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# Functional integral approach to the kinetic theory of inhomogeneous systems

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## Abstract

We present a derivation of the kinetic equation describing the secular evolution of spatially inhomogeneous systems with long-range interactions, the so-called inhomogeneous Landau equation, by relying on a functional integral formalism. We start from the BBGKY hierarchy derived from the Liouville equation. At the order $1/N$, where $N$ is the number of particles, the evolution of the system is characterised by its 1-body distribution function and its 2-body correlation function. Introducing associated auxiliary fields, the evolution of these quantities may be rewritten as a traditional functional integral. By functionally integrating over the 2-body autocorrelation, one obtains a new constraint connecting the 1-body DF and the auxiliary fields. When inverted, this constraint allows us to obtain the closed non-linear kinetic equation satisfied by the 1-body distribution function. This derivation provides an alternative to previous methods, either based on the direct resolution of the truncated BBGKY hierarchy or on the Klimontovich equation. It may turn out to be fruitful to derive more accurate kinetic equations, e.g. accounting for collective effects, or higher order correlation terms.

*Keywords:* Kinetic Theory, Landau equation, Angle-action variables, Spatially inhomogeneous systems, Long-range interactions

## 1. Introduction

Recently, the dynamics and thermodynamics of systems with long-range interactions has been a subject of active research [1, 2]. The equilibrium properties of these systems, and their specificities such as negative specific heats, various kinds of phase transitions and ensemble inequivalence, are now relatively well understood. However, their dynamical evolution is more complex and many aspects of it need to be improved and exploited in order to obtain explicit predictions. A short historic of the early development of kinetic theory for plasmas, stellar systems, and other systems with long-range interactions is presented in [3, 4, 5]. The main lines of this historic are recalled below, with some complements, in order to place our work in a general context. We show in particular how the necessity to develop a kinetic theory for spatially inhomogeneous systems such as those considered in the present paper progressively emerged.

The first kinetic theory describing the statistical evolution of a large number of particles was developed by Boltzmann for a dilute neutral gas [6]. In that case, the particles do not interact except during strong local collisions. The gas is spatially homogeneous and the Boltzmann kinetic equation describes the evolution of the velocity distribution function $f(\boldsymbol{v}, t)$ of the particles under the effect of strong collisions. It can be shown to satisfy a H-Theorem corresponding to an increase of Boltzmann's entropy.

Boltzmann's kinetic theory was extended to charged gases (plasmas) by Landau [7]. In that case, the particles interact via long-range Coulombian forces but, because of electroneutrality and Debye shielding [8, 9], the interaction is screened on a lengthscale of the order of the Debye length, so that the collisions are essentially local. A neutral plasma is spatially homogeneous and the kinetic equation again describes the evolution of the velocity distribution function $f(\boldsymbol{v}, t)$ of the charges under the effect of close encounters (electrostatic deflections). Since these encounters are weak, one can expand the Boltzmann equation in the limit of small deflections and make a linear trajectory approximation. This leads to the so-called Landau equation [7] which is valid in such a weak coupling approximation.

The Landau equation exhibits a logarithmic divergence at small scales due to the neglect of strong collisions (that are rare but that cannot be totally neglected) and a logarithmic divergence at large scales due to the neglect of collective effects, i.e., the dressing of particles by their polarisation cloud (because two like sign charges repel each other and two opposite charges attract each other, a particle of a given charge has the tendency to be surrounded by a cloud of particles of opposite charge). Landau regularised these divergences by introducing rather arbitrarily a lower cut-off at the impact parameter producing a deflection at 90° (Landau length) and an upper cut-off at the Debye length. Collective effects were rigorously taken into account later by Balescu [10] and Lenard [11], leading to the Balescu-Lenard equation. They showed that this equation is valid at the order $1/\Lambda$, where $\Lambda$ is the plasma parameter (number of charges in the Debye sphere). The Balescu-Lenard equation is similar to the Landau equation except that it includes the square of the dielectric function in the denominator of the potential of interaction (in Fourier space). The dielectric function first appeared as a probe of the dynamical stability of plasmas based on the linearised Vlasov equation [12, 13]. In the Balescu-Lenard equation the dielectric function accounts for Debye shielding and removes the logarithmic divergence at large scales present in the Landau equation. This amounts to replacing the bare potential of interaction by a dressed potential of interaction. The Landau equation is recovered from the Balescu-Lenard equation by replacing the dielectric function by unity, i.e., by neglecting collective effects. In addition to including the dielectric function, the form of the kinetic equation given by Balescu and Lenard exhibits a local condition of resonance, encapsulated in a Dirac $\delta_D-$function. Resonant contributions are the drivers of diffusion on secular timescales (collisional evolution), as they do not average out. When integrating over this condition of resonance, we recover the original form of the kinetic equation given by Landau.

Self-gravitating systems are spatially inhomogeneous but the early kinetic theories pioneered by Jeans [14] and Chandrasekhar [15, 16, 17] were based on the assumption that the collisions (close encounters) between stars can be treated with a local approximation as if the system were infinite and homogeneous. Since a star experiences a large number of weak deflections, Chandrasekhar [18] developed an analogy with Brownian motion. He started from the Fokker-Planck equation and computed the diffusion and friction coefficients in a binary collision theory. This leads to a kinetic equation (usually called the Fokker-Planck equation by astrophysicists) that is equivalent to the Landau equation.[1] The gravitational Landau equation exhibits a logarithmic divergence at small scales due to the neglect of strong collisions and a logarithmic divergence at large scales due to the local approximation or to the assumption that the system is infinite and homogeneous. Strong collisions are taken into account in the treatment of Chandrasekhar which shows, without having to introduce a cut-off, that the small-scale divergence is regularised at the gravitational Landau length. The large-scale divergence is usually regularised by introducing a cut-off at the Jeans length which is the gravitational analogue of the Debye length. The gravitational Landau equation is often thought to be sufficient to describe the collisional dynamics of spherical stellar systems such as globular clusters. However, the treatment based on the local approximation, or on the assumption that the system is infinite and homogeneous, is not fully satisfactory since it leads to a logarithmic divergence. Furthermore, it prevents one from taking into account collective effects, i.e., the dressing of stars by polarisation clouds (because of the gravitational attraction, a star has the tendency to be surrounded by a cloud of stars which increases its effective gravitational mass and reduces its collisional relaxation time). Indeed, if we naively take into account collective effects by introducing the gravitational "dielectric function" in the homogeneous Balescu-Lenard equation (with the sign $-Gm^2$ instead of $+e^2$) we get a strong, linear, divergence at large scales related to the Jeans instability of an artificial infinite homogeneous medium. If we enclose the system within a box, this divergence suggests that collective effects accelerate the relaxation (i.e., reduce the relaxation time) when the size of the system approaches the Jeans length (see [20] and Appendix E of [5]). However, since the size of real stellar systems is precisely of the order of the Jeans scale where the divergence occurs, this approach is not

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[1]The Landau equation only involves the square of the potential of interaction, so that it keeps the same form for Coulombian and gravitational interactions, except for a change in the prefactor: $(-e^2)^2$ has to be replaced by $(Gm^2)^2$. The kinetic equation derived by Chandrasekhar (see also [19]), albeit physically equivalent to the Landau equation, did not appear under the same mathematical form because he started from the Fokker-Planck equation $\partial_t f = \partial_{v_i} \partial_{v_j} (D_{ij} f) + \partial_{v_i} (F_i^{\mathrm{fric}} f)$ in which the diffusion tensor is placed after the two velocity derivatives, while the Landau equation can be viewed as a Fokker-Planck equation $\partial_t f = \partial_{v_i} (D_{ij} \partial_{v_j} f) + \partial_{v_i} (F_i^{\mathrm{pol}} f)$ where the diffusion tensor is placed between the two velocity derivatives. From this second rewriting, Landau obtained a symmetric expression of the collision operator from which one can directly deduce all the conservation laws of the system and derive the $H$-theorem for the Boltzmann entropy. Furthermore, Landau derived simultaneously the diffusion and friction coefficients, while Chandrasekhar obtained them from two different calculations and showed a posteriori that they were connected at equilibrium by the Einstein relation. Let us emphasise, however, that the friction force $\boldsymbol{F}^{\mathrm{fric}}$ computed by Chandrasekhar is the true friction force while the friction force $\boldsymbol{F}^{\mathrm{pol}}$ appearing in the Landau equation is the friction due to the polarisation [5].

rigorous and not fully conclusive (no divergence should occur for a stable spatially inhomogeneous system).

