Period spacings in red giants
B. Mosser, C. Gehan, K. Belkacem, R. Samadi, E. Michel, M.-J. Goupil

To cite this version:

HAL Id: hal-01321683
https://hal.sorbonne-universite.fr/hal-01321683
Submitted on 17 Dec 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Period spacings in red giants

IV. Toward a complete description of the mixed-mode pattern

B. Mosser¹, C. Gehan¹, K. Belkacem¹, R. Samadi¹, E. Michel¹, M-J. Goupil¹

LESIA, Observatoire de Paris, PSL Research University, CNRS, Université Pierre et Marie Curie, Université Paris Diderot, 92195 Meudon, France; benoit.mosser@obspm.fr

ABSTRACT

Context. Oscillation modes with a mixed character, as observed in evolved low-mass stars, are highly sensitive to the physical properties of the innermost regions. Measuring their properties is therefore extremely important to probe the core, but requires some care, due to the complexity of the mixed-mode pattern.

Aims. The aim of this work is to provide a consistent description of the mixed-mode pattern of low-mass stars, based on the asymptotic expansion. We also study the variation of the gravity offset $\varepsilon_g$ with stellar evolution.

Methods. We revisit previous works about mixed modes in red giants and empirically test how period spacings, rotational splittings, mixed-mode widths, and heights can be estimated in a consistent view, based on the properties of the mode inertia ratios.

Results. From the asymptotic fit of the mixed-mode pattern of a large set of red giants at various evolutionary stages, we derive unbiased and precise asymptotic parameters. As the asymptotic expansion of gravity modes is verified with a precision close to the frequency resolution for stars on the red giant branch ($10^{-4}$ in relative values), we can derive accurate values of the asymptotic parameters. We decipher the complex pattern in a rapidly rotating star, and explain how asymmetrical splittings can be inferred. We also revisit the stellar inclinations in two open clusters, NGC 6819 and NGC 6791: our results show that the stellar inclinations in these clusters do not have privileged orientation in the sky. The variation of the asymptotic gravity offset with stellar evolution is investigated in detail. We also derive generic properties that explain under which conditions mixed modes can be observed.

Key words. Stars: oscillations - Stars: interiors - Stars: evolution

1. Introduction

Probing the cores of stars is difficult since, generally, stellar information arises from their photosphere. Fortunately, asteroseismology of evolved stars reveals stellar interiors in a unique and powerful way: gravity waves that propagate throughout the core couple with pressure waves and construct mixed modes that can be observed (Beck et al. 2011; Bedding et al. 2011; Benomar et al. 2014). The measurement of the global seismic properties of these mixed modes then carries unique information on the core structure (e.g., Montalbán et al. 2013; Lagarde et al. 2016; Bossini et al. 2015, 2017). Observations with the space missions CoRoT and Kepler provided the measurement of the asymptotic period spacings (Mosser et al. 2012b; Vrard et al. 2016), of the differential-rotation profile in red giants (Beck et al. 2012; Deheuvels et al. 2014, 2015), and of the core rotation for about 300 stars analyzed by Mosser et al. (2012c).

Most of the previous studies are based on the measurement and analysis of global seismic parameters, such as the asymptotic large separation $\Delta\nu$ and the asymptotic period spacings $\Delta P_1$ (e.g., Miglio et al. 2017). It is now time to access the properties of individual frequencies in red giants. Up to now, most of the studies (e.g., Baudin et al. 2012; Di Mauro et al. 2016) were limited to stars on the red giant branch (RGB). Two main reasons explain this restriction: first, the oscillation spectra benefit from a better relative frequency resolution for this evolutionary stage; second, the oscillation spectra remain simple, with rotational splittings smaller than period spacings. When stars evolve, these features become intricate, so that confusion is possible. For the most evolved stars, mixed modes are no longer observable (e.g., Baudin et al. 2012; Mosser et al. 2013; Stello et al. 2014).

The understanding of any complicated mixed-mode oscillation pattern must be based on an unambiguous identification of the modes. Up to now, the most efficient method has relied on the use of the asymptotic expansion, completed by a clear description of the influence of rotation (Mosser et al. 2015). New insights on rotation were provided by the analysis depicted in Gehan et al. (2016), who have developed a methodology to measure rotational splittings in an automated way; Gehan et al. (2017) and Gehan et al. (2018) showed how rapid rotation can be addressed efficiently. This efficiency derives from the use of stretched oscillation spectra.

In this work, we first examine in Section 2 how the different frequency spacings in the asymptotic mixed-mode expansion can be expressed as a function of the mode inertia. New expressions are proposed for the mixed-mode spacings and rotational splittings. Case studies are examined in Section 3 to test and validate these expressions. In Section 4, we take advantage of the precision of the fits to derive accurate asymptotic period spacings and gravity offsets; for the first time, we can exhibit the global evolution of
these gravity offsets as a function of stellar evolution. New insights on the rotational splittings are proposed in Section 5; in particular, we show how the asymptotic expansion can be used to provide priors based upon physical assumptions for any fitting code used later in the analysis. Finally, we assess the conditions for observing mixed modes, based on global asymptotic parameters only (Section 6). Section 7 is devoted to our conclusions.

2. Mixed-mode parameters

Following the work of Shibahashi (1979) and Unno et al. (1989), we derived asymptotic expansions of mixed modes for different seismic parameters: eigenfrequencies (Mosser et al. 2012b), period spacings (Christensen-Dalsgaard 2012), rotational splittings (Goupil et al. 2013; Deheuvels et al. 2015), and mode widths and mode heights (Grosjean et al. 2014; Belkacem et al. 2015b,a; Mosser et al. 2017a). Here, we intend to revisit all these parameters that depict the mixed-mode spectrum in order to provide a more precise and unified view.

2.1. Asymptotic expansion

The asymptotic expansion of mixed modes is an implicit relation between the phases $\theta_p$ and $\theta_g$ of the pressure- and gravity-wave contributions to the mixed modes, respectively. It reads

$$\tan \theta_p = q \tan \theta_g, \hspace{1cm} (1)$$

where $q$ is the coupling factor (Mosser et al. 2017b). The phases are related to the large separation $\Delta \nu$ and the period spacing $\Delta \Pi_1$. The most convenient expressions of the phase refer respectively to the pure$^1$ $p$ and $g$ mode spectra

$$\theta_g = \frac{\pi}{\Delta \Pi_1} \left( 1 - \frac{1}{\nu} - \nu_g \right), \hspace{1cm} (2)$$

$$\theta_p = \frac{\pi}{\Delta \nu_p} \nu_p - \nu_p, \hspace{1cm} (3)$$

where $\nu_p$ and $\nu_g$ are the asymptotic frequencies of pure pressure and gravity modes, respectively, and $\Delta \nu_p$ is the frequency difference between the consecutive pure pressure radial modes with radial orders $n_p$ and $n_p+1$. In this work, we consider that the radial modes and pure dipole pressure modes obey the universal red giant oscillation pattern (Mosser et al. 2011b) and that the dipole gravity modes follow the asymptotic comb-like pattern

$$\frac{1}{\nu_g} = (-n_g + \epsilon_g) \Delta \Pi_1, \hspace{1cm} (4)$$

where $\Delta \Pi_1$ is the period spacing and $\epsilon_g$ is the gravity offset.

Mosser et al. (2015) derived that the variation of the oscillation period $P$ with the mixed radial order $n$ writes

$$\frac{dP}{dt} = \zeta \Delta \Pi_1, \hspace{1cm} (5)$$

A convenient way to write the parameter $\zeta$ is (Hekker & Christensen-Dalsgaard 2017)

$$\zeta(\nu) = \left[ 1 + \frac{q}{N \nu^2 \cos^2 \theta_p + \sin^2 \theta_p} \right]^{-1}, \hspace{1cm} (6)$$

where $N = \Delta \nu/(\nu^2 \Delta \Pi_1)$ is the density of gravity modes compared to pressure modes, in other words the number of mixed modes in a $\Delta \nu$-wide interval. Compared to the original form presented in Mosser et al. (2015), the rapidly varying phase $\theta_g$ has been replaced by a function of $\theta_p$ that varies in a smooth way.

As demonstrated by Goupil et al. (2013) and used by subsequent work (Benomar et al. 2014; Deheuvels et al. 2015), the function $\zeta$ is connected to the inertia of mixed modes. Introducing the contributions of the envelope and of the core,

$$\zeta = \frac{I_{\text{core}}}{I_{\text{env}} + I_{\text{core}}}, \hspace{1cm} (7)$$

and assuming that the envelope contribution of a mixed mode is similar to the inertia of the closest radial mode ($I_{\text{env}} \equiv I_{n_n,0}$), we find that the inertia of the dipole mode with mixed radial order $n$ varies as

$$I_{n,1} = \frac{I_{n_n,0}}{1 - \zeta} \hspace{1cm} (8)$$

For the sake of simplicity, we use hereafter the abridged notation $I$ for the inertia of the dipole mixed modes and $I_0$ for the closest radial modes, and follow the same convention for the mode heights and widths.

2.2. Seismic parameters

With $\zeta$, we now intend to express the different seismic parameters.

2.2.1. Period spacing

Following Christensen-Dalsgaard (2012) and Mosser et al. (2015), period spacings can be expressed as

$$\Delta P = P_n - P_{n+1} = \zeta \Delta \Pi_1. \hspace{1cm} (9)$$

This expression is however ambiguous, since $\zeta$ may vary significantly between the periods $P_{n+1}$ and $P_n$ ($P_{n+1} > P_n$). Therefore, we prefer to consider the expression resulting from the integration of Eq. (5)

$$\Delta P = P_n - P_{n+1} = \Delta \Pi_1 \int_n^{n+1} \zeta(\nu) \, d\nu = \Delta \Pi_1 \langle \zeta \rangle_n, \hspace{1cm} (10)$$

where we consider that the mixed-mode radial order $n$ is a continuous variable defined by $d\nu = d\tau/\Delta \Pi_1$, where $\tau$ is the stretched period introduced by Mosser et al. (2015); i.e.,

$$d\tau = \frac{d\nu}{\zeta \nu^2}. \hspace{1cm} (11)$$

In fact, $n$ takes consecutive integer values for each mixed mode. In this work, we use an estimate of $n = n_p + n_g$ derived from the pressure and gravity radial orders; $n_p$
is derived from the universal red giant oscillation pattern (Mosser et al. 2011b), whereas \( n_g \) is given by

\[
\frac{1}{\nu \Delta \Pi_1} - \frac{1}{4}
\]

on the RGB, \( n_g = \frac{1}{\nu \Delta \Pi_1} + \frac{1}{4} \) in the red clump, where the correcting terms ±1/4 that depend on the evolutionary stage are justified in Section 4.4. They differ by where the correcting terms ± notation = \( n \in \mathbb{R} \), as depicted by the asymptotic relation (e.g., Tassoul 1980; Benomar et al. 2013). In red giants, the high density \( N \) of mixed modes implies that \( |n_g| \gg n_p \), so that the mixed-mode orders are negative.

From the definition of the stretched period, Eq. (10) reduces to

\[
\Delta P = \int_{\nu_n}^{\nu_{n+1}} \frac{d\nu}{\nu^2}. \tag{14}
\]

This evident relation justifies the relevance of Eq. (10) instead of Eq. (9): using \( (\zeta)_n \) is necessarily more accurate than using \( \zeta \) for computing period spacings.

### 2.2.2. Rotational splitting

As introduced by Goupil et al. (2013), the function \( \zeta \) is used to express the mixed-mode rotational splitting as a function of the mean rotational splittings related to pure gravity or pure pressure modes:

\[
\delta \nu_{\text{rot}} = \zeta \delta \nu_{\text{rot}, g} + (1 - \zeta) \delta \nu_{\text{rot}, p}. \tag{15}
\]

As shown by subsequent works (e.g., Deheuvels et al. 2014; Di Mauro et al. 2016; Triana et al. 2017), it is difficult to derive from the observed rotational splittings more than these two mean quantities.

Again, we have to solve the ambiguity of the meaning of \( \zeta \) in Eq. (15), since we can either consider the value\(^2\) \( \zeta(\nu_{n,0}) \), in the framework of the perturbation of the non-rotating frequency \( \nu_{n,0} \), or \( \zeta(\nu_{n,m}) \), considering that the inertia to be considered corresponds to the actual frequency \( \nu_{n,m} \). By analogy with the equation dealing with the period spacing, we propose to rewrite the rotational splitting \( \delta \nu_{\text{rot}} = \nu_{n,m} - \nu_{n,0} \), in the limit case where the mean envelope rotation is negligible compared to the core rotation, as:

\[
\delta \nu_{\text{rot}} = \delta \nu_{\text{rot, core}} \int_{\nu_{n,0}}^{\nu_{n,m}} \zeta \, d\nu = \delta \nu_{\text{rot, core}} (\zeta)_m, \tag{16}
\]

where \( \delta \nu_{\text{rot, core}} \equiv \delta \nu_{\text{rot}, g} \). As for the radial order \( n \) in Eq. (10), we consider the azimuthal order \( m \) as a continuous variable varying from 0 to ±1. So, we have introduced two mean values of \( \zeta \),

\[
(\zeta)_n = \int_n^{n+1} \zeta \, d\nu = \int_{\nu_{n,0}}^{\nu_{n,m}} \frac{d\nu}{\Delta \Pi_1 \nu^2}, \tag{17}
\]

\[
(\zeta)_m = \int_0^{\pm 1} \zeta \, dm = \int_{\nu_{n,0}}^{\nu_{n,m} \pm 1} \frac{d\nu}{\Delta \Pi_1 \nu^2}, \tag{18}
\]

to account for the period spacings and rotational splittings. The relevance of \( (\zeta)_n \) is already proven by Eqs. (10) and\(^2\) since we consider dipole modes only, we use a simplified notation \( \nu_{n,m} \) instead of \( \nu_{n,\ell,m} \).

