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ARTICLE IN PRESS
JID:NUPHB AID:13760 /FLA [m1+; v1.232; Prn:6/06/2016; 14:42] P.1 (1-15)
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Chains of topological oscillators with instantons and calculable topological observables in topological quantum mechanics

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Received 24 May 2016; accepted 28 May 2016

Editor: Hubert Saleur

Abstract

We extend to a possibly infinite chain the conformally invariant mechanical system that was introduced earlier as a toy model for understanding the topological Yang–Mills theory. It gives a topological quantum model that has interesting and computable zero modes and topological invariants. It confirms the recent conjecture by several authors that supersymmetric quantum mechanics may provide useful tools for understanding robotic mechanical systems (Vitelli et al.) and condensed matter properties (Kane et al.), where trajectories are allowed or not by the conservation of topological indices. The absences of ground state and mass gaps are special features of such systems.

1. Introduction

Topological Quantum Field Theories TQFT's are possible realisations of the invariance under general local field transformations general coordinates invariant symmetries. Such an invariance goes beyond that of current gauge theories. The first non-trivial example of a TQFT was introduced by Witten [1] showing that the genuine $N = 2$ supersymmetric gauge theories contains observables that describe the Donaldson invariants. The reinterpretation [2] of this theory ap-

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http://dx.doi.org/10.1016/j.nuclphysb.2016.05.030
0550-3213/

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peared soon after, as a suitably gauge-fixed quantum field theory stemming from a classical topological invariant that explores the BRST cohomology of general Yang–Mills field transformations modulo ordinary gauge transformations. Because the pattern of TQFT's is that of an ultimate type of gauge-fixing, and because they can be solved, they greatly interested Raymond Stora, who wrote himself a very interesting article on the subject [3].

At the heart of TQFT's, the topological BRST nilpotent operator Q plays a fundamental role. It is such that the TQFT Hamiltonian is basically $H = \frac{1}{2}[Q, \overline{Q}]$. One often defines the physical Hilbert space as the cohomology of Q (states which are annihilated by Q without being the Q transformation of other states). This unambiguous definition of observables from the cohomology of a BRST operator is perfectly suited for the gauge theories of elementary particles (where the expression of Q is more restricted in comparison to that of a TQFT and the relation between Q and H is different). The cohomology of a TQFT is often contained in another cohomology, in which case it is called an equivalent cohomology [2–4]. There were doubts for a while on the validity of this construction of TQFT's, so [5] defined and explored a solvable quantum mechanical supersymmetric example to check very precisely all the details and confirm the construction. The model was that of particle moving in a punctured plane, where the closed trajectories carry topology because of their non-trivial winding numbers. Instantons exist in this case because one choses in this case a potential that yields a supersymmetric action with twisted scale and vector supersymmetries, in fact a superconformal supersymmetry. Strikingly, the ingredients for constructing the model completely reproduce those of the much more involved Yang–Mills topological theory and we completely solved it in [5]. The goal of this article is to generalise this model to a more physical multiparticle case with conformal interactions. To do so, we need first to review [5].

Afterward we will show that [5] can be extended into a very intriguing model, which is quite beautiful and might furthermore have richer applications in practical domains. It is an explicit example of what was foreseen long time ago in [12], for building some robots with rotational constrained degrees of freedom, and more recently by condensed matter physicists, for instance [13]. Our model generalises [5] and gives a sort of conformally vibrating lattice where each site is a particle interacting by superconformal interactions with its nearest neighbours (two in this present case). This model exhibits non-trivial instanton solutions and has some topological observables.

2. The one-particle conformal supersymmetric topological model

The model is a quantum mechanical system of a particle moving in a 2D-plane where one excludes the origin and submitted to a potential we will shortly display. One has a non-trivial topological structure because of trajectories with different possible winding numbers $0 \leq N \leq \infty$ around the origin. The classical topological symmetry is the group of arbitrary local deformations of each particle trajectory. They can be possibly defined modulo local dilatations of the distance of the particle to the origin. We will see that the model is a conformal one.

We call the time by the real variable t and the Euclidian time by τ, with $t = i\tau$. The cartesian coordinates on the plane are q_i, with $i = 1, 2$, and we often use complex coordinates $z = q_1 + q_2$.

We select trajectories with periodic conditions. Namely, the particle does a closed (multi-)loop between the initial and final times $t = 0$ and $t = T$ (we will choose $T = 1$). An integer winding number $0 \leq N \leq \infty$ is assigned to all trajectories which can be classified in equivalence classes according to N.

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As in the Yang–Mills TQFT [2], one starts from a topological classical action $\mathcal{I}_{cl}[\vec{q}]$, (with the above enlarged gauge symmetry). It must be gauge-fixed in a BRST invariant way to define a path integral so as one can compute some topological observables.

$\mathcal{I}_{cl}[\vec{q}]$ must be independent on the time metric and on local reformations of trajectories. Such conditions are satisfied for

$$\mathcal{I}_{cl}[\vec{q}] = \frac{1}{g}\int d\theta = \frac{1}{g}\int_0^T d\tau\ \dot{\theta}(\tau) = \frac{1}{g}\int_0^T d\tau\ \frac{\epsilon^{ij}\dot{q}_i q_j}{\vec{q}^2} = \frac{2\pi N}{g} \tag{1}$$

where g is a real number that will become a coupling constant. In fact, $\mathcal{I}_{cl}[\vec{q}]$ measures the winding number N of the particle (times $\frac{2\pi}{g}$). It is a tremendously simplified version of the second Chern class $\int d^4x \operatorname{tr} F \wedge F$, F being the curvature of a Yang–Mills field. Here and in what follows the symbol $\dot{X}$ denotes $\frac{dX}{d\tau}$.

