HAL open science

Patterns of deformations of Peregrine breather of order 3 and 4 solutions to the NLS equation with multi parameters

Pierre Gaillard, Mickaël Gastineau

► To cite this version:

Pierre Gaillard, Mickaël Gastineau. Patterns of deformations of Peregrine breather of order 3 and 4 solutions to the NLS equation with multi parameters. Journal of Theoretical and Applied Physics, 2016, 10 (2), pp.83-89. 10.1007/s40094-015-0204-6 . hal-01321436

HAL Id: hal-01321436

https://hal.sorbonne-universite.fr/hal-01321436

Submitted on 25 May 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution 4.0 International License

J Theor Appl Phys (2016) 10:83–89
DOI 10.1007/s40094-015-0204-6

RESEARCH

Patterns of deformations of Peregrine breather of order 3 and 4 solutions to the NLS equation with multi parameters

Pierre Gaillard[1] · **Mickaël Gastineau**[2]

Received: 28 September 2015 / Accepted: 19 December 2015 / Published online: 14 January 2016
© The Author(s) 2016. This article is published with open access at Springerlink.com

Abstract In this article, one gives a classification of the solutions to the one dimensional nonlinear focusing Schrödinger equation (NLS) by considering the modulus of the solutions in the (x, t) plan in the cases of orders 3 and 4. For this, we use a representation of solutions to NLS equation as a quotient of two determinants by an exponential depending on t. This formulation gives in the case of the order 3 and 4, solutions with, respectively 4 and 6 parameters. With this method, beside Peregrine breathers, we construct all characteristic patterns for the modulus of solutions, like triangular configurations, ring and others.

Keywords NLS equation · Peregrine breathers · Rogue waves

Introduction

The rogue waves phenomenon currently exceed the strict framework of the study of ocean's waves [1–4] and play a significant role in other fields; in nonlinear optics [5, 6], Bose–Einstein condensate [7], superfluid helium [8], atmosphere [9], plasmas [10], capillary phenomena [11] and even finance [12]. In the following, we consider the one-dimensional focusing nonlinear equation of Schrödinger (NLS) to describe the phenomena of rogue waves. The first results concerning the NLS equation date from the Seventies. Precisely, in 1972 Zakharov and Shabat solved it using the inverse scattering method [13, 14]. The case of periodic and almost periodic algebro-geometric solutions to the focusing NLS equation was first constructed in 1976 by Its and Kotlyarov [15, 16]. The first quasi rational solutions of NLS equation were constructed in 1983 by Peregrine [17]. In 1986 Akhmediev, Eleonskii and Kulagin obtained the two-phase almost periodic solution to the NLS equation and obtained the first higher order analogue of the Peregrine breather [18–20]. Other analogues of the Peregrine breathers of order 3 and 4 were constructed in a series of articles by Akhmediev et al. [21–23] using Darboux transformations. The present paper presents multi-parametric families of quasi rational solutions of NLS of order N in terms of determinants of order $2N$ dependent on $2N-2$ real parameters. The aim of this paper was to try to distinguish among all the possible configurations obtained by different choices of parameters, one those which have a characteristic in order to try to give a classification of these solutions.

Expression of solu-tions of NLS equation in terms of a ratio of two determinants

We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{1}$$

To solve this equation, we need to construct two types of functions $f_{j,k}$ and $g_{j,k}$ depending on many parameters. Because of the length of their expressions, one defines the functions $f_{\nu,\mu}$ and $g_{\nu,\mu}$ of argument A_ν and B_ν only in the appendix. We have already constructed solutions of equation NLS in terms of determinants of order $2N$ which we call solution of order N depending on $2N-2$ real parameters. It is given in the following result [24–27]:

✉ Pierre Gaillard
pierre.gaillard@u-bourgogne.fr

Mickaël Gastineau
gastineau@imcce.fr

1 Université de Bourgogne, 9 Av. Alain Savary, Dijon, France

2 IMCCE, Observatoire de Paris, PSL Research University, CNRS, Sorbonne Universités UPMC Univ, Paris 06 Univ. Lille 77 Av. Denfert-Rochereau, 75014 Paris, France