\( (\zeta)_m \) has yet to be demonstrated. If we succeed, we will also have understood the relevance of the use of stretched periods for analyzing the mixed modes (Eq. 11).

### 2.3. Mixed-mode width, height, and amplitude

The work performed by the gas during one oscillation cycle is the same for all modes, associated with surface damping, when the radiative damping in the Brunt-Väisälä cavity is considered as negligible. Hence, Benomar et al. (2014) have estimated that the mode width of the mixed modes writes

\[
(\Gamma_n) = \frac{\Gamma_0}{\Gamma_{\text{res}}} = (\Gamma_0 - \zeta), \tag{19}
\]

From this relation, we verify that mixed modes have smaller mode widths than radial modes. However, we recall that a family of stars behave differently, when mixed modes are depressed because of an extra damping in the radiative inner region (Mosser et al. 2012a; García et al. 2014; Mosser et al. 2017a). From Belkacem et al. (2015a) we also derive that the amplitude of a resolved dipole mixed mode is

\[
A_2^n = A_0^2 (1 - \zeta), \tag{20}
\]

when the geometrical factor that conducts to a visibility of about 1.54 for red giant dipole modes (Mosser et al. 2012a, 2017a) is omitted. Such amplitudes correspond to similar heights for radial and dipole modes since \( A_2^2 = \pi \Gamma H /2 \). When, for non-resolved mixed modes, the width \( \Gamma_n \) is less than the frequency resolution \( \delta \nu_{\text{res}} \), a dilution factor must be considered (Dupret et al. 2009). It expresses

\[
H_n = \frac{\pi}{2} \frac{\Gamma_n}{\delta \nu_{\text{res}}} \cdot H_0, \tag{21}
\]

when radial modes are resolved, which is the common case.

### 2.4. Synthetic mixed-mode pattern

The previous ingredients can be used to depict an oscillation pattern. Figure 1 shows the synthetic spectrum of a typical star on the low RGB, based on Eq. (16) for the rotational splittings, on Eq. (19) for the mode widths, and on Eq. (21) for the mode heights of unresolved modes. This spectrum resembles the description derived by Grosjean.
et al. (2014) from non-adiabatic computations, with a time-dependent treatment of convection which provides the lifetimes of radial and non-radial mixed modes.

3. Case studies

In this Section, we use RGB stars showing clear oscillation spectra as case studies, in order to test the description of the mixed-mode spacings, widths, heights, and rotational splittings, which were previously introduced. The first steps consist in identifying their oscillation spectra and in fitting as many dipole mixed modes as possible. One of the two stars considered here, KIC 6144777 was already investigated in many previous articles (e.g., Corsaro et al. 2015; García Saravia Ortiz de Montellano et al. 2018). The other one, KIC 3955033, was less studied since it shows a complicated mixed-mode spectrum; it belongs to the list of red giants with period spacings automatically computed by Vrard et al. (2016). We used data downloaded from the KASOC site³, processed using the Kepler pipeline developed by Jenkins et al. (2010), and corrected from outliers, occasional jumps, and drifts (see García et al. 2011, for details).

3.1. Identification of the mixed modes

The location of the mixed modes primarily relies on the firm identification of the pure pressure-mode spectrum. The determination of the large separation $\Delta \nu$, first derived from the envelope autocorrelation function (Mosser & Appourchaux 2009), is based on the universal red giant oscillation pattern. This method provides the efficient identification of the radial modes and helps to locate the frequency ranges where mixed modes cannot be mistaken for radial or quadrupole modes. For $\ell = 1$ modes, the second-order asymptotic expansion writes

$$\nu_p = \left( n_\ell + \varepsilon_p + \frac{1}{2} + d_0 + \frac{\alpha}{2} \left[ n_\ell - n_{\text{max}} \right]^2 \right) \Delta \nu,$$

where $\varepsilon_p$ is the acoustic offset, $n_{\text{max}} = \nu_{\text{max}} / \Delta \nu - \varepsilon_p$, and $\alpha = 0.076 / \nu_{\text{max}}$. The parameter $d_0$ is function of the large separation, under the form $A + B \log \Delta \nu$ (where $\Delta \nu$ is expressed in $\mu$Hz), with $A = 0.0553$ and $B = -0.036$, as determined from the large-scale analysis along the RGB conducted by Mosser et al. (2014). The accurate determination of $d_0$ is crucial for the determination of the pure dipole pressure modes, hence for the determination of the minima of the function $\zeta$. In that respect, the small modulation of the radial-mode pattern induced by the sound-speed glitches (Miglio et al. 2010; Vrard et al. 2015) must be considered also. Therefore, we fit the actual position of the radial modes first, then use them to refine the pure pressure dipole-mode frequencies according to

$$\nu_p = \left( \nu_{n_\ell,0} + \nu_{n_\ell+1,0} \right) / 2 + d_0 \left( \nu_{n_\ell+1,0} - \nu_{n_\ell,0} \right).$$

The background parameters, derived as in Mosser et al. (2012a), are used to correct the granulation contribution in the frequency range around $\nu_{\text{max}}$. Hence, mixed modes can be automatically identified in frequency ranges that have no radial and quadrupole modes when their heights are significantly above the background. The automatic selection of the modes relies on a statistical test: the height-to-background ratio of the modes must be higher than a threshold level $R_p$, in order to reject the null hypothesis to a low probability $p$. According to Appourchaux et al. (2006), the relation between $R_p$ and $p$ depends for long-lived modes on the observation duration $T_{\text{obs}}$ and on the width $\Delta \nu$ of the frequency range where a mode is expected. This relation expresses

$$R_p \simeq \ln \frac{T_{\text{obs}} \Delta \nu}{\nu}.$$

when expressed in noise unit. This situation applies here, since the precise identification of the mixed-mode pattern is based on gravity-dominated mixed modes. With 4-year observations and the search of a couple of modes in a frequency range $\Delta \nu = \Delta \nu / N$, the threshold is typically 10 for a secure probability rejection at the $10^{-2}$ level. In practice, mixed modes with a height-to-background ratio higher than 10 are used to initiate the fit. A lower threshold is enough for the final agreement, when the synthetic mixed-mode pattern based on secure modes can be used to search for long-lived mixed modes in narrow frequency ranges. We benefit from the fact that the asymptotic fit is precise and enables to search for thin modes in a frequency range $\Delta \nu$ narrower than 0.1 $\mu$Hz. Therefore, a threshold of 7 is enough for rejecting the null hypothesis at the 1%-level for these modes whose detection benefits from the information gained by larger peaks. The thin mode widths (Eq. 19) are of great use to map the observed spectrum: a thin gravity-dominated mixed mode must be found in the close vicinity, less than 4 times the mode width, of its expected position. For unresolved peaks, this condition is relaxed to 4 times the frequency resolution. The global seismic parameters of the gravity component are then derived from the methods described in Vrard et al. (2016) and Mosser et al. (2017b), with a least-square fit between the observed and asymptotic patterns.

At this stage, global seismic parameters are measured and mixed modes are identified, so that it is possible to measure their individual properties.

3.2. Individual fitting procedure

When fitting individually mixed modes, we aim at testing the validity of the asymptotic expression, but not at reaching the ultimate precision, which is the role of a dedicated fit of individual modes (e.g., Gaulme et al. 2009). Therefore, in order to simplify the fit, we supposed (and checked a posteriori) that all multiplets can be fitted independently. This is not the case in all red giant spectra, but it is verified for most stars on the early RGB or in the red clump.

From the asymptotic fit, we identify in the background-corrected spectrum the power excess associated to each mode. Then, we determine the central frequency of the peak as the barycenter of the power excess. The height $H$ and full width at half maximum $\Gamma$ are simultaneously derived from the Lorentzian fit of the mode. We use Eqs. (19) and (21) as priors. Modes are fitted individually when the mode density is low, or simultaneously when the Lorentzians used as priors overlap.

The fitted spectrum and the seismic parameters of KIC 6144777, used as a first study case, are given in Fig. 2 and

³ http://kasoc.phys.au.dk
Fig. 2. Fit of the oscillation pattern of the RGB star KIC 6144777, showing the pressure radial orders \( n_p \) from 9 to 12. The power spectrum density has been divided by the fit of the background. Radial and quadrupole modes are highlighted in red and green. The expected locations of dipole mixed modes are labelled with their mixed radial orders. When detected, mixed modes are highlighted in dark blue \((m = -1)\), light blue \((m = 0)\), or purple \((m = 1)\). \( \ell = 3 \) modes, which are also mixed, are located near the abscissa 0.22; extra peaks in the range \([-0.2, -0.05]\) are mixed quadrupole modes. The gray dashed lines indicate the two thresholds used in this work, corresponding to height-to-background ratios of 7 and 10.

Fig. 3. Relative residuals, multiplied by 1000, between the observed and asymptotic mixed-mode frequencies in KIC 6144777. The color of the symbols indicates the azimuthal order: dark blue squares for \( m = -1 \), light blue diamonds for \( m = 0 \), or purple triangles for \( m = 1 \); 1-\( \sigma \) uncertainties are also shown. The dashed line corresponds to a perfect fit. The dotted lines show the frequency resolution plus an extra-modulation \( \Delta \nu (1 - \zeta) / 100 \), which is empirically used to define the quality of the fit.

in Table A.1. We note the large agreement between the observed and asymptotic peaks. As in other stars showing a seismic signal with a high signal-to-noise ratio \((S/R)\), outliers with a height-to-background value \( R \) higher than 7 are present. Their detection does not invalidate the method presented above: they correspond either to \( \ell = 2 \) or 3 mixed modes, possibly also to \( \ell = 4 \) modes, or to aliases (since the duty cycle is about 93\%), or even to noise since the detection of 1 noisy peak with \( R \geq 8 \) is expected in a 30-\( \mu \)Hz frequency range after 4 years of observation, assuming that the noise statistic is a \( \chi^2 \) with two degrees of freedom.

The quality of the fit is shown by the small residuals between the observed frequencies and the asymptotic fits (Fig. 3); we note that these residuals are comparable to the uncertainties, derived from Libbrecht (1992) or slightly larger when the quality of the fit may be affected by the high mode density. These residuals are of about the frequency resolution. When pressure-dominated mixed modes are excluded, the standard deviation of the asymptotic fit is 11 nHz. This value represents 1.3 times the frequency resolution \( \delta f_{\text{res}} \), or \( \Delta \nu / 1000 \), or a relative precision at \( \nu_{\text{max}} \) of about \( 10^{-4} \). The quality of the fits is based on a small number of parameters: the radial mode frequencies, the mean location \( d_{01} \) of the expected pure pressure dipole modes, and four asymptotic parameters: the period spacing \( \Delta \Pi_1 \), the coupling factor \( q \), gravitational offset \( \varepsilon_g \), and the mean core rotational splitting \( \delta \nu_{\text{rot}} \). Residuals reach maximum values near the pressure-dominated mixed modes: there, deviations of about \( \Delta \nu / 200 \) are observed, to be compared to the amplitudes of pressure glitches of about \( \Delta \nu / 40 \) (Vrard et al. 2015). We suspect that these residuals are mostly due to the variation of the parameter \( d_{01} \) with frequency.
Fig. 4. Period spacings of the RGB star KIC 6144777. Top: plot as a function of the arithmetical mean value \( \langle \zeta \rangle_n + \zeta_{n+1} \)/2. Bottom: plot as a function of the mean value \( \langle \zeta \rangle_n \). The colors code the azimuthal orders, as in Fig. 2; the dashed line indicates the 1:1 relation; 1-σ uncertainties on both the spacings and the mean values of \( \zeta \) are indicated by vertical and horizontal error bars.

3.3. Relationships with \( \zeta \)

With the identification of the mixed-mode pattern, we aim to verify the relevance of the use of \( \langle \zeta \rangle_n \) for the period spacings, to test the relevance of \( \langle \zeta \rangle_m \) for the rotational splittings, and further test the predictions for the mode widths and heights.

3.3.1. Period spacings

Period spacings were fitted with different functions of \( \zeta \), according either to the integrated value \( \langle \zeta \rangle_n \) (Eq. 10) or to the arithmetical mean \( \zeta = (\zeta_n + \zeta_{n+1})/2 \). The resulting plots are shown in Fig. 4. When \( \langle \zeta \rangle_n \) is not used, one remarks that the \( \Delta P(\zeta) \) relation shows a modulation that results from the concavity of \( \zeta \). When \( \zeta \) is close to unity, the period spacings are larger than predicted; below 0.7, where the function is concave, the period spacings are smaller than expected. The relation between \( \Delta P \) and \( \langle \zeta \rangle_n \) does not show such a modulation. Furthermore, the fit with \( \langle \zeta \rangle_n \) is nearly linear, with residuals two times smaller. From this comparison, we confirm that the use of \( \langle \zeta \rangle_n \) is preferable for fitting the period spacings.

Fig. 5. Mean rotation splittings of the RGB star KIC 3955033. Top: plot as a function of \( \zeta \). Bottom: plot as a function of the mean value \( \langle \zeta \rangle_m \). The colors code the azimuthal orders; the dashed line indicates the 1:1 relation; 1-σ uncertainties on both the splittings and the mean values of \( \zeta \) are indicated by vertical and horizontal error bars.

