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Vladimir S. Dotsenko. Analytic continuations of 3-point functions of the conformal field theory. Nuclear Physics B - Proceedings Supplements, 2016, 10.1016/j.nuclphysb.2016.03.037 . hal-01299185

**HAL Id: hal-01299185**

**<https://hal.sorbonne-universite.fr/hal-01299185>**

Submitted on 7 Apr 2016

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# Analytic continuations of 3-point functions of the conformal field theory

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Received 29 January 2016; accepted 29 March 2016

Editor: Hubert Saleur

## Abstract

It is shown that the general 3-point function  $\langle \Phi_c \Phi_b \Phi_a \rangle$ , with continuous values of charges  $a, b, c$  of a statistical model operators, and the 3-point function of the Liouville model, could all be obtained by successive analytical continuations starting from the 3-point function of the minimal model.

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## 1. Introduction

Recent interest in the 3-point functions  $\langle \Phi_c \Phi_b \Phi_a \rangle$  with continuous values of charges  $a, b, c$ , which do not satisfy the neutrality conditions of the Coulomb gas minimal models, is, principally, due to recently found realisations of these correlation functions in the context of statistical models, on the lattice: Potts model 3 spin correlation function [1], loop models [2].

On the other side, the interest in the Liouville model correlation function was always present, since 1981 [3].

The Liouville 3-point function was defined in [4,5]. The statistical model general 3-point function (of imaginary Liouville or Coulomb gas) was defined in [6].

In the present paper we rederive these results somewhat differently, by a sequence of analytical continuations, starting with the minimal model 3-point function [7–9].

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<http://dx.doi.org/10.1016/j.nuclphysb.2016.03.037>

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The present study does not provide new results but gives some new methods and provides some unification, hopefully.

## 2. Analytic continuation of $(1, n)$ operators correlation function towards the general $(n', n)$ operators 3-point function

The structure constant of the  $(1, n)$  minimal model subalgebra, which is the 3-point function of  $(1, n)$  operators, is of the form [8]:

$$\begin{aligned} & \langle V_{1,p}^+(\infty) V_{1,n}(1) V_{1,m}(0) \rangle \\ &= \prod_{j=1}^k \frac{\Gamma(j, \rho)}{\Gamma(1-j\rho)} \times \prod_{j=0}^{k-1} \frac{\Gamma(1+\alpha+j\rho)\Gamma(1+\beta+j\rho)\Gamma(1+\gamma+j\rho)}{\Gamma(-\alpha-j\rho)\Gamma(-\beta-j\rho)\Gamma(-\gamma-j\rho)} \end{aligned} \quad (2.1)$$

where  $V_{1,m}$ ,  $V_{1,n}$ ,  $V_{1,p}^+$ , are the Coulomb gas vertex operators,

$$\begin{aligned} V(z, \bar{z})_{1,m} &= V_{\alpha_{1,m}}(z, \bar{z}) = e^{i\alpha_{1,m}\varphi(z, \bar{z})}, \\ V_{1,n}(z, \bar{z}) &= V_{\alpha_{1,n}}(z, \bar{z}) = e^{i\alpha_{1,n}\varphi(z, \bar{z})} \\ \alpha_{1,m} &= \frac{1-m}{2}\alpha_+, \quad \alpha_{1,n} = \frac{1-n}{2}\alpha_+ \end{aligned} \quad (2.2)$$

$V_{1,p}^+$  is the Coulomb gas conjugate operator:

$$\begin{aligned} V_{1,p}^+(z, \bar{z}) &= V_{\alpha_{1,p}^+}(z, \bar{z}) = e^{i\alpha_{1,p}^+\varphi(z, \bar{z})} \\ \alpha_{1,p}^+ &= 2\alpha_0 - \alpha_{1,p} = 2\alpha_0 - \frac{(1-p)}{2}\alpha_+ = \alpha_- + \frac{1+p}{2}\alpha_+ \end{aligned} \quad (2.3)$$

$\varphi(z, \bar{z})$  is the Coulomb gas field.

Parameters  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\rho$  in (2.1) are defined as:

$$\begin{aligned} \alpha &= 2\alpha_+\alpha_{1,m} = (1-m)\rho, \quad \beta = 2\alpha_+\alpha_{1,n} = (1-n)\rho, \\ \gamma &= 2\alpha_+\alpha_{1,p}^+ = 2\alpha_+(2\alpha_0 - \alpha_{1,p}) = 2\alpha_+(\alpha_- + \frac{1+p}{2}\alpha_+) = -2 + (1+p)\rho, \quad \rho = \alpha_+^2 \end{aligned} \quad (2.4)$$

$\alpha_+$ ,  $\alpha_-$  are the charges of the screening operators

$$V_+(z, \bar{z}) = e^{i\alpha_+\varphi(z, \bar{z})}, \quad V_-(z, \bar{z}) = e^{i\alpha_-\varphi(z, \bar{z})} \quad (2.5)$$

$\alpha_0$  is the Coulomb gas background charge,  $2\alpha_0 = \alpha_+ + \alpha_-$ , and  $\alpha_+ \cdot \alpha_- = -1$ .

The parameter  $k$  in (2.1) is the number of screening operators  $V_+$ , required by the function on the l.h.s. of (2.1), to satisfy the neutrality condition:

$$\alpha_{1,p}^+ + \alpha_{1,n} + \alpha_{1,m} + k\alpha_+ = 2\alpha_0 \quad (2.6)$$

On finds that

$$k = \frac{m+n-p-1}{2} \quad (2.7)$$

If (2.1) is compared with the integral (B.9) in [8], which is the expression for the 3-point function  $\langle V_{1,p}^+(\infty) V_{1,n}(1) V_{1,m}(0) \rangle$  with the parameters  $\alpha, \beta, \gamma$  given in (2.4), it is observed that we have removed, in (2.1), the factor

$$\pi^k \left( \frac{\Gamma(1-\rho)}{\Gamma(\rho)} \right)^k \quad (2.8)$$

which the normalisation factor. It could be removed by renormalising the screening operator constant  $\mu_+$ , in the Coulomb gas action for the field  $\varphi(z, \bar{z})$ . Its general form is

$$A[\varphi] = \int d^2x \left( \frac{1}{4\pi} \partial_z \varphi \cdot \partial_{\bar{z}} \varphi - \mu_+ V_+ - \mu_- V_- \right) \quad (2.9)$$

Also, the last product in (B.9), [8], has been expressed slightly differently: multiplying (2.6) by  $2\alpha_+$  one gets

$$\begin{aligned} \gamma + \beta + \alpha + 2\rho k &= 2\rho - 2, \\ \gamma &= -2 - \alpha - \beta - 2\rho(k-1) \end{aligned} \quad (2.10)$$

and the last product in (2.1) takes the form

$$\begin{aligned} \prod_{j=0}^{k-1} \frac{\Gamma(1+\gamma+j\rho)}{\Gamma(-\gamma-j\rho)} &= \prod_{j=0}^{k-1} \frac{\Gamma(-1-\alpha-\beta-(2k-2-j)\rho)}{\Gamma(2+\alpha+\beta+(2k-2-j)\rho)} \\ &= \prod_{j=0}^{k-1} \frac{\Gamma(-1-\alpha-\beta-(k-1+j)\rho)}{\Gamma(2+\alpha+\beta+(k-1+j)\rho)} \end{aligned} \quad (2.11)$$

which agrees with (B.9), [8].

The general structure constant of the minimal model, which is the 3-point function of the general  $[n', n]$  operators, is of the form [8]:

$$\begin{aligned} &\langle V_{p',p}^+(\infty) V_{n',n}(1) V_{m',m}(0) \rangle \\ &= \rho^{-4lk} \prod_{i=1}^l \frac{\Gamma(i\rho' - k)}{\Gamma(1 - i\rho' + k)} \prod_{j=1}^k \frac{\Gamma(j\rho)}{\Gamma(1 - j\rho)} \\ &\quad \times \prod_{i=0}^{l-1} \frac{\Gamma(1 - k + \alpha' + i\rho') \Gamma(1 - k + \beta' + i\rho') \Gamma(1 - k + \gamma' + i\rho')}{\Gamma(k - \alpha' - i\rho') \Gamma(k - \beta' - i\rho') \Gamma(k - \gamma' - i\rho')} \\ &\quad \times \prod_{j=0}^{k-1} \frac{\Gamma(1 + \alpha + j\rho) \Gamma(1 + \beta + j\rho) \Gamma(1 + \gamma + j\rho)}{\Gamma(-\alpha - j\rho) \Gamma(-\beta - j\rho) \Gamma(-\gamma - j\rho)} \end{aligned} \quad (2.12)$$

Here  $V_{m',m}, V_{n',n}, V_{p',p}^+$  are the Coulomb gas vertex operators:

$$\begin{aligned} V_{m',m}(z, \bar{z}) &= V_{\alpha_{m',m}}(z, \bar{z}) = e^{i\alpha_{m',m}\varphi(z, \bar{z})}, \\ V_{n',n}(z, \bar{z}) &= V_{\alpha_{n',n}}(z, \bar{z}) = e^{i\alpha_{n',n}\varphi(z, \bar{z})}, \\ V_{p',p}^+(z, \bar{z}) &= e^{i\alpha_{p',p}^+\varphi(z, \bar{z})} \end{aligned} \quad (2.13)$$

with

$$\begin{aligned}\alpha_{m',m} &= \frac{1-m'}{2}\alpha_- + \frac{1-m}{2}\alpha_+, & \alpha_{n',n} &= \frac{1-n'}{2}\alpha_- + \frac{1-n}{2}\alpha_+, \\ \alpha_{p',p}^+ &= 2\alpha_0 - \alpha_{p',p} = \frac{1+p'}{2}\alpha_- + \frac{1+p}{2}\alpha_+\end{aligned}\quad (2.14)$$

The parameters  $\alpha, \beta, \gamma$  in (2.12) are now different from those in (2.1), (2.4). They are given by:

$$\begin{aligned}\alpha &= 2\alpha_+\alpha_{m',m} = -(1-m') + (1-m)\rho, & \beta &= 2\alpha_+\alpha_{n',n} = -(1-n') + (1-n)\rho \\ \gamma &= 2\alpha_+\alpha_{p',p}^+ = -(1+p') + (1+p)\rho, & \rho &= \alpha_+^2\end{aligned}\quad (2.15)$$

and

$$\begin{aligned}\alpha' &= 2\alpha_-\alpha_{m',m} = (1-m')\rho' - (1-m), & \beta' &= 2\alpha_-\alpha_{n',n} = (1-n')\rho' - (1-n) \\ \gamma' &= 2\alpha_-\alpha_{p',p}^+ = (1+p')\rho' - (1+p), & \rho' &= \alpha_-^2\end{aligned}\quad (2.16)$$

It is seen that  $(\alpha_+\alpha_- = -1, \rho'\alpha_+ = \alpha_-^2\alpha_+ = -\alpha_-)$ :

$$\begin{aligned}\alpha' &= -\rho'\alpha, & \beta' &= -\rho'\beta, & \gamma' &= -\rho'\gamma, \\ \alpha &= -\rho\alpha', & \beta &= -\rho\beta', & \gamma &= -\rho\gamma'\end{aligned}\quad (2.17)$$

The parameters  $l, k$  in (2.12), the numbers of screening operators, they satisfy the Coulomb gas neutrality condition:

$$\alpha_{p',p}^+ + \alpha_{n',n} + \alpha_{m',m} + l\alpha_- + k\alpha_+ = 2\alpha_0 \quad (2.18)$$

By collecting the coefficients of  $\alpha_+$  and  $\alpha_-$ , separately, assuming that there is no compensation between the two ( $\rho = \alpha_+^2$  and  $\rho' = \alpha_-^2 = \rho^{-1}$  are being non-rational), one finds:

$$l = \frac{m' + n' - p' - 1}{2}, \quad k = \frac{m + n - p - 1}{2} \quad (2.19)$$

As compared to the integral (B.10) of [8], which is the expression for the 3-point function  $\langle V_{p',p}(\infty)V_{n',n}(1)V_{m',m}(0) \rangle$ , we have removed in (2.12) the normalisation factors

$$\pi^{l+k} \times \left( \frac{\Gamma(1-\rho')}{\Gamma(\rho')} \right)^l \left( \frac{\Gamma(1-\rho)}{\Gamma(\rho)} \right)^k \quad (2.20)$$

Again, these factors could be removed by renormalising the constants  $\mu_+, \mu_-$  in (2.9).

We have also reorganised the last two factors in the products over  $i$  and over  $j$ , by using the neutrality condition (2.18): multiplying (2.18) by  $2\alpha_+$ , or by  $2\alpha_-$ , one gets, respectively,

$$\gamma + \beta + \alpha - 2l + 2k\rho = 2\rho - 2 \quad (2.21)$$

$$\gamma' + \beta' + \alpha' + 2l\rho' - 2k = -2 + 2\rho' \quad (2.22)$$

which gives

$$\gamma = -2 - \alpha - \beta + 2l - 2(k-1)\rho \quad (2.23)$$

$$\gamma' = -2 - \alpha' - \beta' - 2(l-1)\rho' + 2k \quad (2.24)$$

By manipulating the products in (2.12), those with  $\gamma$  and  $\gamma'$ , in a way similar to that in (2.11), one gets the agreement of (2.12) with the expression in (B.10), [8].

The objective of this section is to show that one gets the general 3-point function (2.12), for the general degenerate operators of the minimal model (the operators producing degenerate

representations, saying it properly), by replacing  $\alpha, \beta, \gamma$  in (2.1), (2.4) by  $\alpha, \beta, \gamma$  in (2.15) and by continuing (2.1) to the fractional value of  $k$ :

$$k \rightarrow k - \rho' l \quad (2.25)$$

Saying it shortly: (2.12) is obtained by the analytic continuation of (2.1).

On general remark is in order.

We are talking in this section about the correlation functions of degenerate operators, in (2.1) and in (2.12), with  $\alpha, \beta, \gamma$  having special values in (2.4) and in (2.15), to have the objects which are well defined physically, as minimal model correlation functions. But the demonstration given below implies in fact that the Coulomb gas integral in (B.10), [8], with general values of  $\alpha, \beta, \gamma$ , is obtained from the integral in (B.9) by the analytic continuation in  $k$ , by eq. (2.25). This is up to the normalisation factors (2.8) and (2.20).

More precisely, in (2.1) and (2.12)  $\alpha, \beta, \gamma$  would not be totally general. They will still be subjects to one constraint, the neutrality condition: (2.10) for (2.1), with  $k$  being integer, and (2.21) for (2.12), with  $l, k$  being integers. As  $k$  moves by (2.25), the values of  $\alpha, \beta, \gamma$  are being moved also, from the values satisfying (2.10) to the values satisfying (2.21), like in the case of the degenerate values of the parameters, (2.4) and (2.15).

Going back to our correlation functions, we shall continue the logarithm of the expression in (2.1), by using the integral representation of the logarithm of  $\Gamma$ -functions.

Let us define

$$g_k(\rho) = \prod_{j=1}^k \frac{\Gamma(j\rho)}{\Gamma(1-j\rho)} \quad (2.26)$$

$$G_{lk}(\rho) = \prod_{i=1}^l \frac{\Gamma(i\rho' - k)}{\Gamma(1 - i\rho' + k)} \prod_{j=1}^k \frac{\Gamma(j\rho)}{\Gamma(1 - j\rho)} \quad (2.27)$$

$$g_k^{(\alpha)}(\rho) = \prod_{j=1}^{k-1} \frac{\Gamma(1 + \alpha + j\rho)}{\Gamma(-\alpha - j\rho)} \quad (2.28)$$

$$G_{lk}^{(\alpha)}(\rho) = \prod_{i=0}^{l-1} \frac{\Gamma(1 - k + \alpha' + i\rho')}{\Gamma(k - \alpha' - i\rho')} \prod_{j=0}^{k-1} \frac{\Gamma(1 + \alpha + j\rho)}{\Gamma(-\alpha - j\rho)} \quad (2.29)$$

and similarly for  $g_k^{(\beta)}(\rho)$ ,  $g_k^{(\gamma)}(\rho)$ ,  $G_{lk}^{(\beta)}(\rho)$ ,  $G_{lk}^{(\gamma)}(\rho)$ .

With these notations, the function in (2.1), which we shall note as  $C_{n,m}^p$ , takes the form:

$$\begin{aligned} \langle V_{1,p}(\infty) V_{1,n}(1) V_{1,m}(0) \rangle &\equiv C_{n,m}^p(\rho) \\ &= g_k(\rho) g_k^{(\alpha)}(\rho) g_k^{(\beta)}(\rho) g_k^{(\gamma)}(\rho) \end{aligned} \quad (2.30)$$

and the function in (2.12), which we shall note as  $C_{(n',n)(m',m)}^{(p',p)}$ , takes the form:

$$\begin{aligned} \langle V_{p',p}^+(\infty) V_{n',n}(1) V_{m',m}(0) \rangle &\equiv C_{(n',n)(m',m)}^{(p',p)}(\rho) \\ &= \rho^{-4lk} G_{lk}(\rho) G_{lk}^{(\alpha)}(\rho) G_{lk}^{(\beta)}(\rho) G_{lk}^{(\gamma)}(\rho) \end{aligned} \quad (2.31)$$

It is shown in the Appendix A that  $\log g_k(\rho)$ , analytically continued in  $k$ ,  $k \rightarrow k - \rho' l$ , eq. (2.25), is given by:

$$\log g_{k-\rho'l}(\rho) = \log G_{lk}(\rho) - \log \rho \cdot (2kl + l - \rho'l - \rho'l^2) \quad (2.32)$$

For the analytic continuation of  $\log g_k^{(\alpha)}(\rho)$  we obtain, [Appendix A](#):

$$\log g_{k-\rho'l}^{(\alpha)}(\rho) = \log G_{lk}^{(\alpha)}(\rho) - \log \rho \cdot (2kl - l(2\alpha' - \rho' + 1) - l^2 \rho') \quad (2.33)$$

and similar expressions for  $\log g_{k-\rho'l}^{(\beta)}(\rho)$  and  $\log g_{k-\rho'l}^{(\gamma)}(\rho)$ .

Putting them together, by eq. (2.30), we obtain

$$\begin{aligned} & (\log C_{n,m}^p(\rho))_{continued,k-\rho'l} \\ &= \log(G_{lk}^{(\alpha)} G_{lk}^{(\beta)} G_{lk}^{(\gamma)}) + \log \rho \cdot (-4kl + 2l(\rho' - 1)) \end{aligned} \quad (2.34)$$

To get the coefficient of  $\log \rho$ , in its form above, we have used the neutrality condition on the parameters, eq. (2.22).

Finally one obtains:

$$(C_{n,m}^p(\rho))_{continued,k-\rho'l} = C_{(n',n)(m',m)}^{(p',p)}(\rho) \times \rho^{2l(\rho'-1)} \quad (2.35)$$

$C_{(n',n)(m',m)}^{(p',p)}(\rho)$  has been reconstructed according to its form in (2.31).

The factor  $\rho^{2l(\rho'-1)}$  is another normalisation factor, being an exponent linear in  $l$ , produced this time in the process of analytic continuation. It could also be “symmetrized”, so that  $l$  and  $k$  would appear on equal footing, by using the neutrality condition (2.21) on  $l$  and  $k$ . We shall do it later, because one extra factor of this type is still coming, will be obtained in the next section.