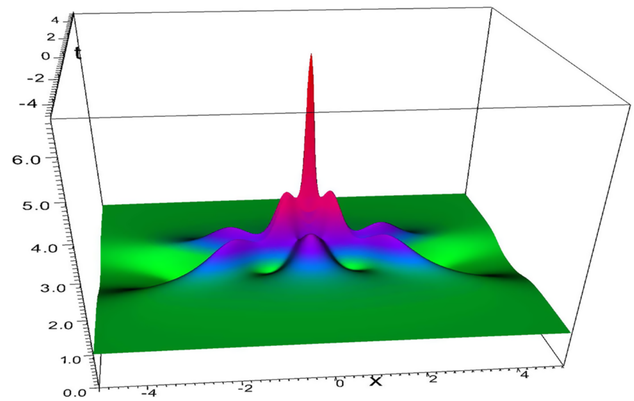

Fig. 1 Solution P_3 to NLS, N=3, $\tilde{a}_1 = \tilde{b}_1 = \tilde{a}_2 = \tilde{b}_2 = 0$

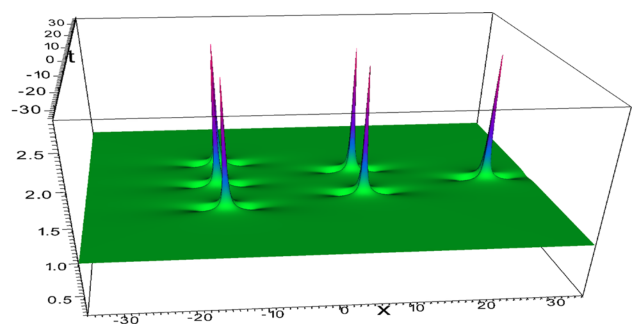

Fig. 2 Solution $1T6$ to NLS, N=3, $\tilde{a}_1 = 10^4$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$

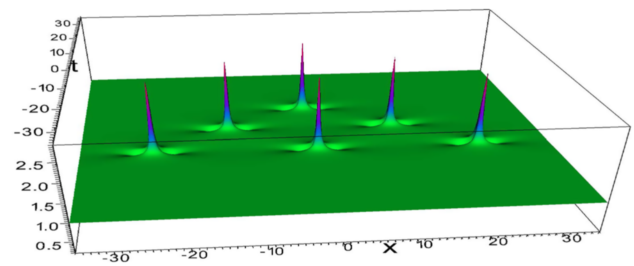

Fig. 3 Solution $1T6$ to NLS, N=3, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^4$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$

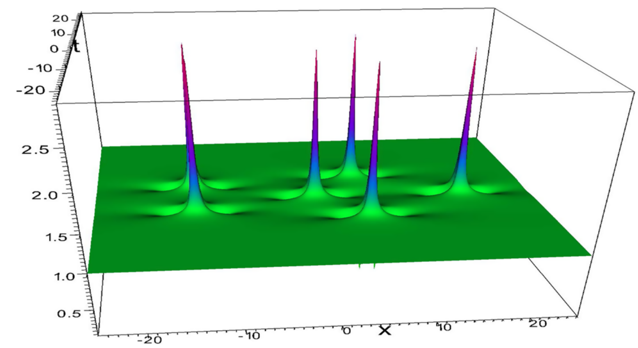

Fig. 4 Solution $1R5 + 1$ to NLS, N=3, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$

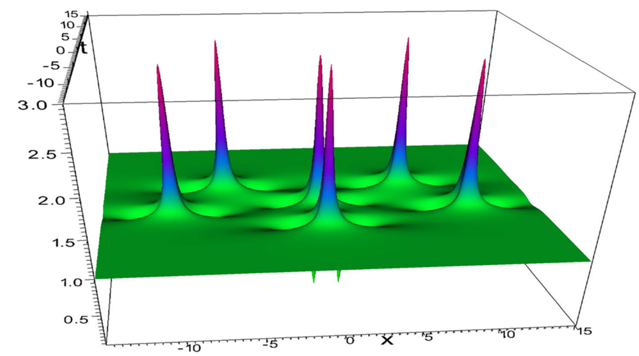

Fig. 5 Solution $1R5 + 1$ to NLS, N=3, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 10^5$

Theorem 1 *The functions v defined by*

$$v(x,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} e^{2it-i\varphi} \tag{2}$$

are quasi-rational solution of the NLS Eq. (1) depending on $2N-2$ *parameters* $\tilde{a}_j$, $\tilde{b}_j$, $1 \le j \le N-1$, *where*

$$\begin{aligned}
n_{j1} &= f_{j,1}(x,t,0),\\
n_{jk} &= \frac{\partial^{2k-2} f_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
n_{jN+1} &= f_{j,N+1}(x,t,0),\\
n_{jN+k} &= \frac{\partial^{2k-2} f_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0),
\end{aligned}$$

$$\begin{aligned}
d_{j1} &= g_{j,1}(x,t,0),\\
d_{jk} &= \frac{\partial^{2k-2} g_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
d_{jN+1} &= g_{j,N+1}(x,t,0),\\
d_{jN+k} &= \frac{\partial^{2k-2} g_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
&2 \le k \le N,\ 1 \le j \le 2N
\end{aligned} \tag{3}$$

The functions f and g are defined in (9), (10), (11), (12).