The TQFT path integral is defined from a BRST invariant gauge-fixed action added to $\mathcal{I}_{cl}$ as in [2]:

$$\int \mathcal{D}[\vec{q}] \exp -\mathcal{I}_{cl}[\vec{q}] \to \int \mathcal{D}[\vec{q}] \exp -\Big(\mathcal{I}_{cl}[\vec{q}] + gauge - fixing\Big) \tag{2}$$

This way of proceeding is called nowadays a localisation procedure.

Once the details of "$gauge - fixing$" have been determined, one can compute topological quantities from Green functions of well chosen composite operators O.

$$\text{Topological information} = \int \mathcal{D}[\vec{q}] O \exp\Big(\mathcal{I}_{cl}[\vec{q}] + gauge - fixing\Big) \neq 0 \tag{3}$$

The way it goes is as follows. The action $\mathcal{I}_{cl}[\vec{q}]$ must be invariant under the gauge symmetry

$$\vec{q(t)} \to \vec{q(t)} + \vec{\epsilon(t)} \tag{4}$$

where $\epsilon(t)$ is any given local shift of the particle position $q(t)$ with appropriate boundary conditions – it cannot change the winding number of the trajectory. Such shifts can be decomposed in radial and angular deformations, and one foresees an interesting decomposition between angular and radial shifts, already noticed in [5].

The BRST transformation laws associated to the symmetry (4) are basically found by changing the parameters $\epsilon(t)$ of arbitrary shifts into an anticommuting ghost $\Psi(t)$ and an anticommuting antighost $\overline{\Psi}(t)$, and introducing a Lagrange multiplier $\vec{\lambda}(t)$, with

$$s\vec{q} = \vec{\Psi} \quad s\vec{\Psi} = 0 \quad s\vec{\overline{\Psi}} = \vec{\lambda} \quad s\vec{\lambda} = 0 \tag{5}$$

The operation s acts on field functions as a differential operator graded by the ghost number. A formal superfield unification exists that unifies $\vec{\Psi}(t)$, $\vec{\overline{\Psi}}(t)$ and $d\vec{q(t)}$ in a single quantity and s can be interpreted as a differential operator.

To get a gauge-fixed action with a quadratic dependence on the velocity $\dot{\vec{q}}$, one chooses a localisation gauge function $\dot{q}_i + \frac{\delta V}{\delta q_i}$. The prepotential $V[q]$ is a priori arbitrary, but the equivariance with respect to dilatations determines its dependence on $\vec{q}$, as it will be shown shortly. The gauge-fixing term is s-exact, and gives the (supersymmetric) BRST invariant action $\mathcal{I}_{gf}$

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$$
\begin{aligned}
\mathcal{I}_{gf}[\vec{q}, \vec{\Psi}, \vec{\overline{\Psi}}, \vec{\lambda}] &= \int_0^T d\tau \left(\frac{1}{g}\dot{\theta} - s\overline{\Psi}(\frac{1}{2}\lambda_i + \dot{q}_i + \frac{\delta V}{\delta q_i}) \right) \\
&= \int_0^T d\tau \left(\frac{1}{g}\dot{\theta} - \frac{1}{2}\lambda_i^2 + \lambda_i \left(\dot{q}_i + \frac{\delta V}{\delta q_i} \right) - \overline{\Psi}_i \left(\dot{\Psi}_i + \frac{\delta^2 V}{\delta q_i \delta q_j} \Psi_j \right) \right)
\end{aligned} \tag{6}
$$

Notice that the boundary terms $\dot{V}$ add up to $\dot{\theta}$ after the elimination of the auxiliary field by its algebraic equation of motion, so that their combination can cancel out.

The partition function is

$$
Z = \int \mathcal{D}[\vec{q}]\mathcal{D}[\vec{\Psi}]\mathcal{D}[\vec{\overline{\Psi}}]\mathcal{D}[\vec{\lambda}] \exp -\mathcal{I}_{gf} \tag{7}
$$

and, given some functionals $O(\vec{q})$ one has well-defined Euclidian path integrals

$$
< O > = \int \mathcal{D}[\vec{q}]\mathcal{D}[\vec{\Psi}]\mathcal{D}[\vec{\overline{\Psi}}]\mathcal{D}[\vec{\lambda}] O \exp -\mathcal{I}_{gf} \tag{8}
$$

A Faddeev–Popov field theory interpretation of the gauge-fixing can be done by considering $\dot{q}_i + \frac{\delta V}{\delta q_i}$ as a gauge function for the quantum variable $\vec{q}$ and by inspecting the Faddeev–Popov determinant obtained by the path integration over the ghosts. The topological non-triviality of the theory occurs when this determinant has zero modes.