For the time being we are dealing with correlation functions of Coulomb gas vertex operators, which is simpler.

They have their own nontrivial normalisation which will be specified later. Further down we shall normalise the operators by 1. With that universal normalisation the extra normalisation factors in our analytic continuation formulas, like the one in (2.35), will disappear, as we shall see later. But for the time being we shall still stay with vertex operators, like the ones in (2.2), (2.3), (2.13).

### 3. Analytic continuation to the general, unconstrained values of charges of the vertex operators in the 3-point function

To start, we shall reinterpret the results of the previous section in the opposite direction. We shall consider that it has been shown that the general minimal model 3-point correlation function  $\langle V_{p',p}^+(\infty) V_{n',n}(1) V_{m',m}(0) \rangle \equiv C_{(n',n)(m',m)}^{(p',p)}$  is equal to the function  $\langle V_{1,p}^+(\infty) V_{1,n}(1) V_{1,m}(0) \rangle \equiv C_{n,m}^p$  analytically continued, eq. (2.35) read from right to left:

$$C_{(n',n)(m',m)}^{(p',p)}(\rho) = (C_{n,m}^p(\rho))_{continued,k-\rho'l} \times \rho^{-2l(\rho'-1)} \quad (3.1)$$

And we shall continue further, the function  $(C_{n,m}^p(\rho))_{continued}$  to the values of charges, of the operators in it,  $V_{p',p}^+ = e^{i\alpha_{p',p}^+ \varphi}$ ,  $V_{n',n} = e^{i\alpha_{n',n} \varphi}$ ,  $V_{m',m} = e^{i\alpha_{m',m} \varphi}$ , to the unconstrained, continuous values,  $a, b, c$ :

$$(V_{p',p}^+) \rightarrow V_c = e^{ic\varphi}, \quad V_{n',n} \rightarrow V_b = e^{ib\varphi}, \quad V_{m',m} \rightarrow V_a = e^{ia\varphi} \quad (3.2)$$

We remind that in the process of analytic continuation in Section 2, which results in the equality (3.1),  $k$  has been replaced with  $k - \rho'l$ , but also the charges  $\alpha_{1,m}$ ,  $\alpha_{1,n}$ ,  $\alpha_{1,p}^+$  has been replaced by  $\alpha_{m',m}$ ,  $\alpha_{n',n}$ ,  $\alpha_{p',p}^+$ , comp. the comment preceding eq. (2.25).

It is much easier to continue this way, in two steps, the general minimal model correlator  $C_{(n',n)(m',m)}^{(p',p)}$ ,

$$C_{(n',n)(m',m)}^{(p',p)} \rightarrow (C_{n,m}^p)_{\text{continued}} \rightarrow C_{b,a}^c \equiv \langle V_c(\infty) V_b(1) V_a(0) \rangle \quad (3.3)$$

instead of performing the continuation directly

$$\langle V_{p',p}^+(\infty) V_{n',n}(1) V_{m',m}(0) \rangle \rightarrow \langle V_c(\infty) V_b(1) V_a(0) \rangle \quad (3.4)$$

with  $a, b, c$ , being unconstrained.

To perform the second step in (3.3), we shall need the detailed expressions for  $\log g_{k-\rho'l}(\rho)$ ,  $\log g_{k-\rho'l}^{(\alpha)}(\rho)$ ,  $\log g_{k-\rho'l}^{(\beta)}(\rho)$ ,  $\log g_{k-\rho'l}^{(\gamma)}(\rho)$ , obtained in Appendix A. They are as follows:

$$\begin{aligned} \log g_{k-\rho'l}(\rho) = & \int_0^\infty \frac{dt}{t} \{ [(k-\rho'l)(k-\rho'l+1) - (k-\rho'l)] e^{-t} \\ & + \frac{(1 - e^{-(k-\rho'l)\rho t}) e^{-\rho t} + (1 - e^{(k-\rho'l)\rho t}) e^{-t}}{(1 - e^{-t})(1 - e^{-\rho t})} \} \end{aligned} \quad (3.5)$$

$$\begin{aligned} \log g_{k-\rho'l}^{(\alpha)}(\rho) = & \int_0^\infty \frac{dt}{t} \{ [(2\alpha+1)(k-\rho'l) + (k-\rho'l-1)(k-\rho'l)\rho] e^{-t} \\ & + \frac{e^{-(1+\alpha)t} (1 - e^{-(k-\rho'l)\rho t}) + e^{\alpha t - \rho t} (1 - e^{(k-\rho'l)\rho t})}{(1 - e^{-t})(1 - e^{-\rho t})} \} \end{aligned} \quad (3.6)$$

and similar expressions for  $\log g_{k-\rho'l}^{(\beta)}(\rho)$  and  $\log g_{k-\rho'l}^{(\gamma)}(\rho)$ .

For  $(C_{n,m}^p(p))$  continued we get the following expression:

$$\begin{aligned} \log(C_{n,m}^p(\rho))_{\text{continued}} &= \log g_{k-\rho'l}(\rho) + \log g_{k-\rho'l}^{(\alpha)}(\rho) + \log g_{k-\rho'l}^{(\beta)}(\rho) + \log g_{k-\rho'l}^{(\gamma)}(\rho) \\ &= \int_0^\infty \frac{dt}{t} \{ [(k-\rho'l)(k-\rho'l+1)\rho - (k-\rho'l) \\ &+ (2\alpha+1+2\beta+1+2\gamma+1)(k-\rho'l) + 3(k-\rho'l-1)(k-\rho'l)\rho] e^{-t} \\ &+ \frac{1}{(1 - e^{-t})(1 - e^{-\rho t})} [(1 - e^{-(k-\rho'l)\rho t}) \times (e^{-\rho t} + e^{-(1+\alpha)t} + e^{-(1+\beta)t} + e^{-(1+\gamma)t}) \\ &+ (1 - e^{(k-\rho'l)\rho t}) (e^{-t} + e^{\alpha t - \rho t} + e^{\beta t - \rho t} + e^{\gamma t - \rho t})] \} \end{aligned} \quad (3.7)$$

First we shall simplify the “polynomial” part in (3.7), the first part of it, the coefficient of  $e^{-t}$ .

By using the neutrality conditions (2.21), on  $\alpha, \beta, \gamma$ , we get (we remind that  $\rho\rho' = 1$ ):

$$\begin{aligned} & (k-\rho'l)(k-\rho'l+1)\rho - (k-\rho'l) \\ &+ (2\alpha+1+2\beta+1+2\gamma+1)(k-\rho'l) + 3(k-\rho'l-1)(k-\rho'l)\rho \\ &= (k-\rho'l)[(k-\rho'l+1)\rho - 1 + 2 \cdot (2\rho - 2 + 2l - 2k\rho) + 3 + 3(k-\rho'l-1)\rho] \\ &= (k-\rho'l) \cdot 2(\rho - 1) \end{aligned} \quad (3.8)$$



To continue eventually to the general unconstrained values of charges, of the operators, we have to express everything in terms of these charges in particular the combination of the numbers of screenings  $k - \rho' l$  in (3.8). From (2.22):

$$2(k - \rho' l) = \alpha' + \beta' + \gamma' + 2 - 2\rho' \quad (3.9)$$

By eq. (2.16)

$$\alpha' = 2\alpha_- a, \quad \beta' = 2\alpha_- b, \quad \gamma' = 2\alpha_- c \quad (3.10)$$

where we have replaced  $\alpha_{m',m}, \alpha_{n',n}, \alpha_{p',p}^+$  by  $a, b, c$ . For (3.9) we obtain

$$2(k - \rho' l) = 2\alpha_- (a + b + c) + 2(1 - \rho') \quad (3.11)$$

For (3.8) we get:

$$\begin{aligned} 2(k - \rho' l)(\rho - 1) &= 2\alpha_- (a + b + c)(\rho - 1) + 2(1 - \rho')(\rho - 1) \\ &= -2\alpha_+ (a + b + c) - 2\alpha_- (a + b + c) + 2(\rho - 2 + \rho') \\ &= -4\alpha_0 (a + b + c) + 8\alpha_0^2 \end{aligned} \quad (3.12)$$

We remind that

$$\rho = \alpha_+^2, \quad \rho' = \alpha_-^2, \quad \alpha_+ \alpha_- = -1, \quad \rho \alpha_- = -\alpha_+ \quad (3.13)$$

For the polynomial part of (3.7) we obtain:

$$\int_0^\infty \frac{dt}{t} [8\alpha_0^2 - 4\alpha_0(a + b + c)] e^{-t} \quad (3.14)$$

We shall simplify next the “exponential” part in (3.7), its second part. It has to be observed that, separately, the integrals of the polynomial part and of the exponential part in (3.7), they are divergent at  $t \rightarrow 0$ . To manipulate them separately we should introduce the limit  $\epsilon > 0$ , in the integrals, instead of 0, and assume that, finally, we shall take the limit  $\epsilon \rightarrow 0$  when everything is put together, as it has been done already in Appendix A.

For the first part we have done no transformations for the integration variable  $t$ , so we could keep it as it is in (3.14), although, more properly, we could have assumed that the lower limit of integration in (3.14) is  $\epsilon$ .

But for the second part of (3.7) we do intend to transform the integration variable  $t$ , so that the explicit introduction of  $\epsilon$ , for the lower limit of the integration, will be necessary at some point.

In fact, we shall start simplifying (or reorganising) the second part of (3.7) by transforming the variable  $t$ :

$$t = \sqrt{\rho'} \tilde{t} \quad (3.15)$$

We shall do all the transformations by ignoring, at first, the divergence at  $t = 0$ . But afterward we shall take specific care of the extra terms, the “anomaly” terms, being produced by this divergent limit.

With the change of the variable in (3.15), the second, exponential part of (3.7) takes form:

$$\begin{aligned} &\int_0^\infty \frac{d\tilde{t}}{\tilde{t}} \frac{1}{(1 - e^{-\sqrt{\rho'} \tilde{t}})(1 - e^{-\sqrt{\rho} \tilde{t}})} \\ &\times [(1 - e^{-(k-\rho' l)\sqrt{\rho'} \tilde{t}})(e^{-\sqrt{\rho} \tilde{t}} + e^{-(1+\alpha)\sqrt{\rho'} \tilde{t}} + e^{-(1+\beta)\sqrt{\rho'} \tilde{t}} + e^{-(1+\gamma)\sqrt{\rho'} \tilde{t}}) \\ &+ (1 - e^{(k-\rho' l)\sqrt{\rho'} \tilde{t}})(e^{-\sqrt{\rho'} \tilde{t}} + e^{\alpha\sqrt{\rho'} \tilde{t} - \sqrt{\rho} \tilde{t}} + e^{\beta\sqrt{\rho'} \tilde{t} - \sqrt{\rho} \tilde{t}} + e^{\gamma\sqrt{\rho'} \tilde{t} - \sqrt{\rho} \tilde{t}})] \end{aligned} \quad (3.16)$$

We shall simplify next the notations and we shall express everything  $(k - \rho'l, \alpha, \beta, \gamma)$  in terms of the charges  $a, b, c$ .

The Coulomb gas parameter  $\sqrt{\rho} = \alpha_+$ , corresponds (is proportional) to the parameter  $b$  of the Liouville model, or to the parameter  $\beta$  in the imaginary Liouville [5,6]. As  $b$  and  $\beta$  are already in use, and the notations  $\sqrt{\rho} = \alpha_+$ ,  $\sqrt{\rho'} = -\alpha_-$  would be slightly heavy, we shall use a single parameter, as in the Liouville model, but we shall note it  $h$ , so that

$$\sqrt{\rho} = \alpha_+ = h, \quad \sqrt{\rho'} = \frac{1}{h} = -\alpha_-, \quad \alpha_0 = \frac{\alpha_+ + \alpha_-}{2} = \frac{h}{2} - \frac{1}{2h} \quad (3.17)$$

Next:

$$\begin{aligned} \alpha &= 2\alpha_+ a = 2ha, & \beta &= 2hb, & \gamma &= 2hc \\ \alpha' &= 2\alpha_- a = -\frac{2}{h}a & \beta' &= -\frac{2}{h}b, & \gamma' &= -\frac{2}{h}c \\ \alpha\sqrt{\rho'} &= 2a, & \beta\sqrt{\rho'} &= 2b, & \gamma\sqrt{\rho'} &= 2c \\ \alpha'\sqrt{\rho} &= -2a, & \beta'\sqrt{\rho} &= -2b, & \gamma'\sqrt{\rho} &= -2c \end{aligned} \quad (3.18)$$

and, according to (3.9), or (2.22),

$$k - \rho'l = \frac{\alpha' + \beta' + \gamma'}{2} + 1 - \rho' \quad (3.19)$$

then

$$\begin{aligned} (k - \rho'l)\sqrt{\rho} &= \frac{1}{2}(\alpha' + \beta' + \gamma')\sqrt{\rho} + \sqrt{\rho} - \sqrt{\rho'} \\ &= -(a + b + c) + \alpha_+ + \alpha_-, \\ (k - \rho'l)\sqrt{\rho} &= 2\alpha_0 - (a + b + c) \end{aligned} \quad (3.20)$$

We shall suppress also the tilde of  $\tilde{t}$ ,  $\tilde{t} \rightarrow t$ . Then the expression in (3.16) takes the form:

$$\begin{aligned} &\int_0^\infty \frac{dt}{t} \frac{1}{(1 - e^{-\frac{t}{h}})(1 - e^{-ht})} \\ &\quad \times [(1 - e^{-(2\alpha_0 - a - b - c)t})(e^{-ht} + e^{-\frac{t}{h} - 2at} + e^{-\frac{t}{h} - 2bt} + e^{-\frac{t}{h} - 2ct}) \\ &\quad + (1 - e^{(2\alpha_0 - a - b - c)t})(e^{-\frac{t}{h}} + e^{-ht + 2at} + e^{-ht + 2bt} + e^{-ht + 2ct})] \end{aligned} \quad (3.21)$$

Multiplying the numerator and the denominator in (3.21) by  $\exp\{\frac{t}{2h} + \frac{ht}{2}\}$  we get:

$$\begin{aligned} &\int_0^\infty \frac{dt}{t} \frac{1}{(e^{\frac{t}{2h}} - e^{-\frac{t}{2h}})(e^{\frac{ht}{2}} - e^{-\frac{ht}{2}})} \\ &\quad \times [(1 - e^{-(2\alpha_0 - a - b - c)t})(e^{-\alpha_0 t} + e^{\alpha_0 t - 2at} + e^{\alpha_0 t - 2bt} + e^{\alpha_0 t - 2ct}) \\ &\quad + (1 - e^{(2\alpha_0 - a - b - c)t})(e^{\alpha_0 t} + e^{-\alpha_0 t + 2at} + e^{-\alpha_0 t + 2bt} + e^{-\alpha_0 t + 2ct})] \end{aligned} \quad (3.22)$$

Next we obtain, by regrouping the terms:

$$\begin{aligned}
& \int_0^\infty \frac{dt}{t} \frac{1}{4 \sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}} \\
& \times [e^{\alpha_0 t} + e^{-\alpha_0 t} - e^{(3\alpha_0 - a - b - c)t} - e^{-(3\alpha_0 - a - b - c)t} \\
& + e^{(\alpha_0 - 2a)t} + e^{-(\alpha_0 - 2a)t} + e^{(\alpha_0 - 2b)t} + e^{-(\alpha_0 - 2b)t} + e^{(\alpha_0 - 2c)t} + e^{-(\alpha_0 - 2c)t} \\
& - e^{(\alpha_0 + a - b - c)t} - e^{-(\alpha_0 + a - b - c)t} - e^{(\alpha_0 - a + b - c)t} - e^{-(\alpha_0 - a + b - c)t} \\
& - e^{(\alpha_0 - a - b + c)t} - e^{-(\alpha_0 - a - b + c)t}] \quad (3.23)
\end{aligned}$$

It can be presented as:

$$\begin{aligned}
& \int_0^\infty \frac{dt}{t} \frac{1}{\sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}} \times [\sinh^2(\alpha_0 \frac{t}{2}) \\
& + \sinh^2((\alpha_0 - 2a) \frac{t}{2}) + \sinh^2((\alpha_0 - 2b) \frac{t}{2}) + \sinh^2((\alpha_0 - 2c) \frac{t}{2}) \\
& - \sinh^2((3\alpha_0 - a - b - c) \frac{t}{2}) \\
& \times \sinh^2((\alpha_0 + a - b - c) \frac{t}{2}) - \sinh^2((\alpha_0 - a + b - c) \frac{t}{2}) - \sinh^2((\alpha_0 - a - b + c) \frac{t}{2})] \quad (3.24)
\end{aligned}$$

For  $t \rightarrow 0$ , the above integral takes the asymptotic form:

$$\begin{aligned}
& \int_0^\infty \frac{dt}{t} \frac{4}{t^2} \times \frac{t^2}{4} [(\alpha_0)^2 + (\alpha_0 - 2a)^2 + (\alpha_0 - 2b)^2 + (\alpha_0 - 2c)^2 - (3\alpha_0 - a - b - c)^2 \\
& - (\alpha_0 + a - b - c)^2 - (\alpha_0 - a + b - c)^2 - (\alpha_0 - a - b + c)^2] \\
& = \int_0^\infty \frac{dt}{t} [-8\alpha_0^2 + 4\alpha_0(a + b + c)] \quad (3.25)
\end{aligned}$$

This divergence, at  $t \rightarrow 0$ , is compensated by the polynomial part (3.14), of the integral (3.7). But, saying it differently, the equality of the expressions under integrals in (3.14) and (3.25) implies that the polynomial part (3.14) could be distributed as in (3.25):

$$\begin{aligned}
& - \int_0^\infty \frac{dt}{t} [(\alpha_0)^2 + (\alpha_0 - 2a)^2 + (\alpha_0 - 2b)^2 + \alpha_0 - 2c)^2 - 3(\alpha_0 - a - b - c)^2 \\
& - (\alpha_0 + a - b - c)^2 - (\alpha_0 - a + b - c)^2 - (\alpha_0 - a - b + c)^2] \times e^{-t} \quad (3.26)
\end{aligned}$$

so that the full integral (3.7), which is the sum of (3.14) = (3.26) and (3.24), takes the form:

$$\begin{aligned}
& \int_0^\infty \frac{dt}{t} \left\{ -[(\alpha_0)^2 e^{-t} - \frac{\sinh^2(\alpha_0 \frac{t}{2})}{\sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}}] - [(\alpha_0 - 2a)^2 e^{-t} - \frac{\sinh^2((\alpha_0 - 2a) \frac{t}{2})}{\sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}}] \right. \\
& \left. - [(\alpha_0 - 2b)^2 e^{-t} - \frac{\sinh^2((\alpha_0 - 2b) \frac{t}{2})}{\sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}}] - [(\alpha_0 - 2c)^2 e^{-t} - \frac{\sinh^2((\alpha_0 - 2c) \frac{t}{2})}{\sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}}] \right\}
\end{aligned}$$

$$\begin{aligned}
& + [(3\alpha_0 - a - b - c)^2 e^{-t} - \frac{\sinh^2((3a - a - b - c)\frac{t}{2})}{\sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}}] \\
& + [(\alpha_0 + a - b - c)^2 e^{-t} - \frac{\sinh^2((\alpha_0 + a - b - c)\frac{t}{2})}{\sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}}] \\
& + [(\alpha_0 - a + b - c)^2 e^{-t} - \frac{\sinh^2((\alpha_0 - a + b - c)\frac{t}{2})}{\sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}}] \\
& + [(\alpha_0 - a - b + c)^2 e^{-t} - \frac{\sinh^2((\alpha_0 - a - b + c)\frac{t}{2})}{\sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}}] \} \quad (3.27)
\end{aligned}$$

We find that everything is expressed in terms of the function  $\Upsilon(x, h)$  [5,6]

$$\log \Upsilon_M(x, h) = \int_0^\infty \frac{dt}{t} \left\{ (\alpha_0 - x)^2 e^{-t} - \frac{\sinh^2((\alpha_0 - x)\frac{t}{2})}{\sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}} \right\} \quad (3.28)$$

so that  $\log(C_{n,m}^p)_{\text{continued}}$ , which is the integral (3.7), takes the form:

$$\begin{aligned}
\log(C_{n,m}^p)_{\text{continued}} &= -\log \Upsilon_M(2\alpha_0, h) \\
&- \log \Upsilon_M(2a, h) - \log \Upsilon_M(2b, h) - \log \Upsilon_M(2c, h) \\
&+ \log \Upsilon_M(a + b + c - 2\alpha_0, h) \\
&+ \log \Upsilon_M(-a + b + c, h) + \log \Upsilon_M(a - b + c, h) + \log \Upsilon_M(a + b - c, h) \quad (3.29)
\end{aligned}$$

and

$$\begin{aligned}
& (C_{n,m}^p)_{\text{continued}} \\
&= \frac{\Upsilon_M(a + b + c - 2\alpha_0, h) \Upsilon_M(-a + b + c, h) \Upsilon_M(a - b + c, h) \Upsilon_M(a + b - c, h)}{\Upsilon_M(2\alpha_0, h) \Upsilon_M(2a, h) \Upsilon_M(2b, h) \Upsilon_M(2c, h)} \quad (3.30)
\end{aligned}$$

This is the function in [6], though not completely so.