Patterns of quasi rational solutions to the NLS equation

The solutions v_N to NLS Eq. (2) of order N depending on $2N-2$ parameters $\tilde{a}_j$, $\tilde{b}_j$ (for $1 \le j \le N-1$) have been already explicitly constructed and can be written as

$$v_N(x,t) = \frac{n(x,t)}{d(x,t)} \exp(2it)$$

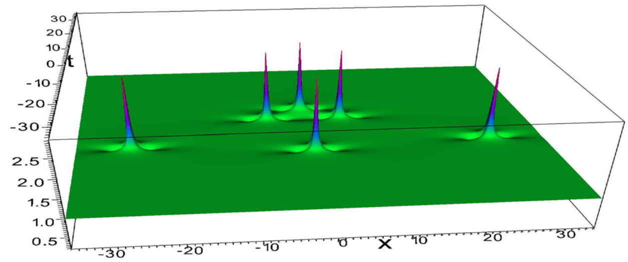

Fig. 6 Solution $1A3 + 1T3$ to NLS, N=3, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^4$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 5 \times 10^6$

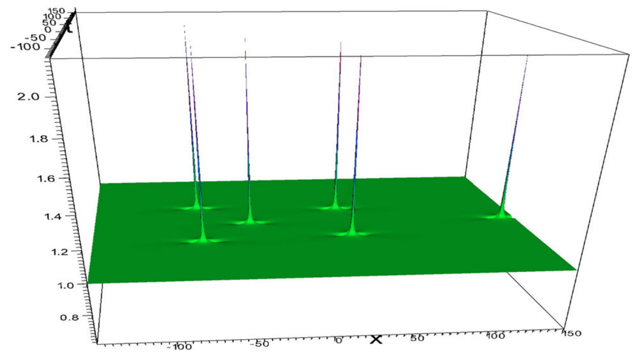

Fig. 7 Solution $1A3 + 1T3$ to NLS, N=3, $\tilde{a}_1 = 10^6$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^{10}$, $\tilde{b}_2 = 0$

$$= \left(1 - \alpha_N \frac{G_N(2x,4t) + iH_N(2x,4t)}{Q_N(2x,4t)}\right) e^{2it}$$

with

$$G_N(X,T) = \sum_{k=0}^{N(N+1)} g_k(T)X^k,$$
$$H_N(X,T) = \sum_{k=0}^{N(N+1} h_k(T)X^k,$$
$$Q_N(X,T) = \sum_{k=0}^{N(N+1} q_k(T)X^k.$$

For order 3 these expressions can be found in [28]; in the case of order 4, they can be found in [29]. In the following, based on these analytic expressions, we give a classification of these solutions by means of patterns of their modulus in the plane (x; t).

Patterns of quasi-rational solutions of order 3 with 4 parameters

P_3 breather

If we choose all parameters equal to 0, $\tilde{a}_1 = \tilde{b}_1 = \ldots = \tilde{a}_{N-1} = \tilde{b}_{N-1} = 0$, we obtain the classical Peregrine breather given by

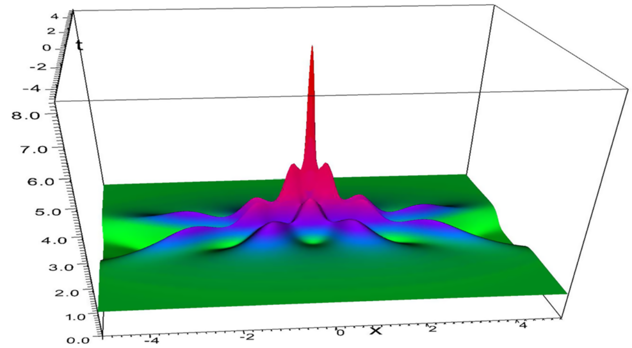

Fig. 8 Solution P_4 to NLS, N=4, $\tilde{a}_1 = \tilde{b}_1 = \tilde{a}_2 = \tilde{b}_2 = \tilde{a}_3 = \tilde{b}_3 = 0$