3.3.2. Rotational splittings

We performed similar test for the rotational splittings. We a priori excluded a dependence on \( \zeta_{(n,0)} \), since we clearly observe asymmetrical splittings (see below, Section 5.1) that cannot be reproduced with \( \zeta_{(n,0)} \). In fact, the rotation rate of KIC 6144777 is not important enough to observe any difference between the variations with either \( \zeta \) or \( \langle \zeta \rangle_n \). We therefore performed the fit of the star KIC 3955033 (Fig. A.3), which shows a much more rapid rotation (Fig. 5). From the comparison of \( \delta \nu_{\text{rot}}(\zeta) \) and \( \delta \nu_{\text{rot}}(\langle \zeta \rangle_m) \), we derive that this latter expression is more convenient since it provides a \( \chi^2 \) ten times smaller than when using \( \zeta \); associated with a much more precise estimate of the core rotation: \( \delta \nu_{\text{rot,core}} = 765 \pm 10 \text{ nHz} \) with \( \langle \zeta \rangle_m \), versus \( \delta \nu_{\text{rot,core}} = 730 \pm 50 \text{ nHz} \) with \( \zeta \). From this test, we conclude positively about the relevance of the use of \( \langle \zeta \rangle_m \) for the rotational splittings.

3.3.3. Widths, amplitudes, and heights

As expected from Eq. (19), the mixed-mode width shows large variations: pressure-dominated mixed modes have widths comparable to those of the radial modes, contrary to gravity-dominated modes that are much thinner (Fig. 6, top panel). Figure 6 also shows the validity of Eq. (19), with
the mixed-mode width proportional to \((1 - \zeta)\), except for low values where the observations resolution hampers the measurement of very thin widths. The precision of the fit is limited by the stochastic excitation, especially for long-lived peaks: the presence or absence of signal in a single frequency bin can modify the width in large proportion. This limit added to the limitation in frequency resolution does not allow us to test if small additional radiative damping affects the gravity-dominated mixed modes (Dupret et al. 2009; Grosjean et al. 2014).

As shown by Mosser et al. (2015), Eq. (20) has a strong theoretical justification, since it expresses the conservation of energy: the sum of all the energy distributed in the mixed modes corresponds to the energy expected in the single pure pressure mode that should exist in absence of any coupling. So, our result is in line with the findings of Mosser et al. (2012a), who measured that, except for depressed modes, the observed total visibility of dipole modes matches the theoretical expectations.

Due to the stochastic nature of the excitation, the mode heights show a large spread (Fig. 7). Dips in the distributions occur when modes are not resolved. It is however clear in Fig. 7 bottom that the dipole mode heights follow the radial distribution according to the trend of Eq. (21). We note that all mixed modes associated with a given pressure radial order show a systematic behavior. For instance, all mixed modes of KIC 6144777 in the frequency range \([137, 143 \mu Hz]\) associated with the pressure mode \(n_p = 11\) show lower amplitudes than expected from the Gaussian fit of the power excess. Such a behavior recalls us that the excitation of a mixed mode is due to its acoustic component.

### 3.4. Validation

From these two case studies and from other examples shown in Appendix, we can conclude that the asymptotic fits are relevant at all evolutionary stages, when the signal-to-noise ratio is high enough. So, the equations developed in Section 2 allow us to depict the mixed-mode spectrum with a very high accuracy, when the integrated values \(\zeta_n\) and \(\zeta_m\) are considered for the period spacings and the rotational splittings, respectively. Up to now, only red clump stars showing buoyancy glitches cannot be fitted with a single set of parameters.
4. Asymptotic period spacings and gravity offsets

In this section, we show how previous findings can be used to derive accurate period spacings. We also explore the variation of the gravity offsets $\epsilon_g$ with stellar evolution. These studies rely on the determination of the pure-gravity mode pattern.

4.1. Observations

Our analysis was conducted over 372 red giants at various evolutionary stages, mainly from Mosser et al. (2014) and Vrard et al. (2016), with stars also considered in Beck et al. (2012), Kallinger et al. (2012), Deheuvels et al. (2014), and Corsaro et al. (2015). Data were obtained as for the two stars considered in Section 3. When available, effective temperatures are from APOGEE spectra (Albareti et al. 2017). Selection criteria are mainly based upon the noise level, with Kepler magnitudes brighter than 12 on the low RGB or 14 for more evolved stars. Following the method exposed in Section 3.1, we need data with a S/R high enough to allow the identification of gravity-dominated mixed modes. When such modes are too few, measurements are impossible. This condition induces a selection bias, specifically addressed in Section 6.

The 372 stars that were analyzed are shown in a seismic diagram (Fig. 8). We considered stars from the low RGB (Fig. A.1) to more evolved RGB stars (Fig. A.2). The spectrum of the evolved RGB star KIC 2443903 (Fig. A.4) corresponds to a case near the limit of visibility of gravity-dominated mixed modes, with a mode density $N \approx 22.4$ close to the limit value above which the detection is impossible (Section 6). The fitting process for red clump stars can be achieved only when the amplitude of the buoyancy glitches remains limited (Fig. A.5); the same limitation appears in the secondary red clump (Fig. A.6). In fact, except for red-clump stars with large buoyancy glitches (Cunha et al. 2015; Mosser et al. 2015), the asymptotic expansion provides a relevant fit. We could then obtain precise measurements of the asymptotic gravity parameters in Eq. (4) and of their uncertainties for a large number of stars. We must report one exception: KIC 3216736 is the only RGB star of our sample where we found buoyancy glitches and could not provide a relevant fit of the spectrum, but only an échelle diagram based on stretched periods (Fig. 9). Since
we have tested more than 160 stars on the RGB, with a systematic approach, we can conclude that the most common case on the RGB is the absence of buoyancy glitches, as expected theoretically (Cunha et al. 2015).

Characterizing the sample we studied in terms of bias is difficult. Apart from the RGB stars that were already studied in detail in previous works, we have mostly treated the stars with increasing KIC numbers. This systematic method implies that we did not introduce any further bias compared to the Kepler sample of red giants. Considering a high enough S/R, which is almost equivalent to select bright stars in the red giant domain, is not supposed to introduce biases either. Contrary to many previous studies, we are not limited to stars showing rotational splittings smaller than the confusion limit \(\delta \nu \lesssim \nu_{\text{max}}\). However, the presence of a strong cutoff (Section 6) limits the sample, when gravity-dominated mixed modes disappear. Red-clump stars with strong buoyancy glitches are absent in our data set since the fitting process requires then to account for the extra-modulation, which can be quite large (about \(\Delta \Pi/10\)). When mixed modes are depressed, the low height-to-background ratio of the mixed modes allows the measurement of \(\Delta \Pi\) (Mosser et al. 2017a) but is not enough for fitting the pattern. Both cases deserve specific care beyond the scope of this work.

### 4.2. Pure gravity modes

The identification of the mixed modes depicted in Section 3.1 allows us to retrieve the periods of the pure gravity modes and to infer global asymptotic parameters of the gravity components. We compute these periods from the mixed-mode frequencies \(\nu\), using Eqs. (1) and (2),

\[
\frac{1}{\nu_g} = \frac{1}{\nu} - \frac{\Delta \Pi_1}{\pi} \tan^{-1}\left(\frac{\tan \theta_p}{q}\right). \tag{25}
\]

Close to each radial mode, when \(\theta_p\) varies from values less than but close to \(\pi/2\) to values higher than but close to \(\pi/2\), the atan correcting term introduces an offset of \(\Delta \Pi_1\), which in fact allows to relate the \((\nu + 1)\) mixed modes in the \(\Delta \nu\)-wide interval to \(\nu\) only gravity modes. In order to use all mixed modes, including the \(|m| = 1\) components, we corrected first the rotational splittings, using Eq. (16) in order to obtain \(\nu\) values that are corrected from the rotational splitting.

From the periods of the gravity modes \(1/\nu_g\), we could then derive the asymptotic parameters \(\Delta \Pi_1\) and \(\varepsilon_g\), assuming the first-order asymptotic expression for pure gravity modes (Eq. 4). In practice, a first estimate of \(\Delta \Pi_1\) derived from the formalism of Mosser et al. (2015) and Vrard et al. (2016) is used in Eq. (25), then iterated with a least-square fit of the linear variation of the gravity modes (Eq. 4).

### 4.3. Asymptotic period spacings

Up to now, measurements of \(\Delta \Pi_1\) considering that \(\varepsilon_g\) is a free parameter have been obtained for a few stars only (Buysschaert et al. 2016; Hekker et al. 2018, for 3 and 22 observed stars, respectively). The offset \(\varepsilon_g\) being arbitrarily fixed, Mosser et al. (2012b) and Mosser et al. (2014) reported a very high precision for the period spacings, of typically 0.1 s for stars on the RGB and 0.3 s in the red clump. However, owing to the choice of \(\varepsilon_g = 0\), their period spacings were slightly affected by a bias of about a fraction of \(\nu_{\text{max}}\). The values reported by Vrard et al. (2016), free of any hypothesis on \(\varepsilon_g\), are not biased but show uncertainties typically five to fifteen times higher than the new values. Their comparison with our data confirms the absence of systematic offsets (Fig. 10). So, the new method ensures accuracy, in the sense that the measurements of \(\Delta \Pi_1\) are now free of any hypothesis on \(\varepsilon_g\) and prove that the asymptotic expansion for gravity modes (Eq. 4) is relevant. The relative accuracy we obtained for the period

---

**Table 1. Period spacings and gravity offsets**

<table>
<thead>
<tr>
<th>KIC</th>
<th>(\nu_{\text{max}}) ((\mu)Hz)</th>
<th>(\Delta \nu) ((\mu)Hz)</th>
<th>(\Delta \Pi_1) (s)</th>
<th>(\varepsilon_g)</th>
<th>(\sigma_{\nu_{\text{rot}}}) (nHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1576469</td>
<td>90.60 ± 0.98</td>
<td>7.41 ± 0.04</td>
<td>284.80 ± 0.64</td>
<td>0.23 ± 0.03</td>
<td>−0.102 ± 0.096</td>
</tr>
<tr>
<td>1723700</td>
<td>39.42 ± 0.57</td>
<td>4.48 ± 0.04</td>
<td>323.40 ± 0.17</td>
<td>0.24 ± 0.04</td>
<td>0.066 ± 0.043</td>
</tr>
<tr>
<td>2437976</td>
<td>89.37 ± 1.10</td>
<td>8.22 ± 0.05</td>
<td>74.70 ± 1.00</td>
<td>0.10 ± 0.02</td>
<td>−0.006 ± 0.095</td>
</tr>
<tr>
<td>2443903</td>
<td>66.76 ± 0.90</td>
<td>7.01 ± 0.04</td>
<td>71.10 ± 0.02</td>
<td>0.12 ± 0.02</td>
<td>−0.184 ± 0.078</td>
</tr>
<tr>
<td>3955033</td>
<td>106.10 ± 1.24</td>
<td>9.23 ± 0.05</td>
<td>74.65 ± 0.06</td>
<td>0.13 ± 0.02</td>
<td>0.207 ± 0.115</td>
</tr>
<tr>
<td>5024476</td>
<td>68.66 ± 0.75</td>
<td>5.73 ± 0.04</td>
<td>299.60 ± 1.00</td>
<td>0.27 ± 0.03</td>
<td>−0.199 ± 0.102</td>
</tr>
<tr>
<td>5112373</td>
<td>43.82 ± 0.59</td>
<td>4.63 ± 0.04</td>
<td>240.30 ± 0.14</td>
<td>0.19 ± 0.02</td>
<td>−0.246 ± 0.058</td>
</tr>
<tr>
<td>6144773</td>
<td>128.23 ± 1.50</td>
<td>11.03 ± 0.05</td>
<td>79.05 ± 0.04</td>
<td>0.13 ± 0.02</td>
<td>0.210 ± 0.055</td>
</tr>
<tr>
<td>10272858</td>
<td>341.45 ± 0.16</td>
<td>22.71 ± 0.14</td>
<td>96.90 ± 0.30</td>
<td>0.19 ± 0.02</td>
<td>0.338 ± 0.098</td>
</tr>
<tr>
<td>11353313</td>
<td>127.29 ± 1.46</td>
<td>10.75 ± 0.05</td>
<td>76.95 ± 0.06</td>
<td>0.14 ± 0.02</td>
<td>0.290 ± 0.088</td>
</tr>
</tbody>
</table>

The list of the full data set with 372 red giants showing an uncertainty in \(\varepsilon_g\) less than 0.15 is available on line as a CDS/VizieR document.

**Fig. 11.** Relative precision of the asymptotic period spacings. Same style as in Fig. 8.
Fig. 12. Left: variation of $\varepsilon_g$ with $\Delta \nu$, with the same style as Fig. 11. The horizontal dark gray domain corresponds to the expected range predicted for RGB stars by Takata (2016a), whereas the dot-dashed line shows the value $\varepsilon_{g,as} = 1/4$ derived from the asymptotic expansion (Provost & Berthomieu 1986). Uncertainties on $\varepsilon_g$ are indicated by vertical lines; uncertainties on $\Delta \nu$ are smaller than the symbol size. Right: histograms of the distributions of $\varepsilon_g$ on the RGB (blue curve) and in the red clump (red curve). The dot-dashed line and the gray domain have the same meaning as indicated above.