First, the normalisation of operators used in [6] is different.

Second, we have putted the index “ $M$ ” for  $\Upsilon_M$ , for “matter” (statistical model) which is slightly different from  $\Upsilon(x, h)$  for Liouville, gravity, which has been introduced in [5]. The difference is in:

$$\alpha_0 = \frac{h}{2} - \frac{1}{2h}, \text{ instead of } b_0 = \frac{h}{2} + \frac{1}{2h}, \text{ for } \Upsilon(x, h) \text{ of Liouville} \quad (3.31)$$

In [6], the function which note  $\Upsilon_M(x, h)$  has been used in the form  $\Upsilon(x + \frac{1}{h}, h)$ .

And third, there is one additional factor missing in (3.30), the way we derived it. We haven’t calculated yet the anomaly term, which is produced because of our manipulations with the second, exponential part integral, which is divergent at  $t \rightarrow 0$  when taken separately. So far, in our derivation of (3.30), we have ignored this point. We shall take care of it now.

The second, exponential part in (3.7) should have been taken with the lower integration limit  $\epsilon$ , instead of 0, before the change of the variable  $t$  in (3.15). We reproduce this integral somewhat symbolically:

$$\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} \frac{dt}{t} \frac{1}{(1 - e^{-t})(1 - e^{-\rho t})} [t, \dots] \quad (3.32)$$

After the change of the variable in (3.15) we obtain

$$\lim_{\epsilon \rightarrow 0} \int_{\frac{\epsilon}{\sqrt{\rho'}}}^{\infty} \frac{d\tilde{t}}{\tilde{t}} \cdot \frac{1}{(1 - e^{-\sqrt{\rho'}\tilde{t}})(1 - e^{-\sqrt{\rho''}\tilde{t}})} [\sqrt{\rho'}\tilde{t}, \dots] \quad (3.33)$$

The integral can be decomposed as follows:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} \frac{d\tilde{t}}{\tilde{t}} \cdot \frac{1}{(1 - e^{-\sqrt{\rho'}\tilde{t}})(1 - e^{-\sqrt{\rho''}\tilde{t}})} [\sqrt{\rho'}\tilde{t}, \dots] \\ & - \int_{\frac{\epsilon}{\sqrt{\rho'}}}^{\epsilon} \frac{d\tilde{t}}{\tilde{t}} \cdot \frac{1}{(1 - e^{-\sqrt{\rho'}\tilde{t}})(1 - e^{-\sqrt{\rho''}\tilde{t}})} [\sqrt{\rho'}\tilde{t}, \dots] \end{aligned} \quad (3.34)$$

The first integral in above goes to join the first, polynomial part of (3.7) and gives finally the function in (3.30). But the second integral in (3.34) gives an additional term, which has been missed in our derivation of (3.30).

To calculate the second integral in (3.34) we could use all the transformations, for the expression under the integral, which has been done above. It could be taken in the form in (3.24), but with the limits of integration  $(\epsilon, \epsilon/\sqrt{\rho'})$  instead of  $(0, \infty)$ . As  $t$  stays small, in the limits  $(\epsilon, \epsilon/\sqrt{\rho'})$ , we can replace the expression under the integral by its limiting form, for  $t \rightarrow 0$ , which has already been obtained in (3.25). In this way we get, for the second integral in (3.34), the following result:

$$\begin{aligned} & - \int_{\epsilon}^{\epsilon/\sqrt{\rho'}} \frac{dt}{t} [-8\alpha_0^2 + 4\alpha_0(a + b + c)] \\ & = [8\alpha_0^2 - 4\alpha_0(a + b + c)] \cdot (\log \frac{\epsilon}{\sqrt{\rho'}} - \log \epsilon) \\ & = \log \rho \cdot (4\alpha_0^2 - 2\alpha_0(a + b + c)) \end{aligned} \quad (3.35)$$

This is our anomaly. It has to be added to (3.29). With it, the formula in (3.30) takes the form:

$$\begin{aligned} & (C_{n,m}^p)_{continued} \\ & = \frac{\Upsilon_M(a + b + c - 2\alpha_0)\Upsilon_M(-a + b + c)\Upsilon_M(a - b + c)\Upsilon_M(a + b - c)}{\Upsilon_M(2\alpha_0)\Upsilon_M(2a)\Upsilon_M(2b)\Upsilon_M(2c)} \\ & \times \rho^{4\alpha_0^2 - 2\alpha_0(a+b+c)} \end{aligned} \quad (3.36)$$

We remind that our objective was to continue the general minimal model 3-point function

$$C_{(n',n)(m',m)}^{(p',p)} = \langle V_{p',p}^+(\infty) V_{n',n}(1) V_{m',m}(0) \rangle \quad (3.37)$$

towards the function

$$\langle V_c(\infty) V_b(1) V_a(0) \rangle \quad (3.38)$$

with  $a, b, c$  unconstrained. By (3.1), the result of the first step of continuation, and (3.36), the result of the second step of continuation, we find so far:

$$\begin{aligned}
C_{(n',n)(m',m)}^{(p',p)} &= \langle V_{p',p}^+(\infty) V_{n',n}(1) V_{m',m}(0) \rangle \\
&= \frac{\Upsilon_M(a+b+c-2\alpha_0) \Upsilon_M(-a+b+c) \Upsilon_M(a-b+c) \Upsilon_M(a+b-c)}{\Upsilon_M(2a) \Upsilon_M(2b) \Upsilon_M(2c)} \\
&\quad \times \rho^{4\alpha_0^2-2\alpha_0(a+b+c)} \times \rho^{2l(1-\rho')}
\end{aligned} \tag{3.39}$$

We have suppressed the factor  $\Upsilon_M(2\alpha_0)$  in the denominator of (3.36), because  $\Upsilon_M(2\alpha_0) = 1$ , as can be checked directly using the integral definition of the function  $\Upsilon_M(x)$  in (3.28). Some specific values of  $\Upsilon_M(x)$  are listed in Appendix B. With respect to the notations, we suppress sometimes the dependence of  $\Upsilon_M$  on  $h$ , which is implicit, by writing  $\Upsilon_M(x)$  for  $\Upsilon_M(x, h)$ , as in (3.36) and in (3.39).

In eq. (3.39), the charges  $a, b, c$  in the r.h.s. are still having the discretized values of the degenerate charges:

$$a = \alpha_{m',m}, \quad b = \alpha_{n',n}, \quad c = \alpha_{p',p}^+ \tag{3.40}$$

The remaining obstacle to fully continue to the continuous values of  $a, b, c$ , is the factor

$$\rho^{2l(1-\rho')} \tag{3.41}$$

in (3.39). In particular,  $l$  is still given by (2.19).

The total  $\rho$ -factor in (3.39) could be symmetrised. It is easy to check, by using the equation (3.20) for  $k - \rho' l$ , that

$$\rho^{4\alpha_0^2-2\alpha_0(a+b+c)+2l(1-\rho')} = (\rho)^{l(1-\rho')} \times (\rho')^{k(1-\rho)} \tag{3.42}$$

Still, in the  $\rho$ -factor in (3.39), or in (3.42), there appear the numbers of screenings,  $k$  and  $l$ , in the form which could not be expressed fully by the charges  $a, b, c$ .

But it is clear, by the form of the  $\rho$ -factor in (3.42), that it could be removed by the renormalisation of the constants  $\mu_+$  and  $\mu_-$  in (2.9).

Specifically, if we give the following values for  $\mu_+, \mu_-$ :

$$\mu_+ = \frac{1}{\pi} \frac{\Gamma(\rho)}{\Gamma(1-\rho)} \times (\rho')^{-(1-\rho)}, \quad \mu_- = \frac{1}{\pi} \frac{\Gamma(\rho')}{\Gamma(1-\rho')} \times \rho^{-(1-\rho)} \tag{3.43}$$

then the normalisation factor (2.20) will disappear, from the result for the integral (B.10) in [8], and the factor

$$\rho^{-l(1-\rho')} \times (\rho')^{-k(1-\rho)} \tag{3.44}$$

will appear, in front of the expression for  $\langle V_{p',p}^+(\infty) V_{n',n}(1) V_{m',m}(0) \rangle$  in (2.12). Then we perform the transformations of Section 2 and of the present section, to arrive to (3.39), but because of the extra factor (3.44) always present, unchanged during our transformations, the extra  $\rho$  factor in (3.39), in its form in (3.42), will be cancelled. We shall get, with the choice (3.43) for the values of the Coulomb gas constants  $\mu_+, \mu_-$ , the formula (3.39), without the  $\rho$ -factor. At this point we could finally continue  $a, b, c$  to the continuous values and we obtain, finally, the formula for 3-point function in the form

$$\begin{aligned}
&\langle V_c(\infty) V_b(1) V_a(0) \rangle \\
&= \frac{\Upsilon_M(a+b+c-2\alpha_0) \Upsilon_M(-a+b+c) \Upsilon_M(a-b+c) \Upsilon_M(a+b-c)}{\Upsilon_M(2a) \Upsilon_M(2b) \Upsilon_M(2c)}
\end{aligned} \tag{3.45}$$

with  $a, b, c$  taking general, continuous values.

We summarise that the formula (3.45) has all been obtained by the analytic continuation from general 3-point function for degenerate operators of the minimal model. Though a specific normalisation of the Coulomb gas screening operators, or of the constants  $\mu_+$ ,  $\mu_-$ , was required.

The formula (3.45) is that for the vertex operators

$$V_a(z, \bar{z}) = e^{ia\varphi(z, \bar{z})}, \quad V_b(z, \bar{z}) = e^{ib\varphi(z, \bar{z})}, \quad V_c(z, \bar{z}) = e^{ic\varphi(z, \bar{z})} \quad (3.46)$$

with their nontrivial normalisation,  $N_a$  for  $V_a(z, \bar{z})$ , etc., which will be specified in the next section.

In the next section we shall get a slightly different formula, compared to (3.45), for the 3-point function  $\langle \Phi_c(\infty) \Phi_b(1) \Phi_a(0) \rangle$  of the normalised operators:

$$\Phi_a(z, \bar{z}) = \frac{1}{N_a} V_a(z, \bar{z}), \quad \Phi_b(z, \bar{z}) = \frac{1}{N_b} V_b(z, \bar{z}), \quad \Phi_c(z, \bar{z}) = \frac{1}{N_c} V_c(z, \bar{z}) \quad (3.47)$$

In the case of normalised operators (3.47) the  $\rho$  factors get cancelled automatically, independently of the choice of normalisation of the screening operators, in the course of our derivation from the original Coulomb gas formulas.

#### 4. Normalisations. 3-point function of normalised operators

We shall fix the normalisation of Coulomb gas vertex operators, to normalise them finally as in (3.47), by analysing the values of the correlation functions calculated for  $a, b, c$  having discrete, degenerate values

$$a = \alpha_{m', m}, \quad b = \alpha_{n', n}, \quad c = \alpha_{p', p} \quad (4.1)$$

We shall do it by using the original expression for the correlation functions, in terms of products of  $\Gamma$  functions, in its symmetrised form given below, and also by using the new expression, in terms of products of  $\Upsilon$  functions, which should give the same values, when  $a, b, c$  are degenerate, eq. (4.1).

The fact that we have kept, in Sections 2 and 3, the charge  $\alpha_{p', p}$  always in its conjugate form

$$c = \alpha_{p', p}^+ = 2\alpha_0 - \alpha_{p', p} \quad (4.2)$$

is not actually important for our derivations. We can relax now to the values of  $c$  in (4.1). The case of  $c = \alpha_{p', p}^+$  will correspond, with the definition of  $c$  in (4.1), to  $\alpha_{-p', -p}$ , instead of  $\alpha_{p', p}$ . Which means that we shall allow for the indices to take also the negative values. We have kept, in the derivations of Sections 2 and 3, one of the operators, in its conjugate form,  $V_{p', p}^+(z, \bar{z})$ , in part for historical reasons, to make the transition from the original formulas of [8] smoother, not to become excessively general from the start, which is not needed.

We shall go back, in this section, to the normalisation of  $\mu_+$ ,  $\mu_-$ :

$$\mu_+ = \frac{1}{\pi} \frac{\Gamma(\rho)}{\Gamma(1-\rho)}, \quad \mu_- = \frac{1}{\pi} \frac{\Gamma(\rho')}{\Gamma(1-\rho')} \quad (4.3)$$

which has been taken at the start, in Section 2, with which the correlation function of vertex operators is of the form:

$$\langle V_c(\infty) V_b(1) V_a(0) \rangle = \rho^{-4lk} \prod_{i=1}^l \frac{\Gamma(i\rho' - k)}{\Gamma(1 - i\rho' + k)} \times \prod_{j=1}^k \frac{\Gamma(j\rho)}{\Gamma(1 - j\rho)}$$

$$\begin{aligned}
& \times \prod_{i=0}^{l-1} \frac{\Gamma(1-k+\alpha'+i\rho')\Gamma(1-k+\beta'+i\rho')\Gamma(i-k\gamma'+i\rho')}{\Gamma(k-\alpha'-i\rho')\Gamma(k-\beta'-i\rho')\Gamma(k-\gamma'-i\rho')} \\
& \times \prod_{j=0}^{k-1} \frac{\Gamma(1+\alpha+j\rho)\Gamma(1+\beta+j\rho)\Gamma(1+\gamma+j\rho)}{\Gamma(-\alpha-j\rho)\Gamma(-\beta-j\rho)\Gamma(-\gamma-j\rho)} \quad (4.4)
\end{aligned}$$

when  $a, b, c$  take the degenerate values (4.1). We remind that

$$\begin{aligned}
\alpha &= 2\alpha_+ a, & \beta &= 2\alpha_+ b, & \gamma &= 2\alpha_+ c \\
\alpha' &= 2\alpha_- a, & \beta' &= 2\alpha_- b, & \gamma' &= 2\alpha_- c
\end{aligned} \quad (4.5)$$

and, by the neutrality condition for  $a, b, c$  in (4.1),

$$l = \frac{m' + n' + p' - 1}{2}, \quad k = \frac{m + n + p - 1}{2} \quad (4.6)$$

The expression in (4.4) could additionally be symmetrised by transforming  $\Gamma$ 's with  $-k$ , as follows:

$$\frac{\Gamma(x-k)}{\Gamma(1-x+k)} = \prod_{j=1}^k \frac{(-1)}{(x-j)^2} \times \frac{\Gamma(x)}{\Gamma(1-x)} \quad (4.7)$$

This gives:

$$\begin{aligned}
& \langle V_c(\infty) V_b(1) V_a(0) \rangle \\
&= \rho^{-4lk} \times \prod_{i=1}^l \prod_{j=1}^k \frac{1}{(i\rho' - j)^2} \times \prod_{i=1}^l \frac{\Gamma(i\rho')}{\Gamma(1-i\rho')} \times \prod_{j=1}^k \frac{\Gamma(j\rho)}{\Gamma(1-j\rho)} \\
& \times \prod_{i=0}^{l-1} \prod_{j=0}^{k-1} \frac{1}{(\alpha' + i\rho' - j)^2 (\beta' + i\rho' - j)^2 (\gamma' + i\rho' - j)^2} \\
& \times \prod_{i=0}^{l-1} \frac{\Gamma(1+\alpha' + i\rho')\Gamma(1+\beta' + i\rho')\Gamma(1+\gamma' + i\rho')}{\Gamma(-\alpha' - i\rho')\Gamma(-\beta' - i\rho')\Gamma(-\gamma' - i\rho')} \\
& \times \prod_{j=0}^{k-1} \frac{\Gamma(1+\alpha + j\rho)\Gamma(1+\beta + j\rho)\Gamma(1+\gamma + j\rho)}{\Gamma(-\alpha - j\rho)\Gamma(-\beta - j\rho)\Gamma(-\gamma - j\rho)} \quad (4.8)
\end{aligned}$$

Sometimes the formula (4.8) is more convenient to make various check.

We remind also that this same correlation function, expressed in terms of the function  $\Upsilon_M(x)$ , is of the form

$$\begin{aligned}
& \langle V_c(\infty) V_b(1) V_a(0) \rangle \\
&= \frac{\Upsilon_M(a+b+c-2\alpha_0)\Upsilon_M(-a+b+c)\Upsilon_M(a-b+c)\Upsilon_M(a+b-c)}{\Upsilon_M(2a)\Upsilon_M(2b)\Upsilon_M(2c)} \\
& \times (\rho)^{l(1-\rho')} \times (\rho')^{k(1-\rho)} \quad (4.9)
\end{aligned}$$

– comp. (3.39), with  $\rho$  factor expressed as in (3.42). We remind that we are actually in the normalisation (4.3) of the screening constants  $\mu_+, \mu_-$ .



Going back to the problem of normalisation of vertex operators  $V_a, V_b, V_c$ , we shall base our arguments on several simple examples.

It is shown in [Appendix B](#), by using the formulas in (4.8) and (4.9), that we have the following particular results:

1.

$$\langle III \rangle = Z \quad (4.10)$$

where

$$Z = \frac{-\rho}{(1-\rho)^2} \gamma(\rho') \gamma(\rho) = \Upsilon(-2\alpha_0) \rho^{\rho-\rho'} \quad (4.11)$$

$I$  is the identity operator,  $I = 1$ .  $\gamma(\rho) = \Gamma(\rho)/\Gamma(1-\rho)$ .  $Z$  could be considered as the partition function of the Coulomb gas, because the function  $\langle III \rangle = \langle 1 \rangle$  is given by the Coulomb gas functional integral [7,8], without normalisation. So we note it as  $Z$ . The result (4.10) is obtained in [Appendix B](#), both with (4.8), by the analytic continuation of  $\langle V_c V_b V_a \rangle$  in its charges,  $a \rightarrow 0$ ,  $b \rightarrow 0$ ,  $c \rightarrow 0$ , and also with (4.9).