In order to solve these difficulties, the kinetic theory of stellar systems has recently been generalised to fully inhomogeneous systems, either when collective effects are neglected, leading to the inhomogeneous Landau equation [3, 5], or when they are taken into account, leading to the inhomogeneous Balescu-Lenard equation [21, 22]. These equations are valid at the order $1/N$, where $N$ is the number of stars in the system. These equations do not present any divergence at large scale since they take into account the finite extension of the system. They are written with angle-action variables that are appropriate to describe the intricate dynamics of stars when the system is spatially inhomogeneous and multi-periodic. They also include a condition of resonance encapuslated in a Dirac $\delta_{\mathrm{D}}$−function that generalises the one occurring in the homogeneous Balescu-Lenard equation. This condition of resonance is written with angle-action variables so that it accounts for possibly distant encounters between stars. This is a crucial difference with plasma physics where the encounters between charges are essentially local because of electroneutrality and Debye shielding. Finally, when collective effects are taken into account in stellar systems, the inhomogeneous Balescu-Lenard equation includes a response matrix written with angle-action variables that generalises the dielectric function appearing in the homogeneous Balescu-Lenard equation of plasma physics. This amounts once again to replacing the bare potential of interaction by a dressed one. This effective potential accounts for anti-shielding (the fact that the gravitational mass of a star is enhanced by its polarisation cloud) and for the reduction of the relaxation time. The inhomogeneous Landau and Balescu-Lenard equations, describing the collisional evolution of stellar systems, have been recently used in the astrophysical context [23, 24, 25], and have proven fruitful to probe complex secular regimes. In particular, these works have demonstrated that, in the case of cold stellar discs, accounting for spatial inhomogeneity and collective effects is crucial to correctly explain the results of $N$-body simulations [26]. In particular, they clearly established that collective effects cause cool discs to have shorter two-body relaxation time that one might expect, because each real particle is accompanied by a cloud of correlated particles.

There are two standard methods to derive kinetic equations for a Hamiltonian $N$−body system with long-range interactions. The first one is based on the Liouville equation for the $N$−body distribution function. One writes the first two equations of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy and close the hierarchy by neglecting three-body correlation functions. One then solves the second equation of the BBGKY hierarchy to express the two-body correlation function in terms of the one-body distribution function. Subsequently, one substitutes the result in the first equation of the BBGKY hierarchy to obtain a self-consistent kinetic equation. The same results can be obtained by using projection operator technics. The second method is based on the Klimontovich equation [27] for the discrete distribution function written as a sum of Dirac $\delta_{\mathrm{D}}$−functions. One decomposes the exact distribution function into a smooth component plus fluctuations. One then writes two evolution equations, one for the smooth component and one for the fluctuations and closes this system of equations by neglecting nonlinear terms in the equation for the fluctuations (quasilinear approximation). One then solves the equation for the fluctuations and computes the two-body correlation function in terms of the smooth one-body distribution function. Finally, one substitutes the result in the first equation to obtain a self-consistent kinetic equation. These two methods are physically equivalent, although technically different. It is usually agreed that the method based on the Klimontovich equation is simpler to implement.

In a little-known seven-page paper, [28] presented a general functional integral framework suited to the study of classical kinetic theory. Using this formalism, starting from the Liouville equation, they derived the entire Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy. More interestingly, they showed in an Appendix how this approach allowed them to derive in a simple way the homogeneous Balescu-Lenard equation [10, 11] of plasma physics. In the present paper, we propose to show how one may use the functional integral approach introduced in [28] to derive the inhomogeneous Landau equation, hence presenting a new method to obtain this kinetic equation. This equation describes the long-term evolution of isolated stable systems with long-range interactions, which evolve under the effect of their own discreteness, when collective effects are neglected. In this collisional context (i.e., where finite-$N$ effects are taken into account), one of the main difficulty is to deal with non-local resonances between distant orbits, as the upcoming calculations will emphasise. Although we have in mind the application of the kinetic theory to self-gravitating systems (this will transpire in our presentation), our results are more general, and may find application for other systems with long-range interactions.

The paper is organised as follows. Section 2 sketches a brief derivation of the BBGKY hierarchy. Section 3 presents the functional integral formalism from [28]. Section 4 illustrates how one may use this formalism to derive through a new route the inhomogeneous Landau equation. Finally, section 5 wraps up.

## 2. Derivation of the BBGKY hierarchy

In order to introduce the basic equations of the problem, we first present a brief derivation of the BBGKY hierarchy, following the notations from [5]. We consider a system made of $N$ identical particles, of individual mass $\mu = M_{\text{tot}}/N$, where $M_{\text{tot}}$ is the total mass of the system. The dynamics of these particles is fully described by Hamilton's equations which read

$$\mu \frac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} = \frac{\partial H}{\partial \boldsymbol{v}_i} \quad ; \quad \mu \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\frac{\partial H}{\partial \boldsymbol{x}_i}, \tag{1}$$

where $(\boldsymbol{x}_i, \boldsymbol{v}_i)$ stands for the position and velocity of particle $i$. In equation (1), the Hamiltonian $H$ is given by

$$H = \frac{\mu}{2} \sum_{i=1}^{N} \boldsymbol{v}_i^2 + \mu^2 \sum_{i<j} U(|\boldsymbol{x}_i - \boldsymbol{x}_j|), \tag{2}$$

where $U(|\boldsymbol{x}|)$ corresponds to the interaction potential, e.g., $U(|\boldsymbol{x}|) = -G/|\boldsymbol{x}|$ in the gravitational context. In order to obtain a statistical description of this system, we may now introduce the $N$−body probability distribution function (PDF) $P_N(\boldsymbol{x}_1, \boldsymbol{v}_1, ..., \boldsymbol{x}_N, \boldsymbol{v}_N, t)$ which gives the probability of finding at time $t$, particle 1 at position $\boldsymbol{x}_1$ with velocity $\boldsymbol{v}_1$, particle 2 at position $\boldsymbol{x}_2$ with velocity $\boldsymbol{v}_2$, etc. More precisely, $P_N$ is normalised such that

$$\int \mathrm{d}\Gamma_1 \mathrm{d}\Gamma_2 ... \mathrm{d}\Gamma_N \, P_N(\Gamma_1, \Gamma_2, ... \Gamma_N, t) = 1, \tag{3}$$

where $\Gamma_i = (\boldsymbol{x}_i, \boldsymbol{v}_i)$. The evolution of $P_N$ is governed by Liouville's equation which reads

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \left[ \boldsymbol{v}_i \cdot \frac{\partial P_N}{\partial \boldsymbol{x}_i} + \mu \mathcal{F}_i^{\text{tot}} \cdot \frac{\partial P_N}{\partial \boldsymbol{v}_i} \right] = 0, \tag{4}$$

where the total force $\mathcal{F}_i^{\text{tot}}$ exerted on particle $i$ is given by

$$\mathcal{F}_i^{\text{tot}} = \sum_{j \neq i} \mathcal{F}_{ij} = -\sum_{j \neq i} \frac{\partial U_{ij}}{\partial \boldsymbol{x}_i}. \tag{5}$$