4.4. Gravity offsets $\varepsilon_g$

We could measure $\varepsilon_g$ for a large set of stars. We however have to face the indetermination of $\varepsilon_g$ modulo 1: we simply assume that $\varepsilon_g$ is in the range $[-0.5, 0.5]$. The $\varepsilon_g$ values computed for the set of stars presented in the paper is given in Table 1 and plotted in Fig. 12, where an histogram is also given. The complete table is given online only. Uncertainties on $\varepsilon_g$ are small, related to the uncertainties in $\Delta \Pi_1$ by

$$\delta \varepsilon_g = \frac{\delta \Delta \Pi_1}{\nu_{\text{max}} \Delta \Pi_1^2}.\quad (26)$$

This relation comes from the derivative of Eq. (4). As a result, the median uncertainties are of about 0.08 on the RGB and 0.06 in the red clump.

We noticed that the median value of $\varepsilon_g$ on the RGB is in fact close to 1/4, which is the expected asymptotic value in absence of stratification below the convection zone (Provost & Berthomieu 1986), derived from the contribution $\ell/2 - \varepsilon_{as}$ with $\ell = 1$ and $\varepsilon_{as} = 1/4$. Hence, we inferred that the degeneracy on the determination of $\varepsilon_g$ is removed. We then noted a slight decrease in $\varepsilon_g$ when stars evolve on the RGB, with an accumulation of values close to 0 for red-clump stars. Hekker et al. (2018) reported values of $\varepsilon_g$ in the range $[-0.2, 0.5]$ for 21 stars on the RGB, but did not identify the accumulation of values in the range $[0.20, 0.35]$ predicted by Takata (2016a) for stars on the low RGB. Our measurements fully confirm this prediction. From a check of their data set, we interpret the differences in $\varepsilon_g$ as resulting from less precise gravity spacings when large rotational splittings apparently modify the period spacings. As made clear by the recent theoretical developments of the asymptotic expansion (Takata 2016b,a), the accurate measurement of the leading-order term $\Delta \Pi_1$ is necessary to provide reliable estimates of $\varepsilon_g$.

We can study the variation of $\varepsilon_g$ along stellar evolution. On the RGB, the asymptotic expansion predicts $\varepsilon_g = 1/4 - \vartheta$ (Provost & Berthomieu 1986), where $\vartheta$ is a measure of the stratification just below the convection zone. From this dependence, we can infer that the term $\vartheta$ is certainly very small for most stars on the low RGB. Higher values are suspected for evolved RGB stars, but with too few stars to firmly conclude, whereas lower values are seen for evolved RGB. We checked that the change of regime of $\varepsilon_g$ is not associated with the luminosity bump since it occurs for more evolved stars than our sample (Khan et al. 2018). In the red clump, the $\vartheta$ correction seems important, on the order of 0.3, with a larger spread than observed on the RGB.

An extended study of $\varepsilon_g$ can now be performed to use this parameter as a probe of the stratification occurring in the radiative region. This study is however beyond the scope of this work.
the function \( \zeta \) orders from \(-d\)ial and azimuthal orders. The rotational splittings of the radial
orders from \(-142\) to \(-140\), plotted with diamonds, do no match
the function \( \zeta \). Only the multiplet with \( n = -141 \) is complete:
the \( m = +1 \) splitting is much larger than the \( m = -1 \) splitting;
the colored regions indicate the ranges over which the function
\( \zeta \) is integrated for the components of the multiplet \( n = -141 \).
The dashed lines indicate height-to-background values of 7 and
10.

5. Rotation

The fits based on the function \( \zeta \) also allow us to analyze
rotational splittings in detail.

5.1. Splitting asymmetry

Recently, asymmetries in the rotational splittings were
reported by Deheuvels et al. (2017), as the signature of the
combined effects of rotation and mode mixing. Using both
perturbative and non-perturbative approaches, they com-
puted near-degeneracy effects and could fit the data. In fact,
the asymptotic development of mixed modes also describes
the combined effects of rotation and mode mixing, so that
the rotational splittings based on \( \langle \zeta \rangle_m \) (Eqs. 16 and 18) are
not symmetric. Inversely, the symmetrical rotational splitting
based on \( \zeta \) (Eq. 15) does not reproduce the observed asymmetry.
Hence, observing asymmetrical triplets is a way
to prove the relevance of the use of \( \langle \zeta \rangle_m \) instead of \( \zeta \).

Observe the asymmetry is challenging but possible
for stars with a rapid rotation rate. As explained by Gehan
et al. (2017, 2018), rapid rotation means \( \Delta \nu_{\text{rot}} \geq \Delta \nu/N \)
for seismology. This rotation is however very slow in terms
of interior structure, so that the formalism developed by
Goupil et al. (2013) and Deheuvels et al. (2014), summa-
rized by Eq. (16), remains relevant. It simplifies the study,
as shown by Ouazzani et al. (2013) who treated the case
where rotational splittings can be as large as \( \Delta \nu \). We fitted
the mixed-mode spectrum of KIC 3955033 with both the
symmetrical and asymmetrical splitting. At high radial or-
der \( n_p \), it is hard to distinguish them. At low orders, when
the rotational splittings exceed the mixed-mode spacings,
the symmetrical splittings fail whereas the asymmetrical
one provides a consistent solution along the whole spec-
trum. The radial order \( n_p = 8 \) is shown in Fig. 13, the
whole spectrum is shown in Fig. A.3.

5.2. Surface rotation

For stars on the low RGB, surface rotation can be inferred
from the rotational splittings (Eq. 15). The measurement
is however difficult, since it results from an extrapolation
at \( \zeta = 0 \), when values are mostly obtained above \( \zeta = 0.6 \)
only (Fig. 5). The highest level of precision, hence the use
of \( \langle \zeta \rangle_m \) instead of \( \zeta \), is required for deriving a correct es-
timate of the surface rotation. The case of KIC 3955033
is illustrative, with a negative surface rotation when using
\( \zeta \); the use of \( \langle \zeta \rangle_m \) provides a null value (5 \( \pm 20 \text{ nmHz} \)). This
case also confirms the general situation shown by previous
works (Goupil et al. 2013; Di Mauro et al. 2016; Triana
et al. 2017): deriving surface rotation can be achieved for
the low RGB only.

5.3. Stellar inclination

From its ability to fit the gravity-dominated modes that
carry useful information, the asymptotic fit can be used
to derive the stellar inclination \( i \). The amplitude of the
\( m = 0 \) component of the dipole multiplet is proportional
to \( \sin^2 i \) whereas the sum of the amplitudes of the \( m = \pm 1 \)
mode is proportional to \( \cos^2 i \). From Eq. (20), a correction
factor of \( 1/(1 - \zeta) \) should be applied on the amplitudes: its
differential effect is however much below the precision one
can get on \( i \).

We tested our results on a set of stars for which the incli-
nations measured with other methods have been obtained.
We checked that our results are relevant, with a precision
limited by the uncertainties on the amplitude measure-
ments. In order to avoid bias, we consider only peaks with a
height-to-background ratio larger than 8. Nevertheless, we
noted that the stochastic excitation of the modes induces
a small bias for large inclinations. Equator-on inclinations,
near 90°, cannot be retrieved precisely, with measurements
reduced toward the range 70–80°. As a consequence, they
are rare in our analysis. However, many stars show inclina-
tions that, according to the uncertainties, are compatible
with equator-on measurement, so that the bias does not
affect the following analysis.

We measured inclinations of red giants in the open clus-
ters NGC 6819 observed by Kepler (e.g., Basu et al. 2011;
Stello et al. 2011; Miglio et al. 2012). We selected the stars
that exhibit mixed modes and could fit 20 mixed-mode
patterns with the asymptotic expansion. In one case, the asymptotic fit is impossible, due to a low S/R. In two other cases, different possible solutions exist, based either on different period spacings, or on different rotational splittings, but without any ambiguity for the inclination measurement: when two peaks dominate per period spacing, the inclination is necessarily high, whereas it is low when one single peak only is present. We completed this list with other NGC 6819 members listed in Handberg et al. (2017) and could fit two additional stars, which incidentally show a large inclination. Results for the inclinations and rotational splittings are given in Table 2. As shown in Fig. 14, the distribution of the stellar inclinations mimics the sin $i$ relation expected for random inclinations, except near 90$^\circ$, due to bias mentioned above. A similar test performed on the open cluster NGC 6791 reaches the same conclusion.

Low stellar inclinations in NGC 6819 and 6791 were observed. In fact, the asymptotic fit is impossible, due to a low S/R. In two other cases, different possible solutions exist, based either on different period spacings, or on different rotational splittings, but without any ambiguity for the inclination measurement: when two peaks dominate per period spacing, the inclination is necessarily high, whereas it is low when one single peak only is present. We completed this list with other NGC 6819 members listed in Handberg et al. (2017) and could fit two additional stars, which incidentally show a large inclination. Results for the inclinations and rotational splittings are given in Table 2. As shown in Fig. 14, the distribution of the stellar inclinations mimics the sin $i$ relation expected for random inclinations, except near 90$^\circ$, due to bias mentioned above. A similar test performed on the open cluster NGC 6791 reaches the same conclusion.

Low stellar inclinations in NGC 6819 and 6791 were observed. In fact, the asymptotic fit is impossible, due to a low S/R. In two other cases, different possible solutions exist, based either on different period spacings, or on different rotational splittings, but without any ambiguity for the inclination measurement: when two peaks dominate per period spacing, the inclination is necessarily high, whereas it is low when one single peak only is present. We completed this list with other NGC 6819 members listed in Handberg et al. (2017) and could fit two additional stars, which incidentally show a large inclination. Results for the inclinations and rotational splittings are given in Table 2. As shown in Fig. 14, the distribution of the stellar inclinations mimics the sin $i$ relation expected for random inclinations, except near 90$^\circ$, due to bias mentioned above. A similar test performed on the open cluster NGC 6791 reaches the same conclusion.

Low stellar inclinations in NGC 6819 and 6791 were observed. In fact, the asymptotic fit is impossible, due to a low S/R. In two other cases, different possible solutions exist, based either on different period spacings, or on different rotational splittings, but without any ambiguity for the inclination measurement: when two peaks dominate per period spacing, the inclination is necessarily high, whereas it is low when one single peak only is present. We completed this list with other NGC 6819 members listed in Handberg et al. (2017) and could fit two additional stars, which incidentally show a large inclination. Results for the inclinations and rotational splittings are given in Table 2. As shown in Fig. 14, the distribution of the stellar inclinations mimics the sin $i$ relation expected for random inclinations, except near 90$^\circ$, due to bias mentioned above. A similar test performed on the open cluster NGC 6791 reaches the same conclusion.

Low stellar inclinations in NGC 6819 and 6791 were observed. In fact, the asymptotic fit is impossible, due to a low S/R. In two other cases, different possible solutions exist, based either on different period spacings, or on different rotational splittings, but without any ambiguity for the inclination measurement: when two peaks dominate per period spacing, the inclination is necessarily high, whereas it is low when one single peak only is present. We completed this list with other NGC 6819 members listed in Handberg et al. (2017) and could fit two additional stars, which incidentally show a large inclination. Results for the inclinations and rotational splittings are given in Table 2. As shown in Fig. 14, the distribution of the stellar inclinations mimics the sin $i$ relation expected for random inclinations, except near 90$^\circ$, due to bias mentioned above. A similar test performed on the open cluster NGC 6791 reaches the same conclusion.

Table 2. Asymptotic and rotational parameters in NGC 6819

<table>
<thead>
<tr>
<th>KIC ID</th>
<th>$\Delta\nu$ (µHz)</th>
<th>$\Delta\Pi_1$ (s)</th>
<th>$q$</th>
<th>$\delta\nu_{\text{rot}}$ (nHz)</th>
<th>$i$ (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4937056</td>
<td>4.76</td>
<td>291.0</td>
<td>0.21</td>
<td>90</td>
<td>60±15</td>
</tr>
<tr>
<td>4937257</td>
<td>4.13</td>
<td>292.1</td>
<td>0.19</td>
<td>27</td>
<td>72±13</td>
</tr>
<tr>
<td>4937770</td>
<td>5.82</td>
<td>261.0</td>
<td>0.18</td>
<td>51</td>
<td>12±14</td>
</tr>
<tr>
<td>4937775</td>
<td>7.33</td>
<td>226.3</td>
<td>0.21</td>
<td>110</td>
<td>28±15</td>
</tr>
<tr>
<td>5023953</td>
<td>4.74</td>
<td>293.9</td>
<td>0.24</td>
<td>50</td>
<td>51±28</td>
</tr>
<tr>
<td>5024327</td>
<td>4.72</td>
<td>269.5</td>
<td>0.20</td>
<td>55</td>
<td>56±13</td>
</tr>
<tr>
<td>5024404</td>
<td>4.78</td>
<td>242.6</td>
<td>0.25</td>
<td>110</td>
<td>80±10</td>
</tr>
<tr>
<td>5024414</td>
<td>6.47</td>
<td>283.0</td>
<td>0.30</td>
<td>90</td>
<td>45±20</td>
</tr>
<tr>
<td>5024414</td>
<td>5.73</td>
<td>299.5</td>
<td>0.24</td>
<td>56</td>
<td>71±11</td>
</tr>
<tr>
<td>5024582</td>
<td>4.76</td>
<td>323.5</td>
<td>0.25</td>
<td>70</td>
<td>55±18</td>
</tr>
<tr>
<td>5111718</td>
<td>10.59</td>
<td>88.4</td>
<td>0.12</td>
<td>410</td>
<td>69±21</td>
</tr>
<tr>
<td>5111949</td>
<td>4.81</td>
<td>319.0</td>
<td>0.28</td>
<td>35</td>
<td>66±15</td>
</tr>
<tr>
<td>5112072</td>
<td>10.08</td>
<td>91.9</td>
<td>0.15</td>
<td>350</td>
<td>72±12</td>
</tr>
<tr>
<td>5112361</td>
<td>6.19</td>
<td>99.0</td>
<td>0.12</td>
<td>350</td>
<td>70±20</td>
</tr>
<tr>
<td>5112373</td>
<td>4.63</td>
<td>240.2</td>
<td>0.19</td>
<td>37</td>
<td>47±18</td>
</tr>
<tr>
<td>5112387</td>
<td>4.70</td>
<td>267.2</td>
<td>0.28</td>
<td>84</td>
<td>25±17</td>
</tr>
<tr>
<td>5112401</td>
<td>4.03</td>
<td>311.0</td>
<td>0.26</td>
<td>50</td>
<td>54±13</td>
</tr>
<tr>
<td>5112467</td>
<td>4.68</td>
<td>324.3</td>
<td>0.30</td>
<td>150</td>
<td>31±16</td>
</tr>
<tr>
<td>5112730</td>
<td>4.56</td>
<td>320.0</td>
<td>0.25</td>
<td>45</td>
<td>56±18</td>
</tr>
<tr>
<td>5112938</td>
<td>4.73</td>
<td>320.0</td>
<td>0.30</td>
<td>65</td>
<td>45±11</td>
</tr>
<tr>
<td>5112950</td>
<td>4.35</td>
<td>319.5</td>
<td>0.38</td>
<td>38</td>
<td>61±18</td>
</tr>
<tr>
<td>5112974</td>
<td>4.32</td>
<td>309.6</td>
<td>0.19</td>
<td>37</td>
<td>47±18</td>
</tr>
<tr>
<td>5113441</td>
<td>11.75</td>
<td>88.4</td>
<td>0.12</td>
<td>730</td>
<td>18±18</td>
</tr>
<tr>
<td>5200152</td>
<td>4.73</td>
<td>327.2</td>
<td>0.28</td>
<td>50</td>
<td>70±15</td>
</tr>
</tbody>
</table>

