Reduced-size formulations for metric and cut polyhedra in sparse graphs

Viet Hung Nguyen, Michel Minoux, Dang Phuong Nguyen

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Reduced-size formulations for metric and cut polyhedra in sparse graphs

Viet Hung Nguyen*, Michel Minoux* and Dang Phuong Nguyen†
*Sorbonne Universités, UPMC Univ Paris 06, UMR 7606, LIP6. 4 place Jussieu, Paris, France
†CEA, LIST, Embedded Real Time System Laboratory, Point Courrier 172, 91191 Gif-sur-Yvette, France

ABSTRACT
Given a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$, we consider the metric cone $\mathrm{MET}(G)$ and the metric polytope $\mathrm{METP}(G)$ defined on $\mathbb{R}^E$. These polyhedra are relaxations of several important problems in combinatorial optimization such as the max-cut problem and the multicommodity flow problem. They are known to have non-compact formulations via the cycle inequalities in the original space $\mathbb{R}^E$ and compact (i.e., polynomial size) extended formulations via the triangle inequalities defined on the complete graph K_n. In this paper, we show that one can reduce the number of triangle inequalities to $O(nm)$ and still have extended formulations for $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$. This is particularly interesting for sparse graphs when $m = O(n)$, since formulations of size $O(n^2)$ variables and constraints are thus obtained. Moreover, the possibility of achieving further reduction in size for special classes of sparse graphs is investigated; it is shown that for the case of *series-parallel graphs*, for which the max-cut problem can be solved in linear time (Barahona (1986)), one can refine the above reduction to obtain extended formulations for $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$ fearturing $O(n)$ variables and constraints.

KEY WORDS: sparse graph, metric polyhedra, triangle inequalities, max-cut problem, extended formulation, series-parallel graph.

1 Introduction

Let $G = (V, E)$ be an undirected graph with $n = |V|$ and $m = |E|$. We denote by ij, the edge between the two nodes i and j of V. A *chordless cycle* C in G is a cycle whose induced subgraph is the cycle itself. Let $\mathcal{C}$ be the set of the chordless cycles in G. Let $\mathbb{R}^E$ be the real space of dimension $|E|$ indexed by the edges in E. For a vector $x \in \mathbb{R}^E$, x_e with $e \in E$ denotes the component of x associated with the edge $e \in E$ and for any subset $F \subseteq E$, let $x(F) = \sum_{e \in E} x_e$.
Let us recall the definition of the two polyhedra that will be discussed in the paper. The first is *the metric polytope* $\mathrm{METP}(G)$ associated with G in $\mathbb{R}^E$, which can be defined as follows:

$$x(F) - x(C \setminus F) \leq |F| - 1, \quad \forall C \in \mathcal{C} \text{ and } F \subseteq C \text{ with } |F| \text{ odd}, \tag{1}$$

$$\begin{aligned} x_e &\geq 0 \quad \forall e \in E \text{ s.t. } e \text{ does not belong to any triangle} \\ x_e &\leq 1 \quad \forall e \in E \text{ s.t. } e \text{ does not belong to any triangle} \end{aligned} \tag{2}$$

Note that Inequalities (1) are called *cycle inequalities*. Inequalities (2) are applied only for the edges in G which do not belong to any triangle as those for the other edges can be derived from the cycle inequalities. These inequalities were introduced in the seminal paper by Barahona and Mahjoub (1986) on the cut polytope. The second polyhedron is *the metric cone*

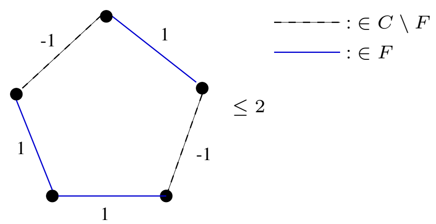

Figure 1: A cycle inequality with $|F| = 3$.

$\mathrm{MET}(G)$ which consists of the cycle inequalities with sets F such that $|F| = 1$, the nonnegativity inequalities and the trivial inequalities (2). More precisely, $\mathrm{MET}(G) = \{x \in \mathbb{R}^E$ such that

$$x_e - x(C \setminus \{e\}) \leq 0, \qquad \forall C \in \mathcal{C} \text{ and } e \in C \,, \tag{3}$$

$$x_e \geq 0 \quad \forall e \in E \text{ s.t. } e \text{ does not belong to any triangle,}$$
$$x_e \leq 1 \quad \forall e \in E$$

Note that $\mathrm{MET}(G)$ is a polytope, not a cone. However, we use here the standard terminology used by Deza and Laurent (1994a) which was proposed in a context where the basic space considered was the hypercube $[0, 1]^n$.