In equation (5), we defined as $\mathcal{F}_{ij}$ the force exerted by particle $j$ on particle $i$. Introducing the potential of interaction $U_{ij} = U(|\boldsymbol{x}_i - \boldsymbol{x}_j|)$, one has $\mathcal{F}_{ij} = -\partial U_{ij}/\partial \boldsymbol{x}_i$. It is important to emphasise that the Liouville equation (4) contains the same information as the set of Hamilton's equations (1). One may now define the reduced PDFs $P_n$ for $1 \leq n \leq N$ by

$$P_n(\Gamma_1, ..., \Gamma_n, t) = \int \mathrm{d}\Gamma_{n+1} ... \mathrm{d}\Gamma_N \, P_N(\Gamma_1, ..., \Gamma_N, t). \tag{6}$$

Using the symmetry of $P_N$ w.r.t. permutations of its arguments, one can integrate equation (4) w.r.t. to $\mathrm{d}\Gamma_{n+1} ... \mathrm{d}\Gamma_N$ to obtain the evolution equation satisfied by $P_n$. This gives the general term of the BBGKY hierarchy which reads

$$\frac{\partial P_n}{\partial t} + \sum_{i=1}^{n} \boldsymbol{v}_i \cdot \frac{\partial P_n}{\partial \boldsymbol{x}_i} + \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mu \mathcal{F}_{ik} \cdot \frac{\partial P_n}{\partial \boldsymbol{v}_i} = -(N-n) \sum_{i=1}^{n} \int \mathrm{d}\Gamma_{n+1} \, \mu \mathcal{F}_{i,n+1} \cdot \frac{\partial P_{n+1}}{\partial \boldsymbol{v}_i}. \tag{7}$$

Here, one should note that the l.h.s. of equation (7) only involves the first $n$ particles, while the collision term from the r.h.s. involves the reduced PDF $P_{n+1}$ of higher order, i.e., the hierarchy is not closed. We now restrict ourselves to the first two equations of this hierarchy which read

$$\frac{\partial P_1}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial P_1}{\partial \boldsymbol{x}_1} = -(N-1) \int \mathrm{d}\Gamma_2 \, \mu \mathcal{F}_{12} \cdot \frac{\partial P_2}{\partial \boldsymbol{v}_1}, \tag{8}$$

and

$$\frac{\partial P_2}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial P_2}{\partial \boldsymbol{x}_1} + \boldsymbol{v}_2 \cdot \frac{\partial P_2}{\partial \boldsymbol{x}_2} + \mu \mathcal{F}_{12} \cdot \frac{\partial P_2}{\partial \boldsymbol{v}_1} + \mu \mathcal{F}_{21} \cdot \frac{\partial P_2}{\partial \boldsymbol{v}_2} = -(N-2) \int \mathrm{d}\Gamma_3 \, \mu \left[ \mathcal{F}_{13} \cdot \frac{\partial P_3}{\partial \boldsymbol{v}_1} + \mathcal{F}_{23} \cdot \frac{\partial P_3}{\partial \boldsymbol{v}_2} \right]. \tag{9}$$

We may now introduce the reduced distribution functions $f_n$ as

$$f_n(\Gamma_1, ..., \Gamma_n, t) = \mu^n \frac{N!}{(N-n)!} P_n(\Gamma_1, ..., \Gamma_n, t). \tag{10}$$

Equations (8) and (9) immediately take the form

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{x}_1} = -\int d\Gamma_2 \, \mathcal{F}_{12} \cdot \frac{\partial f_2}{\partial \boldsymbol{v}_1}, \tag{11}$$

and

$$\frac{\partial f_2}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_2}{\partial \boldsymbol{x}_1} + \boldsymbol{v}_2 \cdot \frac{\partial f_2}{\partial \boldsymbol{x}_2} + \mu \, \mathcal{F}_{12} \cdot \frac{\partial f_2}{\partial \boldsymbol{v}_1} + \mu \, \mathcal{F}_{21} \cdot \frac{\partial f_2}{\partial \boldsymbol{v}_2} = -\int d\Gamma_3 \left[ \mathcal{F}_{13} \cdot \frac{\partial f_3}{\partial \boldsymbol{v}_1} + \mathcal{F}_{23} \cdot \frac{\partial f_3}{\partial \boldsymbol{v}_2} \right]. \tag{12}$$

In order to emphasise the importance of the correlations between particles, we define the cluster representation of the reduced distribution functions. We introduce the $2-$body correlation $g_2$ as

$$f_2(\Gamma_1, \Gamma_2) = f_1(\Gamma_1) f_1(\Gamma_2) + g_2(\Gamma_1, \Gamma_2). \tag{13}$$

Similarly, introducing the irreducible $3-$body correlation $g_3$, one can express $f_3$ as

$$f_3(\Gamma_1, \Gamma_2, \Gamma_3) = f_1(\Gamma_1) f_1(\Gamma_2) f_1(\Gamma_3) + f_1(\Gamma_1) g_2(\Gamma_2, \Gamma_3) + f_1(\Gamma_2) g_2(\Gamma_1, \Gamma_3) + f_1(\Gamma_3) g_2(\Gamma_1, \Gamma_2) + g_3(\Gamma_1, \Gamma_2, \Gamma_3). \tag{14}$$

Within this representation, one can study the scalings of the functions $f_1$, $g_2$ and $g_3$ with the number of particles. Thanks to the definition from equation (6), one has $|P_n| \sim 1$. Since $\mu = M_{\text{tot}}/N$, the definition from equation (10) immediately gives $|f_n| \sim 1$, and in particular

$$|f_1| \sim 1. \tag{15}$$

Integrating equation (13) w.r.t. to $(\Gamma_1, \Gamma_2)$, we obtain $\int d\Gamma_1 d\Gamma_2 \, g_2(\Gamma_1, \Gamma_2) = -\mu^2 N$. Similarly, integrating equation (14) w.r.t. $(\Gamma_1, \Gamma_2, \Gamma_3)$, we obtain $\int d\Gamma_1 d\Gamma_2 d\Gamma_3 \, g_3(\Gamma_1, \Gamma_2, \Gamma_3) = 2\mu^3 N$. As a consequence, we have the scalings

$$|g_2| \sim \frac{1}{N} \quad ; \quad |g_3| \sim \frac{1}{N^2}. \tag{16}$$

Using the decompositions from equations (13) and (14), after some simple calculations, one can rewrite equations (11) and (12) as

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{x}_1} + \left[ \int d\Gamma_2 \, \mathcal{F}_{12} f_1(\Gamma_2) \right] \cdot \frac{\partial f_1}{\partial \boldsymbol{v}_1} = -\frac{\partial}{\partial \boldsymbol{v}_1} \cdot \left[ \int d\Gamma_2 \, \mathcal{F}_{12} \, g_2(\Gamma_1, \Gamma_2) \right], \tag{17}$$

and

$$\frac{1}{2} \frac{\partial g_2}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial g_2}{\partial \boldsymbol{x}_1} + \left[ \int d\Gamma_3 \, \mathcal{F}_{13} f_1(\Gamma_3) \right] \cdot \frac{\partial g_2}{\partial \boldsymbol{v}_1} + \mu \, \mathcal{F}_{12} \cdot \frac{\partial f_1}{\partial \boldsymbol{v}_1} f_1(\Gamma_2) + \left[ \int d\Gamma_3 \, \mathcal{F}_{13} \, g_2(\Gamma_2, \Gamma_3) \right] \cdot \frac{\partial f_1}{\partial \boldsymbol{v}_1}$$

$$+ \mu \, \mathcal{F}_{12} \cdot \frac{\partial g_2}{\partial \boldsymbol{v}_1} + \frac{\partial}{\partial \boldsymbol{v}_1} \cdot \left[ \int d\Gamma_3 \mathcal{F}_{13} \, g_3(\Gamma_1, \Gamma_2, \Gamma_3) \right] + (1 \leftrightarrow 2) = 0. \tag{18}$$