2.

$$\langle I^+ III \rangle = 1 \quad (4.12)$$

Here  $I^+(z, \bar{z}) = V_{2\alpha_0}(z, \bar{z})$  is the conjugate identity operator.

3.

$$\langle I^+ I^+ I \rangle = \frac{1}{Z} \quad (4.13)$$

4.

$$\langle I^+ I^+ I^+ \rangle = \frac{1}{Z^2} \quad (4.14)$$

5.

$$\langle I^+ V_{1,2} V_{1,2} \rangle = (N_{1,2})^2 \quad (4.15)$$

where

$$(N_{1,2})^2 = \frac{\gamma(2\rho-1)}{\gamma(\rho)} \quad (4.16)$$

is the “naive” norm squared of the operator  $V_{1,2}$ . More generally [10, Section 9.1]:

$$\begin{aligned} (N_{1,n})^2 &= \prod_{j=1}^{n-1} \frac{\Gamma(1-j\rho)\Gamma(-1+(1+j)\rho)}{\Gamma(j\rho)\Gamma(2-(1+j)\rho)} \\ &= \prod_{j=1}^{n-1} \frac{\gamma((1+j)\rho-1)}{\gamma(j\rho)} \end{aligned} \quad (4.17)$$

6.

$$\langle IV_{1,2} V_{1,2} \rangle = Z(N_{1,2})^2 \quad (4.18)$$

7.

$$\langle IV_{1,2}^+ V_{1,2} \rangle = 1 \quad (4.19)$$

8.

$$\langle I^+ V_{1,2}^+ V_{1,2} \rangle = Z^{-1} \quad (4.20)$$

9.

$$\langle I^+ V_{1,2}^+ V_{1,2}^+ \rangle = Z^{-2} (N_{1,2})^{-2} \quad (4.21)$$

By comparing  $\langle III \rangle$  in (4.10) and  $\langle I^+ II \rangle$  in (4.12) we have to conclude that

$$I^+ = \frac{1}{Z} I \quad (4.22)$$

Similarly, by comparing (4.18) for  $\langle IV_{1,2} V_{1,2} \rangle$  and (4.19) for  $\langle IV_{1,2}^+ V_{1,2} \rangle$  we have to conclude that

$$V_{1,2}^+ = \frac{1}{Z(N_{1,2})^2} V_{1,2} \quad (4.23)$$

The identifications in (4.22) and (4.23) are not in the sense of the Coulomb gas theory, where these operators are different, but in the sense of the corresponding statistical model (matter) theory, where we assume that  $V$  and  $V^+$  represent the same statistical model operator, like spin in the  $q$  – component Potts model, for general, real values of  $q$ . In the sense that the two operators, in the equalities (4.22), (4.23), should give the same correlation functions.

Next, if we assume, naturally, that the result in (4.10) for  $\langle III \rangle$  is in fact the partition function of the Coulomb gas, then (4.10) could be rewritten as:

$$\langle III \rangle = Z \cdot \frac{\langle III \rangle}{Z} = Z \cdot \langle\langle III \rangle\rangle \quad (4.24)$$

where

$$\langle\langle III \rangle\rangle = \frac{\langle III \rangle}{Z} \quad (4.25)$$

is the properly normalised correlation function. Then the result in (4.10) for  $\langle III \rangle$  implies that

$$\langle\langle III \rangle\rangle = 1 \quad (4.26)$$

which assumes that  $I = 1$  is the properly normalised identity operator,

$$N(I) = 1 \quad (4.27)$$

In this case, by (4.22),

$$N(I^+) = \frac{1}{Z} \quad (4.28)$$

This is consistent with (4.12), (4.13), (4.14). For instance:

$$\begin{aligned} \langle I^+ I^+ I^+ \rangle &= Z \cdot \langle\langle I^+ I^+ I^+ \rangle\rangle \\ &= Z \cdot \langle\langle \frac{1}{Z^3} III \rangle\rangle = \frac{1}{Z^2} \end{aligned} \quad (4.29)$$

Next, the result in (4.15) could be interpreted as:

$$\begin{aligned} \langle I^+ V_{1,2} V_{1,2} \rangle &= \frac{1}{Z} \langle IV_{1,2} V_{1,2} \rangle \\ &= \langle\langle IV_{1,2} V_{1,2} \rangle\rangle = \langle\langle V_{1,2} V_{1,2} \rangle\rangle \end{aligned} \quad (4.30)$$

since  $I = 1$ . By (4.15)

$$\langle\langle V_{1,2} V_{1,2} \rangle\rangle = N_{1,2}^2 \quad (4.31)$$

so that the “naive” norm of  $V_{1,2}$  is in fact its actual norm:

$$N(V_{1,2}) = N_{1,2} \quad (4.32)$$

$(N_{1,2})^2$  given by (4.16).

Again, because of the relation (4.23), the norm of the conjugate operator is given by:

$$\begin{aligned} N(V_{1,2}^+) &= \frac{1}{Z \cdot (N_{1,2})^2} \cdot N_{1,2} \\ N(V_{1,2}^+) &= \frac{1}{Z \cdot N_{1,2}} \end{aligned} \quad (4.33)$$

The consistency with (4.18)–(4.21) could readily be verified.

Now, in general,

$$N(V_a) = N_a, \quad N(V_a^+) = \frac{1}{Z \cdot N_a} \quad (4.34)$$

where  $(N_a)^2$  is given by:

$$(N_a)^2 = \langle I^+ V_a V_a \rangle \quad (4.35)$$

The “naive” norms of vertex operators  $V_a, V_a^+$  have been defined in [10, Section 9.1]. They differ from the actual norms in (4.34) by the absence of the partition function  $Z$ , of the Coulomb gas, in the norm of  $V_a^+$ ,  $N(V_a^+)_{naive} = 1/N_a$ .

By the formula (4.9) we obtain:

$$\begin{aligned} \langle I^+ V_a V_a \rangle &= \frac{\Upsilon_M(2a) \Upsilon_M(2\alpha_0) \Upsilon_M(2\alpha_0) \Upsilon_M(2a - 2\alpha_0)}{\Upsilon_M(2a) \Upsilon_M(2a) \Upsilon_M(4\alpha_0)} \\ &\times \rho^{(n'-1)(1-\rho')} \times (\rho')^{(n-1)(1-\rho)} \end{aligned} \quad (4.36)$$

Here we assume that  $a = \alpha_{n',n}$ , and then, by (4.6)  $l = n' - 1, k = n - 1$ . Since  $\Upsilon_M(2\alpha_0) = 1$ ,  $\Upsilon_M(x) = \Upsilon_M(2\alpha_0 - x)$ , Appendix B, we obtain:

$$(N_a)^2 = \frac{\Upsilon_M(2a - 2\alpha_0)}{\Upsilon_M(2a) \Upsilon_M(-2\alpha_0)} \times \rho^{(n'-1)(1-\rho')} \times (\rho')^{(n-1)(1-\rho)} \quad (4.37)$$

Consistency could readily be checked (Appendix B) that:

$$(N(V_a^+))^2 = (N_{a^+})^2 = \langle I^+ V_a V_a^+ \rangle = \langle I^+ V_{2\alpha_0-a} V_{2\alpha_0-a} \rangle = \frac{1}{Z^2 (N_a)^2} \quad (4.38)$$

– consistent with (4.34). We could conclude that the formula (4.37), for the norm squared of the Coulomb gas vertex operator  $V_a$ , is perfectly general. Though still limited to the degenerate values of  $a$ ,  $a = \alpha_{n',n}$ , as we are still working with the formula (4.9) which contains the  $\rho$  factors, in which the numbers  $l, k$  are still present.

We shall now define the correlation function, properly normalised by  $\frac{1}{Z}$ , and which is defined for the normalised operators (3.47). We find:

$$\begin{aligned}
& \langle\langle \Phi_c(\infty) \Phi_b(1) \Phi_a(0) \rangle\rangle \\
&= \frac{1}{Z} \langle V_c(\infty) V_b(1) V_a(0) \rangle \cdot \frac{1}{N_a N_b N_c} \\
&= \frac{\rho^{-\rho+\rho'}}{\Upsilon_M(-2\alpha_0)} \frac{\Upsilon_M(a+b+c-2\alpha_0) \Upsilon_M(-a+b+c) \Upsilon_M(a-b+c) \Upsilon_M(a+b-c)}{\Upsilon_M(2a) \Upsilon_M(2b) \Upsilon_M(2c)} \\
&\quad \times \rho^{l(1-\rho')} (\rho')^{k(1-\rho)} \left[ \frac{\Upsilon_M(2a) \Upsilon(-2\alpha_0)}{\Upsilon_M(2a-2\alpha)} \times \frac{\Upsilon_M(2b) \Upsilon_M(-2\alpha_0)}{\Upsilon_M(2b-2\alpha_0)} \right. \\
&\quad \times \left. \frac{\Upsilon_M(2c) \Upsilon_M(-2\alpha_0)}{\Upsilon_M(2c-2\alpha_0)} \right]^{1/2} \times (\rho)^{-\frac{m'+n'+p'-3}{2}(1-\rho')} (\rho')^{-\frac{m+n+p-3}{2}(1-\rho)} \quad (4.39)
\end{aligned}$$

We have assumed that  $a = \alpha_{m',m}$ ,  $b = \alpha_{n',n}$ ,  $c = \alpha_{p',p}$ . We obtain:

$$\begin{aligned}
& \frac{\Upsilon_M(a+b+c-2\alpha_0) \Upsilon_M(-a+b+c) \Upsilon_M(a-b+c) \Upsilon_M(a+b-c)}{\sqrt{\Upsilon_M(2a) \Upsilon_M(2a-2\alpha_0)} \times \sqrt{\Upsilon_M(2b) \Upsilon_M(2b-2\alpha_0)} \times \sqrt{\Upsilon_M(2c) \Upsilon_M(2c-2\alpha_0)}} \\
& \times \sqrt{\Upsilon_M(-2\alpha_0)} \times \rho^{-\rho+\rho'} \times \rho^{l(1-\rho')} (\rho')^{k(1-\rho)} \\
& \times (\rho)^{-l(1-\rho')+(1-\rho')} \times (\rho')^{-k(1-\rho')+(1-\rho)} \quad (4.40)
\end{aligned}$$

We have used the formulas (4.6) for  $l$  and  $k$ . We observe that all the  $\rho$ -factors get cancelled and we find, finally, the normalised 3-point function for the normalised operators in the form:

$$\begin{aligned}
& \langle\langle \Phi_c(\infty) \Phi_b(1) \Phi_a(0) \rangle\rangle \\
& \times \frac{\Upsilon_M(a+b+c-2\alpha_0) \Upsilon_M(-a+b+c) \Upsilon_M(a-b+c) \Upsilon_M(a+b-c) \sqrt{\Upsilon_M(-2\alpha_0)}}{\sqrt{\Upsilon_M(2a) \Upsilon_M(2a-2\alpha_0)} \times \sqrt{\Upsilon_M(2b) \Upsilon_M(2b-2\alpha_0)} \times \sqrt{\Upsilon_M(2c) \Upsilon_M(2c-2\alpha_0)}} \quad (4.41)
\end{aligned}$$

In this formula everything is expressed, analytically, in terms of charges  $a, b, c$ , so that we can continue the formula to the general, continuous values of charges.

The expression (4.41) is the formula (5.1) of [6], obtained there differently. Our point is that we have derived everything, by a series of analytic continuations, from the general 3-point function of the minimal model [8].

## 5. Analytic continuation of the function $\langle\langle \Phi_c \Phi_b \Phi_a \rangle\rangle$ towards the 3-point function of Liouville

The 3-point function  $\langle\langle \Phi_c \Phi_b \Phi_a \rangle\rangle$ , which have been defined in Section 4, could further be analytically continued to give the 3-point function of the Liouville model. We have to continue the charges

$$a \rightarrow -ia, \quad b \rightarrow -ib, \quad c \rightarrow -ic \quad (5.1)$$

so that the vertex operators  $V_a = e^{ia\varphi}$ ,  $V_b = e^{ib\varphi}$ ,  $V_c = e^{ic\varphi}$ , eventually normalised,  $V_a \rightarrow \Phi_a = \frac{1}{N_a} V_a$ , etc., would go to the Liouville model vertex operators  $e^{a\varphi}$ ,  $e^{b\varphi}$ ,  $e^{c\varphi}$ . The central charge parameter  $h(=\alpha_+)$  of the corresponding conformal theory, has also to be continued:

$$h \rightarrow -ih \quad (5.2)$$

Since the function  $\langle\langle \Phi_c \Phi_b \Phi_a \rangle\rangle$  is expressed as a product of  $\Upsilon_M(x, h)$  functions, eq. (4.41), we have to continue first the function  $\Upsilon_M(x, h)$ , towards  $\Upsilon_M(-ix, -ih)$ , and then we shall have to construct with it the analytic continuation of the 3-point function  $\langle\langle \Phi_c \Phi_b \Phi_a \rangle\rangle$ .

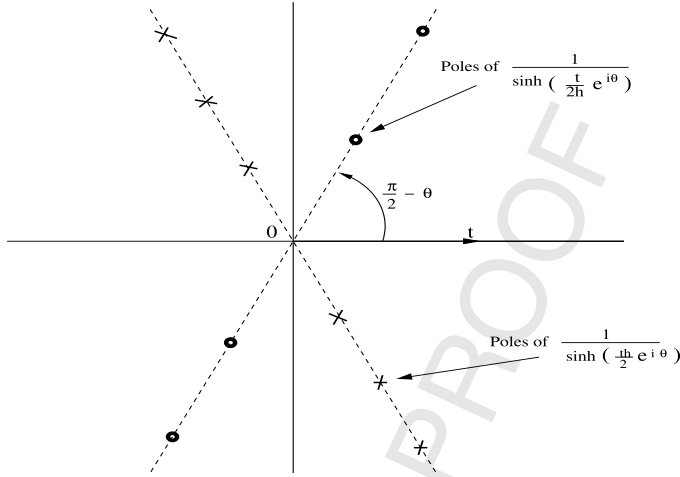


Fig. 1. The start of the analytic continuation of the function  $\Upsilon_M$ :  $\Upsilon_M(x, h) \rightarrow \Upsilon_M(\tilde{x}e^{-i\theta}, \tilde{h}e^{-i\theta})$ . The plane of the figure is the complex plane of  $t$ .

### 5.1. Analytic continuation of the function $\Upsilon_M(x, h)$

For convenience, we reproduce here the integral definition of  $\log \Upsilon_M(x, h)$ , eq. (3.28):

$$\log \Upsilon_M(x, h) = \int_0^\infty \frac{dt}{t} \left\{ (\alpha_0 - x)^2 e^{-t} - \frac{\sinh^2[(\alpha_0 - x)\frac{t}{2}]}{\sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \right\},$$

$$\alpha_0 = \frac{h}{2} - \frac{1}{2h} \quad (5.1.1)$$

We shall put

$$x = \tilde{x}e^{-i\theta}, \quad h = \tilde{h}e^{-i\theta}; \quad \theta: 0 \rightarrow \frac{\pi}{2} \quad (5.1.2)$$

where  $\tilde{x}, \tilde{h}$  are considered to be real, positives, for the moment. Eq. (5.1.1) takes the form:

$$\begin{aligned} \log \Upsilon_M(\tilde{x}e^{-i\theta}, \tilde{h}e^{-i\theta}) &= \int_0^\infty \frac{dt}{t} \left\{ \left( \frac{\tilde{h}}{2} e^{-i\theta} - \frac{1}{2\tilde{h}} e^{i\theta} - \tilde{x}e^{-i\theta} \right)^2 e^{-t} \right. \\ &\quad \left. - \frac{\sinh[(\frac{\tilde{h}}{2} e^{-i\theta} - \frac{1}{2\tilde{h}} e^{i\theta} - \tilde{x}e^{-i\theta})\frac{t}{2}]}{\sinh(\frac{t\tilde{h}}{2} e^{-i\theta}) \sinh(\frac{t}{2\tilde{h}} e^{i\theta})} \right\} \end{aligned} \quad (5.1.3)$$

In Fig. 1 are shown the poles of the expression under the integral in (5.1.3) considered as a function in the complex plane of  $t$ . We observe that there are no poles at  $t = 0$ .

The poles are due to the factors

$$\frac{1}{\sinh(\frac{t\tilde{h}}{2} e^{-i\theta})}, \quad \frac{1}{\sinh(\frac{t}{2\tilde{h}} e^{i\theta})} \quad (5.1.4)$$

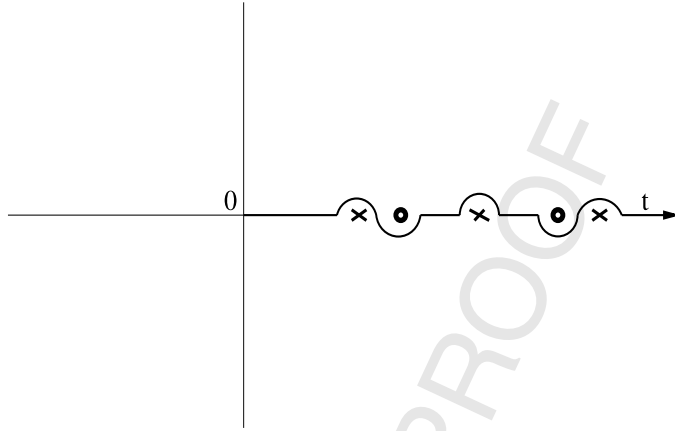


Fig. 2. The Fig. 1 when  $\theta$  becomes equal to  $\pi/2$ ,  $\Upsilon_M(\tilde{x}e^{-i\theta}, \tilde{h}e^{-i\theta}) \rightarrow \Upsilon_M(-i\tilde{x}, -i\tilde{h})$ .

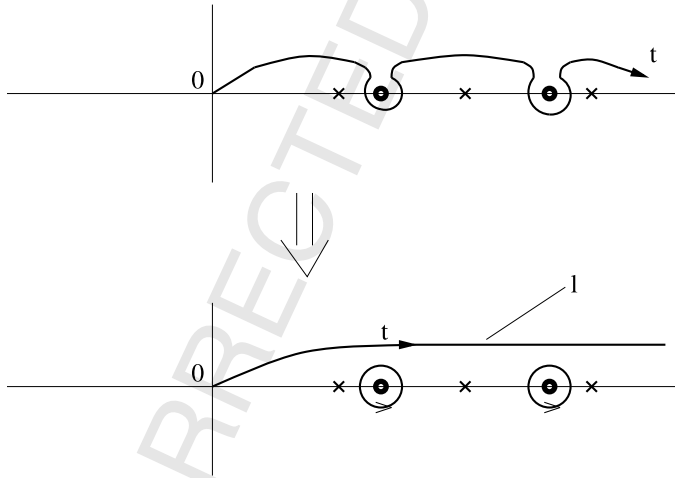


Fig. 3. Successive deformations of the integration contour over  $t$ , in the integral for  $\log \Upsilon_M(-i\tilde{x}, -i\tilde{h})$ .