The two polyhedra $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$ are strongly related to *the maximum cut problem* which is one of the basic problems in combinatorial optimization. Actually, the metric cone $\mathrm{MET}(G)$ is a relaxation of $\mathrm{CUTB}(G)$, the intersection of the unit hypercube with the cone generated by all the cut vectors $\delta(S)$ for $S \subset V$ (with abuse of notation, by $\delta(S)$ we denote both the edge set of the cut defined by the node set S and its incidence vector). Similarly, the metric polytope is a relaxation of *the cut polytope* $\mathrm{CUTP}(G)$, the convex hull of all the cut vectors $\delta(S)$ for $S \subset V$. If we replace the trivial inequalities by the 0/1 constraints $x \in \{0, 1\}^E$ in the formulation of the two polyhedra, we obtain respectively integer formulations for $\mathrm{CUTB}(G)$ and $\mathrm{CUTP}(G)$.

Note that since there is a priori no known polynomial upper bound (in terms of n and m) on the number of chordless cycles and there may be also an exponential number of choices for the set F given a chordless cycle C, the above formulations of $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$ have a priori an exponential number of inequalities. Nevertheless, when $G = K_n$, the complete graph of n nodes, $\mathrm{MET}(K_n)$ and $\mathrm{METP}(K_n)$ are of polynomial size since in this case $\mathcal{C}$ reduces to the set of the triples $\{i \neq j \neq k \in V\}$ and F can have only 1 or 3 edges. Concretely, let $\mathcal{T}$ be the set of all the (unordered) triples of distinct nodes $i, j, k \in V$, the following system:

$$x_{ij} + x_{ik} + x_{jk} \leq 2 \text{ for all } i, j, k \in \mathcal{T}. \tag{4}$$

$$\begin{aligned} &x_{ij} - x_{ik} - x_{jk} \leq 0, \\ &x_{ik} - x_{ij} - x_{jk} \leq 0, \\ &x_{jk} - x_{ij} - x_{ik} \leq 0 \text{ for all } i, j, k \in \mathcal{T}. \end{aligned} \tag{5}$$

defines $\mathrm{METP}(K_n)$. Inequalities (4) are called *the non-homogeneous triangle inequalities* and the ones in (5) are called *the homogenous triangle inequalities*. They are all commonly called *the triangle inequalities*. The cone $\mathrm{MET}(K_n)$ is defined only by the homogeneous inequalities (5) and the trivial inequalites (2). The number of inequalities in $\mathrm{MET}(K_n)$ and in $\mathrm{METP}(K_n)$ is clearly in $O(n^3)$, and thus polynomial in terms of n. In fact, Barahona (1993) showed that the projections of $\mathrm{MET}(K_n)$ and $\mathrm{METP}(K_n)$ on $\mathbb{R}^E$ are exactly $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$. Hence, $\mathrm{MET}(K_n)$ and $\mathrm{METP}(K_n)$ respectively represent compact extended formulations for $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$.

The metric cone and metric polytope have several important applications in combinatorial optimization, e.g., the max-cut problem and the multicommodity flow problem. An overview of these applications can be found in Deza and Laurent (1994a,b) and Ben-Ameur et al. (2013). In these applications, optimizing a linear function over $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$ usually appears as a subproblem and thus the latter has to be solved repeatedly. In this situation, the compact formulations $\mathrm{MET}(K_n)$ and $\mathrm{METP}(K_n)$ are usually preferred to the non-compact ones for optimizing over $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$ since they can be directly transmitted to a linear programming solver. However, the number of triangle inequalities in $\mathrm{MET}(K_n)$ and $\mathrm{METP}(K_n)$, which is in $O(n^3)$, can be huge even for medium values of n making the optimization over compact formulations computationally difficult (Frangioni et al. (2005) is a typical reference reporting such