$\Delta\nu$, $\Delta\Pi_1$, and $\delta\nu_{\text{rot}}$ are measured by Corsaro et al. (2017), using a Bayesian analysis, from which aligned spins were inferred. Our measurements however contradict their claim, as shown in Fig. 15 of Vrard et al. (2018). Figure 1 of the supplementary material of Corsaro et al. (2017) provides an explanation of the discrepant Bayesian values. Their fit of the star KIC 5112373 in NGC 6819 provides nearly uniform large mode widths, relevant for the pressure-dominated mixed modes but much too high for gravity-dominated modes, in contradiction with the physical variation indicated by Eq. (19). As a consequence, their fit assumes that all the power is concentrated in the $m = 0$ mode; the resulting stellar inclination is 20±8$^\circ$. We show the asymptotic solution of KIC 5112373 in Fig. 16, with thin gravity dominated mixed modes and the clear identification of triplets. Since $m \pm 1$ modes are observed all along the spectrum, our solution for the inclination is larger, about 47±18$^\circ$.

We provide another example with the star KIC 2437976, a NGC 6791 member. As shown in Fig. 17, rotational splittings are explained in a consistent way with thin unresolved gravity-dominated mixed modes and a rotation rate rapid enough to ensure that close modes do not belong to the same multiplets. All peaks can be explained by the $m = \pm 1$ modes. In practice, $m = 0$ modes are absent, that this star has necessarily an inclination close to 90$^\circ$, whereas Corsaro et al. (2017) measured $i \approx 90^\circ$. We conclude that some of the low inclinations reported in Corsaro et al. (2017) are incompatible with the analysis presented here. It seems that the difference is due to a too low range of the linewidth priors in the Bayesian analysis, which favors a solution with a low inclination angle and a high splitting. As a result, stellar spins in old open clusters are neither aligned nor quasi parallel to the line of sight. Our study emphasizes a major role for the asymptotic analysis: pro-
6. Observability of the mixed modes

All the information derived from mixed modes relies on their observability. The properties of the function $\zeta$ can be used to assess under which conditions mixed modes can be actually observed. To achieve this, we investigate first the domain where pressure-dominated mixed modes are observed, then the condition for observing gravity-dominated mixed modes.

### 6.1. Pressure-dominated mixed modes

We can define the frequency range where mixed modes are pressure-dominated (pm) from the full width at half minimum of the $\zeta$ function. So, these modes cover a range, expressed in terms of the pressure phase $\theta_p$ (Eq. 6), verifying

$$\delta \theta_p |_{\text{pm}} = 2q \sqrt{\frac{1}{1 + \frac{1}{Nq}}}$$

(27)

under the assumption that $q$ is small, which is verified for all stars except at the transition between subgiants and red giants (Mosser et al. 2017b). When expressed in frequency and compared to the large separation, this condition corresponds to a frequency range surrounding each pure pressure modes with a width $\delta \nu_{\text{pm}}$ defined by

$$\frac{\delta \nu_{\text{pm}}}{\Delta \nu} = \frac{2q}{\pi} \sqrt{1 + \frac{1}{Nq}}.$$  

(28)

The variations in $q$ and $N$ explain the narrowing of the region with pressure-dominated mixed modes when stars evolve on the RGB. An example is shown in the Appendix (Fig. A.4). The expression of $\delta \nu_{\text{pm}}$ also shows that red-clump stars, with larger $q$ show pressure-dominated mixed modes in a broader region than RGB stars.

### 6.2. Visible gravity-dominated mixed modes

The non-dilution of the mode height expressed by Eq. 21 can be used to define a criterion of visibility of the gravity-dominated (gm) mixed modes. So, they are clearly visible when they show heights similar to those of the pressure modes ($H_g = H_p$), hence when $\Gamma_0(1 - \zeta) \geq 2\delta f_{\text{res}}/\pi$ (Eq. 19). This condition translates into

$$(1 - q^2) \sin^2 \theta_p \leq \frac{q}{N} \left( \frac{\pi}{2} \frac{\Gamma_0}{\delta f_{\text{res}}} - 1 \right) - q^2.$$  

(29)

Except at the transition from subgiants to red giants, where mixed modes are unambiguously visible (Benomar et al. 2013; Deheuvels et al. 2014), the terms $q^2$ are negligible, so that modes are clearly visible if

$$|\sin \theta_p| |_{\text{gm}} \leq \sqrt{\frac{q}{N} \left( \frac{\pi}{2} \frac{\Gamma_0}{\delta f_{\text{res}}} - 1 \right)}.$$  

(30)

This condition for observing gravity-dominated mixed modes has many consequences:
- It can be fulfilled only if the definition of the right term is ensured, which requires a frequency resolution low enough compared to the radial mode width. With $\Gamma_0$ in the range $[100, 150 \, \text{mHz}]$, the observation must last 50-75 days at least.
- In fact, mixed modes were observable with CoRoT runs lasting about 150 days (Mosser et al. 2011a), but are hardly observable with K2 80-day time series (Stello et al. 2017).
- When stars evolve on the RGB, the decrease in $q$ and increase in $N$ contribute to the narrowing of observable modes. Mixed modes are more easily visible in the red clump, owing to larger $q$ values. This criterion is implicitly used by Elsworth et al. (2017) for their determination of the evolutionary state of red-giant stars.
- All mixed modes are clearly visible when the condition expressed by Eq. 30 is always met, that is when

![Fig. 16. Fit of the mixed modes corresponding to $n_p = 7$ in KIC 5112373 (NGC 6819 member). The color codes the azimuthal order: $m = +1$ in purple, $m = -1$ in blue. The gray dashed lines indicate the two thresholds used in this work, corresponding to height-to-background ratios of 7 and 10. Contrary to the analysis conducted by Corsaro et al. (2017), modes with $m = \pm 1$ are clearly identified.](image1)

![Fig. 17. Fit of the mixed modes corresponding to $n_p = 9$ in KIC 2437976 (NGC 6791 member). The color codes the azimuthal order: $m = +1$ in purple, $m = -1$ in blue. The location of $m = 0$ modes is indicated in light blue, but none shows a large height for this star seen equator-on. The gray dashed lines indicate the two thresholds used in this work, corresponding to height-to-background ratios of 7 and 10. Many peaks above the threshold value 5.5 that rejects the null hypothesis at the 5%-level follow the mixed-mode pattern.](image2)
\( N \leq q(\pi \Gamma_0/2\Delta f_{\text{res}} - 1) \). This condition is met for subgiants, on the lower RGB, and for secondary-clump stars (Mosser et al. 2014).

- No mixed mode can be observed when the condition is so drastic that only pressure-dominated mixed modes can be observed. The combination of the conditions expressed by Eq. (27) and Eq. (30) yields the limit of visibility of gravity-dominated mixed modes, expressed by a condition on the mixed-mode density

\[
N \leq \frac{1}{4q} \left( \frac{\pi}{2} \frac{\Gamma_0}{\Delta f_{\text{res}}} - 5 \right).
\]  

(31)

In the conditions of observation of \textit{Kepler}, with typical parameters defined as in Mosser et al. (2017a), this limit corresponds to a mode density of about 25, for RGB and clump stars, over which no gravity-dominated mixed modes can be identified. This theoretical estimate is observed in practice, with a few exceptions with larger \( N \) (Fig. 8). On the RGB, observation of mixed modes with \textit{Kepler} is limited to \( \Delta \nu \gtrsim 6 \mu \text{Hz} \), whereas the limit is around 3 \( \mu \text{Hz} \) for clump stars. As a consequence, visible mixed modes in an oscillation spectrum with \( \Delta \nu \) in the range [3, 6 \( \mu \text{Hz} \)] most often indicate a red-clump star. Incidentally, the location of the RGB bump was recently identified by Khan et al. (2018) in the range [5, 6 \( \mu \text{Hz} \)], depending on the stellar mass and metallicity. This means that sounding the bump with mixed modes will be very difficult, if not impossible.

7. Conclusion

The asymptotic analysis allows us to depict the whole properties of the mixed-mode spectrum in a consistent way. Period spacings, rotational splittings, mode widths, and mode heights, all depend on the mode inertia so that all are related to the parameter \( \zeta \). We could derive interesting properties:

- The asymptotic fit of the mixed modes proves to be precise and unbiased. Its precision for the RGB stars is so high that the asymptotic expansion of gravity modes can be validated when buoyancy glitches are absent. This ensures the delivery of accurate asymptotic parameters \( \Delta \Pi_1, q, \) and \( \varepsilon_\zeta \). We found only one RGB star with such buoyancy glitches; on the contrary, buoyancy glitches are often present in red-clump stars.

- The period spacings and rotational splittings are better estimated with integrated values of the function \( \zeta \). The use of these mean values \( \langle \zeta \rangle_n \) and \( \langle \zeta \rangle_m \) is useful for evolved RGB stars and mandatory for stars with intricate splittings and spacings. Using the stretched period (Mosser et al. 2015) is in fact equivalent.

- The gravity asymptotic parameters \( \Delta \Pi_1 \) and \( \varepsilon_\zeta \) can now be accurately determined, with typical accuracy of respectively 0.06 s and 0.1 on the RGB, and 0.22 s and 0.08 in the red clump. This opens the way to a fruitful dialogue with theoretical developments (Takata 2006, 2016b,a) and modeling (e.g., Bossini et al. 2015; Cunha et al. 2015).

- We have made clear that observing mixed modes in evolved red giants requires an observation duration longer than \( \approx 100 \) days. However, gravity-dominated mixed modes are no longer observable when the stars are more evolved than \( \Delta \nu \approx 6 \mu \text{Hz} \) on the RGB, or \( \Delta \nu \approx 3 \mu \text{Hz} \) in the red clump. These thresholds are indicative values: the natural spread of the seismic parameters with respect to their mean values explain slight differences.

- We have demonstrated the non-alignment of the rotation axis of the stars belonging to the old open clusters NGC 6791 and NGC 6819. These results contradict previous findings by Corsaro et al. (2017) and illustrate how useful the asymptotic fit will be in the future when used to define priors to any Bayesian or other type of fit of mixed modes.

Acknowledgements. We thank the entire \textit{Kepler} team, whose efforts made these results possible. BM warmly thanks Yvonne Elsworth, James Kuszlewicz and Masao Takata for their comments on the draft submitted for internal review on the website of the \textit{Kepler} Asteroseismic Science Operations Center. We acknowledge financial support from the Programme National de Physique Stellaire (CNRS/INSU). BM acknowledges the support of the International Space Institute (ISSI) for the program AsteroSTEP (Asteroseismology of STEllar Populations)

References


Appendix A: Seismic parameters

We used KIC 6144777 as a case study (Fig. 2). Table A.1 provides the fit of its radial dipole mixed modes. Our results are in agreement with those published by Corsaro et al. (2015) and derive a similar number of modes (about 100), but also show differences:

- The determination of the frequencies in Corsaro et al. (2015) can be as precise as 0.3 nHz. This precision of about $\delta f_{\text{res}}/30$ was corrected into about $\delta f_{\text{res}}/10$ in their corrigendum (Corsaro et al. 2018), which remains surprisingly good; the frequencies we obtain are given with a precision that is at best about half the frequency resolution ($\approx 4$ nHz).
- Their mode widths are quite different and, most often, larger than ours;
- Heights also differ, which can come from a different treatment of the time series.

A large agreement is also met with the results obtained by García Saravia Ortiz de Montellano et al. (2018) with a peak detection algorithm that works in a fully blind manner, if we relax their uncertainties that can be as low as $\delta f_{\text{res}}/20$.