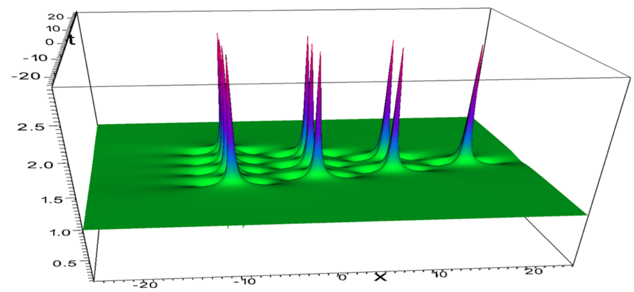

Fig. 9 Solution $1T10$ to NLS, N=4, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 0$, $\tilde{b}_3 = 0$

Triangles

To shorten, the following notations are used: for example, the sequence $1A3 + 1T3$ means that the structure has one arc of 3 peaks and one triangle of 3 peaks. If we choose $\tilde{a}_1$ or $\tilde{b}_1$ not equal to 0 and all other parameters equal to 0, we obtain triangular configuration with 6 peaks.

Rings

If we choose $\tilde{a}_2$ or $\tilde{b}_2$ not equal to 0, all other parameters equal to 0, we obtain ring configuration with peaks.

Arcs

If we choose $\tilde{a}_1$ and $\tilde{a}_2$ not equal to 0 and all other parameters equal to 0 (and vice versa, $\tilde{b}_1$ and $\tilde{b}_2$ not equal to 0 and all other parameters equal to 0), we obtain deformed triangular configuration which we can call as arc structure.

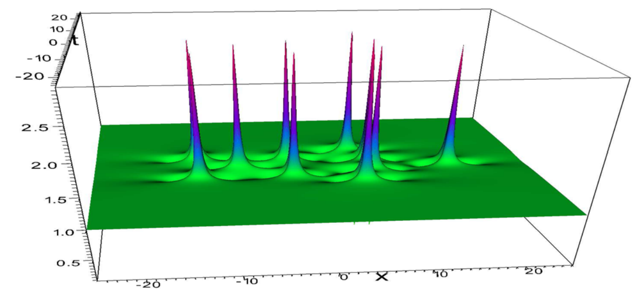

Fig. 10 Solution $2R5$ / 5 to NLS, N=4, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^5$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 0$, $\tilde{b}_3 = 0$

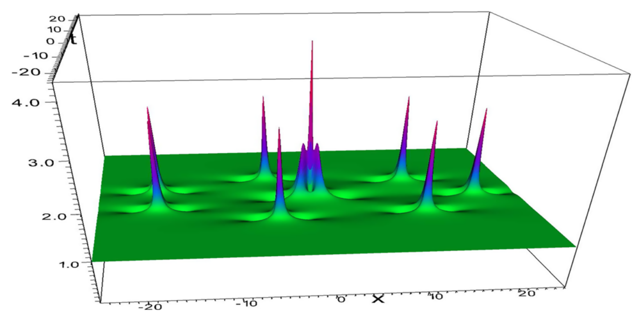

Fig. 11 Solution $1R7 + P_2$ to NLS, N=4, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 10^8$, $\tilde{b}_3 = 0$

Patterns of quasi rational solutions of order 4 with 6 parameters

P_4 breather

If we choose all parameters equal to 0, $\tilde{a}_1 = \tilde{b}_1 = \ldots = \tilde{a}_{N-1} = \tilde{b}_{N-1} = 0$, we obtain the classical Peregrine breather given in the following figure.

Triangles

To shorten, we use the notations defined in the previous section. If we choose $\tilde{a}_1$ or $\tilde{b}_1$ not equal to 0 and all other parameters equal to 0, we obtain triangular configuration with 10 peaks.

Rings

If we choose $\tilde{a}_2$ or $\tilde{a}_3$ not equal to 0 and all other parameters equal to 0 (or vice versa $\tilde{b}_2$ or $\tilde{b}_3$ not equal to 0 and all other parameters equal to 0), we obtain ring configuration with 10 peaks.