We may now perform a truncation at the order $1/N$ of the two equations (17) and (18). To do so, we rely on the scalings from equations (15) and (16), and on the fact that $\mu \sim 1/N$ and $|\mathcal{F}_{12}| \sim 1$. In equation (17), all the terms are of order $1/N$ or larger so that they should all be kept. In equation (18), one can note that the terms from the first line are all of order $1/N$ and have to be conserved, while all the terms from the second line are of order $1/N^2$, and may therefore be neglected.[2] In addition to these truncations, and in order to consider quantities of order 1, we finally introduce the $1-$body DF $F$ and the $2-$body correlation function $C$ as

$$F = f_1 \quad ; \quad C = \frac{g_2}{\mu}. \tag{19}$$

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[2]There is, however, a subtlety with the first term on the second line of equation (18). Indeed, even if this term is of order $1/N^2$ "in average", it can become very large when particle 2 approaches particle 1 due to the divergence of the Coulombian or gravitational force at small separations. This term accounts for the effect of strong collisions. Even if strong collisions are not dominant for systems with long-range interactions, they have to be taken into account for 3D Coulombian or gravitational systems otherwise a logarithmic divergence occurs at small scales. This implies that the $1/N$ expansion is not uniformly convergent. More details can be found in [5]. In the present paper, for the sake of simplicity, we shall ignore this difficulty.

It is straighforward to note that these functions scale like $|F| \sim 1$ and $|C| \sim 1$. Using these new functions, the first two equations (17) and (18) of the BBGKY hierarchy when truncated at the order $1/N$ take the form

$$\frac{\partial F}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial F}{\partial \boldsymbol{x}_1} + \left[ \int \mathrm{d}\Gamma_2 \, \mathcal{F}_{12} F(\Gamma_2) \right] \cdot \frac{\partial F}{\partial \boldsymbol{v}_1} = -\mu \frac{\partial}{\partial \boldsymbol{v}_1} \cdot \left[ \int \mathrm{d}\Gamma_2 \, \mathcal{F}_{12} \, C(\Gamma_1, \Gamma_2) \right], \tag{20}$$

and

$$\frac{1}{2} \frac{\partial C}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial C}{\partial \boldsymbol{x}_1} + \left[ \int \mathrm{d}\Gamma_3 \, \mathcal{F}_{13} F(\Gamma_3) \right] \cdot \frac{\partial C}{\partial \boldsymbol{v}_1} + \mathcal{F}_{12} \cdot \frac{\partial F}{\partial \boldsymbol{v}_1} F(\Gamma_2) + \left[ \int \mathrm{d}\Gamma_3 \, \mathcal{F}_{13} \, C(\Gamma_2, \Gamma_3) \right] \cdot \frac{\partial F}{\partial \boldsymbol{v}_1} + (1 \leftrightarrow 2) = 0 \,. \tag{21}$$

These two evolution equations, which only involve $F$ and $C$, are the two coupled equations which are central to the upcoming functional integral formalism. The physical interpretation of the terms appearing in these equations can be found in [5].

## 3. Functional integral formalism

[28] relied on the general functional integral formalism [29] to derive the entire BBGKY hierarchy as well as the homogeneous Balescu-Lenard equation for plasma physics. The main result used is the following one. Let us consider a dynamical quantity $f$ depending on the time $t$ and defined on a generic phase-space $\Gamma = (\boldsymbol{q}, \boldsymbol{p})$. We assume that this quantity evolves according to an equation of the form

$$[\partial_t + L]f = 0 \,, \tag{22}$$

where $L$ is a differential operator. We denote as $f_0$ the solution of equation (22). Our starting point is to rewrite the dynamical constraint from equation (22) under a functional integral of the form

$$1 = \int \mathcal{D}f \, \delta_D(f - f_0) \,. \tag{23}$$

We recall that the composition of a function and a $\delta_D-$functional satisfies

$$\begin{aligned} \delta_D([\partial_t + L]f) &= \delta_D([\partial_t + L](f - f_0)) \\ &= \frac{\delta_D(f - f_0)}{\det \left| \partial([\partial_t + L]f)/\partial f \right|} \,. \end{aligned} \tag{24}$$

As the determinant appearing in equation (24) is only a pure number, independent of the dynamical quantity $f$, it may be dropped in the next calculations. As a consequence, equation (23) becomes

$$1 = \int \mathcal{D}f \, \delta_D([\partial_t + L]f) \,. \tag{25}$$

Finally, we recall that the $\delta_D-$functional satisfies the general identity

$$\delta_D(g) = \int \mathcal{D}\lambda \exp \left[ i \int \mathrm{d}t \mathrm{d}\Gamma \, \lambda \, g \right], \tag{26}$$

where the auxiliary field $\lambda$ is defined on the same space as $g$. As a consequence, the evolution constraint on $f$ from equation (25) may be rewritten under the form

$$1 = \int \mathcal{D}f \, \mathcal{D}\lambda \exp \left\{ i \int \mathrm{d}t \, \mathrm{d}\Gamma \, \lambda \, [\partial_t + L]f \right\} . \tag{27}$$

In analogy with the classical limit of quantum field theory, the argument of the exponential $S[f, \lambda] = i \int \mathrm{d}t \mathrm{d}\Gamma \lambda [\partial_t + L]f$ in equation (27) is called the action[3], while equation (27) is the corresponding classical path integral. Finally, in

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[3]It should not be mixed up with the action coordinates from inhomogeneous dynamics, see section 4.1.

equation (27), one can note that the evolution equation of $f$ corresponds to the quantity by which the auxiliary field $\lambda$ is multiplied in the action.

In the present paper, we are interested in the long-term collisional evolution of an inhomogeneous system made of $N$ particles. As detailed in section 2, to describe such a system, one has to consider simultaneously two dynamical quantities, namely the $1-$body distribution function (DF) $F(t, \Gamma)$ and the $2-$body autocorrelation function $C(t, \Gamma_1, \Gamma_2)$. Here $F$ satisfies the normalisation constraint

$$\int d\Gamma \, F(t, \Gamma) = N \mu = M_{\text{tot}} \,, \tag{28}$$

where $M_{\text{tot}}$ is the total mass of the system, and $\mu = M_{\text{tot}}/N$ is the mass of the individual particles. As presented in section 2, at first-order in $\varepsilon = 1/N$, the evolution of the system is entirely characterised by the dynamical quantities $F$ and $C$. The truncated first two equations of the BBGKY hierarchy (20) and (21) then form a pair of coupled evolution equations which describe the simultaneous evolutions of these dynamical quantities. Introducing the auxiliary fields $\lambda_1(t, \Gamma_1)$ and $\lambda_2(t, \Gamma_1, \Gamma_2)$ respectively associated with $F$ and $C$, these coupled evolution equations may straightforwardly be rewritten under the functional form

$$1 = \int \mathcal{D}F \, \mathcal{D}C \, \mathcal{D}\lambda_1 \, \mathcal{D}\lambda_2 \, \exp\left\{ i \left[ \int dt \, d\Gamma_1 \, \lambda_1(A_1 F + B_1 C) + \frac{1}{2} \int dt \, d\Gamma_1 \, d\Gamma_2 \, \lambda_2(A_2 C + D_2 C + S_2) \right] \right\}. \tag{29}$$

