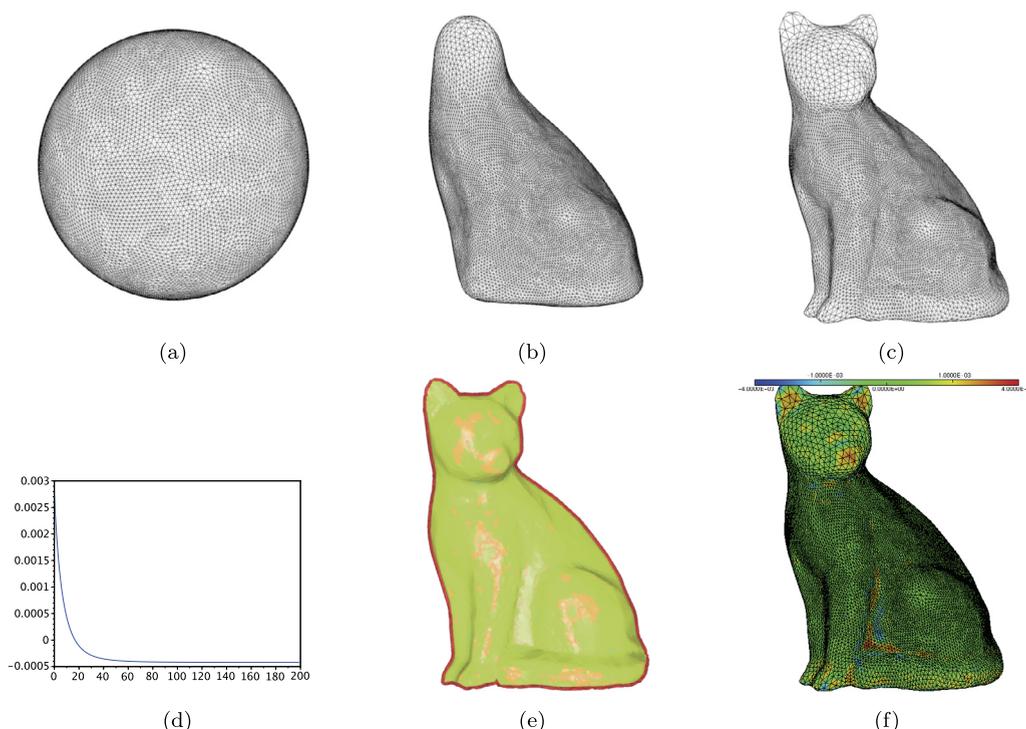
Starting from the template shape  $\Omega_0$ , we perform a gradient descent algorithm with adaptive step size in order to get a sequence of pairs  $(\Omega_k, \mathcal{T}_k)$  of domains and their corresponding meshes with decreasing values of  $J(\Omega_k)$ . The algorithm stops when the step size is smaller than a fixed tolerance  $\varepsilon$ .

#### Remark 2.

- (i) The only information required about the target shape  $\Omega_T$  is the datum of its signed distance function which can be defined on a possibly non-conforming mesh (e.g., showing small gaps, overlapping entities, etc.).
- (ii) The global mapping from  $\Omega_0$  to  $\Omega_T$  is easily recovered by the composition of the different displacements between each iteration.
- (iii) The computational meshes used to perform the calculation are non uniform; they are refined in the vicinity of the boundaries according to a curvature-based sizing function and coarsened in the interior of the domain. This has proven to prevent severe distortion/tangling of the elements (avoiding the need to remesh the domain) and hence to increase the efficiency of the overall algorithm.

#### 3.1. Numerical examples

In the proposed examples, the calculation of the signed distance function to  $\Omega_T$  is performed using the algorithm described in [8]. At first, Fig. 2 depicts a 2D test case. Both target and template meshes are embedded in a unit computational



**Fig. 3.** An example in 3D. (a) Template shape  $\Omega_0$ . (b) Deformed shape  $\Omega_k$  for  $k = 300$ . (c) Deformed shape  $\Omega_T$  for  $k = 2250$ . (d) Objective functional  $J(\Omega_k)$  versus number of iterations  $k$ . (e) Target shape  $\Omega_T$  as the zero-level set of the signed distance function. (f) Discrepancy between  $\Omega_T$  and  $\Omega_{2250}$ .

box with dimensions  $[0, 1]^2$ . The set  $\omega$  chosen for aligning  $\Omega_0$  and  $\Omega_T$  is a small disk located in the interior of both shapes.

The template mesh  $\mathcal{T}_0$  has about 1200 edges, and the convergence of the gradient descent procedure is obtained in 2100 iterations for a tolerance  $\varepsilon = 1 \cdot e^{-6}$ . The  $L^2$  norm of the distance  $d_{\Omega_T}$  calculated on the boundary of the resulting shape  $\Omega_{2100}$  equals  $5.73 \cdot e^{-4}$  (much smaller than the minimal mesh size), revealing an excellent matching of  $\Omega_T$ .

Next, we consider a 3D example; see Fig. 3. Both the target and the template meshes are embedded in a unit computational box  $D = [0, 1]^3$ . The shapes  $\Omega_0$  and  $\Omega_T$  are aligned by choosing a small ball  $\omega$  in  $\Omega_0 \cap \Omega_T$  as for the subset  $\omega$ . The template mesh  $\mathcal{T}_0$  has about 9000 triangles, and 2250 iterations of the gradient descent algorithm have been performed to achieve convergence for a tolerance  $\varepsilon = 1 \cdot e^{-6}$ , running in a few minutes on a standard laptop computer. The  $L^2$  norm of the distance  $d_{\Omega_T}$  calculated on the boundary of the final shape  $\Omega_{2250}$  is  $5.04 \cdot e^{-4}$  (again, much smaller than the minimal mesh size).

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