Arcs

If we choose two parameters non equal to 0, $\tilde{a}_1$ and $\tilde{a}_2$, or $\tilde{a}_1$ and $\tilde{a}_3$ not equal to 0, or $\tilde{a}_2$ and $\tilde{a}_3$ and all other

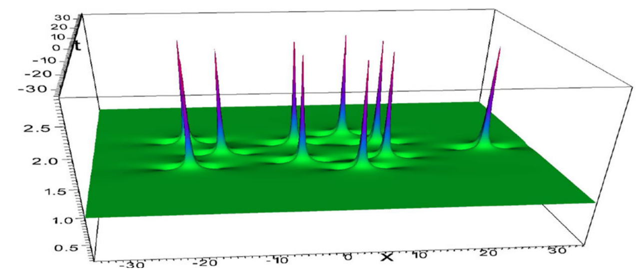

Fig. 12 Solution $2A3/4I + T3$ to NLS, N=4, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 0$, $\tilde{b}_3 = 0$

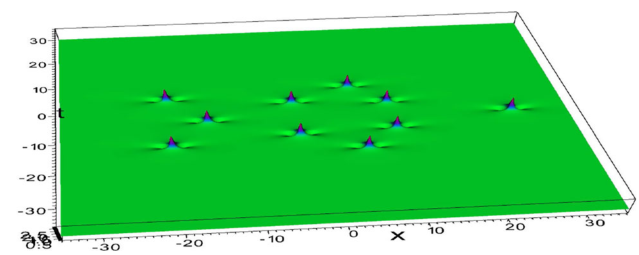

Fig. 13 Solution $2A3/4I + T3$ to NLS, N=4, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 0$, $\tilde{b}_3 = 0$, sight top

parameters equal to 0 (or vice versa for parameters b), we obtain arc configuration with 10 peaks.[1]

Triangles inside rings

If we choose three parameters non equal to 0, $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ and all other parameters equal to 0 (or vice versa for parameters b), we obtain ring with inside triangle.

Conclusion

We have presented here patterns of modulus of solutions to the NLS focusing equation in the (x, t) plane. This study can be useful at the same time for hydrodynamics as well for nonlinear optics; many applications in these fields have been realized, as it can be seen in recent works of Chabchoub et al. [30] or Kibler et al. [31]. This study tries to bring all possible types of patterns of quasi-rational solutions to the NLS equation. We see that we can obtain 2^{N-1} different structures at the order N. Parameters a or b give the same type of structure. For $a_1 \neq 0$ (and other parameters equal to 0), we obtain triangular rogue wave; for $a_j \neq 0$ ($j \neq 1$ and other parameters equal to 0) we get ring rogue wave; in the other choices of parameters, we get in particular arc structures (or claw structure). This type of study has been realized in preceding works. Akhmediev

[1] In the following notations $2A4$ / $3I$, I meaning Reversed.

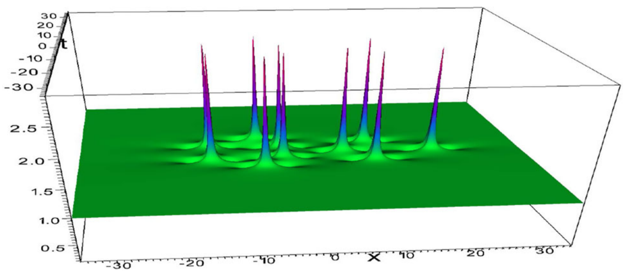

Fig. 14 Solution $2A4/3 + 1T3$ to NLS, N=4, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 5 \times 10^7$, $\tilde{b}_3 = 0$

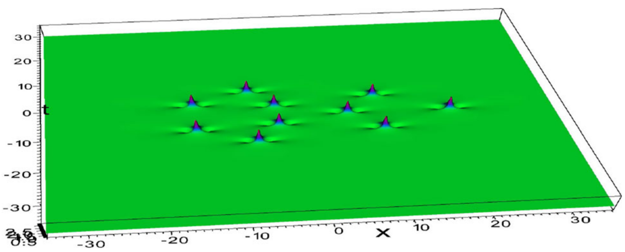

Fig. 15 Solution $2A4/3 + 1T3$ to NLS, N=4, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 5 \times 10^7$, $\tilde{b}_3 = 0$, sight top

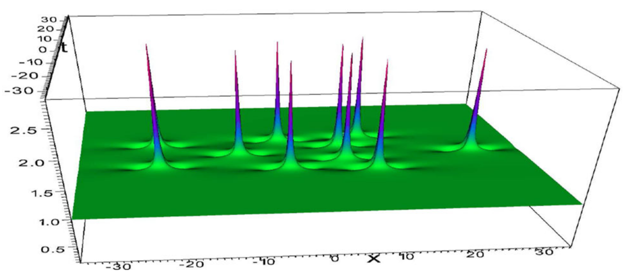

Fig. 16 Solution $2A3/4 + 1T3$ to NLS, N=4, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 3 \times 10^8$, $\tilde{b}_3 = 0$