The BRST invariance of the field polynomial O, if any, allows one to prove the topological properties of $< O >$, that is the fact it only depends on the winding number N of the trajectories. Any given topological observable must be first computed in a given topological sector N, and one can possibly sum over N, sometimes with a relevant regularisation. Our knowledge of supersymmetric quantum mechanics tells us that this mean value may depend on the class of the function V. What happens is that in the case of topological field theories, the Euclidian path integral explores the moduli space of the equation $\dot{q}_i + \frac{\delta V}{\delta q_i} = 0$, as a result of the gauge fixing, which is non-trivial only for relevant choices of V. The topological observables are defined from the cohomologies of the BRST operator with all possible ghost numbers.

A possible way to select the prepotential $V(\vec{q})$ leading to interesting topological information has been investigated in [6]. One asks for a larger invariance of the action that is more restrictive than the topological BRST symmetry, namely a local version of it, for which the parameter becomes an affine function of the time, with arbitrary infinitesimal coefficients. It is called a vector BRST symmetry. In fact, in our case, it can be identified with the requirement of conformal symmetry.

The "local" BRST transformations δ_l are

$$
\delta_l \mathcal{I}_{gf}[\vec{q}, \vec{\Psi}, \vec{\overline{\Psi}}, \vec{\lambda}] = 0 \tag{9}
$$

where

$$
\delta l\vec{q} = \eta(t)\vec{\Psi} \quad \delta_l\vec{\Psi} = 0 \quad \delta_l\vec{\overline{\Psi}} = \eta(t)\vec{\lambda} - \dot{\eta(t)}\vec{q} \quad \delta_l\vec{\lambda} = \dot{\eta(t)}\vec{\Psi} \tag{10}
$$

and $\eta(t) = a + bt$ where a and b are constant anticommuting parameters. The idea of local BRST symmetry was earlier considered in a paper with Raymond Stora [9], for the sake of interpreting higher order cocycles which occurs when solving the anomaly consistency conditions. This symmetry has been shown to play a role in topological field theories in [7].

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V satisfies the following constraint due to this local symmetry, [6]

$$\frac{\delta V}{\delta q_i} + q_j \, \frac{\delta^2 V}{\delta q_i \, \delta q_j} = 0 \tag{11}$$

This constraint is solved for $V(\vec{q}) \sim \theta$ or $V(\vec{q}) \sim \log|\vec{q}|$ where θ is the angle such that $z = q_1 + iq_2 = |\vec{q}| \exp i\theta$. We introduce again a multiplicative scale with a real number g. By putting this value of $\dot{q}_i + \frac{\delta V}{\delta q_i}$ in (6) and eliminating the Lagrange multiplier λ by its equation of motion we obtain (all boundary terms compensate against each other thanks to the choice of V):

$$\mathcal{I}_{gf}[\vec{q}, \vec{\Psi}, \vec{\overline{\Psi}}] = \int_0^T d\tau \left(\frac{1}{2}\dot{q}_i^2 + \frac{1}{2g^2\vec{q}^2} - \overline{\Psi}_i \left(\dot{\Psi}_i + \frac{\delta^2\theta}{g\delta q_i \delta q_j}\Psi_j \right) \right) \tag{12}$$

One has

$$\begin{aligned} \frac{\delta^2\theta}{\delta q_i \delta q_j} &= \frac{1}{\vec{q}^2} \begin{pmatrix} -\sin 2\theta & \cos 2\theta \\ \cos 2\theta & \sin 2\theta \end{pmatrix}_{ij} \\ &= \frac{1}{\vec{q}^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}_{ij} \end{aligned} \tag{13}$$

The signal that the theory truly carries some topological information is the existence of an instanton structure, leading to fermionic zero modes.

Our gauge-fixing gives an action whose bosonic part is the square of the gauge function or localisation function. Its Euclidian equations of motion are thus obtained when this gauge function vanishes

$$\dot{q}_i + \frac{1}{g}\frac{\epsilon^{ij} q_j}{\vec{q}^2} = 0 \tag{14}$$

$$\dot{\Psi}_i + \frac{1}{g}\frac{\delta^2\theta}{\delta q_i \delta q_j}\Psi_j = 0 \tag{15}$$

In the complex number representation with $z = q_1 + iq_2$ and $\Psi_z = \Psi_1 + i\Psi_2$, one has $sz = \Psi_z$ and one can write these equations as

$$-ig\dot{z} = \frac{1}{\bar{z}} \tag{16}$$

$$-ig\dot{\Psi}_z^* = \frac{1}{z\bar{z}}\Psi_z \tag{17}$$

(we use the symbol * for the complex conjugation on the ghost Ψ_z^* not to do a confusion with antighost $\overline{\Psi}_z$). Assuming periodic boundary conditions, one can easily solve the first-order equation for z as a particle making N cycles at constant angular speed with radius $R \sim 1/\sqrt{N}$. More precisely:

$$2\pi Ng - \frac{1}{\bar{z}z} = 0 \quad z = \frac{1}{\sqrt{2\pi Ng}} \exp i\theta \quad \theta = 2\pi Nt \tag{18}$$

Therefore, since the action vanishes for such trajectories, one finds that the model has instantons indexed by an integer N, which are circles described at fixed angular speed with frequency $2\pi Ng$, a tremendously simplified version of the Yang–Mills instantons.