The potential of the comparison between methods based on different principles is very high: coupling the physics of the asymptotic expansion and the power of a pure numerical approach is the next step for delivering duly identified mixed modes.

The échelle diagrams of the stars mentioned in the main text are also presented:
- KIC 10272858 lies on the low part of the RGB (Fig. A.1);
- KIC 1135313 is on the RGB (Fig. A.2);
- KIC 3950353 is a RGB star with a rapid core rotation (Fig. A.3); its frequencies are given in Table A.2;
- KIC 2443903 is more evolved on the RGB, at the limit of detection of mixed modes (Fig. A.4);
- and KIC 1723700 is in the red clump star (Fig. A.5);
- and KIC 1725190 is a secondary red clump star (Fig. A.6).

Appendix A: Seismic parameters

We used KIC 6144777 as a case study (Fig. 2). Table A.1 provides the fit of its radial dipole mixed modes. Our results are in agreement with those published by Corsaro et al. (2015) and derive a similar number of modes (about 100), but also show differences:

- The determination of the frequencies in Corsaro et al. (2015) can be as precise as 0.3 nHz. This precision of about $\delta f_{\text{res}}/30$ was corrected into about $\delta f_{\text{res}}/10$ in their corrigendum (Corsaro et al. 2018), which remains surprisingly good; the frequencies we obtain are given with a precision that is at best about half the frequency resolution ($\approx 4$ nHz).
- Their mode widths are quite different and, most often, larger than ours;
- Heights also differ, which can come from a different treatment of the time series.

A large agreement is also met with the results obtained by García Saravia Ortiz de Montellano et al. (2018) with a peak detection algorithm that works in a fully blind manner, if we relax their uncertainties that can be as low as $\delta f_{\text{res}}/20$.

The potential of the comparison between methods based on different principles is very high: coupling the physics of the asymptotic expansion and the power of a
<table>
<thead>
<tr>
<th>( n_p )</th>
<th>( n )</th>
<th>( m )</th>
<th>( \xi )</th>
<th>( \nu_{\text{rad}} ) (( \mu \text{Hz} ))</th>
<th>( \nu ) (( \mu \text{Hz} ))</th>
<th>( x )</th>
<th>( \nu_{\text{de}} ) (( \mu \text{Hz} ))</th>
<th>( H ) (ppm( \mu \text{Hz}^{-1} ))</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>111</td>
<td>89</td>
<td>0.7111</td>
<td>106.957</td>
<td>0.111</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
</tr>
<tr>
<td>9</td>
<td>109</td>
<td>0.7439</td>
<td>106.957</td>
<td>0.232</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.7439</td>
<td>106.957</td>
<td>0.232</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.7439</td>
<td>106.957</td>
<td>0.232</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.7439</td>
<td>106.957</td>
<td>0.232</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0.7439</td>
<td>106.957</td>
<td>0.232</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Radial modes**

<table>
<thead>
<tr>
<th>( n_p )</th>
<th>( n )</th>
<th>( m )</th>
<th>( \xi )</th>
<th>( \nu_{\text{rad}} ) (( \mu \text{Hz} ))</th>
<th>( \nu ) (( \mu \text{Hz} ))</th>
<th>( x )</th>
<th>( \nu_{\text{de}} ) (( \mu \text{Hz} ))</th>
<th>( H ) (ppm( \mu \text{Hz}^{-1} ))</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>111</td>
<td>89</td>
<td>0.7111</td>
<td>106.957</td>
<td>0.111</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
</tr>
<tr>
<td>9</td>
<td>109</td>
<td>0.7439</td>
<td>106.957</td>
<td>0.232</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.7439</td>
<td>106.957</td>
<td>0.232</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.7439</td>
<td>106.957</td>
<td>0.232</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.7439</td>
<td>106.957</td>
<td>0.232</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0.7439</td>
<td>106.957</td>
<td>0.232</td>
<td>0.011</td>
<td>0.004</td>
<td>1305</td>
<td>148.6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Dipole mixed modes**
Table A.1. continued.

<table>
<thead>
<tr>
<th>$\eta_p$</th>
<th>$n$</th>
<th>$m$</th>
<th>$\zeta$</th>
<th>$\nu_{as}$ ((\mu)Hz)</th>
<th>$\nu$ ((\mu)Hz)</th>
<th>$x$</th>
<th>$\Gamma_{as}$ ((\mu)Hz)</th>
<th>$\Gamma$ ((\mu)Hz)</th>
<th>$H$ (ppm(^2)(\mu)Hz(^{-1}))</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>-80</td>
<td>-1</td>
<td>0.864</td>
<td>139.007</td>
<td>139.015 ± 0.007</td>
<td>0.403</td>
<td>0.024</td>
<td>0.012 ± 0.004</td>
<td>2324 ± 388</td>
<td>36.5</td>
</tr>
<tr>
<td>11</td>
<td>-80</td>
<td>0</td>
<td>0.8183</td>
<td>139.247</td>
<td>139.230 ± 0.006</td>
<td>0.422</td>
<td>0.033</td>
<td>0.020 ± 0.005</td>
<td>2879 ± 427</td>
<td>50.3</td>
</tr>
<tr>
<td>11</td>
<td>-80</td>
<td>1</td>
<td>0.7594</td>
<td>139.431</td>
<td>139.438 ± 0.007</td>
<td>0.441</td>
<td>0.043</td>
<td>0.029 ± 0.006</td>
<td>942 ± 147</td>
<td>18.2</td>
</tr>
<tr>
<td>11</td>
<td>-79</td>
<td>-1</td>
<td>0.4672</td>
<td>140.063</td>
<td>140.116 ± 0.009</td>
<td>0.503</td>
<td>0.096</td>
<td>0.078 ± 0.015</td>
<td>2479 ± 552</td>
<td>102.8</td>
</tr>
<tr>
<td>11</td>
<td>-79</td>
<td>0</td>
<td>0.4628</td>
<td>140.175</td>
<td>140.185 ± 0.012</td>
<td>0.509</td>
<td>0.097</td>
<td>0.128 ± 0.031</td>
<td>1534 ± 491</td>
<td>74.4</td>
</tr>
<tr>
<td>11</td>
<td>-79</td>
<td>1</td>
<td>0.4939</td>
<td>140.295</td>
<td>140.352 ± 0.009</td>
<td>0.524</td>
<td>0.091</td>
<td>0.071 ± 0.012</td>
<td>1831 ± 330</td>
<td>49.4</td>
</tr>
<tr>
<td>11</td>
<td>-78</td>
<td>-1</td>
<td>0.8100</td>
<td>140.998</td>
<td>141.019 ± 0.006</td>
<td>0.585</td>
<td>0.034</td>
<td>0.013 ± 0.004</td>
<td>3675 ± 536</td>
<td>61.6</td>
</tr>
<tr>
<td>11</td>
<td>-78</td>
<td>0</td>
<td>0.8559</td>
<td>141.194</td>
<td>141.202 ± 0.006</td>
<td>0.601</td>
<td>0.026</td>
<td>0.009 ± 0.004</td>
<td>3583 ± 541</td>
<td>58.3</td>
</tr>
<tr>
<td>11</td>
<td>-78</td>
<td>1</td>
<td>0.8894</td>
<td>141.409</td>
<td>141.419 ± 0.009</td>
<td>0.621</td>
<td>0.020</td>
<td>0.027 ± 0.007</td>
<td>1200 ± 332</td>
<td>44.1</td>
</tr>
<tr>
<td>11</td>
<td>-77</td>
<td>-1</td>
<td>0.9561</td>
<td>142.436</td>
<td>142.432 ± 0.006</td>
<td>0.713</td>
<td>0.008</td>
<td>0.007 ± 0.004</td>
<td>1831 ± 358</td>
<td>30.4</td>
</tr>
<tr>
<td>11</td>
<td>-77</td>
<td>0</td>
<td>0.9617</td>
<td>142.668</td>
<td>142.663 ± 0.006</td>
<td>0.734</td>
<td>0.007</td>
<td>0.006 ± 0.004</td>
<td>1912 ± 278</td>
<td>33.3</td>
</tr>
<tr>
<td>11</td>
<td>-77</td>
<td>1</td>
<td>0.9667</td>
<td>142.962</td>
<td>142.895 ± 0.009</td>
<td>0.755</td>
<td>0.006</td>
<td>0.009 ± 0.004</td>
<td>434 ± 133</td>
<td>7.3</td>
</tr>
<tr>
<td>12</td>
<td>-74</td>
<td>-1</td>
<td>0.9778</td>
<td>147.312</td>
<td>147.297 ± 0.007</td>
<td>0.154</td>
<td>0.004</td>
<td>0.005 ± 0.004</td>
<td>528 ± 146</td>
<td>9.4</td>
</tr>
<tr>
<td>12</td>
<td>-74</td>
<td>0</td>
<td>0.9761</td>
<td>147.549</td>
<td>147.551 ± 0.007</td>
<td>0.177</td>
<td>0.004</td>
<td>0.006 ± 0.004</td>
<td>557 ± 102</td>
<td>11.1</td>
</tr>
<tr>
<td>12</td>
<td>-74</td>
<td>1</td>
<td>0.9739</td>
<td>147.785</td>
<td>147.778 ± 0.006</td>
<td>0.197</td>
<td>0.005</td>
<td>0.006 ± 0.004</td>
<td>665 ± 120</td>
<td>12.0</td>
</tr>
<tr>
<td>12</td>
<td>-73</td>
<td>-1</td>
<td>0.9511</td>
<td>148.996</td>
<td>148.980 ± 0.009</td>
<td>0.306</td>
<td>0.009</td>
<td>0.010 ± 0.004</td>
<td>747 ± 193</td>
<td>18.1</td>
</tr>
<tr>
<td>12</td>
<td>-73</td>
<td>0</td>
<td>0.9422</td>
<td>149.226</td>
<td>149.215 ± 0.007</td>
<td>0.327</td>
<td>0.010</td>
<td>0.009 ± 0.004</td>
<td>756 ± 160</td>
<td>13.9</td>
</tr>
<tr>
<td>12</td>
<td>-73</td>
<td>1</td>
<td>0.9306</td>
<td>149.452</td>
<td>149.441 ± 0.006</td>
<td>0.348</td>
<td>0.013</td>
<td>0.011 ± 0.004</td>
<td>726 ± 122</td>
<td>14.3</td>
</tr>
<tr>
<td>12</td>
<td>-72</td>
<td>-1</td>
<td>0.7456</td>
<td>150.556</td>
<td>150.584 ± 0.008</td>
<td>0.452</td>
<td>0.046</td>
<td>0.041 ± 0.008</td>
<td>545 ± 112</td>
<td>18.2</td>
</tr>
<tr>
<td>12</td>
<td>-72</td>
<td>0</td>
<td>0.6672</td>
<td>150.737</td>
<td>150.764 ± 0.011</td>
<td>0.468</td>
<td>0.060</td>
<td>0.051 ± 0.011</td>
<td>298 ± 79</td>
<td>19.8</td>
</tr>
<tr>
<td>12</td>
<td>-72</td>
<td>1</td>
<td>0.5911</td>
<td>150.880</td>
<td>150.951 ± 0.011</td>
<td>0.485</td>
<td>0.074</td>
<td>0.060 ± 0.012</td>
<td>444 ± 99</td>
<td>23.6</td>
</tr>
<tr>
<td>12</td>
<td>-71</td>
<td>-1</td>
<td>0.4722</td>
<td>151.509</td>
<td>151.598 ± 0.010</td>
<td>0.543</td>
<td>0.095</td>
<td>0.067 ± 0.012</td>
<td>650 ± 126</td>
<td>27.4</td>
</tr>
<tr>
<td>12</td>
<td>-71</td>
<td>0</td>
<td>0.5300</td>
<td>151.624</td>
<td>151.720 ± 0.009</td>
<td>0.554</td>
<td>0.085</td>
<td>0.026 ± 0.006</td>
<td>1788 ± 329</td>
<td>66.9</td>
</tr>
<tr>
<td>12</td>
<td>-71</td>
<td>1</td>
<td>0.6122</td>
<td>151.772</td>
<td>151.861 ± 0.012</td>
<td>0.567</td>
<td>0.070</td>
<td>0.077 ± 0.018</td>
<td>380 ± 113</td>
<td>20.2</td>
</tr>
<tr>
<td>12</td>
<td>-70</td>
<td>1</td>
<td>0.5393</td>
<td>153.268</td>
<td>153.289 ± 0.007</td>
<td>0.697</td>
<td>0.012</td>
<td>0.007 ± 0.004</td>
<td>572 ± 100</td>
<td>12.3</td>
</tr>
<tr>
<td>12</td>
<td>-69</td>
<td>0</td>
<td>0.9717</td>
<td>154.828</td>
<td>154.849 ± 0.007</td>
<td>-0.162</td>
<td>0.005</td>
<td>0.005 ± 0.004</td>
<td>587 ± 160</td>
<td>11.7</td>
</tr>
<tr>
<td>12</td>
<td>-69</td>
<td>1</td>
<td>0.9731</td>
<td>155.064</td>
<td>155.083 ± 0.007</td>
<td>-0.141</td>
<td>0.005</td>
<td>0.006 ± 0.004</td>
<td>546 ± 97</td>
<td>11.8</td>
</tr>
</tbody>
</table>

$\zeta$ is derived from the best asymptotic fit; $\nu_{as}$ are the asymptotic frequencies, whereas $\nu$ correspond to the observed values; $x = \nu/\Delta \nu - (\eta_p - \varepsilon_p)$ is the reduced frequency; $\Gamma_{as}$ are the asymptotic mode widths, whereas $\Gamma$ correspond to the observed values; $H$ are the observed heights, and $R$ is the height-to-background ratio.
Appendix B: Stars in open clusters

All stars studied by Corsaro et al. (2017) were investigated. The fitting process is challenging, due to the dim magnitudes of such dim stars in open clusters. However, the combination of all pressure radial orders near $\nu_{\text{max}}$ provides in most cases an unambiguous fit, and at least a few mixed-mode radial orders provide clear splittings.