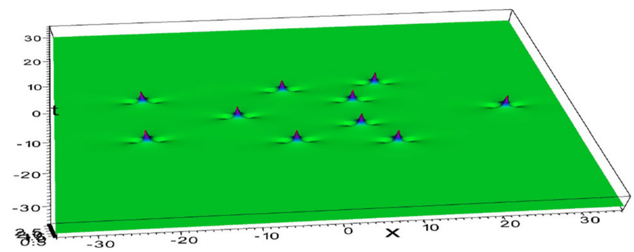

Fig. 17 Solution $2A3/4 + 1T3$ to NLS, N=4, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 3 \times 10^8$, $\tilde{b}_3 = 0$, sight top

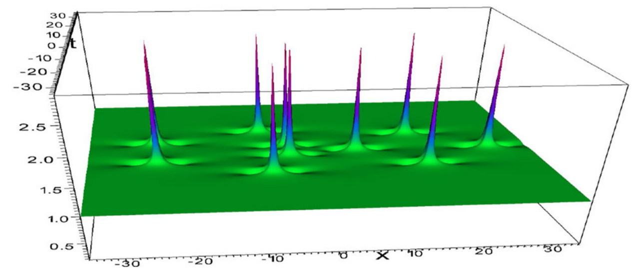

Fig. 18 Solution $1A7 + 1T3$ to NLS, N=4, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^3$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 10^9$, $\tilde{b}_3 = 0$

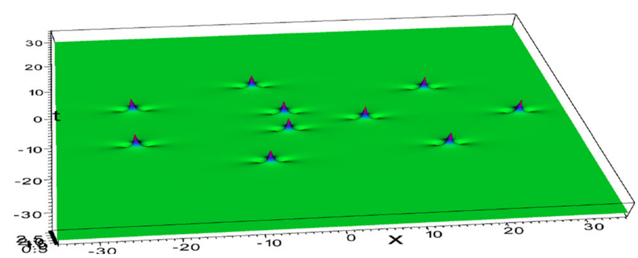

Fig. 19 Solution $1A7 + 1T3$ to NLS, N=4, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^3$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 10^9$, $\tilde{b}_3 = 0$, sight top

et al, study the order $N = 2$ in [32], $N = 3$ in [33]; the case $N = 4$ was studied in particular ($N = 5$, 6 were also studied) in [34, 35] showing triangle and arc patterns; only one type of ring was presented. The extrapolation was done until the order $N = 9$ in [36]. Ohta and Yang [37] presented the study of the case cas $N = 3$ with rings and triangles. Recently, Ling and Zhao [38] presented the cases $N = 2, 3, 4$ with rings, triangle and also claw structures.

In the present study, one sees appearing richer structures, in particular the appearance of a triangle of 3 peaks inside a ring of 7 peaks in the case of order $N = 4$; to the best of my knowledge, it is the first time that this configuration for order 4 is presented. In this way, we try to bring a better understanding to the hierarchy of NLS rogue wave solutions. It will be relevant to go on this study with higher orders.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Appendix: Parameters and functions

We consider the terms λ_ν satisfying the relations for $1 \leq j \leq N$

$$0 < \lambda_j < 1,\ \lambda_{N+j} = -\lambda_j,$$
$$\lambda_j = 1 - 2\epsilon^2 j^2,$$

with ϵ a small number intended to tend towards 0. The terms κ_ν, δ_ν, γ_ν are functions of the parameters λ_ν,

$1 \leq v \leq 2N$. They are given by the following equations, for $1 \leq j \leq N$:

$$\kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \\ \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, \tag{4}$$

$$\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \\ \gamma_{N+j} = 1/\gamma_j. \tag{5}$$

The terms $x_{r,v}$ $r = 3, 1$ are defined by

$$x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \\ x_{r,N+j} = (r-1)\ln\frac{\gamma_{N+j} - i}{\gamma_{N+j} + i}. \tag{6}$$

The parameters e_v are given by

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \tag{7}$$

where a_j and b_j are chosen in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a_k}\epsilon^{2k+1} j^{2k+1}, \\ b_j = \sum_{k=1}^{N-1} \tilde{b_k}\epsilon^{2k+1} j^{2k+1}, \\ 1 \leq j \leq N, \tag{8}$$

with $\tilde{a_j}$, $\tilde{b_j}$,, $1 \leq j \leq N-1$, $2N-2$, arbitrary real numbers.