In equation (29), the operators $A_1$, $B_1$, $A_2$, $D_2$ and $S_2$ (see equations (20) and (21)) are given by

$$A_1 F = \left[ \frac{\partial}{\partial t} + v_1 \cdot \frac{\partial}{\partial x_1} + \left[ \int d\Gamma_2 \, \mathcal{F}_{12} \, F(\Gamma_2) \right] \cdot \frac{\partial}{\partial v_1} \right] F(\Gamma_1) \,,$$

$$B_1 C = \mu \int d\Gamma_2 \, \mathcal{F}_{12} \cdot \frac{\partial C(\Gamma_1, \Gamma_2)}{\partial v_1} \,,$$

$$A_2 C = \left[ \frac{\partial}{\partial t} + v_1 \cdot \frac{\partial}{\partial x_1} + v_2 \cdot \frac{\partial}{\partial x_2} + \int d\Gamma_3 \, F(\Gamma_3) \left[ \mathcal{F}_{13} \cdot \frac{\partial}{\partial v_1} + \mathcal{F}_{23} \cdot \frac{\partial}{\partial v_2} \right] \right] C(\Gamma_1, \Gamma_2) \,,$$

$$D_2 C = \left[ \int d\Gamma_3 \, \mathcal{F}_{13} \, C(\Gamma_2, \Gamma_3) \right] \cdot \frac{\partial F}{\partial v_1} + (1 \leftrightarrow 2) \,,$$

$$S_2 = F(\Gamma_2) \mathcal{F}_{12} \cdot \frac{\partial F}{\partial v_1} + (1 \leftrightarrow 2) \,, \tag{30}$$

where we represented our phase space canonical variables $\Gamma$ as $\Gamma = (x, v)$, and did not write explicitly the dependence w.r.t. $t$ so as to simplify the notations. In the expression of $B_1 C$, one should note the presence of the small factor $\mu = M_{\text{tot}}/N$, illustrating the fact that we are considering a kinetic development at first order in $\varepsilon = 1/N$. Finally, in equation (29), one may note the presence of a factor $1/2$ in front of the second action term. This was only added for later convenience; it does not play any role on the final expression of the evolution equations since it was added as a global prefactor to the constraints given by the dynamical equations. One can now detail the physical content of each of the terms appearing in equation (29). Here, $A_1 F$ corresponds to the usual $1-$body Vlasov advection term, $B_1 C$ is a first-order term (because of the presence of the mass factor $\mu = M_{\text{tot}}/N$) which corresponds to the $1/N-$sourcing of the $1-$body DF's evolution under the effect of the $2-$body autocorrelation $C$. Similarly, $A_2 C$ corresponds to the usual $2-$body Vlasov advection term, $D_2 C$ corresponds to the collective effects, e.g., the Debye shielding for plasmas or the self-gravity for stellar systems, while $S_2$ is a source term, depending only on $F$, which sources the dynamics of the $2-$body autocorrelation.

In order to obtain a closed kinetic equation describing the long term evolution of $F$, the traditional approach [10, 11], as discussed above, is the following one. One can first integrate equation (29) functionally over $\lambda_2$. As in equation (27), this gives an evolution constraint $(A_2 C + D_2 C + S_2) = 0$ which couples $C$ and $F$. One must then invert this equation so as to obtain $C = C[F]$. By functionally integrating equation (29) w.r.t. to $\lambda_1$, one obtains an additional evolution constraint $(A_1 + B_1 C) = 0$, which involves both $F$ and $C$. Injecting the previously obtained expression of $C$ in this new constraint, one finally obtains a closed evolution equation involving $F$ only: this is the Balescu-Lenard equation.

However, thanks to the functional rewriting from equation (29), [28] suggested a different strategy. This is based on a rewriting of equation (29) under the form

$$1 = \int \mathcal{D}F \mathcal{D}C \mathcal{D}\lambda_1 \mathcal{D}\lambda_2 \exp\left\{ i \int dt\, d\Gamma_1\, \lambda_1(\Gamma_1)\, A_1 F(\Gamma_1) + \frac{i}{2} \int dt\, d\Gamma_1 d\Gamma_2\, \lambda_2(\Gamma_1, \Gamma_2)\, G(\Gamma_1, \Gamma_2) \right.$$
$$\left. - \frac{i}{2} \int dt\, d\Gamma_1 d\Gamma_2\, C(\Gamma_1, \Gamma_2)\, E(\Gamma_1, \Gamma_2) \right\}, \tag{31}$$

where we defined the quantity $G(\Gamma_1, \Gamma_2)$ as

$$G(\Gamma_1, \Gamma_2) = \mathcal{F}_{12} \cdot \left[ F(\Gamma_2) \frac{\partial F}{\partial v_1} - F(\Gamma_1) \frac{\partial F}{\partial v_2} \right], \tag{32}$$

which corresponds to the contribution from the source term $\lambda_2 S_2$ in equation (29), from which $C$ is absent. In equation (31), we also introduced $E(\Gamma_1, \Gamma_2)$ as

$$E(\Gamma_1, \Gamma_2) = A_2 \lambda_2(\Gamma_1, \Gamma_2) + \int d\Gamma_3 \left[ \mathcal{F}_{13} \lambda_2(\Gamma_2, \Gamma_3) + \mathcal{F}_{23} \lambda_2(\Gamma_1, \Gamma_3) \right] \cdot \frac{\partial F}{\partial v_3} + \mu \mathcal{F}_{12} \cdot \left[ \frac{\partial \lambda_1}{\partial v_1} - \frac{\partial \lambda_1}{\partial v_2} \right]. \tag{33}$$

The three terms present in equation (33) can straightforwardly be obtained from equation (29) through the following manipulations. The first term comes from the component $\lambda_2 A_2 C$ in equation (29). One has to use an integration by parts and get rid of the boundary terms. To invert the time derivative with $t \in [0; T]$, where $T$ is an arbitrary temporal upper bound, we assume that $C(t = 0) = 0$ (the system is supposed to be initially uncorrelated) and $\lambda_2(T) = 0$ (we are free to impose a condition on $\lambda_2$). The second term comes from the component $\lambda_2 D_2 C$ in equation (29), where the only operation required is to permute accordingly the indices $(1, 2, 3)$. Finally, the third term comes from $\lambda_1 B_1 C$ in equation (29). One has to use the integration by parts formula, get rid of the boundary terms, and use the fact that $\mathcal{F}_{12}$ is independent of $v_1$ so that $\partial/\partial v_1 \cdot [\lambda_1 \mathcal{F}_{12}] = \mathcal{F}_{12} \cdot \partial \lambda_1/\partial v_1$. One also has to use the permutation $1 \leftrightarrow 2$, for which $\mathcal{F}_{12} = -\mathcal{F}_{21}$, and recovers the factor $1/2$ present in equation (31). At this stage, it is crucial to note that in the rewriting of equation (31) all the dependences on $C$ have been put in the prefactor $C(\Gamma_1, \Gamma_2)$ multiplying the quantity $E(\Gamma_1, \Gamma_2)$.

[28] then suggested the following steps. By first integrating functionally equation (31) w.r.t. to $C$, one obtains a new dynamical constraint $E[F, \lambda_1, \lambda_2] = 0$. This constraint may then be inverted so as to obtain $\lambda_2 = \lambda_2[F, \lambda_1]$. The final step of the calculation is then to make this substitution in equation (31), which now only involves $\lambda_1$ and $F$. By functionally integrating it w.r.t. $\lambda_1$, one obtains a closed kinetic equation involving $F$ only: this is the Balescu-Lenard equation.