As  $\theta \rightarrow \frac{\pi}{2}$ , the poles approach the real axes, the integration line of  $t$ . For  $\theta = \frac{\pi}{2}$  the poles put themselves on the real axes and the integration line, over  $t$ , gets deformed accordingly, to avoid poles, Fig. 2.

Next, the contour of integration could be deformed as shown in Fig. 3.

Accordingly, the integration breaks into two part: the sum of integrals around poles, the first part, and the integral along the line  $l$ , Fig. 3, the second part. We shall calculate them successively.

$$Poles = \sum_{n=1}^{\infty} \oint_{C_n} \frac{dt}{t} \left( - \frac{\sinh^2 \left[ \left( \frac{-i\tilde{h}}{2} - \frac{i}{2\tilde{h}} + i\tilde{x} \right) \frac{t}{2} \right]}{\sinh \left( -\frac{i\tilde{h}}{2} \right) \cdot \sinh \left( i \frac{t}{2\tilde{h}} \right)} \right) \quad (5.1.5)$$

$C_n$  is a small closed contour around  $t_n = 2\pi\tilde{h} \cdot n$ , the pole due to the factor  $1/\sinh(\frac{it}{2\tilde{h}})$ . We get:

$$\begin{aligned}
Poles &= \sum_{n=1}^{\infty} \oint_{C_n} \frac{dt}{t} \left( - \frac{e^{(-\frac{i\hbar}{2} - \frac{i}{2\hbar} + i\tilde{x})2\pi\tilde{h}n} + c.c. - 2}{2(e^{-\frac{i\hbar}{2}2\pi\tilde{h}n} - e^{\frac{i\hbar}{2}2\pi\tilde{h}n})i(-1)^n \cdot \frac{1}{2\hbar}(t - 2\pi\tilde{h}n)} \right) \\
&= \sum_{n=1}^{\infty} \frac{2\pi i}{2\pi\tilde{h}n} \cdot \frac{i}{2}(-1)^n \cdot 2\tilde{h} \times \frac{e^{-i\pi\tilde{h}^2n - i\pi n + 2\pi i\tilde{h}\tilde{x}n} + c.c. - 2}{(e^{-i\pi\tilde{h}^2n} - e^{i\pi\tilde{h}^2n})} \\
&= - \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{e^{-i\pi\tilde{h}^2n + 2\pi i\tilde{h}\tilde{x}n} + e^{i\pi\tilde{h}^2n - 2\pi i\tilde{h}\tilde{x}n} - 2 \cdot (-1)^n}{(e^{-i\pi\tilde{h}^2n} - e^{i\pi\tilde{h}^2n})} \quad (5.1.6)
\end{aligned}$$

$$Poles = - \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{e^{2\pi i\tilde{h}\tilde{x}n} + q^{2n} \cdot e^{-2\pi i\tilde{h}\tilde{x}n} - 2(-1)^n \cdot q^n}{1 - q^{2n}} \quad (5.1.7)$$

with

$$q = e^{i\pi\tilde{h}^2} \quad (5.1.8)$$

The series in (5.1.7) is almost that for the log of ratio of two  $\vartheta$ -functions:

$$- \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{e^{2iun} + q^{2n}e^{-2iun} - 2(-1)^n q^n}{1 - q^n} = -\frac{1}{4} \log q - i\left(\frac{\pi}{2} - u\right) + \log \frac{\vartheta_1(u, q)}{\vartheta_3(0, q)} \quad (5.1.9)$$

This formula is obtained in Appendix C. Using (5.1.9), the result (5.1.7), for *Poles*, could be given as:

$$Poles = -\frac{1}{4} \log q - i\left(\frac{\pi}{2} - \pi\tilde{h}\tilde{x}\right) + \log \frac{\vartheta_1(\pi\tilde{h}\tilde{x}, q)}{\vartheta_3(0, q)}, \quad q = e^{i\pi\tilde{h}^2} \quad (5.1.10)$$

The second part of (5.1.3) (with  $\theta = \frac{\pi}{2}$ ) is given by the integral along the line  $l$ , Fig. 3:

$$\begin{aligned}
I_l &= \int_0^{\infty} \frac{dt}{t} \left\{ \left( -\frac{i\hbar}{2} - \frac{i}{2\hbar} + i\tilde{x} \right)^2 e^{-t} \right. \\
&\quad \left. - \frac{\sinh^2 \left[ \left( -\frac{i\hbar}{2} - \frac{i}{2\hbar} + i\tilde{x} \right) \frac{t}{2} \right]}{\sinh \left( -\frac{i\hbar}{2} t \right) \cdot \sinh \left( \frac{i}{2\hbar} t \right)} \right\} \quad (5.1.11)
\end{aligned}$$

We shall break it into pieces as:

$$\begin{aligned}
I_l &= \lim_{\epsilon \rightarrow 0} \left\{ - \int_{\epsilon}^{\infty} \frac{dt}{t} \left( \frac{\tilde{h}}{2} + \frac{1}{2\tilde{h}} - \tilde{x} \right)^2 e^{-t} \right. \\
&\quad + \int_{C_{\epsilon}} \frac{dt}{t} \frac{\left( \frac{\tilde{h}}{2} + \frac{1}{2\tilde{h}} - \tilde{x} \right)^2 \cdot \frac{t^2}{4}}{\frac{\tilde{h}}{2} t \cdot \frac{1}{2\tilde{h}} t} \\
&\quad \left. + \int_{\epsilon}^{\infty} \frac{d\tilde{t}}{\tilde{t}} \frac{\sinh^2 \left[ \left( \frac{\tilde{h}}{2} + \frac{1}{2\tilde{h}} - \tilde{x} \right) \frac{\tilde{t}}{2} \right]}{\sinh \left( \frac{\tilde{h}\tilde{t}}{2} \right) \cdot \sinh \left( \frac{\tilde{t}}{2\tilde{h}} \right)} \right\} \quad (5.1.12)
\end{aligned}$$

In the above, we have introduced the lower limit of integration  $\epsilon$ , with the limit  $\epsilon \rightarrow 0$ , because the integrals in (5.1.12), taken separately, are divergent at  $t \rightarrow 0$ . Then we kept the integration

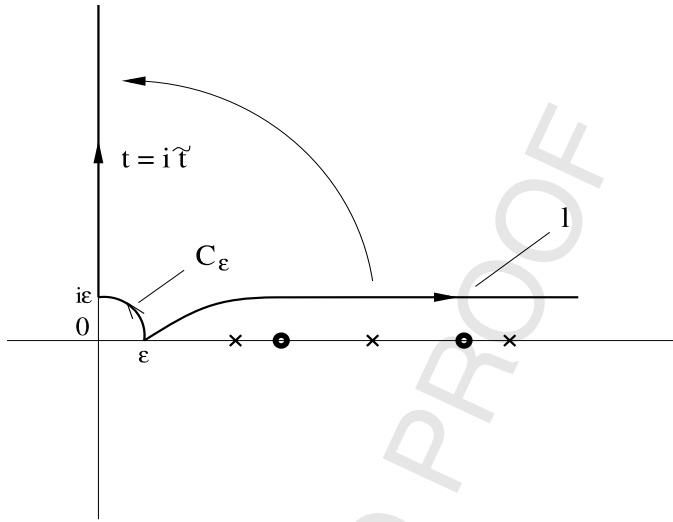


Fig. 4. Final deformation of the integration contour along the line  $l$ , in the integral  $I_l$ .

contour as it is,  $t : \epsilon \rightarrow \infty$ , in the first integral. For this part, there are no poles and the integration line  $l$ , Fig. 3, could be put back on the real axes. While for the integral of the second term in (5.1.11), the part which contains poles in the vicinity of  $l$ , Fig. 3, we have turned the contour of integration, the line  $l$ , towards the imaginary axes, as is shown in Fig. 4.

This contour then break in two, the small contour  $C_\epsilon$  around the origin, and the integral along the imaginary axes, for which we have changed the variable  $t = i\tilde{t}$ , Fig. 4. This last integral is the third one in (5.1.12). We shall drop the tildes, of  $\tilde{t}$ , in third term of (5.1.12), in the following. In the second integral, over  $C_\epsilon$  in (5.1.12), we have already developed the  $\sinh$ 's in the numerator and in the denominator. This integral is equal to

$$i \frac{\pi}{2} (\beta_0 - \tilde{x})^2 \quad (5.1.13)$$

where

$$\beta_0 = \frac{\tilde{h}}{2} + \frac{1}{2\tilde{h}} \quad (5.1.14)$$

The first and the third integrals could be put together and the limit  $\epsilon \rightarrow 0$  could be lifted. Altogether, we get

$$I_l = i \frac{\pi}{2} (\beta_0 - \tilde{x})^2 - \log \Upsilon_L(\tilde{x}, \tilde{h}) \quad (5.1.15)$$

Here  $\Upsilon_L(\tilde{x}, \tilde{h})$  is the  $\Upsilon$  function for the Liouville model, which was introduced in [5]:

$$\log \Upsilon_L(\tilde{x}, \tilde{h}) = \int_0^\infty \frac{dt}{t} \left\{ (\beta_0 - \tilde{x})^2 e^{-t} - \frac{\sinh^2[(\beta_0 - \tilde{x})\frac{\tilde{t}}{2}]}{\sinh(\frac{\tilde{h}t}{2}) \cdot \sinh(\frac{t}{2\tilde{h}})} \right\} \quad (5.1.16)$$

$\beta_0$  is given by (5.1.14).

Now, putting together the two parts of (5.1.3) (with  $\theta = \frac{\pi}{2}$ ), the *Poles*, eq. (5.1.10), and  $I_l$ , eq. (5.1.15), we obtain:



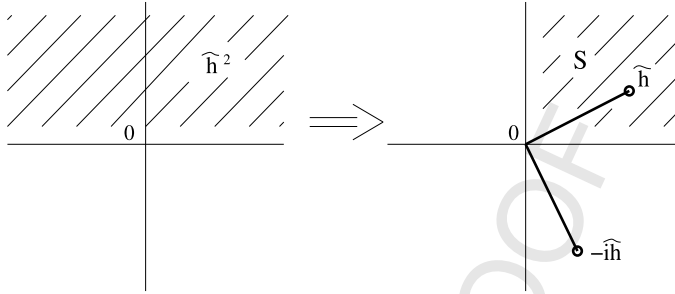


Fig. 5. Domains in the complex planes of  $\tilde{h}^2$  and of  $\tilde{h}$  where the functions  $\vartheta(x, q)$ , with  $q = e^{i\tilde{h}^2}$ , are well defined.

$$\begin{aligned} \log \Upsilon_M(-i\tilde{x}, -i\tilde{h}) &= -\frac{1}{4} \log q - i\left(\frac{\pi}{2} - \pi\tilde{h}\tilde{x}\right) + \log \frac{\vartheta_1(\pi\tilde{h}\tilde{x}, q)}{\vartheta_3(0, q)} \\ &\quad + i\frac{\pi}{2}(\beta_0 - \tilde{x})^2 - \log \Upsilon_L(\tilde{x}, \tilde{h}) \end{aligned} \quad (5.1.17)$$

or

$$\Upsilon_M(-i\tilde{x}, -i\tilde{h})\Upsilon_L(\tilde{x}, \tilde{h}) = \frac{1}{q^{1/4}} e^{i\frac{\pi}{2}(\alpha_0 + \tilde{x})^2} \times \frac{\vartheta_1(\pi\tilde{h}\tilde{x}, q)}{\vartheta_3(0, q)} \quad (5.1.18)$$

We remind that

$$\alpha_0 = \frac{\tilde{h}}{2} - \frac{1}{2\tilde{h}}, \quad \beta_0 = \frac{\tilde{h}}{2} + \frac{1}{2\tilde{h}}, \quad q = e^{i\pi\tilde{h}^2} \quad (5.1.19)$$

When passing from (5.1.17) to (5.1.18) we have grouped together two terms in (5.1.17):

$$\begin{aligned} i\frac{\pi}{2}(\beta_0 - \tilde{x})^2 - i\left(\frac{\pi}{2} - \pi\tilde{h}\tilde{x}\right) &= i\frac{\pi}{2}(\beta_0^2 - 2\beta_0\tilde{x} + \tilde{x}^2 - 1 + 2\tilde{h}\tilde{x}) \\ &= i\frac{\pi}{2}\left(\frac{\tilde{h}^2}{4} + \frac{1}{2} + \frac{1}{4\tilde{h}^2} - \left(\tilde{h} + \frac{1}{\tilde{h}}\right)\tilde{x} + \tilde{x}^2 - 1 + 2\tilde{h}\tilde{x}\right) \\ &= i\frac{\pi}{2}\left(\frac{\tilde{h}^2}{4} - \frac{1}{2} + \frac{1}{4\tilde{h}^2} + \left(\tilde{h} - \frac{1}{\tilde{h}}\right)\tilde{x} + \tilde{x}^2\right) = i\frac{\pi}{2}(\alpha_0^2 + 2\alpha_0\tilde{x} + \tilde{x}^2) = i\frac{\pi}{2}(\alpha_0 + \tilde{x})^2 \end{aligned} \quad (5.1.20)$$

The relation (5.1.18) is the formula (6.2) of [6], derived there by different methods.

One comment is in order, with respect to our derivation of the formula (5.1.18).

The  $\vartheta$  functions in (5.1.18) are not defined for  $q = e^{i\pi\tilde{h}^2}$  with  $\tilde{h}$  real. We need to have

$$\text{Im} \tilde{h}^2 > 0 \quad (5.1.21)$$

and then  $\tilde{h}$  ought be in the sector  $S$ , Fig. 5.

Also the argument  $-i\tilde{h}$  of  $\Upsilon_M(-i\tilde{x}, -i\tilde{h})$  in (5.1.18) will, in this case, be in the sector below, Fig. 5, such that the poles of the factors

$$\frac{1}{\sinh\left(\frac{-i\tilde{h}t}{2}\right) \sinh\left(\frac{it}{2\tilde{h}}\right)} \quad (5.1.22)$$

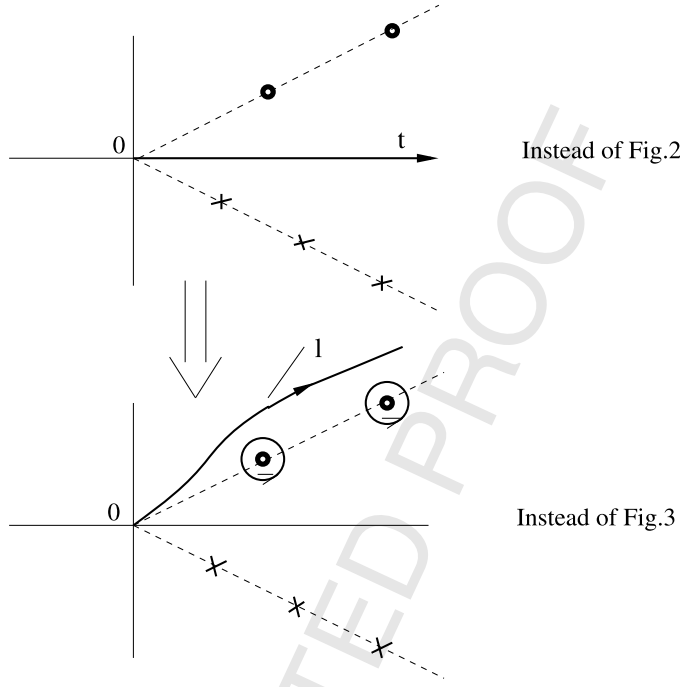


Fig. 6. Modifications of Figs. 2 and 3 when  $\tilde{h}$  is complex, i.e.  $h = \tilde{h}e^{-i\theta}$  is complex initially, at the start of the analytic continuation, and  $0 < \arg \tilde{h} < \pi/2$ .

in the integral form of  $\log \Upsilon_M(-i\tilde{x}, -i\tilde{h})$ , would stay away, still, from the integration line over  $t$  (real axes):  $-i\tilde{h}$  should stay away, to the right, from the lower part of the imaginary axes, for the integral representation of  $\log \Upsilon_M(-i\tilde{x}, -i\tilde{h})$  were well defined.

In summary,  $\tilde{h}$  should be in the sector  $S$ , Fig. 5, of its complex plane, for our analytic continuation had to make sense.

This implies that, at the start of our analytic continuation,  $h = \tilde{h}$  (eq. (5.1.2) for  $\theta = 0$ ) had to have a “small” imaginary part, positive. This implies in turn that our figures should slightly be deformed, as is indicated in Fig. 6, Fig. 7. Otherwise, the derivation stays as has been presented above, though in a somewhat (artificially) simplified context.

By the way, the fact that  $\log \Upsilon_M(-i\tilde{x}, -i\tilde{h})$ , in (5.1.18), is well defined, by its integral, as was discussed above, is seen by the first figure in Fig. 6: the poles stay away from the initial integration line, the real axes of  $t$ .

## 5.2. Analytic continuation of the function $\langle\langle \Phi_c \Phi_b \Phi_a \rangle\rangle$

The properly normalised 3-point function  $\langle\langle \Phi_c \Phi_b \Phi_a \rangle\rangle$  in (4.41) is all expressed in terms of the function  $\Upsilon_M(x, h)$ . To continue it to the Liouville sector we just have to replace  $\Upsilon_M(x, h)$  by its analytically continued form, given by the formula (5.1.18):

$$\Upsilon_M(-ix, -ih) = \frac{1}{q^{1/4}} \cdot \frac{1}{\Upsilon_L(x, h)} e^{i\frac{\pi}{2}(\alpha_0+x)^2} \times \frac{\vartheta_1(\pi hx, q)}{\vartheta_3(0, q)} \quad (5.2.1)$$

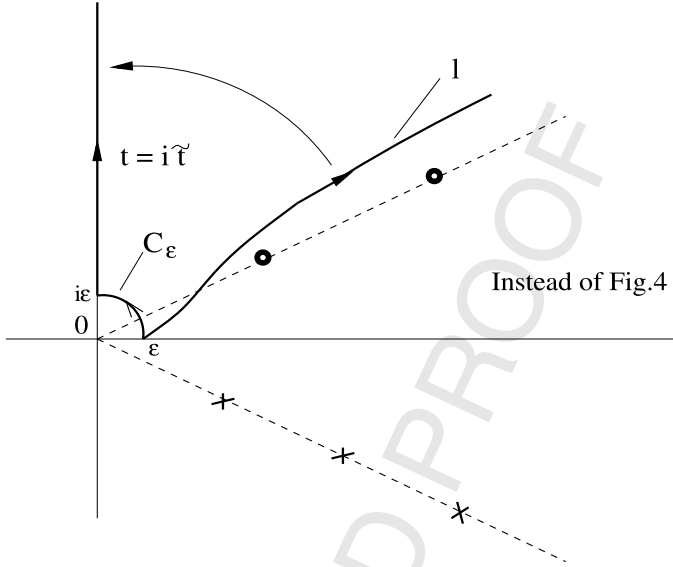


Fig. 7. The final deformation of the contour of integration along the line  $l$ , in the case when  $\tilde{h}$  is complex.