The equation of motion of the fermion can be written as

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$$ig\dot{\Psi}_z^* + \frac{1}{\bar{z}z} U_\theta \Psi_z = 0 \tag{19}$$

$$U_\theta = R_\theta C R_\theta \tag{20}$$

U_θ is in fact the subgroup of the large rotation group $O(2)$ elements connected to the inversion/conjugation matrix C with $\det C = -1$. The other components of $O(2)$, the $SO(2)$ "small" rotation matrices R, with $\det R = 1$, are connected to the identity. In matrix notations, one has for the vector representation:

$$C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{21}$$

Thus the meaning of Eq. (13) is

$$\frac{1}{\bar{z}z}\Big(U_\theta\Big)_{ij} = \frac{\delta^2\theta}{\delta q_i \delta q_j} = \frac{\delta^2 Log\sqrt{z\bar{z}}}{\delta q_i \delta q_j} = \frac{1}{\vec{q}^2} R_\theta C R_\theta \tag{22}$$

In the complex number representation the rotation R_θ is $z \to \exp i\theta\ z$ and the matrix U_θ is the complex conjugation followed by the multiplication by $\exp 2i\theta$, $z \to \exp 2i\theta\ \bar{z}$.

The solution of the equation of fermionic zero mode in the field of the instanton with winding number N is obtained by defining

$$\Psi(t) = \Psi_0 \exp -i2\pi N t \tag{23}$$

that implies that Ψ_0 is time independent and satisfies

$$\Psi_0^* - C\Psi_0 = 0 \tag{24}$$

Thus Ψ_0 is an eigen-vector of the operator C. Depending on the orientation of the winding number N of the instanton, we have a fermionic zero mode

$$\Psi_0 = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{25}$$

The fermionic zero modes can be drawn as constant vectors attached to the particle and running along the circles of radius $\sqrt{2\pi N g}$ at a constant angular speed $2\pi N g$.

To summarise, for each instanton (here we reinstall the period T previously rescaled at $T = 1$),

$$z^{(n)} = \sqrt{\frac{T}{2\pi N}} \exp -i\frac{2\pi N t}{T} \qquad N \in Z \tag{26}$$

one has a ghost zero mode

$$\Psi_z^{(n)} = \Psi_{0z} \exp -i\frac{2\pi N t}{T} \tag{27}$$

where Ψ_{0z} is a constant Grassmann variable. The Euclidian energy and angular momentum of the action evaluated for these field configurations vanish for all values of N.

Because of these degenerate zero modes, BRST-exact observables exist with non-vanishing mean values. They are in fact independent on energy and time because of the BRST-exactness of the action. Moreover, they can be computed as a series over the topological sector index N.

We verified these properties in [5] by explicit computation in Hamiltonian formalism, using the technics developed in [8,10].

We found a continuum spectrum of states that are normalisable as plane waves in one dimension. This is in fact a consequence of the continuity of the spectrum of the Hamiltonian in the

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radial direction. They build an appropriate basis of stationary solutions since, with the appropriate normalisations factor, one has $\sum_n \int_{E>0} dE|E,n><E,n| = 1$, where n is the angular momentum quantum number. On the other hand, for $E = 0$, the solvable Schrödinger has no admissible normalisable solution. Thus we have a continuum spectrum, bounded from below, with a “spin” degeneracy equal to 4 and an infinite degeneracy in n. The peculiarity of this spectrum is that there is no ground state, since we have states with energy as little as we want, but we cannot have $E = 0$. This is a consequence of the conformal property of the potential $\frac{1}{|\vec{q}|^2}$.

Since we cannot reach the energy zero which would be the only Q and $\overline{Q}$ invariant state, we concluded that supersymmetry is broken, and that the model is for a non-trivial topological supersymmetric quantum mechanics.

As for the computation of BRST invariant observables, we found by dimensional arguments that the candidates are (in polar coordinates)

$$O_\theta = [Q, r\overline{\Psi}_\theta]_+ = [\overline{Q}, r\Psi_\theta]_+^\dagger \qquad O_r = [Q, r\overline{\Psi}_r]_+ = [\overline{Q}, r\Psi_r]_+^\dagger \tag{28}$$

The mean values of these operators between normalised states could be computed as

$$\frac{<E,n|[Q, r\overline{\Psi}_\theta]_+|E,n>}{<E,n|E,n>} = n + i\frac{1}{g} \tag{29}$$

and

$$\frac{<E,n|[Q, r\overline{\Psi}_r]_+|E,n>}{<E,n|E,n>} = \lim_{L\to\infty} \frac{L^2 J^2_{\sqrt{n^2+f^2}}(L)}{\int_0^L dr\, J_{\sqrt{n^2+f^2}}(r)} \tag{30}$$

The last quantity is bounded but ill-defined, so we rejected it. Therefore, for any normalised state $|\phi_n> = \int dE\rho(E)|E,n>$ with a given angular momentum n, the expectation value of $[Q, r\overline{\Psi}_\theta]_+$ is

$$<\phi_n|[Q, r\overline{\Psi}_\theta]_+|\phi_n> = n + i\frac{1}{g} \tag{31}$$

independently of the weighting function ρ.