In their appendix, [28] explicitly applied this new strategy to derive the homogeneous Balescu-Lenard equation, and showed that this approach was not only succesful but fairly simple. In the present paper, we intend to show how one may use the same strategy in the inhomogeneous context. In order to simplify the calculations, we will neglect collective effects and show how one can then recover the inhomogeneous Landau equation (the reduced form of the inhomogeneous Balescu-Lenard equation when collective effects are neglected). The generalisation of these calculations to the case where collective effects are take into account will be the subject of a future work.

## 4. Application to inhomogeneous systems

### 4.1. Angle-action coordinates

When considering an inhomogeneous system, the trajectories of the particles tend to be fairly intricate. We therefore restrict ourselves to symmetric configurations for which the mean gravitational background potential $\psi_0$ associated with the Hamiltonian $H_0$ is quasi-stationary and integrable. As a consequence, one can always remap the physical phase-space coordinates $(x, v)$ to the angle-action ones $(\theta, J)$ [30, 31, 32]. The intrinsic frequencies of motion along the action torus are defined as

$$\Omega(J) = \dot{\theta} = \frac{\partial H_0}{\partial J}. \tag{34}$$

Within these new coordinates, along the unperturbed orbits, the angles $\theta$ are $2\pi-$periodic, evolving with the frequencies $\Omega$, while the actions $J$ are conserved. This change of coordinates is canonical so that the infinitesimal volumes are conserved, i.e.,

$$d\Gamma = dx dv = d\theta dJ. \tag{35}$$

8

Relying on the adiabatic approximation [21, 22, 5], we assume that the 1−body DF $F$ evolves in a quasi-stationary fashion, so that $F(\boldsymbol{\theta}, \boldsymbol{J}) = F(\boldsymbol{J})$, where the dependence w.r.t. $t$ has not been written out explicitly to simplify the notations. Since $\lambda_1$ is the auxiliary field associated with $F$, we also have $\lambda_1(\boldsymbol{\theta}, \boldsymbol{J}) = \lambda_1(\boldsymbol{J})$. The second auxiliary field $\lambda_2(\boldsymbol{\theta}_1, \boldsymbol{J}_1, \boldsymbol{\theta}_2, \boldsymbol{J}_2)$ remains fully dependent on the angle-action coordinates, contrary to the assumption $\lambda_2(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2)$ used in the homogeneous case [28]. Another property of these coordinates comes from the derivatives along the mean motion which take the simple form

$$\boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1} + \left[ \int \mathrm{d}\Gamma_2 \mathcal{F}_{12} F(\Gamma_2) \right] \cdot \frac{\partial}{\partial \boldsymbol{v}_1} = \boldsymbol{\Omega}_1 \cdot \frac{\partial}{\partial \boldsymbol{\theta}_1} \,. \tag{36}$$

Finally, we will also rely on the invariance of the Poisson bracket under the change of coordinates $(\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{\theta}, \boldsymbol{J})$, so that for any functions $L_1(\boldsymbol{x}, \boldsymbol{v})$ and $L_2(\boldsymbol{x}, \boldsymbol{v})$, one may write

$$\frac{\partial L_1}{\partial \boldsymbol{x}} \cdot \frac{\partial L_2}{\partial \boldsymbol{v}} - \frac{\partial L_1}{\partial \boldsymbol{v}} \cdot \frac{\partial L_2}{\partial \boldsymbol{x}} = \frac{\partial L_1}{\partial \boldsymbol{\theta}} \cdot \frac{\partial L_2}{\partial \boldsymbol{J}} - \frac{\partial L_1}{\partial \boldsymbol{J}} \cdot \frac{\partial L_2}{\partial \boldsymbol{\theta}} \,. \tag{37}$$

Using these transformations, one can rewrite in angle-action space the quantities appearing in equation (31). Since we assumed that the 1−body DF is quasi-stationary, one has $\partial F / \partial \boldsymbol{\theta} = 0$, so that thanks to equation (36), equation (30) gives

$$A_1 F = \frac{\partial F}{\partial t} \,. \tag{38}$$

Similarly, thanks to equation (37), the quantity $G$ from equation (32) may be rewritten as

$$G(\Gamma_1, \Gamma_2) = -\left[ F(\boldsymbol{J}_2) \frac{\partial U_{12}}{\partial \boldsymbol{\theta}_1} \cdot \frac{\partial F}{\partial \boldsymbol{J}_1} + F(\boldsymbol{J}_1) \frac{\partial U_{21}}{\partial \boldsymbol{\theta}_2} \cdot \frac{\partial F}{\partial \boldsymbol{J}_2} \right]. \tag{39}$$

Finally, using the fact that the auxiliary field $\lambda_1$ is a quasi-stationary quantity such that $\lambda_1(\Gamma) = \lambda_1(\boldsymbol{J})$, we may rewrite the constraint $E(\Gamma_1, \Gamma_2)$ from equation (33) as

$$\begin{aligned} E(\Gamma_1, \Gamma_2) = &\frac{\partial \lambda_2}{\partial t} + \boldsymbol{\Omega}_1 \cdot \frac{\partial \lambda_2}{\partial \boldsymbol{\theta}_1} + \boldsymbol{\Omega}_2 \cdot \frac{\partial \lambda_2}{\partial \boldsymbol{\theta}_2} \\ &+ \int \mathrm{d}\Gamma_3 \left[ \frac{\partial U_{31}}{\partial \boldsymbol{\theta}_3} \cdot \frac{\partial F}{\partial \boldsymbol{J}_3} \lambda_2(\Gamma_2, \Gamma_3) + \frac{\partial U_{32}}{\partial \boldsymbol{\theta}_3} \cdot \frac{\partial F}{\partial \boldsymbol{J}_3} \lambda_2(\Gamma_1, \Gamma_3) \right] \\ &- \mu \left[ \frac{\partial U_{12}}{\partial \boldsymbol{\theta}_1} \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_1} + \frac{\partial U_{21}}{\partial \boldsymbol{\theta}_2} \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_2} \right]. \end{aligned} \tag{40}$$

As presented in [5], we will now neglect collective effects, i.e., neglect contributions associated with the term $D_2 C$ in equation (30). Under these conditions, equation (40) becomes

$$E(\Gamma_1, \Gamma_2) = \frac{\partial \lambda_2}{\partial t} + \boldsymbol{\Omega}_1 \cdot \frac{\partial \lambda_2}{\partial \boldsymbol{\theta}_1} + \boldsymbol{\Omega}_2 \cdot \frac{\partial \lambda_2}{\partial \boldsymbol{\theta}_2} - \mu \left[ \frac{\partial U_{12}}{\partial \boldsymbol{\theta}_1} \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_1} + \frac{\partial U_{21}}{\partial \boldsymbol{\theta}_2} \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_2} \right], \tag{41}$$

and the constraint $E(\Gamma_1, \Gamma_2) = 0$ implies

$$\frac{\partial \lambda_2}{\partial t} + \boldsymbol{\Omega}_1 \cdot \frac{\partial \lambda_2}{\partial \boldsymbol{\theta}_1} + \boldsymbol{\Omega}_2 \cdot \frac{\partial \lambda_2}{\partial \boldsymbol{\theta}_2} - \mu \left[ \frac{\partial U_{12}}{\partial \boldsymbol{\theta}_1} \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_1} + \frac{\partial U_{21}}{\partial \boldsymbol{\theta}_2} \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_2} \right] = 0 \,. \tag{42}$$

### 4.2. Inverting the constraint

In order to solve equation (42), we will rely on Bogoliubov's Ansatz (adiabatic approximation). We assume that the fluctuations (i.e., $\lambda_2$) evolve rapidly compared to the mean dynamical quantities (i.e., $F$ and $\lambda_1$). Indeed, the 2−body correlation function $C$ tends to its asymptotic value on a timescale of the order of the dynamical time $t_{\mathrm{dyn}}$, while the 1−body DF $F$ evolves on the secular timescale $N t_{\mathrm{dyn}}$ much larger. As a consequence, on the timescale of evolution of $\lambda_2$, we may assume $F$ and $\lambda_1$ to be constant, and at a given secular time $t$, $\lambda_2$ can be considered as equal to the asymptotic value associated with the current frozen values of $\lambda_1$ and $F$. In equation (42), we may therefore assume that $\lambda_1$ is frozen and that only $\lambda_2$ depends on time.