$q = e^{i\pi h^2}$ . We have suppressed the “tildes” of  $\tilde{x}$  and  $\tilde{h}$ , which served us in subsection 5.1 for the presentation purposes of the analytic continuation.

As was discussed above, at the end of subsection 5.1, for the formula (5.2.1) to be valid,  $h$  have to be complex, with

$$0 < \arg h < \frac{\pi}{2} \quad (5.2.2)$$

–  $h$  have to be in the Sector S, Fig. 5,  $\tilde{h} \equiv h$  in (5.2.1) and in Fig. 5.

Replacing every  $\Upsilon_M$  in (4.41) by its analytically continued form, eq. (5.2.1), we obtain:

$$\begin{aligned} & \langle \langle \Phi_c(\infty) \Phi_b(1) \Phi_a(0) \rangle \rangle_{continued} \\ &= \sqrt{\Upsilon_M(-i(-2\beta_0), -ih)} \times [\Upsilon_M(-i(a+b+c-2\beta_0), -ih) \\ & \quad \times \Upsilon_M(-i(-a+b+c), -ih) \Upsilon_M(-i(a-b+c), -ih) \Upsilon_M(-i(a+b-c), -ih)] \\ & \quad / [\Upsilon_M(-2ia, -ih) \Upsilon_M(-i(2a-2\beta_0), -ih) \Upsilon_M(-i2b, -ih) \\ & \quad \times \Upsilon_M(-i(2b-2\beta_0), -ih) \Upsilon_M(-i2c, -ih) \Upsilon_M(-i(2c-2\beta_0), -ih)]^{1/2} \\ &= \frac{1}{q^{1/4}} \times \frac{1}{\vartheta_3(0, q)} \times \sqrt{\Upsilon_M(-i(-2\beta_0), -ih)} \\ & \quad \times \exp\{i\frac{\pi}{2}[(\alpha_0 + a + b + c - 2\beta_0)^2 + (\alpha_0 - a + b + c)^2 + (\alpha_0 + a - b + c)^2 \\ & \quad + (\alpha_0 + a + b - c)^2 - \frac{1}{2}(\alpha_0 + 2a)^2 - \frac{1}{2}(\alpha_0 + 2b)^2 - \frac{1}{2}(\alpha_0 + 2c)^2 \\ & \quad - \frac{1}{2}(\alpha_0 + 2a - 2\beta_0)^2 - \frac{1}{2}(\alpha_0 + 2b - 2\beta_0)^2 - \frac{1}{2}(\alpha_0 + 2c - 2\beta_0)^2]\} \\ & \quad \times \frac{\sqrt{\Upsilon_L(2a, h) \Upsilon_L(2a - 2\beta_0, h) \Upsilon_L(2b, h) \Upsilon_L(2b - 2\beta_0, h) \Upsilon_L(2c, h) \Upsilon_L(2c - 2\beta_0, h)}}{\sqrt{\vartheta_1(\pi h 2a) \vartheta_1(\pi h(2a - 2\beta_0)) \vartheta_1(\pi h 2b) \vartheta_1(\pi h(2b - 2\beta_0)) \vartheta_1(\pi h 2c) \vartheta_1(\pi h(2c - 2\beta_0))}} \end{aligned}$$

$$\times \frac{\vartheta_1(\pi h(a+b+c-2\beta_0))\vartheta_1(\pi h(-a+b+c))\vartheta_1(\pi h(a-b+c))\vartheta_1(\pi h(a+b-c))}{\Upsilon_L(a+b+c-2\beta_0, h)\Upsilon_L(-a+b+c, h)\Upsilon_L(a-b+c, h)\Upsilon_L(a+b-c, h)} \quad (5.2.3)$$

We have suppressed the argument  $q$  in the functions  $\vartheta_1$  ( $\vartheta_1(\pi h 2a, q) \rightarrow \vartheta_1(\pi h 2a)$ , etc.) to compactify the expression a little bit.

With some simple algebra one can reduce  $\exp\{i\frac{\pi}{2}[\dots]\}$  in the r.h.s. of (5.2.3) to

$$\exp\{i\frac{\pi}{2}[-2\beta_0^2 + 2\alpha_0\beta_0 + \alpha_0^2]\} \quad (5.2.4)$$

and one can check, with some manipulations for  $\Upsilon_M(-i(-2\beta_0), -ih)$ , that  $\sqrt{\Upsilon_M(-i(-2\beta_0), -ih)}$  is equal to

$$\frac{1}{4\alpha_0\beta_0} \sqrt{\frac{-1}{\Upsilon_M(-2\alpha_0, h)}} e^{-i2\pi\alpha_0\beta_0} \quad (5.2.5)$$

Finally, the whole expression, in the r.h.s. of (5.2.3), preceding the part with  $\Upsilon$  functions, could be reduced to:

$$\frac{e^{i\frac{\pi}{2}\alpha_0^2}}{q^{3/4}\vartheta_3(0, q)4\alpha_0\beta_0\sqrt{\Upsilon_M(-2\alpha_0, h)}} \quad (5.2.6)$$

We obtain:

$$\begin{aligned} & \langle\langle \Phi_c(\infty)\Phi_b(1)\Phi_a(0) \rangle\rangle_{continued} \\ &= \sqrt{\Upsilon_M(-i(-2\beta_0), -ih)} \times [\Upsilon_M(-i(a+b+c-2\beta_0), -ih) \\ & \quad \times \Upsilon_M(-i(-a+b+c), -ih)\Upsilon_M(-i(a-b+c), -ih)\Upsilon_M(-i(a+b-c), -ih)] \\ & \quad / [\Upsilon_M(-2ia, -ih)\Upsilon_M(-i(2a-2\beta_0), -ih)\Upsilon_M(-i2b, -ih) \\ & \quad \times \Upsilon_M(-i(2b-2\beta_0), -ih)\Upsilon_M(-i2c, -ih)\Upsilon_M(-i(2c-2\beta_0), -ih)]^{1/2} \\ &= \frac{e^{i\frac{\pi}{2}\alpha_0^2}}{q^{3/4}\vartheta_3(0, q)4\alpha_0\beta_0\sqrt{\Upsilon_M(-2\alpha_0, h)}} \\ & \quad \times \frac{\sqrt{\Upsilon_L(2a, h)\Upsilon_L(2a-2\beta_0, h)\Upsilon_L(2b, h)\Upsilon_L(2b-2\beta_0, h)\Upsilon_L(2c, h)\Upsilon_L(2c-2\beta_0, h)}}{\sqrt{\vartheta_1(\pi h 2a)\vartheta_1(\pi h(2a-2\beta_0))\vartheta_1(\pi h 2b)\vartheta_1(\pi h(2b-2\beta_0))\vartheta_1(\pi h 2c)\vartheta_1(\pi h(2c-2\beta_0))}} \\ & \quad \times \frac{\vartheta_1(\pi h(a+b+c-2\beta_0))\vartheta_1(\pi h(-a+b+c))\vartheta_1(\pi h(a-b+c))\vartheta_1(\pi h(a+b-c))}{\Upsilon_L(a+b+c-2\beta_0, h)\Upsilon_L(-a+b+c, h)\Upsilon_L(a-b+c, h)\Upsilon_L(a+b-c, h)} \quad (5.2.7) \end{aligned}$$

Now we shall rewrite the equation (5.2.7) as follows:

$$\begin{aligned} & \langle\langle \Psi_c(\infty)\Psi_b(1)\Psi_a(0) \rangle\rangle_{continued} \\ & \quad / [\vartheta_1(\pi h(a+b+c-2\beta_0), q)\vartheta_1(\pi h(-a+b+c), q) \\ & \quad \times \vartheta_1(\pi h(a-b+c), h)\vartheta_1(\pi h(a+b-c), h)] \\ & \quad \times [\vartheta_1(\pi h 2a, q)\vartheta_1(\pi h(2a-2\beta_0), q)\vartheta_1(\pi h 2b, q)\vartheta_1(\pi h(2b-2\beta_0), q) \\ & \quad \times \vartheta_1(\pi h 2c, q)\vartheta_1(\pi h(2c-2\beta_0), q)]^{1/2} \\ &= \sqrt{\Upsilon_M(-i(-2\beta_0), -ih)} [\Upsilon_M(-i(a+b+c-2\alpha_0)\Upsilon_M(-i(-a+b+c), -ih) \\ & \quad \times \Upsilon_M(-i(a-b+c), -ih)\Upsilon_M(-i(a+b-c), -ih)] \end{aligned}$$

$$\begin{aligned}
& /[\vartheta_1(\pi h(a+b+c-2\beta), q)\vartheta_1(\pi h(-a+b+c), q) \\
& \times \vartheta_1(\pi h(a-b+c), q)\vartheta_1(\pi h(a+b-c), q)] \\
& \times [\vartheta_1(\pi h2a, q)\vartheta_1(\pi h(2a-2\beta_0), q)\vartheta_1(\pi h2b, q)\vartheta_1(\pi h(2b-2\beta_0), q) \\
& \times \vartheta_1(\pi h2c, q)\vartheta_1(\pi h(2c-2\beta_0), q)]^{1/2} \\
& /[\Upsilon_M(-i2a, -ih)\Upsilon_M(-i(2a-2\beta_0), -ih)\Upsilon_M(-i2b, -ih)\Upsilon_M(-i(2b-2\beta_0), -ih) \\
& \times \Upsilon_M(-i2c, -ih)\Upsilon_M(-i(2c-2\beta_0), -ih)]^{1/2} \\
& = \frac{e^{i\frac{\pi}{2}\alpha_0^2}}{q^{3/4}\vartheta_3(0, q)4\alpha_0\beta_0\sqrt{\Upsilon_M(-2\alpha_0, h)}} \\
& \times \frac{[\Upsilon_L(2a, h)\Upsilon_L(2a-2\beta_0, h)\Upsilon_L(2b, h)\Upsilon_L(2b-2\beta_0, h)\Upsilon_L(2c, h)\Upsilon_L(2c-2\beta_0, h)]^{1/2}}{\Upsilon_L(a+b+c-2\beta_0, h)\Upsilon_L(-a+b+c, h)\Upsilon_L(a-b+c, h)\Upsilon_L(a+b-c, h)} \\
& \quad (5.2.8)
\end{aligned}$$

The idea to organise, in this way, the analytically continued expression for  $\langle\langle\Phi_c\Phi_b\Phi_c\rangle\rangle$  is the following.

Passing from the 3-point function, for statistical models, to the 3-point function of Liouville, is delicate. The final test, or the definition, for the 3-point functions is, in fact, given by the 4 point functions, by their decomposition, or factorisation, into a product of two 3-point functions, with the sum over the states in the intermediate channel. As the spectrum, of the intermediate channel, is discrete, in the minimal model (and equally in the generalised minimal model), while the spectrum of the Liouville theory is expected to be continuous [12,13], to pass from the sum, over the intermediate states, to the integral, in the decomposition of the 4-point functions, could be organised by representing the initial sum as a sum over the residues, by adding an appropriate function which produce poles. And then the sum of the residues could be expressed by the appropriate integral.

The role of  $\vartheta_1$  functions in the denominator of (5.2.8), in the l.h.s., might be that of providing the necessary poles.

Saying it differently, the appearance of these  $\vartheta_1$  functions, the ones involving interactions (like  $\vartheta_1(\pi h(a+b+c-2\beta_0), q)$ , etc.), in the analytic continuation from minimal models to Liouville, might be interpreted as a sign, or a proof, that in fact the intermediate states spectrum of the Liouville is going to be continuous.

We are, actually, starting talking of the possibility to obtain the 4 point functions of Liouville by the analytic continuation of the well defined 4 point functions of minimal models. If realised, the associativity, in particular, will not need to be proved, will be automatic.

The task should be more complicated than that of continuing the 3-point functions. The possibility is to be attempted. For the moment we haven't yet much progressed in that direction.

The appearance of a product of "local"  $\vartheta_1$  functions, under the square root, in the l.h.s. of (5.2.8), like  $\vartheta_1(\pi h2a, q)$ ,  $\vartheta_1(\pi h(2a-2\beta_0), q)$ , etc., is related to the question of the appropriate normalisation, of the individual operators. We have putted them to the l.h.s., in (5.2.8), so that they complete the product of  $\Upsilon_M(-i2a, -ih)$ , etc., which are also the normalisation factors, analytically continued. In particular in this way the common zeros, of  $\Upsilon_M(-i2a, -ih)$  and  $\vartheta_1(\pi h2a, q)$ , etc., will be cancelled. But we would not insist on this point for the moment. It is the question of the appropriate normalisation of the Liouville vertex operators. The factor in front, in the r.h.s. of (5.2.8), is also related to the question of normalisation, of the 3-point function of Liouville. Might also to be decided by the proper definition of the 4-point function.

## 6. Discussions

Historically, the first step of the analytical continuation, from the  $(1, n)$  operators 3-point functions to the general  $(n', n)$  minimal model operators 3-point functions, the continuation presented in Section 2, it was realised long ago [14], as a by product of the curiosity, during my work on 3-point amplitudes of minimal models coupled to gravity [15]. At that time I have also defined the “naive” norms of vertex operators,  $V_a(z, \bar{z})$  and  $V_a^+(z, \bar{z})$ , the analyses described in [10, Section 9.1]. This last curiosity was better justified, because in 3-point amplitudes of minimal models coupled to gravity, after cancellations, remain only products of norms of the operators.

At the end of the previous section we have started arguing that the final precisions for the definition of 3-point functions should be given by the 4-point ones, in which the 3-point functions participate “dynamically”, in the sum over the states in the intermediate channel.

For instance, the question was raised in [6] with respect to apparent non-decoupling of some states from the outside of the minimal model (finite) Kac table of primary operators (the actual minimal model, not the generalised one). The problem that the 3-point functions (or operator algebra constants) with particular operators from outside the Kac table, do not vanish, the way they are defined analytically by the direct calculation of the 3-point functions. This breaks the “fusion rules” of minimal models. So that some decouplings have to be added by hand.

We have seen the answer to this question in our work of [7–9], where the operator algebra constants (3-point functions) have been derived from the structure of the 4-point functions of minimal models, which were the principal objects of [7–9]. We have seen that in the sum over the intermediate states, in the case of actual minimal models, it happens that for a particular the primary operator, which is placed outside of the Kac table, its contribution to the sum over the intermediate states gets cancelled by the contribution of a descendent operator of the nearby channel of another primary operator, positioned inside the table. And in this way the “fusion rules” get restored, analytically, not by hand. One example of such “delicate decoupling”, which could be seen only on the level of 4-point functions, is described in Section 9.2 of [10].

Above mentioned is just an example. But in general, we wish to stress again that, most likely, the proper, definite definition of 3-point functions should be provided by the 4-point ones. In particular, for the Liouville model.

## Acknowledgements

I grateful to Marco Picco and Raoul Santachiara for numerous useful discussions.

## Appendix A. Formulas needed for the analytic continuation of $C_{n,m}^P(\rho)$ in Section 2

In Section 2 we have defined the factors  $g_k(\rho)$ ,  $G_{lk}(\rho)$ ,  $g_k^{(\alpha)}(\rho)$ ,  $G_{lk}^{(\alpha)}(\rho)$ , in (2.26)–(2.29), so that the functions  $C_{n,m}^P(\rho)$  and  $C_{(n',n)(m',m)}^{(P,p)}$  were given by the products in (2.30) and (2.31). To prove the formula (2.35), the first step of the analytic continuation, we have used the formulas (2.32), (2.33). We shall prove these relations now.

$g_k(\rho)$ ,  $G_{lk}(\rho)$ .

Taking log of (2.26) we get:

$$\log g_k(\rho) = \sum_{j=1}^k (\log \Gamma(j\rho) - \log \Gamma(1 - j\rho)) \quad (\text{A.1})$$

Next we use the integral representation of  $\Gamma(x)$ :

$$\log \Gamma(x) = \int_0^{\infty} \frac{dt}{t} [(x-1)e^{-t} - \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}] \quad (\text{A.2})$$

$$\log \Gamma(x) - \log \Gamma(1-x) = \int_0^{\infty} \frac{dt}{t} [(2x-1)e^{-t} + \frac{e^{-xt} - e^{-(1-x)t}}{1 - e^{-t}}] \quad (\text{A.3})$$

One finds:

$$\begin{aligned} \log g_k(\rho) &= \int_0^{\infty} \frac{dt}{t} \sum_{j=1}^k [(2j\rho-1)e^{-t} + \frac{e^{-j\rho t} - e^{-(1-j\rho)t}}{1 - e^{-t}}] \\ &= \int_0^{\infty} \frac{dt}{t} [(k(k+1)\rho - k)e^{-t} \\ &\quad + \frac{1}{1 - e^{-t}} (\frac{e^{-\rho t}(1 - e^{-k\rho t})}{1 - e^{-\rho t}} - \frac{e^{-t}e^{\rho t}(1 - e^{k\rho t})}{1 - e^{\rho t}})] \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} \log g_k(\rho) &= \int_0^{\infty} \frac{dt}{t} [(k(k+1)\rho - k)e^{-t} \\ &\quad + \frac{1}{1 - e^{-t}} (\frac{e^{-\rho t}(1 - e^{-k\rho t})}{1 - e^{-\rho t}} + \frac{e^{-t}(1 - e^{k\rho t})}{1 - e^{\rho t}})] \end{aligned} \quad (\text{A.5})$$

In a similar way, by taking log of (2.27) we find:

$$\begin{aligned} \log G_{lk}(\rho) &= \int_0^{\infty} \frac{dt}{t} [(l(l+1)\rho' - 2kl - l + k(k+1)\rho - k)e^{-t} \\ &\quad + \int_0^{\infty} \frac{dt}{t} \frac{1}{(1 - e^{-t})(1 - e^{-\rho' t})} (e^{kt - \rho' t}(1 - e^{-l\rho' t}) + e^{-(1+k)t}(1 - e^{l\rho' t})) \\ &\quad + \int_0^{\infty} \frac{dt}{t} \frac{1}{(1 - e^{-t})(1 - e^{-\rho t})} (e^{-\rho t}(1 - e^{-k\rho t}) + e^{-t}(1 - e^{k\rho t}))] \end{aligned} \quad (\text{A.6})$$

Next, replacing  $k$  by  $k - \rho' l$  in (A.5), we get:

$$\begin{aligned} \log g_{k-\rho' l}(\rho) &= \int_0^{\infty} \frac{dt}{t} [((k - \rho' l)(k - \rho' l + 1)\rho - (k - \rho' l))e^{-t} \\ &\quad + \frac{1}{(1 - e^{-t})(1 - e^{-\rho t})} (e^{-\rho t}(1 - e^{-(k-\rho' l)\rho t}) + e^{-t}(1 - e^{(k-\rho' l)\rho t}))] \end{aligned} \quad (\text{A.7})$$

Our purpose is to compare  $\log G_{lk}(\rho)$ , (A.6), and  $\log g_{k-\rho' l}(\rho)$ , the expression above.

For the polynomial part, the coefficient of  $e^{-t}$  in (A.7), one finds:

$$\begin{aligned} & (k - \rho' l)(k - \rho' l + 1)\rho - (k - \rho' l) \\ & = k(k + 1)\rho - l(2k + 1) + \rho' l^2 - k + \rho' l \\ & = k(k + 1)\rho - 2kl - l + \rho' l(l + 1) - k \end{aligned} \quad (\text{A.8})$$

This agrees with the polynomial part in (A.6).