If we now sum over all values of n, what remains is the topological number

$$<O_\theta> = \sum_n <\phi_n|[Q, r\overline{\Psi}_\theta]_+|\phi_n> = \sum_n n + i\frac{1}{g}\sum_n 1 \tag{32}$$

Our result [5] meant that there are two observables, organized in a complex form, in the cohomology of the punctured plane.

3. What’ new after years: Chains of topological oscillators with conformal properties

We wish a system of equations for the interactions of particles confined in a 2D-surface with planar complex coordinates z_n and possible conformal interactions between next neighbours as a generalisation of what we presented in the previous section. Non-trivial topology can arise because the potential is such that particles cannot sit on top of each other.

For concrete purposes and the sake of simplicity, we ask that the particles can have stable or metastable rest solutions on a line at given locations u_n, with $\frac{du_n}{dt} = 0$. We choose the following particle alternate rest positions on a line

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$$\begin{aligned} u_{2n} &= 2na \\ u_{2n+1} &= 2na + b \end{aligned} \tag{33}$$

In fact one has

$$\begin{aligned} u_{2n+1} - u_{2n} &= b \\ u_{2n+2} - u_{2n+1} &= 2a - b \equiv c \\ u_{2n+3} - u_{2n+2} &= 2na + 2a + b - 2na - 2a = b \\ &etc.\ldots \end{aligned} \tag{34}$$

So the distance between the site u_{2n} and its left neighbour u_{2n-1} is b and between u_n and its right neighbour u_{2n+1} it is $c = 2a - b$, $b \neq c$, and so on

$$\ldots \to (u_{2n-1}) \leftarrow b \to (u_{2n}) \leftarrow c \to (u_{2n+1}) \leftarrow b \to (u_{2n+2}) \leftarrow c \to (u_{2n+3}) \leftarrow b \to \ldots \tag{35}$$

In fact, we wish to build a system that is analogous to a bidimensional crystal of particles that interacts with few (here two) of their neighbours, by demanding some conformal properties and instanton solutions. The use of supersymmetric quantum mechanics is thus desirable to define and compute topological invariants for they system by path integration.

We have in mind to describe, in particular, systems as in [12], and recently, [13], for rotor models, chemical chains, etc...

The potential we will introduce will not give an integrable model. Rather, it has classical solutions that reproduce the behaviour systems with rigid links between points, such as articulate bars with rotation freedom, for instance the rods coupling the wheels of a locomotive, and the parameters can be adjusted such one has a global movement, with a careful adjustments of the articulations for which some indices don't vanish.

In fact, the multivalued prepotential we will introduce is just what is needed to possibly go "off-shell" from the classical behaviour of an articulated classical system, with stable classical trajectories corresponding to rigid links between its elements. One builds a supersymmetric model to calculate indices and/or topological numbers that ensures non-trivial propagations, such as wave packets that are soliton and spin-like waves. The power of a TQFT is that, when one does the path integration and when the bosonic part of the classical systems hits a given instanton, fermionic zero modes can occur, and their path integration contributes by a normalised amount to a topological observables. In fact the Yang–Mills TQFT works this way, but our model just does the same in a much more concrete way.

We will be also concerned on possible limits, when n can be replaced by a continuous variable x, as standard way of dealing with a very large number of particles, with a $1+1$ field theory limit.

3.1. The equations

It is not a restriction to consider a generic value of n that is even. By inspiration from the Yang–Mills self-dual equations and the toy model [5], we choose

$$\begin{aligned} ig\dot{\bar{z}}_n &= \frac{1}{z_n - z_{n-1} - b} + \frac{1}{z_n - z_{n+1} + c} \\ ig\dot{\bar{z}}_{n+1} &= \frac{1}{z_{n+1} - z_n - c} + \frac{1}{z_{n+1} - z_{n+2} + b} \end{aligned} \tag{36}$$

So we have

$$ig(\dot{\bar{z}}_{n+1} - \dot{\bar{z}}_n) = \frac{2}{z_{n+1} - z_n - c} - \frac{1}{z_{n+2} - z_{n+1} - b} - \frac{1}{z_n - z_{n-1} - b} \tag{37}$$

We find indeed that, for these equations, we have static solutions for $z_p = u_p$, with $\dot{u}_p = 0$, for all values of p.

If we define the Z's as

$$z_n(t) = u_n + Z_n(t) \tag{38}$$

we have

$$ig\dot{\bar{Z}}_n = \frac{1}{Z_n - Z_{n-1}} + \frac{1}{Z_n - Z_{n+1}} \tag{39}$$

with

$$ig(\dot{\bar{Z}}_{n+1} - \dot{\bar{Z}}_n) = \frac{2}{Z_{n+1} - Z_n} - \frac{1}{Z_{n+2} - Z_{n+1}} - \frac{1}{Z_n - Z_{n-1}} \tag{40}$$

These equations give solutions to the equations of motion derived from the topological gauge-fixing of a topological term $\int_\Gamma dV_{2-\text{neighbours}}$, where the multivalued prepotential $V_{2-\text{neighbours}}$ is

$$V_{2-\text{neighbours}}(\{z_n\}) = \sum_{p \subset \mathcal{Z}} \Big(Log|z_{2p+1} - z_{2p} - c| + Log|z_{2p} - z_{2p-1} - b| \Big) \tag{41}$$