To simplify the calculations, we rely on the fact that the angles $\theta$ are $2\pi-$periodic. We define the Fourier transform w.r.t. the angles $\theta$ as

$$f(\boldsymbol{\theta}, \boldsymbol{J}) = \sum_{\boldsymbol{m}} f_{\boldsymbol{m}}(\boldsymbol{J}) \, e^{i\boldsymbol{m}\cdot\boldsymbol{\theta}} \quad ; \quad f_{\boldsymbol{m}}(\boldsymbol{J}) = \frac{1}{(2\pi)^d} \int d\boldsymbol{\theta} \, f(\boldsymbol{\theta}, \boldsymbol{J}) \, e^{-i\boldsymbol{m}\cdot\boldsymbol{\theta}} , \tag{43}$$

where $d$ is the dimension of the considered physical space (e.g. $d = 2$ for a razor-thin disc as in [23, 24]). Following [33, 34, 5], we may Fourier transform the interaction potential $U$ as

$$U(\boldsymbol{x}(\boldsymbol{\theta}_1, \boldsymbol{J}_1) - \boldsymbol{x}(\boldsymbol{\theta}_2, \boldsymbol{J}_2)) = \sum_{\boldsymbol{m}_1, \boldsymbol{m}_2} A_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2) \, e^{i(\boldsymbol{m}_1\cdot\boldsymbol{\theta}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\theta}_2)} , \tag{44}$$

where $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathbb{Z}^d$ are integer vectors. The coefficients $A_{\boldsymbol{m}_1, \boldsymbol{m}_2}$ satisfy the symmetry relations

$$A_{\boldsymbol{m}_2, \boldsymbol{m}_1}(\boldsymbol{J}_2, \boldsymbol{J}_1) = A_{-\boldsymbol{m}_1, -\boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2) = [A_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2)]^* . \tag{45}$$

Similarly, we also introduce the Fourier transform of $\lambda_2$ as

$$\lambda_2(\boldsymbol{\theta}_1, \boldsymbol{J}_1, \boldsymbol{\theta}_2, \boldsymbol{J}_2) = \sum_{\boldsymbol{m}_1, \boldsymbol{m}_2} \lambda_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2) \, e^{i(\boldsymbol{m}_1\cdot\boldsymbol{\theta}_1 + \boldsymbol{m}_2\cdot\boldsymbol{\theta}_2)} . \tag{46}$$

Since $\lambda_2$ is real, it satisfies the symmetry property

$$\lambda_{-\boldsymbol{m}_1, -\boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2) = \lambda^*_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2) . \tag{47}$$

We now multiply equation (42) by $1/(2\pi)^{2d} e^{i(\boldsymbol{m}_1\cdot\boldsymbol{\theta}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\theta}_2)}$ and integrate it w.r.t. $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. The constraint $E(\Gamma_1, \Gamma_2) = 0$ then takes the form

$$\frac{\partial \lambda_{-\boldsymbol{m}_1, \boldsymbol{m}_2}}{\partial t} - i\Delta\omega \lambda_{-\boldsymbol{m}_1, \boldsymbol{m}_2} = -i\mu A^*_{\boldsymbol{m}_1, \boldsymbol{m}_2} \left[ \boldsymbol{m}_1 \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2 \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_2} \right] . \tag{48}$$

where we used the shortening notations $\lambda_{-\boldsymbol{m}_1, \boldsymbol{m}_2} = \lambda_{-\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2)$, $A_{\boldsymbol{m}_1, \boldsymbol{m}_2} = A_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2)$, and $\Delta\omega = \boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2$. Thanks to the adiabatic approximation, $\lambda_1$ can be assumed to be frozen, so that the differential equation (48) can be straightforwardly solved. We recall that to obtain equation (33), we had imposed the boundary condition $\lambda_2(T) = 0$, so that equation (48) leads to

$$\lambda_{-\boldsymbol{m}_1, \boldsymbol{m}_2}(t) = \mu A^*_{\boldsymbol{m}_1, \boldsymbol{m}_2} \left[ \boldsymbol{m}_1 \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2 \cdot \frac{\partial \lambda_2}{\partial \boldsymbol{J}_2} \right] \frac{1 - e^{i\Delta\omega(t-T)}}{\Delta\omega} . \tag{49}$$

At this stage, we assume that the arbitrary temporal bound $T$ is large compared to the considered time $t$, so as to consider only the forced regime of evolution, uninfluenced by the temporal boundary condition on $\lambda_2$. We may therefore place ourselves in the limit $T \to +\infty$. Recalling the formula

$$\lim_{T \to +\infty} \frac{e^{iT\Delta\omega} - 1}{\Delta\omega} = i\pi\delta_{\mathrm{D}}(\Delta\omega) , \tag{50}$$

equation (49) immediately gives

$$\lim_{T \to +\infty} \lambda_{-\boldsymbol{m}_1, \boldsymbol{m}_2}(t) = i\pi\mu A^*_{\boldsymbol{m}_1, \boldsymbol{m}_2} \left[ \boldsymbol{m}_1 \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2 \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{J}_2} \right] \delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2) . \tag{51}$$

Thanks to Bogoliubov's Ansatz, we have therefore inverted the constraint $E[F, \lambda_1, \lambda_2] = 0$ from equation (42), so as to obtain $\lambda_2 = \lambda_2[F, \lambda_1]$. This will allow us to recover the expression of the Landau collision operator, as detailed in the next section.

### 4.3. Recovering the inhomogeneous Landau operator

After having inverted equation (42), the expression of $\lambda_2 = \lambda_2[F, \lambda_1]$ may be substituted back in equation (31). In equation (31), it then only remains the quantities $F$ and $\lambda_1$, and we note as $S[F, \lambda_1]$ the remaining action term. It takes the form

$$S[F, \lambda_1] = i \int dt d\Gamma_1 \, \lambda_1 A_1 F + \frac{i}{2} \int dt d\Gamma_1 d\Gamma_2 \, \lambda_2[F, \lambda_1] \, G(\Gamma_1, \Gamma_2) . \tag{52}$$