The functions $f_{v,1}$ and $g_{v,1}$, $1 \leq v \leq N$ are defined by

$$f_{4j+1,1} = \gamma_k^{4j-1}\sin A_1, \\ f_{4j+2,1} = \gamma_k^{4j}\cos A_1, \\ f_{4j+3,1} = -\gamma_k^{4j+1}\sin A_1, \\ f_{4j+4,1} = -\gamma_k^{4j+2}\cos A_1, \tag{9}$$

$$f_{4j+1,N+1} = \gamma_k^{2N-4j-2}\cos A_{N+1}, \\ f_{4j+2,N+1} = -\gamma_k^{2N-4j-3}\sin A_{N+1}, \\ f_{4j+3,N+1} = -\gamma_k^{2N-4j-4}\cos A_{N+1}, \\ f_{4j+4,N+1} = \gamma_k^{2N-4j-5}\sin A_{N+1}, \tag{10}$$

$$g_{4j+1,1} = \gamma_k^{4j-1}\sin B_1, \\ g_{4j+2,1} = \gamma_k^{4j}\cos B_1, \\ g_{4j+3,1} = -\gamma_k^{4j+1}\sin B_1, \\ g_{4j+4,1} = -\gamma_k^{4j+2}\cos B_1, \tag{11}$$

$$g_{4j+1,N+1} = \gamma_k^{2N-4j-2}\cos B_{N+1}, \\ g_{4j+2,N+1} = -\gamma_k^{2N-4j-3}\sin B_{N+1}, \\ g_{4j+3,N+1} = -\gamma_k^{2N-4j-4}\cos B_{N+1}, \\ g_{4j+4,N+1} = \gamma_k^{2N-4j-5}\sin B_{N+1}. \tag{12}$$

The arguments A_v and B_v of these functions are defined by $1 \leq v \leq 2N$:

$$A_v = \kappa_v x/2 + i\delta_v t - ix_{3,v}/2 - ie_v/2, \\ B_v = \kappa_v x/2 + i\delta_v t - ix_{1,v}/2 - ie_v/2.$$

References

1. Kharif, C., Pelinovsky, E., Slunyaev, A.: Rogue waves in the Ocean. Springer (2009)
2. Akhmediev, N., Pelinovsky, E.: Discussion and debate: rogue waves - towards a unifying concept? Eur. Phys. Jour. Spec. Top. V **185** (2010)
3. Kharif, Ch., Pelinovsky, E.: Physical mechanisms of the rogue wave phenomenon. Eur. Jour. Mech. B Fluid **22**(6), 603–634 (2003)
4. Slunyaev, A., Didenkulova, I., Pelinovsky, E.: Rogue waters. Cont. Phys. **52**(6), 571–590 (2003)
5. Solli, D.R., Ropers, C., Koonath, P., Jalali, B.: Optical rogue waves. Nature **450**, 1054–1057 (2007)
6. Dudley, J.M., Genty, G., Eggleton, B.J.: Optical rogue waves. Opt. Express **16**, 3644 (2008)
7. Bludov, Y.V., Konotop, V.V., Akhmediev, N.: Matter rogue waves. Phys. Rev. A **80**(033610), 1–5 (2009)
8. Ganshin, A.N., Efimov, V.B., Kolmakov, G.V., Mezhov-Deglin, L.P., McClintok, P.V.E.: Observation of an inverse energy cascade in developed acoustic turbulence in superfluid helium. Phys. Rev. Lett. **101**, 065303 (2008)
9. Stenflo, L., Marklund, M.: Rogue waves in the atmosphere. J. Plasma Phys. **76**(3–4), 293–295 (2010)
10. Ruderman, M.S.: Freak waves in laboratory and space plasmas. Eur. Phys. J. Spec. Top. **185**, 57–66 (2010)
11. Shats, M., Punzman, H., Xia, H.: Matter rogue waves. Phys. Rev. Lett. **104**(104503), 1–5 (2010)
12. Yan, Z.Y.: Financial rogue waves. Commun. Theor. Phys. V **54**, 5, 947, 1-4, (2010)
13. Zakharov, V.E.: Stability of periodic waves of finite amplitude on a surface of a deep fluid. J. Appl. Tech. Phys **9**, 86–94 (1968)
14. Zakharov, V.E., Shabat, A.B.: Exact theory of two dimensional self focusing and one dimensinal self modulation of waves in nonlinear media. Sov. Phys. JETP **34**, 62–69 (1972)
15. Its, A.R., Kotlyarov, V.P.: Explicit expressions for the solutions of nonlinear Schrödinger equation. Dockl. Akad. Nauk. SSSR S A V **965**, 11 (1976)
16. Its, A.R., Rybin, A.V., Salle, M.A.: Exact integration of nonlinear Schrödinger equation. Teore. i Mat. Fiz. **74**(1), 29–45 (1988)
17. Peregrine, D.: Water waves, nonlinear Schrödinger equations and their solutions. J. Aust. Math. Soc. Ser. B **25**, 16–43 (1983)
18. Akhmediev, N., Eleonsky, V., Kulagin, N.: Generation of periodic trains of picosecond pulses in an optical fiber: exact solutions. Sov. Phys. JETP V **62**, 894–899 (1985)
19. Eleonskii, V., Krichever, I., Kulagin, N.: Rational multisoliton solutions to the NLS equation, Soviet Doklady 1986 sect. Math. Phys. **287**, 606–610 (1986)
20. Akhmediev, N., Eleonskii, V., Kulagin, N.: Exact first order solutions of the nonlinear Schrödinger equation. Theor. Math. Phys. **72**(2), 183–196 (1987)
21. Akhmediev, N., Ankiewicz, A., Soto-Crespo, J.M.: Rogue waves and rational solutions of nonlinear Schrödinger equation. Phys. Rev. E **80**(026601), 1–9 (2009)
22. Akhmediev, N., Ankiewicz, A., Clarkson, P.A.: Rogue waves, rational solutions, the patterns of their zeros and integral relations. J. Phys. A Math. Theor. **43**(122002), 1–9 (2010)