These solutions extremise the following bosonic action

$$\int_\Gamma dt \sum_{i,p} \Big(\frac{1}{2}(\dot{q}^i_p)^2 + \frac{1}{2g^2}\frac{1}{(q_p - q_{p-1})^2} + \frac{1}{2g^2}\frac{1}{(q_p - q_{p+1})^2} \Big) + \int_\Gamma dV_{2-\text{neighbours}} \tag{42}$$

Some of the solutions of these coupled non-linear equations are in particular epicycles that are shifted along successive points on a lines. They are instantons, since they give the same value $S = 0$ for the action. In the next section we indeed give particular solutions. Generalising the steps for the one particle case the supersymmetric action will be

$$\int_\Gamma dt \sum_{i,p} \Big(\frac{1}{2}(\dot{q}^i_p)^2 + \frac{1}{2}\Big(\frac{\delta V_{2-\text{neighbours}}}{\delta \dot{q}^i_p}\Big)^2 \Big) - \sum_{i,j,p,q} \int_\Gamma \overline{\Psi}^i_p \Big(\delta_{ij}\delta^{pq}\frac{d}{dt} + \frac{\delta^2 V_{2-\text{neighbours}}}{\delta q^i_p \delta q^j_q} \Big) \Psi^j_q \tag{43}$$

We might extend the computation for more than two next-neighbour oscillators using more length scales

$$\begin{aligned} V_{4\ neighbours}(\{z_n\}) = \sum_{p \subset \mathcal{Z}} \Big(& Log|z_{2p+1} - z_{2p} - c| + Log|z_{2p} - z_{2p-1} - b| \\ & + Log|z_{2p+2} - z_{2p} - e| + Log|z_{2p} - z_{2p-2} - f| \Big) \end{aligned} \tag{44}$$

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3.2. Epicycles as particular solutions

Let us define

$$\Delta_n = Z_n - Z_{n-1} \tag{45}$$

Eq. (39) reads as

$$ig\dot{\overline{\Delta}}_{2p} = \frac{2}{\Delta_{2p}} - \frac{1}{\Delta_{2p+1}} - \frac{1}{\Delta_{2p-1}}$$
$$ig\dot{\overline{\Delta}}_{2p+1} = \frac{2}{\Delta_{2p+1}} - \frac{1}{\Delta_{2p+2}} - \frac{1}{\Delta_{2p}}$$

and we have no need to distinguish between even and odd sites.

3.2.1. Solutions with constant frequencies and radii

If we take solutions with the same radii $|Z_p| = R$, all particles rotate at the constant angular speed $\omega = \sqrt{2\pi Ng}$. The Z_p are the submits of a regular rotating regular polyedre in a circle, and one gets a representation of a fixed solid that rotates with a speed related to its dimension. As seen in the rest frame of one of the z_p, the particles describe epicycles. The solution is given by:

$$\Delta_n = \frac{1}{\sqrt{2\pi Ng}} \exp i(2\pi Ngt + \delta_n) \tag{46}$$

and the phases δ_n satisfy

$$\exp i\delta_n = 2\exp i\delta_n - \exp i\delta_{n-1} - \exp i\delta_{n+1} \tag{47}$$

that is

$$\exp i\delta_n - \exp i\delta_{n-1} - \exp i\delta_{n+1} = 0 \tag{48}$$

This equation can be solved using determinant techniques.

3.2.2. Solutions with alternate frequencies and radii

We have solutions where the last two off-diagonal terms on the right hand-side compensate each other. They are such that Δ_n describes circles at a constant frequency N^{even} for even n and at a possibly different frequency N^{odd} for odd n: the phase of $Z_n(t)$ differs from that of $Z_{n+2}(t)$ by odd numbers of π, namely

$$\Delta_{2p} = \frac{1}{\sqrt{2\pi N^{even} g}} \exp i(2\pi N^{even} t + \delta_{even})$$
$$\Delta_{2p+2} = -\frac{1}{\sqrt{2\pi N^{even} g}} \exp i(2\pi N^{even} t + \delta_{even}) \tag{49}$$

and

$$\Delta_{2p+1} = \frac{1}{\sqrt{2\pi N^{odd} g}} \exp i(2\pi N^{odd} t + \delta_{odd})$$
$$\Delta_{2p+3} = -\frac{1}{\sqrt{2\pi N^{odd} g}} \exp(2\pi N^{odd} t + \delta_{odd}) \tag{50}$$

The radii are respectively $\frac{1}{\sqrt{2\pi N^{even} g}}$ and $\frac{1}{\sqrt{2\pi N^{odd} g}}$ for the even and odds Δ_n.

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Going back to the variables Z' and z one sees moreover that the coordinates z_n describe involved epicycles.