computational problem).
In Section 2, we show that one can reduce the number of triangle inequalities to $O(nm)$ while preserving equivalence with $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$. This result is of particular interest for the case of sparse graphs, when $m = O(n)$, since this yields much more compact formulation of size $O(n^2)$ variables and constraints. Clearly such reduction in problem size can be exploited computationally e.g. in the solution of the max-cut problem, due to the induced reduction in computational effort devoted to solving the linear relaxations in each node of the Branch-and-Bound tree.
However, beyond its computational interest, this result raises the natural and challenging new question of whether it is possible to further reduce the size of a linear formulation for $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$ in sparse graphs, or at least some subclasses of sparse graphs. And, since $\Omega(m)$ is a lower bound to the size (number of variables and constraints) of any linear formulation (just considering the non negativity constraints, assuming connectivity), it is possible to achieve linear size $O(m) = O(n)$, at least for some subclasses of sparse graphs.
As a first step towards answering such polyhedral issues, Section 3 provides a positive answer to this last question by showing that for the subclass of series-parallel graphs (for which the max-cut problem can be solved in linear time, see Barahona (1986)), it is possible to refine the reduced formulations obtained in section 2 to come up with linear-size formulations for $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$. To the best of our knowledge, this is the first nontrivial subclass of graphs enjoying linear-size representations for the associated metric polyhedra. Furthermore, as explained in the concluding section (Section 4), this result raises several important open research questions related to the existence of other subclasses of sparse graphs with similar polyhedral properties, and thus likely to lend themselves to more efficient resolution of some basic combinatorial problems such as graph partitioning (Nguyen et al. (2016)) or multicommodity flow feasibility testing. The latter often arises, in many network synthesis or discrete network optimization problems, as a subproblem to be solved repeatedly (see e.g. Minoux (1989), Gabrel et al. (1999)), and is most often NP-hard (even in cases when the underlying graph is series-parallel).

2 A $O(nm)$ size formulation for $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$

Let

$$\mathcal{T}' = \{(i, j, k) \in \mathcal{T} | \text{ at least one of } ij, ik \text{ or } jk \in E\}$$

Proposition 1 $|\mathcal{T}'| \leq m \times (n-2)$.

Proof: By definition of $\mathcal{T}'$, every triple $(i, j, k) \in \mathcal{T}'$ can be viewed as a triangle composed by, for example, an edge $ij \in E$ and a node $k \in V$. Hence, the number of such triangles, which is equal to $m \times (n-2)$, is an upper bound of $|\mathcal{T}'|$. □
Let us define $\mathrm{RMETP}(K_n)$ as the polytope defined by the following "reduced" system,

$$x_{ij} + x_{ik} + x_{jk} \leq 2 \text{ for all } i, j, k \in \mathcal{T}'. \tag{6}$$

$$\begin{aligned} x_{ij} - x_{ik} - x_{jk} &\leq 0, \\ x_{ik} - x_{ij} - x_{jk} &\leq 0, \\ x_{jk} - x_{ij} - x_{ik} &\leq 0 \text{ for all } i, j, k \in \mathcal{T}'. \end{aligned} \tag{7}$$

together with the nonnegativity and trivial inequalities (2) for the edges that do not belong to any triangle in $\mathcal{T}'$. We define the reduced metric cone $\mathrm{RMET}(K_n)$ as the one defined by inequalities (7) , the nonnegativity for the edges that do not belong to any triangle in $\mathcal{T}'$ and the trivial inequalities (2) for all the edges in K_n.

Corollary 1 *The number of non trivial inequalities in* $\mathrm{RMETP}(K_n)$ *and* $\mathrm{RMET}(K_n)$ *are respectively at most* $4m(n-2)$ *and* $3m(n-2)$. *The variables in* $\mathrm{RMETP}(K_n)$ *and* $\mathrm{RMET}(K_n)$ *correspond to the edges in* K_n, *their number is thus in* $O(n^2)$.

Let $\mathrm{RMETP}(K_n)_G$ and $\mathrm{RMET}(K_n)_G$ be respectively the projections of $\mathrm{RMETP}(K_n)$ and $\mathrm{RMET}(K_n)$ on $\mathbb{R}^E$. Similarly, the $\mathrm{METP}(K_n)_G$ and $\mathrm{MET}(K_n)_G$ are respectively the projections of $\mathrm{METP}(K_n)$ and $\mathrm{MET}(K_n)$ on $\mathbb{R}^E$. We will prove in this section the following theorem.

Theorem 1 $\mathrm{RMETP}(K_n)_G = \mathrm{METP}(G)$ *and* $\mathrm{RMET}(K_n)_G = \mathrm{MET}(G)$.