23. Chabchoub, A., Hoffmann, H., Onorato, M., Akhmediev, N.: Super rogue waves: observation of a higher-order breather in water waves. Phys. Rev. X **2**(011015), 1–6 (2012)
24. Gaillard, P.: Families of quasi-rational solutions of the NLS equation and multi-rogue waves. J. Phys. A Meth. Theor. **44**, 1–15 (2011)
25. Gaillard, P.: Wronskian representation of solutions of the NLS equation and higher Peregrine breathers. J. Math. Sci. Adv. Appl. **13**(2), 71–153 (2012)
26. Gaillard, P.: Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves. J. Math. Phys. V **54**, 013504-1-32 (2013)
27. Gaillard, P.: Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves. Adv. Res. **4**, 346–364 (2015)
28. Gaillard, P.: Deformations of third order Peregrine breather solutions of the NLS equation with four parameters. Phys. Rev. E V **88**, 042903-1-9 (2013)
29. Gaillard, P.: Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation. J. Math. Phys. V **54**, 073519-1-22 (2013)
30. Chabchoub, A., Hoffmann, N.P., Akhmediev, N.: Rogue wave observation in a water wave tank, Phys. Rev. Lett. V **106**, 204502-1-4 (2011)
31. Kibler, B., Fatome, J., Finot, C., Millot, G., Dias, F., Genty, G., Akhmediev, N., Dudley, J.M.: The Peregrine soliton in nonlinear fibre optics. Nat. Phys. **6**, 790–795 (2010)
32. Ankiewicz, A., Kedziora, D.J., Akhmediev, N.: Rogue wave triplets. Phys Lett. A **375**, 2782–2785 (2011)
33. Kedziora, D.J., Ankiewicz, A., Akhmediev, N.: The phase patterns of higher-order rogue waves. J. Opt. V **15**, 064011-1-9 (2013)
34. Kedziora, D.J., Ankiewicz, A., Akhmediev, N.: Circular rogue wave clusters. Phys. Rev. E **84**(056611), 1–7 (2011)
35. Kedziora, D.J., Ankiewicz, A., Akhmediev, N.: Triangular rogue wave cascades. Phys. Rev. E **86**(056602), 1–9 (2012)
36. Kedziora, D.J., Ankiewicz, A., Akhmediev, N.: Classifying the hierarchy of the nonlinear Schrödinger equation rogue waves solutions. Phys. Rev. E V **88**, 013207-1-12 (2013)
37. Ohta, Y., Yang, J.: General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation. Proc. R. Soc. A **468**(2142), 1716–1740 (2012)
38. Ling, L., Zhao, L.C.: Simple determinant representation for rogue waves of the nonlinear Shrodinger equation. Phys. Rev. E V **88**, 043201-1-9 (2013)