3.2.3. The limiting case for the wheel rods of a steam locomotive

There is interesting limit of the last case, when the frequency of for instance the particles with odd indices is very large, $N^{odd} \to \infty$. Then Z_{2p-1} runs at a very high speed on a circles of radius almost equal to zero around Z_{2p}, and Eq (39) becomes

$$ig\dot{\overline{Z}}_{2p} = \frac{2}{Z_{2p}} \tag{51}$$

modulo terms of order $O(1/N^{odd}g)$. In the limit, one has

$$\begin{aligned} z_{2p} &\sim 2pa + \frac{1}{\sqrt{4\pi N^{even} g}} \exp i(4\pi N^{even} t + \delta_{even}) \\ z_{2p+2} &\sim (2pa + 2a) - \frac{1}{\sqrt{4\pi N^{even} g}} \exp i(4\pi N^{even} t + \delta_{even}) \end{aligned} \tag{52}$$

Each particle with an odd index runs on a very short radius trajectory at a very high frequency, that is, it is basically glued by an almost rigid bar to a particle with an even index that runs itself on a circle at finite frequency with a finite radius $\frac{1}{\sqrt{4\pi N^{even}}}$, centred on a position $u_2 p$, that is the almost exactly in a $\frac{1}{N^{odd}}$ approximation the centre of ration of a very fast running particle and so on. In the limit where the winding number N^{odd} goes to infinite, the odd particle becomes invisible, as a small amplitude local vibration in the middle of a bar! On the other hand the trajectories of the even particles are circles of radius $\sqrt{2\pi N^{even} g}$ around the points $u_{2n} = 2na$, with alternative phase as read in Eq. (52). For these trajectories one can put a rigid articulate bar between z_{2n} and z_{2n+2}.

This interesting solution describes the behaviour of the rods of the wheels of a steam locomotive. The z_p are nothing else than the points where the rods connect two neighbours wheels. This also describes the apparatus of [13].

This is an interesting model where a conformal potential reproduces on-shell articulate mechanical attachments. Since we have a non-linear equations, there are certainly other solutions, with more complicated propagating waves.

3.3. The supersymmetric Lagrangian

It is

$$S = \int_\Gamma dt \sum_p \frac{1}{2}(\dot{q}^i_p)^2 + \frac{1}{2}\Big(\frac{\delta V}{\delta q^i_p}\Big)^2 - \sum_{p,r} \overline{\psi}^i_p \Big(\delta_{pr}\delta_{ij}\frac{d}{dt} + \frac{\delta^2 V}{\delta q^i_p \delta q^j_r}\Big)\psi^j_r \tag{53}$$

that is, modulo the introduction of auxiliary fields and their elimination from the action

$$S = \int_\Gamma dt\ , Q\{\sum_p \overline{\psi}^i_p \Big(\frac{1}{2}b^i + \dot{q}^i_p + \frac{\delta V}{\delta q^i_p}\Big)\} - \int_\Gamma dV \tag{54}$$

Here the (topological) supersymmetric graded differential operator is defined as

$$Qq = \psi, \quad Q\psi = 0, \quad Q\overline{\psi} = b, \quad Qb = 0 \tag{55}$$

The Lagrangian is Q-exact, modulo a topological term. The nilpotency of Q proves the Q-invariance. It is noteworthy that S is also $\overline{Q}$-invariant (and $\overline{Q}$-exact), modulo a boundary term, where the definition of $\overline{Q}$ is obtained from that of Q by exchanging ψ and $\overline{\psi}$ and b and $-b$, $\overline{Q}q = \overline{\psi}$, $\overline{Q}\overline{\psi} = 0$, $\overline{Q}\psi = b$, $\overline{Q}b = 0$. Q and $\overline{Q}$ anticommute. The action can be written as a $\overline{Q}$-exact term. However, the action is not $Q\overline{Q}$-exact.

With our choice where V only depends on linear combinations of $\log|\vec{q}_p - \vec{q}_r|$, the action is (super) conformally invariant and, moreover, the boundary term $\int dV$ is non-trivial, and discretely multi-valued.

3.4. Zero modes for the chain

Let us come back to the equation whose solutions extremise the bosonic part of the action of our conformal oscillator chain lattice

$$ig(\dot{\overline{Z}}_{n+1} - \dot{\overline{Z}}_n) = \frac{2}{Z_{n+1} - Z_n} - \frac{1}{Z_{n+2} - Z_{n+1}} - \frac{1}{Z_n - Z_{n-1}} \tag{56}$$

We defined $\Delta_n = Z_{n+1} - Z_n$ and the topological symmetry operation on the particles is $Qz_n = QZ_n = \psi_n$. So by defining $\Psi_n \equiv \psi_{n+1} - \psi_n$ we have

$$QZ_n = \Psi_n \tag{57}$$

The supersymmetric Lagrangian is

$$Q\sum_n \overline{\Psi}_n \Big(ig\dot{\Delta}_n^* - \frac{2}{\Delta_n} + \frac{1}{\Delta_{n+1}} + \frac{1}{\Delta_{n-1}}\Big) \tag{58}$$

Therefore the zero modes for the fermions are the non-trivial solution the Q variation of the topological gauge function when the positions satisfies it, that is

$$ig\dot{\Psi}_n^* - \frac{2U_{\theta_n}\Psi_n}{|\Delta_n|^2} + \frac{U_{\theta_{n+1}}\Psi_{n+1}}{|\Delta_{n+1}|^2} + \frac{U_{\theta_{n-1}}\Psi_{n-1}}{|\Delta_{n-1}|^2} = 0 \tag{59}$$

Consider the previously found trajectories

$$\Delta_n = \frac{1}{\sqrt{2\pi Ng}} exp\ (i2\pi Nt + \delta_n) \tag{60}$$

There are zero modes

$$\Psi_n(t) = \Psi_{n,0}\ exp - i2\pi Nt \tag{61}$$

where the $\Psi_{n,0}$ are time independent. Indeed, after relevant manipulations using the definition of the matrices U_θ, one gets the following conditions:

$$\Psi_{n,0} = 2(\exp 2i\delta_n)\Psi_{n,0} - (\exp 2i\delta_{n+1})\Psi_{n+1,0} - (\exp 2i\delta_{n-1})\Psi_{n-1,0} \tag{62}$$

that can be solved using determinant techniques.