10

Starting from the expressions (38) and (39) of $A_1$ and $G$, and using the Fourier transform in angles introduced in equation (43), we may rewrite the action from equation (52) as

$$S[F, \lambda_1] = i \int dt d\Gamma_1 \, \lambda_1 \frac{\partial F}{\partial t} - \frac{i}{2} \int dt d\Gamma_1 d\Gamma_2 \sum_{m_1, m_2} \left[ i m_1 \cdot \frac{\partial F}{\partial J_1} F(J_2) A_{m_1, m_2} \lambda_{-m_1, m_2} + i m_2 \cdot \frac{\partial F}{\partial J_2} F(J_1) \left[ A_{m_1, m_2} \lambda_{-m_1, m_2} \right]^* \right], \quad (53)$$

where, for simplicity, we used the notation $A_{m_1, m_2} = A_{m_1, m_2}(J_1, J_2)$, and $\lambda_{-m_1, m_2} = \lambda_{-m_1, m_2}(J_1, J_2)$. Using the symmetry properties from equations (45) and (47), equation (53) immediately becomes

$$S[F, \lambda_1] = i \int dt d\Gamma_1 \, \lambda_1 \frac{\partial F}{\partial t} + \frac{i}{2} \int dt d\Gamma_1 d\Gamma_2 \sum_{m_1, m_2} \text{Im} \left[ A_{m_1, m_2} \lambda_{-m_1, m_2} \right] \left[ m_1 \cdot \frac{\partial F}{\partial J_1} F(J_2) - m_2 \cdot \frac{\partial F}{\partial J_2} F(J_1) \right]. \quad (54)$$

Thanks to the inversion from equation (51), one can write

$$\text{Im} \left[ A_{m_1, m_2} \lambda_{-m_1, m_2} \right] = \pi \mu \, \delta_D(m_1 \cdot \Omega_1 - m_2 \cdot \Omega_2) \left| A_{m_1, m_2} \right|^2 \left[ m_1 \cdot \frac{\partial \lambda_1}{\partial J_1} - m_2 \cdot \frac{\partial \lambda_1}{\partial J_2} \right]. \quad (55)$$

Inserting this result in equation (54), we get

$$S[F, \lambda_1] = i \int dt d\Gamma_1 \, \lambda_1 \frac{\partial F}{\partial t}$$
$$+ \frac{i}{2} \int dt d\Gamma_1 d\Gamma_2 \sum_{m_1, m_2} \pi \mu \, \delta_D(m_1 \cdot \Omega_1 - m_2 \cdot \Omega_2) \left| A_{m_1, m_2} \right|^2 \left[ m_1 \cdot \frac{\partial \lambda_1}{\partial J_1} - m_2 \cdot \frac{\partial \lambda_1}{\partial J_2} \right] \left[ m_1 \cdot \frac{\partial F}{\partial J_1} F(J_2) - m_2 \cdot \frac{\partial F}{\partial J_2} F(J_1) \right]. \quad (56)$$

The last step of the calculation is then to rewrite the second term in equation (56) under the form $\int dt d\Gamma_1 \lambda_1(\Gamma_1) \dots$. This is straightforward thanks to an integration by parts. In the second term of equation (56), let us focus on the term associated with $m_1 \cdot \partial \lambda_1 / \partial J_1$. It reads

$$\frac{i}{2} \pi \mu \int dt d\Gamma_1 d\Gamma_2 \sum_{m_1, m_2} \left| A_{m_1, m_2} \right|^2 \delta_D(m_1 \cdot \Omega_1 - m_2 \cdot \Omega_2) \, m_1 \cdot \frac{\partial \lambda_1}{\partial J_1} \left[ m_1 \cdot \frac{\partial F}{\partial J_1} F(J_2) - m_2 \cdot \frac{\partial F}{\partial J_2} F(J_1) \right]$$
$$= -\frac{i}{2} \pi (2\pi)^d \mu \int dt d\Gamma_1 \lambda_1(J_1) \frac{\partial}{\partial J_1} \cdot \left[ \sum_{m_1, m_2} m_1 \int dJ_2 \, \delta_D(m_1 \cdot \Omega_1 - m_2 \cdot \Omega_2) \left| A_{m_1, m_2} \right|^2 \left[ m_1 \cdot \frac{\partial F}{\partial J_1} F(J_2) - m_2 \cdot \frac{\partial F}{\partial J_2} F(J_1) \right] \right], \quad (57)$$

where the additional prefactor $(2\pi)^d$ comes from the transformation $\int d\Gamma_2 f(J_2) = (2\pi)^d \int dJ_2 f(J_2)$. One can perform the exact same calculation for the second term present in equation (56) associated with $m_2 \cdot \partial \lambda_1 / \partial J_2$. One only has to permute the indices $1 \leftrightarrow 2$, and recovers the exact same contribution as in equation (57). As a consequence, one can get rid of the factor $1/2$ present in equation (56), so that it becomes

$$S[F, \lambda_1] = i \int dt d\Gamma_1 \, \lambda_1(\Gamma_1)$$
$$\times \left\{ \frac{\partial F}{\partial t} - \pi (2\pi)^d \mu \frac{\partial}{\partial J_1} \cdot \left[ \sum_{m_1, m_2} m_1 \int dJ_2 \, \delta_D(m_1 \cdot \Omega_1 - m_2 \cdot \Omega_2) \left| A_{m_1, m_2} \right|^2 \left[ m_1 \cdot \frac{\partial F}{\partial J_1} F(J_2) - m_2 \cdot \frac{\partial F}{\partial J_2} F(J_1) \right] \right] \right\}. \quad (58)$$

By integrating functionally equation (58) w.r.t. $\lambda_1$, one finally obtains the expression of the inhomogeneous Landau equation which reads

$$\frac{\partial F}{\partial t} = \pi (2\pi)^d \mu \frac{\partial}{\partial J_1} \cdot \left[ \sum_{m_1, m_2} m_1 \int dJ_2 \, \delta_D(m_1 \cdot \Omega_1 - m_2 \cdot \Omega_2) \left| A_{m_1, m_2}(J_1, J_2) \right|^2 \left( m_1 \cdot \frac{\partial}{\partial J_1} - m_2 \cdot \frac{\partial}{\partial J_2} \right) F(J_1, t) F(J_2, t) \right]. \quad (59)$$

As a conclusion using the functional integral approach presented in [28], we have been able to recover in an alternative manner the inhomogeneous Landau equation obtained in [5].

## 5. Conclusion

Understanding the long-term evolution of astrophysical dynamical systems is a subject of renewed interest. On galactic scales we now have the well established ΛCDM paradigm for the formation of structures. It allows us to quantify in detail the statistical impacts of cosmic perturbations on self-gravitating systems. These developments offer new clues to address the pressing question of the respective long-term roles of nature vs. nurture in the establishment of the observed properties of these systems. Numerous dynamical challenges are therefore ready to be re-examined in much greater detail than before. Examples include: the secular evolution of the metallicity-dispersion relationship in galactic discs, the mechanisms of disc thickening by molecular clouds and/or spiral waves, the stellar dynamical evolution of the Galactic centre, the evolution of proto-planetary discs of debris, etc. Yet, characterising the secular evolution of such systems remains a difficult task since it requires intricate inhomogeneous kinetic models, complex numerical experiments, and an accurate physical understanding of the involved competing physical processes. Kinetic equations such as the Landau and Balescu-Lenard equations are expected to provide a crucial new lightning on these complex dynamical processes.

Using the functional integral formalism introduced in [28], we showed how one may derive through this approach the inhomogeneous Landau equation (59). This calculation offered new insights on the content of this kinetic equation. A natural next step of this calculation would be to show how one may use the same method to derive the inhomogeneous Balescu-Lenard equation [21, 22]. Such a derivation is expected to be more involved, because one has to take into account the polarisation dressing of the potential fluctuations. In the inhomogeneous context, this requires to rely on the matrix method [35] and to introduce potential-density elements. This will be the subject of a future work. Finally, one could expect this new functional integral approach to be applicable to other kinetic equations. For example, this methodology can be transposed to the kinetic theory of two-dimensional point vortices [36]. On the other hand, because of its alternative point of view, such a method may turn out fruitful to tackle the question of obtaining a closed kinetic equation when higher order correlation terms are taken into account.[4] This is also the topic of ongoing work. More generally, it would be of great interest to identify in which contexts this functional approach could be more successful.

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