- Figure B.1 provides the asymptotic fit of KIC 5024476, member of the open cluster NGC 6819 observed by Kepler. We note that $m = \pm 1$ modes are clearly identified and derive a stellar inclination $i = 79 \pm 11^\circ$ for this star. This result is in disagreement with Corsaro et al. (2017) who found an inclination $i = 20 \pm 7^\circ$.

- Similar conclusions are reached for KIC 2437976 (Fig. B.2), member of the open cluster NGC 6791. Corsaro et al. (2017) found an inclination $i = 0 \pm 10^\circ$, despite the fact $|m| = 1$ modes are clearly identified and indicate $i = 76 \pm 14^\circ$.

These stars are representative of the whole data set treated by Corsaro et al. (2017): the inability of the fitting process to identify thin short-lived mixed modes translates into the identification of a single broad $m = 0$ peak. In such cases, stellar inclinations derived from the Bayesian fits are necessarily underestimated and biased toward low values.
### Table A.2. Oscillation pattern of the RGB star KIC 3955033

<table>
<thead>
<tr>
<th>$n_p$</th>
<th>$n$</th>
<th>$m$</th>
<th>$\zeta$</th>
<th>$\nu_{as}$ (\mu Hz)</th>
<th>$\nu$ (\mu Hz)</th>
<th>$x$</th>
<th>$\Gamma_{as}$ (\mu Hz)</th>
<th>$\Gamma$ (\mu Hz)</th>
<th>$H$ (ppm$^2$/\mu Hz$^{-1}$)</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>8</td>
<td>0</td>
<td>0.077</td>
<td>84.745</td>
<td>84.745 ± 0.077</td>
<td>0.053</td>
<td>0.198 ± 0.089</td>
<td>1015 ± 656</td>
<td>16.2</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>0</td>
<td>0.012</td>
<td>93.958</td>
<td>93.762 ± 0.012</td>
<td>0.010</td>
<td>0.094 ± 0.017</td>
<td>3985 ± 872</td>
<td>27.3</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0.013</td>
<td>102.911</td>
<td>103.010 ± 0.013</td>
<td>0.012</td>
<td>0.138 ± 0.027</td>
<td>7641 ± 1745</td>
<td>30.2</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>0</td>
<td>0.009</td>
<td>112.215</td>
<td>112.143 ± 0.009</td>
<td>0.002</td>
<td>0.067 ± 0.012</td>
<td>1817 ± 3832</td>
<td>114.7</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>0</td>
<td>0.015</td>
<td>121.869</td>
<td>121.474 ± 0.015</td>
<td>0.013</td>
<td>0.122 ± 0.025</td>
<td>2495 ± 657</td>
<td>19.6</td>
<td></td>
</tr>
</tbody>
</table>

**Radial modes**

<table>
<thead>
<tr>
<th>$n_p$</th>
<th>$n$</th>
<th>$m$</th>
<th>$\zeta$</th>
<th>$\nu_{as}$ (\mu Hz)</th>
<th>$\nu$ (\mu Hz)</th>
<th>$x$</th>
<th>$\Gamma_{as}$ (\mu Hz)</th>
<th>$\Gamma$ (\mu Hz)</th>
<th>$H$ (ppm$^2$/\mu Hz$^{-1}$)</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>8</td>
<td>0</td>
<td>0.077</td>
<td>84.745</td>
<td>84.745 ± 0.077</td>
<td>0.053</td>
<td>0.198 ± 0.089</td>
<td>1015 ± 656</td>
<td>16.2</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>0</td>
<td>0.012</td>
<td>93.958</td>
<td>93.762 ± 0.012</td>
<td>0.010</td>
<td>0.094 ± 0.017</td>
<td>3985 ± 872</td>
<td>27.3</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0.013</td>
<td>102.911</td>
<td>103.010 ± 0.013</td>
<td>0.012</td>
<td>0.138 ± 0.027</td>
<td>7641 ± 1745</td>
<td>30.2</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>0</td>
<td>0.009</td>
<td>112.215</td>
<td>112.143 ± 0.009</td>
<td>0.002</td>
<td>0.067 ± 0.012</td>
<td>1817 ± 3832</td>
<td>114.7</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>0</td>
<td>0.015</td>
<td>121.869</td>
<td>121.474 ± 0.015</td>
<td>0.013</td>
<td>0.122 ± 0.025</td>
<td>2495 ± 657</td>
<td>19.6</td>
<td></td>
</tr>
</tbody>
</table>

**Dipole mixed modes**

<table>
<thead>
<tr>
<th>$n_p$</th>
<th>$n$</th>
<th>$m$</th>
<th>$\zeta$</th>
<th>$\nu_{as}$ (\mu Hz)</th>
<th>$\nu$ (\mu Hz)</th>
<th>$x$</th>
<th>$\Gamma_{as}$ (\mu Hz)</th>
<th>$\Gamma$ (\mu Hz)</th>
<th>$H$ (ppm$^2$/\mu Hz$^{-1}$)</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>8</td>
<td>0</td>
<td>0.077</td>
<td>84.745</td>
<td>84.745 ± 0.077</td>
<td>0.053</td>
<td>0.198 ± 0.089</td>
<td>1015 ± 656</td>
<td>16.2</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>0</td>
<td>0.012</td>
<td>93.958</td>
<td>93.762 ± 0.012</td>
<td>0.010</td>
<td>0.094 ± 0.017</td>
<td>3985 ± 872</td>
<td>27.3</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0.013</td>
<td>102.911</td>
<td>103.010 ± 0.013</td>
<td>0.012</td>
<td>0.138 ± 0.027</td>
<td>7641 ± 1745</td>
<td>30.2</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>0</td>
<td>0.009</td>
<td>112.215</td>
<td>112.143 ± 0.009</td>
<td>0.002</td>
<td>0.067 ± 0.012</td>
<td>1817 ± 3832</td>
<td>114.7</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>0</td>
<td>0.015</td>
<td>121.869</td>
<td>121.474 ± 0.015</td>
<td>0.013</td>
<td>0.122 ± 0.025</td>
<td>2495 ± 657</td>
<td>19.6</td>
<td></td>
</tr>
</tbody>
</table>

Radial modes and mixed modes identified in KIC 3955033 with a height-to-background ratio $R$ larger than $7$. Same caption as Table A.1
Owing to the small radial orders, small shifts are seen between observed and asymptotic spectra. Same style as Fig. 2, but \( \ell = 3 \) modes appear near the abscissa 0.28.

**Fig. A.1.** Fit of the oscillation pattern of the low RGB star KIC 10272858, at the limit of validity of the asymptotic pattern.

10272858  \( \Delta \nu = 22.71 \mu \text{Hz} \)  \( \Delta \Pi_1 = 96.90 \text{s} \)  \( q = 0.19 \)  \( \delta \nu_{\text{rot}} = 660 \text{nHz} \)

**Fig. A.2.** Fit of the oscillation pattern of the RGB star KIC 11353313. Same style as Fig. 2.

11353313  \( \Delta \nu = 10.75 \mu \text{Hz} \)  \( \Delta \Pi_1 = 76.95 \text{s} \)  \( q = 0.14 \)  \( \delta \nu_{\text{rot}} = 465 \text{nHz} \)
Fig. A.3. Fit of the oscillation pattern of the RGB star KIC 3955033. The overlap of mixed modes with different mixed-mode orders is the signature of the rapid core rotation. The second rotation crossing, where all components of the multiplets overlap (Gehan et al. 2017), occurs at the mixed-order \( n = -122 \). Same style as Fig. 2.

Fig. A.4. Fit of the oscillation pattern of the evolved RGB star KIC 2443903, near the limit of capability of identification, with a large crowding due to the high mode density. The second rotation crossing, where all \( m \) components apparently coincide, occurs at \( n = -189 \) (with an abscissa \( \simeq 0.1 \) and \( n_p = 9 \)); the third crossing, where \( |m| = 1 \) components apparently coincide with \( m = 0 \) inbetween, occurs at \( n = -233 \) (with an abscissa \( \simeq 0.25 \) and \( n_p = 7 \)). Same style as Fig. 2. Note that the modes with large heights at an abscissa \( \simeq 0.2 \) are \( \ell = 3 \) modes.
Fig. A.5. Fit of the oscillation pattern of the red-clump star KIC 1723700. Buoyancy glitches explain the small shifts between observed and asymptotic spectra but do no hamper the mode identification. Same style as Fig. 2.

1723700 $\Delta \nu = 4.48 \mu \text{Hz}$ $\Delta \Pi_1 = 323.40 \text{s}$ $q = 0.24$ $\delta \nu_{\text{det}} = 57 \mu \text{Hz}$

![Oscillation pattern of KIC 1723700](image)

Fig. A.6. Fit of the oscillation pattern of the secondary-clump star KIC 1725190. Buoyancy glitches explain the small shifts between observed and asymptotic spectra but do no hamper the mode identification. Same style as Fig. 2.

1576469 $\Delta \nu = 7.41 \mu \text{Hz}$ $\Delta \Pi_1 = 284.80 \text{s}$ $q = 0.23$ $\delta \nu_{\text{det}} = 67 \mu \text{Hz}$

![Oscillation pattern of KIC 1725190](image)
Fig. B.1. Fit of the oscillation pattern of the RGB star KIC 5024476, member of the open cluster NGC 6819. The dim magnitude of the cluster stars explains the low S/R. However, unambiguous doublets are identified all along the mixed-mode spectrum; $m = 0$ modes are mostly absent and $|m| = 1$ modes dominate the mixed-mode spectrum, so that a nearly pole-on inclination is not possible. Same style as Fig. 2.

Fig. B.2. Fit of the oscillation pattern of the RGB star KIC 2437976, member of the open cluster NGC 6791. The dim magnitude of the cluster stars explains the low S/R. The identification at radial order 9, supported by the radial orders 8 and 10, is unambiguously conclusive: $m = 0$ modes are mostly absent and $|m| = 1$ modes dominate the mixed-mode spectrum, so that a pole-on inclination is not possible. Same style as Fig. 2.
Table B.1. Oscillation pattern of the red-clump star KIC 5024476 in NGC 6819