To give the exponential parts of (A.6) and (A.7) a similar appearance, we shall change the variable of integration  $t$  in the first integral of the exponential part in (A.6) as  $t \rightarrow \rho t$ , i.e.  $t = \rho \tilde{t}$ , and shall we drop the tilde afterwards.

We ignore for the moment the fact that the integral, of the exponential part taken alone, is divergent at  $t \rightarrow 0$ . We shall take care of the extra terms, the “anomaly” terms, which are due to this divergence, a little bit later.

After the change  $t \rightarrow \rho t$  in the first integral of the exponential part in (A.6), the whole exponential part of (A.6) takes the form:

$$\begin{aligned} & \int_0^\infty \frac{dt}{t} \frac{1}{(1 - e^{-\rho t})(1 - e^{-t})} [e^{k\rho t - t}(1 - e^{-lt}) + e^{-(1+k)\rho t}(1 - e^{lt})] \\ & + \int_0^\infty \frac{dt}{t} \frac{1}{(1 - e^{-t})(1 - e^{-\rho t})} [e^{-\rho t}(1 - e^{-k\rho t}) + e^{-t}(1 - e^{k\rho t})] + \text{anomaly} \\ & = \int_0^\infty \frac{dt}{t} \frac{1}{(1 - e^{-t})(1 - e^{-\rho t})} [e^{k\rho t - t} - e^{k\rho t - t - lt} + e^{-(1+k)\rho t} - e^{-(1+k)\rho t + lt} \\ & + e^{-\rho t} - e^{-\rho t - k\rho t} + e^{-t} - e^{-t + k\rho t}] + \text{anomaly} \\ & = \int_0^\infty \frac{dt}{t} \frac{1}{(1 - e^{-t})(1 - e^{-\rho t})} [e^{-\rho t}(1 - e^{-k\rho t + lt}) + e^{-t}(1 - e^{k\rho t - lt})] + \text{anomaly} \end{aligned} \quad (\text{A.9})$$

Apart from the “anomaly”, the expression above agrees with the exponential part of (A.7). The agreement for the polynomial parts of (A.6) and (A.7) has been established earlier, (A.8). So that we find the relation:

$$\log G_{lk}(\rho) = \log g_{k-\rho' l}(\rho) + \text{anomaly} \quad (\text{A.10})$$

#### Anomaly.

With respect to the divergence at  $t \rightarrow 0$ , the integral in (A.6) for which we have changed the variable of integration,  $t \rightarrow \rho t$ , this integral should have been taken, more properly, in the form:

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_\epsilon^\infty \frac{dt}{t} \frac{1}{(1 - e^{-t})(1 - e^{-\rho' t})} [e^{kt - \rho' t}(1 - e^{-l\rho' t}) + e^{-(1+k)t}(1 - e^{l\rho' t})] \right\}$$

After the change  $t \rightarrow \rho t$ , ( $t = \rho \tilde{t}$ , and we drop the tilde afterwards) we obtain:

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_{\frac{\epsilon}{\rho}}^\infty \frac{dt}{t} \frac{1}{(1 - e^{-\rho t})(1 - e^{-t})} \times [e^{\rho kt - t}(1 - e^{-lt}) + e^{-(1+k)\rho t}(1 - e^{lt})] \right\}$$



$$\begin{aligned}
 &= \lim_{\epsilon \rightarrow 0} \left\{ \int_{\epsilon}^{\infty} \frac{dt}{t} \frac{1}{(1 - e^{-\rho t})(1 - e^{-t})} \times [e^{\rho k t - t}(1 - e^{-lt}) + e^{-(1+k)\rho t}(1 - e^{lt})] \right. \\
 &\quad \left. + \int_{\frac{\epsilon}{\rho}}^{\epsilon} \frac{dt}{t} \frac{1}{(1 - e^{-\rho t})(1 - e^{-t})} \times [e^{\rho k t - t}(1 - e^{-lt}) + e^{-(1+k)\rho t}(1 - e^{lt})] \right\} \quad (\text{A.11})
 \end{aligned}$$

The first integral in the expression above goes to joint the rest, the polynomial part and the second integral of the exponential part in (A.6), which also should have been treated properly with the limit  $\epsilon \rightarrow 0$  for the lower limit of integration.

But the second integral in (A.11) above gives the additional term, which we have called “anomaly”. We find (the limit  $\epsilon \rightarrow 0$  is assumed):

$$\begin{aligned}
 \text{anomaly} &= \int_{\frac{\epsilon}{\rho}}^{\epsilon} \frac{dt}{t} \frac{1}{\rho t^2} \left[ (1 + \rho k t - t) \left( l t - \frac{1}{2} l^2 t^2 \right) + (1 - (1 + k)\rho t) \left( -l t - \frac{1}{2} l^2 t^2 \right) \right] \\
 &= \int_{\frac{\epsilon}{\rho}}^{\epsilon} \frac{dt}{t} \frac{1}{\rho t^2} \left[ (2kl + l)\rho t^2 - l(l + 1)t^2 \right] = \int_{\frac{\epsilon}{\rho}}^{\epsilon} \frac{dt}{t} [2kl + l - \rho' l(l + 1)] \\
 &= \log \rho \cdot (2kl + l - \rho' l(l + 1)) \quad (\text{A.12})
 \end{aligned}$$

Finally, we get, by (A.10) and (A.12),

$$\log G_{k,l}(\rho) = \log g_{k-\rho'l}(\rho) + \log \rho \cdot (2kl + l - \rho' l(l + 1)) \quad (\text{A.13})$$

which is the relation (2.32).

The second relation, eq. (2.33), is derived in the same way, by starting with the log’s of products for  $g_k^{(\alpha)}(\rho)$  and  $G_{lk}^{(\alpha)}(\rho)$  in (2.28) and (2.29).

#### Translation relations of $\Upsilon_M(x, h)$ .

Just for completeness, we shall show how the discrete translation relations for the function  $\Upsilon_M(x, h)$  [5,6], the relations in (B.1), (B.2), could be derived.

We shall show it with the relation (B.1). We get:

$$\begin{aligned}
 \log \Upsilon_M(x + h) &= \int_0^{\infty} \frac{dt}{t} \left\{ (\alpha_0 - x - h)^2 e^{-t} - \frac{\sinh^2[(\alpha_0 - x - h)\frac{t}{2}]}{\sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \right\} \\
 &= \int_0^{\infty} \frac{dt}{t} \left\{ [(\alpha_0 - x)^2 - 2h(\alpha_0 - x) + h^2] e^{-t} - \frac{\sinh^2[(\alpha_0 - x - h)\frac{t}{2}]}{\sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \right\} \quad (\text{A.14})
 \end{aligned}$$

As  $\alpha_0 = \frac{h}{2} - \frac{1}{2h}$ , one gets:

$$\log \Upsilon_M(x + h) = \int_0^{\infty} \frac{dt}{t} \left\{ [(\alpha_0 - x)^2 + 2hx + 1] e^{-t} - \frac{\sinh^2[(\alpha_0 - x - h)\frac{t}{2}]}{\sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \right\} \quad (\text{A.15})$$

We consider

$$\log \gamma(-hx) = \log \Gamma(-hx) - \log \Gamma(1+hx)$$

$$= \int_0^\infty \frac{dt}{t} \{ (-2hx-1)e^{-t} + \frac{e^{hxt} - e^{-(1+hx)t}}{1-e^{-t}} \} \quad (\text{A.16})$$

– to compensate the extra terms in the polynomial part of (A.15). We have used the integral representation in (A.3).

For the sum of (A.15) and (A.16) we obtain:

$$\log \Upsilon_M(x+h) + \log \gamma(-hx)$$

$$= \int_0^\infty \frac{dt}{t} \{ (\alpha_0 - x)^2 e^{-t} + \frac{e^{hxt} - e^{-(1+hx)t}}{1-e^{-t}} - \frac{\sinh^2[(\alpha_0 - x - h)\frac{t}{2}]}{\sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \} \quad (\text{A.17})$$

The first term in the integral above is already that of

$$\log \Upsilon_M(x, h) = \int_0^\infty \frac{dt}{t} \{ (\alpha_0 - x)^2 e^{-t} - \frac{\sinh^2[(\alpha_0 - x)\frac{t}{2}]}{\sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \} \quad (\text{A.18})$$

We shall transform, separately, the second and the third terms in (A.17).

By transforming  $t \rightarrow \frac{t}{h}$  in the integral of the second term in (A.17), we obtain:

$$\int_0^\infty \frac{dt}{t} \frac{e^{xt} - e^{-\frac{t}{h}-xt}}{1 - e^{-\frac{t}{h}}} \quad (\text{A.19})$$

and we continue its transformation as follows:

$$= \int_0^\infty \frac{dt}{t} \frac{(e^{xt} - e^{-\frac{t}{h}-xt})(1 - e^{-ht})}{(1 - e^{-\frac{t}{h}})(1 - e^{-ht})} = \int_0^\infty \frac{dt}{t} \frac{e^{\frac{ht}{2} + \frac{t}{2h}} (e^{xt} - e^{-\frac{t}{h}-xt})(1 - e^{-ht})}{4 \sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}}$$

$$= \int_0^\infty \frac{dt}{t} \frac{e^{\frac{ht}{2} + \frac{t}{2h} + xt} - e^{\frac{ht}{2} - \frac{t}{2h} - xt} - e^{-\frac{ht}{2} + \frac{t}{2h} + xt} + e^{-\frac{ht}{2} - \frac{t}{2h} - xt}}{4 \sinh \frac{t}{2h} \cdot \sinh \frac{ht}{2}} \quad (\text{A.20})$$

Now we shall transform, a little bit, the third term in the integral of (A.17):

$$\int_0^\infty \frac{dt}{t} \left\{ -\frac{e^{(\alpha_0 - x - h)t} + e^{-(\alpha_0 - x - h)t} - 2}{4 \sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \right\}$$

$$= \int_0^\infty \frac{dt}{t} \left\{ -\frac{e^{-\frac{ht}{2} - \frac{t}{2h} - xt} + e^{\frac{ht}{2} + \frac{t}{2h} + xt} - 2}{4 \sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \right\} \quad (\text{A.21})$$

For the sum of (A.20) and (A.21), which is the sum of the second and the third terms in (A.17), we obtain:

$$\int_0^\infty \frac{dt}{t} \left\{ -\frac{e^{\frac{ht}{2} - \frac{t}{2h} - xt} + e^{-\frac{ht}{2} + \frac{t}{2h} + xt} - 2}{4 \sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \right\}$$

$$\begin{aligned}
&= - \int_0^{\infty} \frac{dt}{t} \frac{e^{\alpha_0 t - xt} + e^{-\alpha_0 t + xt} - 2}{4 \sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \\
&= - \int_0^{\infty} \frac{dt}{t} \frac{\sinh^2[(\alpha_0 - x)\frac{t}{2}]}{\sinh \frac{th}{2} \cdot \sinh \frac{t}{2h}} \quad (A.22)
\end{aligned}$$

Together with the first term in (A.17) this gives

$$\log \Upsilon_M(x+h) + \log \gamma(-hx) = \log \Upsilon_M(x) + \text{anomaly} \quad (A.23)$$

Here “anomaly” stands for the term which we have disregarded so far, by ignoring the divergence, at  $t \rightarrow 0$ , of the second integral in (A.17), while transforming  $t \rightarrow \frac{t}{h}$ .

Properly, we should have defined it as

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_{\epsilon}^{\infty} \frac{dt}{t} \frac{e^{hxt} - e^{-(1+hx)t}}{1 - e^{-t}} \right\} \quad (A.24)$$

With  $t = \frac{\tilde{t}}{h}$ , we obtain:

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_{h\epsilon}^{\infty} \frac{d\tilde{t}}{\tilde{t}} \frac{e^{x\tilde{t}} - e^{-\frac{\tilde{t}}{h} - x\tilde{t}}}{1 - e^{-\frac{\tilde{t}}{h}}} \right\} \quad (A.25)$$

We drop the tildes and we rewrite (A.25) as follows:

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_{\epsilon}^{\infty} \frac{dt}{t} \frac{e^{xt} - e^{-\frac{t}{h} - xt}}{1 - e^{-\frac{t}{h}}} + \int_{h\epsilon}^{\epsilon} \frac{dt}{t} \frac{e^{xt} - e^{-\frac{t}{h} - xt}}{1 - e^{-\frac{t}{h}}} \right\} \quad (A.26)$$

The first integral goes to join our derivation of the relation (A.23), with the exception of the anomaly in it, while the second integral in (A.26) gives the anomaly.

We get:

$$\begin{aligned}
\text{anomaly} &= \lim_{\epsilon \rightarrow 0} \left\{ \int_{h\epsilon}^{\epsilon} \frac{dt}{t} \frac{e^{xt} - e^{-\frac{t}{h} - xt}}{1 - e^{-\frac{t}{h}}} \right\} = \lim_{\epsilon \rightarrow 0} \left\{ \int_{h\epsilon}^{\epsilon} \frac{dt}{t} \frac{xt + \frac{t}{h} + xt}{\frac{t}{h}} \right\} \\
&= \lim_{\epsilon \rightarrow 0} \left\{ \int_{h\epsilon}^{\epsilon} \frac{dt}{t} (2xh + 1) \right\} = -(2xh + 1) \cdot \log h \quad (A.27)
\end{aligned}$$

Finally, the relation (A.23) takes the form:

$$\log \Upsilon_M(x+h) = -\log \gamma(-hx) + \log \Upsilon_M(x) - (2xh + 1) \log h \quad (A.28)$$

$$\Upsilon_M(x+h) = \frac{1}{\gamma(-hx)} h^{-2xh-1} \Upsilon_M(x) \quad (A.29)$$

This is the relation (B.1).

## Appendix B. The function $\Upsilon_M(x, h)$ and the list of some of its values. The list of results for particular 3-point functions which are used in Section 4

We shall remind here certain properties of the function  $\Upsilon_M(x)$  [5,6]. We shall list some particular values of this function. And use shall also derive several expressions for the 3-point functions which are used in Section 4.

The function  $\Upsilon_M(x) \equiv \Upsilon_M(x, h)$ , defined in (3.28), satisfies the following “quasi-periodicity” relations, with respect to translations by  $h$  and by  $1/h$ :

$$\Upsilon_M(x+h) = \frac{1}{\gamma(-hx)} h^{-1-2hx} \times \Upsilon_M(x) \quad (\text{B.1})$$

$$\gamma(x) = \Gamma(x) / \Gamma(1-x).$$

$$\Upsilon_M(x - \frac{1}{h}) = \frac{1}{\gamma(\frac{x}{h})} h^{1-\frac{2x}{h}} \times \Upsilon_M(x) \quad (\text{B.2})$$

$$\Upsilon_M = \gamma(-hx) h^{1+2hx} \times \Upsilon_M(x+h) \quad (\text{B.3})$$

$$\Upsilon_M(x) = \gamma(\frac{x}{h}) h^{-1+\frac{2x}{h}} \times \Upsilon_M(x - \frac{1}{h}) \quad (\text{B.4})$$

$$\Upsilon_M(x-h) = \gamma(-h(x-h)) h^{1+2h(x-h)} \times \Upsilon_M(x) \quad (\text{B.5})$$

$$\Upsilon_M(x + \frac{1}{h}) = \gamma(\frac{1}{h}(x + \frac{1}{h})) h^{-1+\frac{2}{h}(x+\frac{1}{h})} \times \Upsilon_M(x) \quad (\text{B.6})$$

We have listed these properties in various forms, which are useful in actual calculations, though, evidently, (B.3)–(B.6) follow directly from (B.1), (B.2). The derivation of (B.1) is given in Appendix A.

The obvious property, by (3.28), is that

$$\Upsilon_M(2\alpha_0 - x) = \Upsilon_M(x) \quad (\text{B.7})$$

Next, one finds directly, from the integral form of  $\log \Upsilon_M(x)$  in (3.28), that

1.

$$\begin{aligned} \Upsilon_M(x) &\simeq x + \frac{1}{h}, \quad x \rightarrow -\frac{1}{h} \\ \Upsilon_M(-\frac{1}{h}) &= 0 \end{aligned} \quad (\text{B.8})$$

2.

$$\begin{aligned} \Upsilon_M(x) &\simeq h - x, \quad x \rightarrow h \\ \Upsilon_M(h) &= 0 \end{aligned} \quad (\text{B.9})$$

The integral in (3.28), which defines  $\Upsilon_M(x, h)$ , is convergent for  $-\frac{1}{h} < x < h$ . Outside, the function is defined by the analytic continuation: by the translations (B.1)–(B.6). For  $x \rightarrow -\frac{1}{h}$ , from above, and for  $x \rightarrow h$ , from below, the integral in (3.28) is logarithmically divergent at  $t \rightarrow \infty$ , which results, when evaluated, in the asymptotics in (B.8), (B.9).

3. In general, the zeros of  $\Upsilon_M(x)$ , which we denote as  $x_{n,m}^{(M)}$ , are located at

$$x_{n,m}^{(M)} = -\frac{1}{h}(n+1) - hm, \quad \text{and at}$$

$$x_{n,m}^{(M)} = \frac{1}{h}n + h(m+1) \quad (\text{B.10})$$

$n, m = 0, 1, 2, 3, \dots$

They could be obtained, with some care, from (B.8), (B.9) by the translations (B.1)–(B.6).

4. Next, it is evident from (3.28) that

$$\Upsilon_M(\alpha_0) = 1 \quad (\text{B.11})$$

5. One also finds that

$$\Upsilon_M(0) = \Upsilon_M(2\alpha_0) = 1 \quad (\text{B.12})$$

In fact, by (B.5) and (B.9), for  $x \rightarrow h$ , one finds:

$$\begin{aligned} \Upsilon_M(x-h) &= \gamma(-h(x-h))h^{1+2h(x-h)} \times \Upsilon_M(x) \\ &\simeq \frac{1}{-h(x-h)}h \times (h-x) = 1, \\ \Upsilon_M(0) &= 1 \end{aligned} \quad (\text{B.13})$$

$\Upsilon_M(2\alpha_0) = 1$  follow by the property (B.7).

The next several values of  $\Upsilon_M(x)$  are obtained by translations (B.1)–(B.6). One finds:

6.

$$\Upsilon_M(2h - \frac{1}{h}) = \Upsilon_M(h + 2\alpha_0) = \Upsilon_M(-h) = \gamma(h^2) \cdot h^{1-2h^2} = \gamma(\rho) \cdot \rho^{\frac{1}{2}-\rho} \quad (\text{B.14})$$

7.

$$\Upsilon_M(\frac{1}{h}) = \gamma(\frac{1}{h^2})h^{\frac{2}{h^2}-1} = \gamma(\rho')\rho^{\rho'-\frac{1}{2}} \quad (\text{B.15})$$

8.

$$\Upsilon_M(-h) = \gamma(\rho)\rho^{\frac{1}{2}-\rho} \quad (\text{B.16})$$

9.

$$\begin{aligned} \Upsilon_M(-2\alpha_0) &= \Upsilon_M(-h + \frac{1}{h}) \\ &= \gamma(\rho-1)\gamma(\rho')\rho^{1-\rho+\rho'} \\ &= -\frac{\rho}{(\rho-1)^2}\gamma(\rho)\gamma(\rho')\rho^{-\rho+\rho'} \end{aligned} \quad (\text{B.17})$$

10.

$$\begin{aligned} \Upsilon_M(-2h + \frac{1}{h}) &= \Upsilon_M(-h - 2\alpha_0) \\ &= -\frac{1}{(1-\rho)^2}\gamma(2\rho-1)\gamma(\rho)\gamma(\rho')\rho^{\frac{5}{2}-3\rho+\rho'} \end{aligned} \quad (\text{B.18})$$

11.

$$\begin{aligned} \Upsilon_M(-h + \frac{2}{h}) &= \Upsilon_M(\frac{1}{h} - 2\alpha_0) \\ &= -\frac{1}{(1-\rho')^2}\gamma(2\rho'-1)\gamma(\rho)\gamma(\rho')\rho^{-\frac{5}{2}-\rho+3\rho'} \end{aligned} \quad (\text{B.19})$$

12.

$$\Upsilon_M(-4\alpha_0) = \gamma(2\rho - 1)\gamma(2\rho' - 1)\rho^{1-2\rho+2\rho'} \times \Upsilon_M(-2\alpha_0) \quad (\text{B.20})$$

We shall derive now several results for the correlation functions, the results listed in Section 4.