The topological invariant are Green-functions of the type

$$\int \Pi[dz_p][d\psi_p][d\overline{\psi}_p][db_p]\mathcal{P}(\psi, z) \tag{63}$$

where $\mathcal{P}$ is Q invariant and has odd fermionic number.

We can compute as in [5] these invariants in the case of one particle as well of the Witten index of the theory, using the Hamiltonian formalism.

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4. The continuous limit $n \to \infty$ and $b, c, a \to 0$

4.1. The bosonic sector

We can consider the limit where the index n becomes a continuous one, and we have a chain of sites that can exhibits propagation of non-trivial waves.

This limit is quite strong: it is not only the difference between the radii of Δ_n and Δ_{n+1} that becomes small; the difference between the angles $\theta(\Delta_n)$ and $\theta(\Delta_{n+1})$ is also small. We could consider more subtle limits.

Let us suppose that we have a solution for each site

$$\Delta_n = R \exp i\theta_n(t) \tag{64}$$

where R is time independent but $\theta_n(t)$ has a non-trivial t dependence. Thus the equation of motion of θ_n is

$$\begin{aligned} -Rg\dot{\theta}_n \exp -i\theta_n(t) &= \frac{1}{R}(2\exp -i\theta_n - \exp -i\theta_{n+1} - \exp -i\theta_{n-1}) \\ &= \frac{\exp -i\theta_n}{R}[2 - \exp -i(\theta_{n+1} - \theta_n) - \exp -i(\theta_{n-1} - \theta_n)] \end{aligned} \tag{65}$$

In the continuous limit, n is replaced by a suitably normalised continuous length variable $n \to x$, $\theta_n \to \theta(x)$, and one can write

$$igR^2\dot{\theta}(x,t) \sim \frac{\partial^2}{\partial^2 x}\theta(x,t) + (\frac{\partial}{\partial x}\theta(x,t))^2 \tag{66}$$

If we change variables, $\theta = \log v(x,t)$, the former equation reads

$$igR^2\dot{v}(x,t) \sim \frac{\partial^2}{\partial^2 x} v(x,t) \tag{67}$$

that is, a Schrödinger type free equations, with periodic boundary conditions for $\theta(x,t)$. In what follows we replace $\sim$ by an equality.

The general solution is easy to find by Fourier transform, using the time periodicity of v,

$$v(x,t) = \sum_N v_N \ \exp -i(2\pi Nt - R\sqrt{2\pi Ng}\ x)) \tag{68}$$

$$\theta(x,t) = \exp \sum_N v_N \ \exp -i(2\pi Nt - R\sqrt{2\pi Ng}\ x)) \tag{69}$$

Finally one gets non-trivial wave packets

$$\Delta(x) = R \exp 2i\pi \exp \sum_N v_N \ \exp -i(2\pi Nt - R\sqrt{2\pi Ng}\ x)) \tag{70}$$

4.2. The fermionic zero modes

The zero modes continuous field is $\Psi_n \to \Psi(x,t)$. The continuous limit of

$$ig\dot{\Psi}^*_n - \frac{2U_{\theta_n}\Psi_n}{|\Delta_n|^2} + \frac{U_{\theta_{n+1}}\Psi_{n+1}}{|\Delta_{n+1}|^2} + \frac{U_{\theta_{n-1}}\Psi_{n-1}}{|\Delta_{n-1}|^2} = 0 \tag{71}$$

is

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$$ig\dot{\Psi}^*_n = \frac{\partial^2}{\partial^2 x}\Big(\exp 2i\theta(x,t)\Psi^*(x,t)\Big) + [\frac{\partial}{\partial x}\Big(\exp 2i\theta(x,t)\Psi^*(x,t)\Big)]^2 \tag{72}$$

or

$$ig\dot{\Psi}^*_n = \frac{\partial^2}{\partial^2 x}\Big(v^2(x,t)\Psi^*(x,t)\Big) + [\frac{\partial}{\partial x}\Big(v^2\theta(x,t)\Psi^*(x,t)\Big)]^2 \tag{73}$$

In short there are zero modes. Most of them will published in a more extended publication [15].

5. Discussion

We have shown a multi-particle example for which the requirement of local BRST symmetry selects a superconformal quantum mechanical system with intriguing non-linear equations. It generalises the more elementary one particle model [5] and seems to provide a model with apparently deeper applications. The spectrum of the theory has no ground state and a supersymmetry breaking mechanism occurs without the presence of a dimensionful parameter. The generalisation of these observations to a 2d quantum field theory by the continuous limit we sketched here is an interesting question.

Acknowledgements

I thank my collaborator F. Toppan who introduced me to the work [13]. This rebooted my interest in the model [5] that I have generalised for the book in the memory of Raymond Stora. Much more on the properties of this non-linear model will be presented in a joint publication with F. Toppan.

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