<table>
<thead>
<tr>
<th>(n_p)</th>
<th>(n)</th>
<th>(m)</th>
<th>(\zeta)</th>
<th>(\nu_{as}) ((\mu)Hz)</th>
<th>(\nu) ((\mu)Hz)</th>
<th>(x)</th>
<th>(\Gamma_{as}) ((\mu)Hz)</th>
<th>(\Gamma) ((\mu)Hz)</th>
<th>(H) (ppm (\mu)Hz(^{-1}))</th>
<th>(R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial modes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-43</td>
<td>-1</td>
<td>0.9495</td>
<td>63.873</td>
<td>63.870 ± 0.009</td>
<td>0.077</td>
<td>0.013</td>
<td>0.019 ± 0.006</td>
<td>1887 ± 532</td>
<td>16.0</td>
</tr>
<tr>
<td>10</td>
<td>-43</td>
<td>1</td>
<td>0.9481</td>
<td>63.979</td>
<td>63.976 ± 0.007</td>
<td>0.095</td>
<td>0.013</td>
<td>0.009 ± 0.004</td>
<td>3002 ± 632</td>
<td>12.7</td>
</tr>
<tr>
<td>10</td>
<td>-42</td>
<td>-1</td>
<td>0.9006</td>
<td>65.035</td>
<td>65.005 ± 0.015</td>
<td>0.275</td>
<td>0.025</td>
<td>0.045 ± 0.012</td>
<td>497 ± 175</td>
<td>11.6</td>
</tr>
<tr>
<td>10</td>
<td>-42</td>
<td>0</td>
<td>0.8955</td>
<td>65.086</td>
<td>65.131 ± 0.012</td>
<td>0.297</td>
<td>0.026</td>
<td>0.028 ± 0.008</td>
<td>576 ± 191</td>
<td>7.6</td>
</tr>
<tr>
<td>10</td>
<td>-42</td>
<td>1</td>
<td>0.8903</td>
<td>65.136</td>
<td>65.131 ± 0.011</td>
<td>0.297</td>
<td>0.027</td>
<td>0.027 ± 0.008</td>
<td>589 ± 186</td>
<td>7.6</td>
</tr>
<tr>
<td>10</td>
<td>-41</td>
<td>-1</td>
<td>0.6178</td>
<td>66.050</td>
<td>66.050 ± 0.013</td>
<td>0.457</td>
<td>0.096</td>
<td>0.095 ± 0.020</td>
<td>1859 ± 503</td>
<td>20.0</td>
</tr>
<tr>
<td>10</td>
<td>-41</td>
<td>0</td>
<td>0.6011</td>
<td>66.085</td>
<td>66.061 ± 0.013</td>
<td>0.459</td>
<td>0.100</td>
<td>0.099 ± 0.022</td>
<td>1959 ± 563</td>
<td>20.0</td>
</tr>
<tr>
<td>10</td>
<td>-41</td>
<td>1</td>
<td>0.5866</td>
<td>66.117</td>
<td>66.083 ± 0.014</td>
<td>0.463</td>
<td>0.103</td>
<td>0.106 ± 0.025</td>
<td>1783 ± 534</td>
<td>18.7</td>
</tr>
<tr>
<td>10</td>
<td>-40</td>
<td>-1</td>
<td>0.6453</td>
<td>66.784</td>
<td>66.788 ± 0.017</td>
<td>0.586</td>
<td>0.089</td>
<td>0.085 ± 0.020</td>
<td>800 ± 240</td>
<td>8.9</td>
</tr>
<tr>
<td>10</td>
<td>-40</td>
<td>1</td>
<td>0.6794</td>
<td>66.858</td>
<td>66.897 ± 0.020</td>
<td>0.605</td>
<td>0.080</td>
<td>0.097 ± 0.023</td>
<td>632 ± 197</td>
<td>14.4</td>
</tr>
<tr>
<td>10</td>
<td>-39</td>
<td>1</td>
<td>0.9132</td>
<td>67.985</td>
<td>68.003 ± 0.007</td>
<td>0.798</td>
<td>0.022</td>
<td>0.009 ± 0.004</td>
<td>4805 ± 844</td>
<td>23.5</td>
</tr>
<tr>
<td>11</td>
<td>-37</td>
<td>-1</td>
<td>0.9084</td>
<td>70.563</td>
<td>70.562 ± 0.007</td>
<td>0.244</td>
<td>0.023</td>
<td>0.009 ± 0.004</td>
<td>2295 ± 521</td>
<td>11.4</td>
</tr>
<tr>
<td>11</td>
<td>-37</td>
<td>1</td>
<td>0.9003</td>
<td>70.665</td>
<td>70.672 ± 0.006</td>
<td>0.264</td>
<td>0.025</td>
<td>0.008 ± 0.004</td>
<td>2474 ± 398</td>
<td>14.8</td>
</tr>
<tr>
<td>11</td>
<td>-36</td>
<td>-1</td>
<td>0.6140</td>
<td>71.776</td>
<td>71.786 ± 0.015</td>
<td>0.458</td>
<td>0.097</td>
<td>0.103 ± 0.022</td>
<td>1228 ± 333</td>
<td>18.4</td>
</tr>
<tr>
<td>11</td>
<td>-36</td>
<td>0</td>
<td>0.5811</td>
<td>71.843</td>
<td>71.884 ± 0.019</td>
<td>0.475</td>
<td>0.105</td>
<td>0.127 ± 0.030</td>
<td>854 ± 264</td>
<td>12.1</td>
</tr>
<tr>
<td>11</td>
<td>-35</td>
<td>-1</td>
<td>0.6100</td>
<td>72.588</td>
<td>72.615 ± 0.010</td>
<td>0.603</td>
<td>0.097</td>
<td>0.072 ± 0.013</td>
<td>2715 ± 560</td>
<td>29.9</td>
</tr>
<tr>
<td>11</td>
<td>-35</td>
<td>1</td>
<td>0.6455</td>
<td>72.658</td>
<td>72.625 ± 0.011</td>
<td>0.605</td>
<td>0.089</td>
<td>0.078 ± 0.015</td>
<td>2287 ± 527</td>
<td>29.9</td>
</tr>
<tr>
<td>11</td>
<td>-34</td>
<td>-1</td>
<td>0.9051</td>
<td>73.870</td>
<td>73.881 ± 0.007</td>
<td>0.176</td>
<td>0.024</td>
<td>0.016 ± 0.005</td>
<td>2320 ± 417</td>
<td>12.5</td>
</tr>
<tr>
<td>11</td>
<td>-34</td>
<td>1</td>
<td>0.9111</td>
<td>73.972</td>
<td>73.977 ± 0.007</td>
<td>0.160</td>
<td>0.022</td>
<td>0.008 ± 0.004</td>
<td>4124 ± 750</td>
<td>22.2</td>
</tr>
<tr>
<td>Dipole mixed modes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-33</td>
<td>-1</td>
<td>0.9314</td>
<td>75.420</td>
<td>75.422 ± 0.006</td>
<td>0.939</td>
<td>0.017</td>
<td>0.007 ± 0.004</td>
<td>5610 ± 987</td>
<td>31.2</td>
</tr>
<tr>
<td>12</td>
<td>-33</td>
<td>1</td>
<td>0.9296</td>
<td>75.524</td>
<td>75.530 ± 0.008</td>
<td>0.112</td>
<td>0.018</td>
<td>0.008 ± 0.004</td>
<td>1498 ± 379</td>
<td>8.3</td>
</tr>
<tr>
<td>12</td>
<td>-32</td>
<td>-1</td>
<td>0.8053</td>
<td>76.972</td>
<td>76.954 ± 0.008</td>
<td>0.360</td>
<td>0.049</td>
<td>0.024 ± 0.006</td>
<td>1836 ± 391</td>
<td>14.5</td>
</tr>
<tr>
<td>12</td>
<td>-32</td>
<td>1</td>
<td>0.7828</td>
<td>77.061</td>
<td>77.060 ± 0.009</td>
<td>0.379</td>
<td>0.054</td>
<td>0.034 ± 0.007</td>
<td>1103 ± 222</td>
<td>13.2</td>
</tr>
<tr>
<td>12</td>
<td>-31</td>
<td>-1</td>
<td>0.4338</td>
<td>78.016</td>
<td>78.029 ± 0.017</td>
<td>0.548</td>
<td>0.142</td>
<td>0.156 ± 0.035</td>
<td>1362 ± 387</td>
<td>32.1</td>
</tr>
<tr>
<td>12</td>
<td>-31</td>
<td>0</td>
<td>0.4373</td>
<td>78.041</td>
<td>78.045 ± 0.018</td>
<td>0.550</td>
<td>0.141</td>
<td>0.160 ± 0.036</td>
<td>1326 ± 379</td>
<td>16.4</td>
</tr>
<tr>
<td>12</td>
<td>-31</td>
<td>1</td>
<td>0.4416</td>
<td>78.066</td>
<td>78.062 ± 0.017</td>
<td>0.553</td>
<td>0.140</td>
<td>0.154 ± 0.034</td>
<td>1354 ± 382</td>
<td>14.3</td>
</tr>
<tr>
<td>12</td>
<td>-30</td>
<td>-1</td>
<td>0.8397</td>
<td>79.185</td>
<td>79.226 ± 0.008</td>
<td>0.757</td>
<td>0.040</td>
<td>0.013 ± 0.004</td>
<td>2127 ± 377</td>
<td>12.8</td>
</tr>
<tr>
<td>12</td>
<td>-30</td>
<td>1</td>
<td>0.8538</td>
<td>79.280</td>
<td>79.310 ± 0.009</td>
<td>0.771</td>
<td>0.037</td>
<td>0.027 ± 0.006</td>
<td>977 ± 236</td>
<td>9.7</td>
</tr>
</tbody>
</table>

Same caption as Table A.1
Table B.2. Oscillation pattern of the RGB star KIC 2437976 in NGC 6791

<table>
<thead>
<tr>
<th>$n_p$</th>
<th>$n$</th>
<th>$m$</th>
<th>$\zeta$</th>
<th>$\nu_{as}$ ((\mu)Hz)</th>
<th>$\nu$ ((\mu)Hz)</th>
<th>$x$</th>
<th>$\Gamma_{as}$ ((\mu)Hz)</th>
<th>$\Gamma$ ((\mu)Hz)</th>
<th>$H$ (ppm(^2)(\mu)Hz(^{-1}))</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>75.045</td>
<td>75.045 ± 0.041</td>
<td>0.013</td>
<td>0.106 ± 0.042</td>
<td>65.20 ± 36.76</td>
<td>7.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>-161</td>
<td>-1</td>
<td>0.8533</td>
<td>78.887</td>
<td>0.481</td>
<td>0.026</td>
<td>0.022 ± 0.006</td>
<td>10581 ± 2376</td>
<td>12.3</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-160</td>
<td>-1</td>
<td>0.6438</td>
<td>79.234</td>
<td>0.514</td>
<td>0.064</td>
<td>0.038 ± 0.007</td>
<td>14647 ± 2841</td>
<td>11.9</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>-161</td>
<td>1</td>
<td>0.6511</td>
<td>79.343</td>
<td>0.527</td>
<td>0.063</td>
<td>0.031 ± 0.007</td>
<td>10930 ± 2616</td>
<td>9.6</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-159</td>
<td>-1</td>
<td>0.7884</td>
<td>79.561</td>
<td>0.557</td>
<td>0.038</td>
<td>0.026 ± 0.006</td>
<td>14986 ± 2978</td>
<td>15.1</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>-160</td>
<td>1</td>
<td>0.8566</td>
<td>79.691</td>
<td>0.575</td>
<td>0.026</td>
<td>0.023 ± 0.006</td>
<td>6508 ± 1802</td>
<td>9.8</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-146</td>
<td>-1</td>
<td>0.9726</td>
<td>86.115</td>
<td>0.362</td>
<td>0.005</td>
<td>0.009 ± 0.004</td>
<td>16564 ± 5717</td>
<td>7.6</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-144</td>
<td>-1</td>
<td>0.7541</td>
<td>87.141</td>
<td>0.489</td>
<td>0.044</td>
<td>0.054 ± 0.013</td>
<td>20608 ± 6137</td>
<td>33.9</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-145</td>
<td>1</td>
<td>0.7180</td>
<td>87.193</td>
<td>0.490</td>
<td>0.051</td>
<td>0.049 ± 0.011</td>
<td>23464 ± 6034</td>
<td>33.9</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-143</td>
<td>-1</td>
<td>0.6110</td>
<td>87.503</td>
<td>0.528</td>
<td>0.070</td>
<td>0.067 ± 0.014</td>
<td>19703 ± 4780</td>
<td>29.1</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-144</td>
<td>1</td>
<td>0.6404</td>
<td>87.550</td>
<td>0.530</td>
<td>0.065</td>
<td>0.050 ± 0.009</td>
<td>26627 ± 5525</td>
<td>29.1</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-142</td>
<td>-1</td>
<td>0.8736</td>
<td>87.940</td>
<td>0.582</td>
<td>0.023</td>
<td>0.009 ± 0.004</td>
<td>21699 ± 5272</td>
<td>10.0</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-143</td>
<td>1</td>
<td>0.8929</td>
<td>88.001</td>
<td>0.587</td>
<td>0.019</td>
<td>0.009 ± 0.004</td>
<td>29926 ± 5102</td>
<td>13.8</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-141</td>
<td>-1</td>
<td>0.9588</td>
<td>88.486</td>
<td>0.649</td>
<td>0.007</td>
<td>0.007 ± 0.004</td>
<td>17180 ± 3838</td>
<td>8.1</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-142</td>
<td>-1</td>
<td>0.9622</td>
<td>88.537</td>
<td>0.654</td>
<td>0.007</td>
<td>0.006 ± 0.004</td>
<td>38360 ± 5733</td>
<td>18.3</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-132</td>
<td>1</td>
<td>0.9463</td>
<td>94.663</td>
<td>0.402</td>
<td>0.010</td>
<td>0.006 ± 0.004</td>
<td>19038 ± 3790</td>
<td>9.7</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-131</td>
<td>-1</td>
<td>0.9440</td>
<td>94.692</td>
<td>0.405</td>
<td>0.010</td>
<td>0.005 ± 0.004</td>
<td>15841 ± 4952</td>
<td>7.6</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-131</td>
<td>1</td>
<td>0.7969</td>
<td>95.259</td>
<td>0.466</td>
<td>0.037</td>
<td>0.034 ± 0.008</td>
<td>21679 ± 5266</td>
<td>18.4</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-130</td>
<td>-1</td>
<td>0.7827</td>
<td>95.284</td>
<td>0.471</td>
<td>0.039</td>
<td>0.033 ± 0.008</td>
<td>19865 ± 4877</td>
<td>20.4</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-130</td>
<td>1</td>
<td>0.5518</td>
<td>95.692</td>
<td>0.517</td>
<td>0.081</td>
<td>0.025 ± 0.005</td>
<td>42908 ± 7274</td>
<td>26.4</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-129</td>
<td>-1</td>
<td>0.5572</td>
<td>95.709</td>
<td>0.517</td>
<td>0.080</td>
<td>0.027 ± 0.006</td>
<td>39045 ± 6386</td>
<td>26.4</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-129</td>
<td>1</td>
<td>0.8453</td>
<td>96.161</td>
<td>0.582</td>
<td>0.028</td>
<td>0.009 ± 0.004</td>
<td>29045 ± 5946</td>
<td>21.6</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-125</td>
<td>0</td>
<td>0.9891</td>
<td>98.575</td>
<td>0.517</td>
<td>0.080</td>
<td>0.027 ± 0.006</td>
<td>15436 ± 11044</td>
<td>7.5</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>-118</td>
<td>-1</td>
<td>0.8082</td>
<td>103.469</td>
<td>0.467</td>
<td>0.034</td>
<td>0.023 ± 0.006</td>
<td>6844 ± 1785</td>
<td>9.8</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>-108</td>
<td>1</td>
<td>0.7958</td>
<td>111.880</td>
<td>0.487</td>
<td>0.037</td>
<td>0.022 ± 0.006</td>
<td>5310 ± 1445</td>
<td>7.9</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>-105</td>
<td>-1</td>
<td>0.9174</td>
<td>113.342</td>
<td>0.670</td>
<td>0.015</td>
<td>0.006 ± 0.004</td>
<td>16746 ± 4518</td>
<td>8.6</td>
</tr>
</tbody>
</table>

Same caption as Table A.1