$\langle III \rangle$ .

$I = V_{1,1}$ .  $a = b = c = 0$ . By (4.6),  $l = k = 1$ .

By the formula (4.9),

$$\begin{aligned} \langle III \rangle &= \frac{\Upsilon_M(-2\alpha_0)(\Upsilon_M(0))^3}{(\Upsilon_M(0))^3} \times \rho^{1-\rho'} \times (\rho')^{1-\rho} \\ &= \Upsilon_M(-2\alpha_0) \times \rho^{\rho-\rho'} = -\frac{\rho}{(1-\rho)^2} \gamma(\rho) \gamma(\rho') \end{aligned} \quad (\text{B.21})$$

– if we use the value (B.17) for  $\Upsilon_M(-2\alpha_0)$ .

By the formula (4.8), if we put  $a = 0$ ,  $b = 0$ ,  $c = 0$  directly, we shall get a problem, the expression will not be defined. But the integral, which gives (4.8), is defined, in fact, with a single condition,

$$a + b + c + l\alpha_- + k\alpha_- = 2\alpha_0 \quad (\text{B.22})$$

with  $l, k$  being integers. Separately,  $a, b, c$  do not have to be degenerate, to make the expression (4.8) valid. This allows to define the function  $\langle III \rangle$ , with (4.8), as a limit  $a \rightarrow 0$ ,  $b \rightarrow 0$ ,  $c \rightarrow 0$ , while keeping  $a + b + c = 0$  to make the condition (B.22) satisfied, with  $l = k = 1$ .

In this way, for  $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$  being small, and  $l = k = 1$ , we get, by (4.8):

$$\begin{aligned} \langle V_c V_b V_a \rangle &= \rho^{-4} \times \frac{1}{(\rho' - 1)^2} \frac{\Gamma(\rho')}{\Gamma(1 - \rho')} \frac{\Gamma(\rho)}{\Gamma(1 - \rho)} \times \frac{1}{(\alpha')^2 (\beta')^2 (\gamma')^2} \\ &\times \frac{\Gamma(1 + \alpha') \Gamma(1 + \beta') \Gamma(1 + \gamma')}{\Gamma(-\alpha') \Gamma(-\beta') \Gamma(-\gamma')} \times \frac{\Gamma(1 + \alpha) \Gamma(1 + \beta) \Gamma(1 + \gamma)}{\Gamma(-\alpha) \Gamma(-\beta) \Gamma(-\gamma)} \\ &\simeq \rho^{-4} \frac{1}{(\rho' - 1)^2} \gamma(\rho') \gamma(\rho) \frac{1}{(\alpha')^2 (\beta')^2 (\gamma')^2} (-\alpha') (-\beta') (-\gamma') \\ &\times (-\alpha) (-\beta) (-\gamma) \end{aligned} \quad (\text{B.23})$$

Since  $\alpha = -\rho\alpha'$ ,  $\alpha'\alpha = -\rho(\alpha')^2$ , etc., we get

$$\begin{aligned} \langle V_c V_b V_a \rangle &\simeq \frac{\rho^{-4}}{(\rho' - 1)^2} \times \gamma(\rho') \gamma(\rho) \times (-\rho^3) \\ &= -\frac{\rho^{-1}}{(1 - \rho)^2 \cdot (\rho')^2} \gamma(\rho') \gamma(\rho) = -\frac{\rho}{(1 - \rho)^2} \gamma(\rho') \gamma(\rho) \end{aligned} \quad (\text{B.24})$$

In the limit  $a \rightarrow 0$ ,  $b \rightarrow 0$ ,  $c \rightarrow 0$  we get

$$\langle V_0 V_0 V_0 \rangle = \langle III \rangle = -\frac{\rho}{(1 - \rho)^2} \gamma(\rho') \gamma(\rho) \quad (\text{B.25})$$

which agrees with (B.21).

$\langle I+II \rangle$ .

$$\langle I+II \rangle = \langle V_{2\alpha_0} V_0 V_0 \rangle = \langle V_{-1,-1} V_{1,1} V_{1,1} \rangle \quad (\text{B.26})$$

By (4.6),  $l = k = 0$ . Then, by (4.8), one gets  $\langle I^+ II \rangle = 1$  immediately, as

$$\prod_{i=1}^0 (\dots) = \prod_{i=0}^{-1} (\dots) = 1 \quad (\text{B.27})$$

By (4.9),

$$\langle I^+ II \rangle = \frac{\Upsilon_M(0) \Upsilon_M(2\alpha_0) \Upsilon_M(2\alpha_0) \Upsilon(-2\alpha_0)}{\Upsilon_M(0) \Upsilon_M(0) \Upsilon(4\alpha_0)} \quad (\text{B.28})$$

Since  $\Upsilon(0) = \Upsilon(2\alpha_0) = 1$ ,  $\Upsilon(4\alpha_0) = \Upsilon(-2\alpha_0)$  by (B.7), we get equally

$$\langle I^+ II \rangle = 1 \quad (\text{B.29})$$

$$\langle I^+ V_{1,2} V_{1,2} \rangle.$$

For this function  $l = 0$ ,  $k = 1$ . By (4.8):

$$\begin{aligned} \langle I^+ V_{1,2} V_{1,2} \rangle &= \frac{\Gamma(\rho)}{\Gamma(1-\rho)} \times \left( \frac{\Gamma(1+\alpha)}{\Gamma(-\alpha)} \right)^2 \times \frac{\Gamma(1+\gamma)}{\Gamma(-\gamma)} \\ \alpha &= 2\alpha_{1,2}\alpha_+ = 2\left(-\frac{\alpha_+}{2}\right)\alpha_+ = -\rho, \\ \gamma &= 2 \cdot 2\alpha_0 \cdot \alpha_+ = 2(\alpha_+ + \alpha_-)\alpha_+ = 2\rho - 2, \\ \langle I^+ V_{1,2} V_{1,2} \rangle &= \frac{\Gamma(\rho)}{\Gamma(1-\rho)} \cdot \left( \frac{\Gamma(1-\rho)}{\Gamma(\rho)} \right)^2 \cdot \frac{\Gamma(2\rho-1)}{\Gamma(2-2\rho)} \\ &= \frac{\Gamma(1-\rho)}{\Gamma(\rho)} \cdot \frac{\Gamma(2\rho-1)}{\Gamma(2-2\rho)} = \frac{\gamma(2\rho-1)}{\gamma(\rho)} \end{aligned} \quad (\text{B.30})$$

– which agrees with (4.16).

By (4.9),  $a = b = -\frac{\alpha_+}{2}$ ,  $c = 2\alpha_0$ ,

$$\begin{aligned} \langle I^+ V_{1,2} V_{1,2} \rangle &= \frac{\Upsilon_M(-\alpha_+) \Upsilon_M(2\alpha_0) \Upsilon_M(2\alpha_0) \Upsilon(-\alpha_+ - 2\alpha_0)}{\Upsilon_M(-\alpha_+) \Upsilon_M(-\alpha_+) \Upsilon(4\alpha_0)} \times (\rho')^{1-\rho} \\ &= \frac{\Upsilon_M(-\alpha_+ - 2\alpha_0)}{\Upsilon_M(-\alpha_+) \Upsilon_M(-2\alpha_0)} \rho^{-1+\rho} = \frac{\Upsilon_M(-h - 2\alpha_0)}{\Upsilon_M(-h) \Upsilon_M(-2\alpha_0)} \rho^{-1+\rho} \end{aligned} \quad (\text{B.31})$$

By (B.5),

$$\begin{aligned} \Upsilon_M(-h - 2\alpha_0) &= \gamma(-h(-h - 2\alpha_0)) h^{1+2h(-h-2\alpha_0)} \times \Upsilon_M(-2\alpha_0) \\ &= \gamma(-h(-2h + \frac{1}{h})) h^{1+2h(-2h+\frac{1}{h})} \times \Upsilon_M(-2\alpha_0) \\ &= \gamma(2\rho - 1) h^{3-4\rho} \times \Upsilon_M(-2\alpha_0) \end{aligned} \quad (\text{B.32})$$

Putting it into (B.31), we get

$$\langle I^+ V_{1,2} V_{1,2} \rangle = \frac{1}{\Upsilon_M(-h)} \times \gamma(2\rho - 1) \cdot \rho^{\frac{3}{2}-2\rho} \times \rho^{-1+\rho} \quad (\text{B.33})$$

We take now the value for  $\Upsilon_M(-h)$  in (B.16). We obtain:

$$\langle I^+ V_{1,2} V_{1,2} \rangle = \frac{1}{\gamma(\rho)} \rho^{-\frac{1}{2}+\rho} \times \gamma(2\rho - 1) \rho^{\frac{3}{2}-2\rho} \times \rho^{-1+\rho} = \frac{\gamma(2\rho - 1)}{\gamma(\rho)} \quad (\text{B.34})$$

This agrees with (B.30) and with (4.15), (4.16).

$$\langle IV_{1,2} V_{1,2} \rangle = \langle V_{1,1} V_{1,2} V_{1,2} \rangle.$$

We shall calculate it with (4.9), which is simpler.

In this case  $a = b = -\frac{\alpha_+}{2}$ ,  $c = 0$ ;  $l = 1$ ,  $k = 2$ .

$$\begin{aligned} \langle IV_{1,2} V_{1,2} \rangle &= \frac{\Upsilon_M(-\alpha_+ - 2\alpha_0) \Upsilon_M(0) \Upsilon_M(0) \Upsilon_M(-\alpha_+)}{\Upsilon_M(-\alpha_+) \Upsilon_M(-\alpha_+) \Upsilon_M(0)} \rho^{1-\rho'} (\rho')^{2(1-\rho)} \\ &= \frac{\Upsilon_M(-\alpha_+ - 2\alpha_0)}{\Upsilon_M(-\alpha_+)} \rho^{1-\rho'} \rho^{-2+2\rho} \\ &= \frac{\Upsilon_M(-\alpha_+ - 2\alpha_0) \times \rho^{-1+\rho}}{\Upsilon_M(-\alpha_+) \times \Upsilon_M(-2\alpha_0)} \Upsilon_M(-2\alpha_0) \rho^{-\rho'+\rho} \end{aligned} \quad (\text{B.35})$$

The first factor is  $\langle I^+ V_{1,2} V_{1,2} \rangle = (N_{1,2})^2$ , according to (B.31), and the second factor is  $Z = \langle III \rangle$ , according to (B.21). So that we get

$$\langle IV_{1,2} V_{1,2} \rangle = Z (N_{1,2})^2 \quad (\text{B.36})$$

which confirms (4.18).

$$\langle I^+ V_{1,2}^+ V_{1,2} \rangle = \langle V_{-1,-1} V_{-1,-2} V_{1,2} \rangle.$$

In this case  $a = -\frac{\alpha_+}{2}$ ,  $b = \frac{3}{2}\alpha_+ + \alpha_-$ ,  $c = 2\alpha_0$ ;  $l = (-1 - 1 + 1 - 1)/2 = -1$ ,  $k = (-1 - 2 + 2 - 1)/2 = -1$ . With  $l = -1$ ,  $k = -1$ , the use of the formula (4.8) is blocked. We shall calculate this function with (4.9).

$$\begin{aligned} \langle I^+ V_{1,2}^+ V_{1,2} \rangle &= \frac{\Upsilon_M(2\alpha_0) \Upsilon_M(-\alpha_+) \Upsilon_M(4\alpha_0 + \alpha_+) \Upsilon_M(0)}{\Upsilon_M(3\alpha_+ + 2\alpha_-) \Upsilon_M(-\alpha_+) \Upsilon_M(4\alpha_0)} \rho^{-(1-\rho')} (\rho')^{-(1-\rho)} \\ &= \frac{\Upsilon_M(3\alpha_+ + 2\alpha_-)}{\Upsilon_M(3\alpha_+ + 2\alpha_-) \Upsilon_M(-2\alpha_0)} \rho^{-1+\rho'} \rho^{1-\rho} \\ &= \frac{1}{\Upsilon_M(-2\alpha_0) \rho^{-\rho'+\rho}} = \frac{1}{Z} \end{aligned} \quad (\text{B.37})$$

– according to (B.21). This agrees with (4.20).

We shall derive still one more result, which is claimed in (4.38).

$$\langle I^+ V_a^+ V_a^+ \rangle = \langle I^+ V_a^+ V_a^+ \rangle.$$

Here

$$\begin{aligned} a^+ &= 2\alpha_0 - \alpha_{n',n} = \frac{1+n'}{2} \alpha_+ + \frac{1+n}{2} \alpha_-, \quad c = 2\alpha_0; \\ \langle I^+ V_a^+ V_a^+ \rangle &= \langle V_{-1,-1} V_{-n',-n} V_{-n',-n} \rangle, \\ l &= \frac{-1-2n'-1}{2} = -n' - 1, \quad k = -n - 1 \end{aligned} \quad (\text{B.38})$$

We get:

$$\begin{aligned} \langle I^+ V_a^+ V_a^+ \rangle &= \frac{\Upsilon_M(4\alpha_0 - 2a) \Upsilon_M(2\alpha_0) \Upsilon_M(2\alpha_0) \Upsilon_M(2\alpha_0 - 2a)}{\Upsilon_M(4\alpha_0 - 2a) \Upsilon_M(4\alpha_0 - 2a) \Upsilon_M(4\alpha_0)} \\ &\quad \times \rho^{-(n'+1)(1-\rho')} \times (\rho')^{-(n+1)(1-\rho)} \\ &= \frac{\Upsilon_M(2a)}{\Upsilon_M(2a - 2\alpha_0) \Upsilon_M(-2\alpha_0)} \times \rho^{-(n'-1)(1-\rho')} \\ &\quad \times (\rho')^{-(n-1)(1-\rho)} \times \rho^{-2(1-\rho')} \times (\rho')^{-2(1-\rho)} \end{aligned} \quad (\text{B.39})$$



We have used the property (B.7) for  $\Upsilon_M(2\alpha_0 - a)$ ,  $\Upsilon_M(4\alpha_0 - 2a)$  and  $\Upsilon_M(4\alpha_0)$ . We find:

$$\begin{aligned} \langle I^+ V_a^+ V_a^+ \rangle &= \frac{\Upsilon_M(2a) \Upsilon_M(-2\alpha_0)}{\Upsilon_M(2a - 2\alpha_0)} \times \rho^{-(n' - 1)(1 - \rho')} \times (\rho')^{-(n - 1)(1 - \rho)} \\ &\times \frac{1}{(\Upsilon_M(-2\alpha_0) \rho^{-\rho' + \rho})^2} = \frac{1}{(N_a)^2 \times Z^2} \end{aligned} \quad (\text{B.40})$$

– according to (4.37) and (B.21). This result agrees with (4.38).

### Appendix C. Series for $\log[\vartheta_1(u)/\vartheta_3(0)]$

$\vartheta$  functions,  $\vartheta_1(u)$  and  $\vartheta_3(u)$ , could be represented by infinite products [11]:

$$\begin{aligned} \vartheta_1(u) &= 2q^{1/4} \cdot \sin u \cdot \prod_{m=1}^{\infty} (1 - 2q^{2m} \cdot \cos 2u + q^{4m})(1 - q^{2m}) \\ &= 2q^{1/4} \cdot \sin u \cdot \prod_{m=1}^{\infty} (1 - q^{2m} \cdot e^{2iu})(1 - q^{2m} \cdot e^{-2iu})(1 - q^{2m}) \end{aligned} \quad (\text{C.1})$$

$$\begin{aligned} \vartheta_3(u) &= \prod_{m=1}^{\infty} (1 + 2q^{2m-1} \cdot \cos 2u + q^{2(2m-1)})(1 - q^{2m}) \\ &= \prod_{m=1}^{\infty} (1 + q^{2m-1} \cdot e^{2iu})(1 + q^{2m-1} \cdot e^{-2iu})(1 - q^{2m}) \end{aligned} \quad (\text{C.2})$$

In particular,

$$\vartheta_3(0) = \prod_{m=1}^{\infty} (1 + q^{2m-1})^2 (1 - q^{2m}) \quad (\text{C.3})$$

For the ratio  $\vartheta_1(u)/\vartheta_3(0)$  we obtain:

$$\frac{\vartheta_1(u)}{\vartheta_3(0)} = 2q^{1/4} \cdot \sin u \cdot \prod_{m=1}^{\infty} \frac{(1 - q^{2m} e^{2iu})(1 - q^{2m} e^{-2iu})}{(1 + q^{2m-1})^2} \quad (\text{C.4})$$

Taking log of (C.4) we get:

$$\begin{aligned} \log \frac{\vartheta_1(u)}{\vartheta_3(0)} &= \frac{1}{4} \log q + \log(i(e^{-iu} - e^{iu})) \\ &+ \sum_{m=1}^{\infty} [\log(1 - q^{2m} e^{2iu}) + \log(1 - q^{2m} e^{-2iu}) - 2 \log(1 + q^{2m-1})] \\ &= \frac{1}{4} \log q + i \frac{\pi}{2} - iu + \log(1 - e^{2iu}) \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} q^{2mn} \cdot e^{2inu} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} q^{2mn} \cdot e^{-2inu} \\ &- 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} q^{2mn-n} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \log q + i \left( \frac{\pi}{2} - u \right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{2inu} \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \frac{q^{2n} \cdot e^{2inu}}{1 - q^{2n}} + \frac{q^{2n} \cdot e^{-2inu}}{1 - q^{2n}} - 2(-1)^n \frac{q^{2n} \cdot q^{-n}}{1 - q^{2n}} \right\} \\
&= \frac{1}{4} \log q + i \left( \frac{\pi}{2} - u \right) \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \frac{e^{2inu} \cdot (1 - q^{2n}) + q^{2n} \cdot e^{2inu} + q^{2n} \cdot e^{-2inu} - 2(-1)^n q^n}{1 - q^{2n}} \right\} \quad (C.5)
\end{aligned}$$

Finally, we obtain:

$$\log \frac{\vartheta_1(u)}{\vartheta_3(0)} = \frac{1}{4} \log q + i \left( \frac{\pi}{2} - u \right) - \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{e^{2inu} + q^{2n} \cdot e^{-2inu} - 2(-1)^n q^n}{(1 - q^{2n})} \quad (C.6)$$

This is the relation in (5.1.9).

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