Note that we can obtain $\mathrm{RMETP}(K_n)_G$ (respectively $\mathrm{RMET}(K_n)_G$) by applying completely the Fourier-Motzkin elimination procedure (see Balas (2001), Conforti et al. (2013)) on $\mathrm{RMETP}(K_n)$ (respectively $\mathrm{RMET}(K_n)$) to eliminate successively the variables in $E_n \setminus E$ (here E_n denotes the edge set of K_n). Before proving Theorem 1, we will show the following lemma.

Lemma 1 *All the inequalities defining* $\mathrm{METP}(G)$ *can be derived by (partial) application of the Fourier-Motzkin elimination procedure on* $\mathrm{RMETP}(K_n)$.

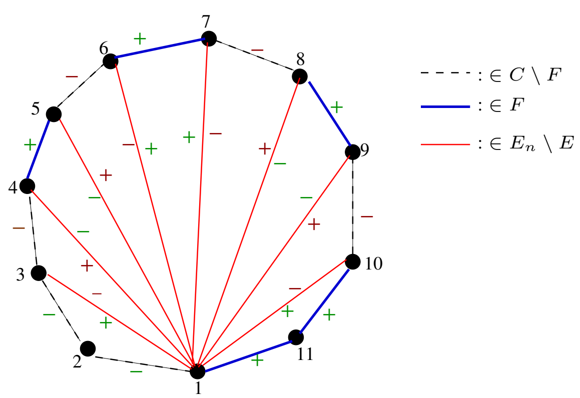

Figure 2: An incomplete Fourier-Motzkin elimination performed on the edges $1j$ ($j = 3, \ldots, k-1$).

Proof: Let us consider any chordless cycle C in G. Let us suppose that the nodes in C are 1, 2, ..., k which are numbered clockwise from 1 (see Figure 2) and its edges are $i(i+1)$ for $i = 1, \ldots, k-1$ and $k1$. Let us take any subset $F = \{f_1, \ldots, f_p\} \subseteq C$ with p odd. The cycle inequality corresponding to C and F reads:

$$x(F) - x(C \setminus F) \leq p - 1 \tag{8}$$

We shall show that this inequality can be deduced from triangle inequalities associated with triples in $\mathcal{T}'$. Consider the triangulation θ of C obtained by adding $k-2$ distinct edges (chords) $1j$ for $j = 3, \ldots, k-1$ (see the dash/dot edges in Figure 2). Each triangle $(1, i, i+1)$ for $i = 2, \ldots, k-1$ corresponds to a triple in $\mathcal{T}'$ (since they all contain at least one edge in E) and the corresponding triangle inequalities read:

$$x_{1i} + x_{1(i+1)} + x_{i(i+1)} \leq 2 \quad \text{for all } i = 2, \ldots, k-1, \tag{a,i}$$

$$x_{1i} - x_{1(i+1)} - x_{i(i+1)} \leq 0 \quad \text{for all } i = 2, \ldots, k-1, \tag{l,i}$$

$$x_{1(i+1)} - x_{1i} - x_{i(i+1)} \leq 0 \quad \text{for all } i = 2, \ldots, k-1, \tag{r,i}$$

$$x_{i(i+1)} - x_{1i} - x_{1(i+1)} \leq 0 \quad \text{for all } i = 2, \ldots, k-1. \tag{m,i}$$

For brevity, we will refer to the triangle $(1, i, i+1)$ as *"triangle i"* with $2 \leq i \leq k-1$. The edges $1i$, $i(i+1)$, $1(i+1)$ will be respectively referred to as the *left edge*, *middle edge*, *right edge* of triangle i (in the system above, the notation "a" stands for "all", and (a,i) refers to the inequality related to triangle i for which all edges are involved with positive coefficients; l, r, and m stand for "left", "right", and "middle" respectively and the inequalties are labelled (l,i), (r,i) or (m,i) depending on which edge is involved with positive coefficient). Now, for each triangle i with $2 \leq i \leq k-1$, let us choose one and exactly one of inequalities (a,i), (l,i), (r,i) and (m,i) according to the following rule:

- if the middle edge $i(i+1)$ is an edge $f_q \in F$ with q odd, choose inequality (m,i),
- if the middle edge $i(i+1)$ is an edge $f_q \in F$ with q even, choose inequality (a,i),
- if the middle edge $i(i+1) \in C \setminus F$, then by scanning clockwise the edges of C from $i(i+1)$ until reaching the node 1, we may or may not meet edges in F. In the former case, let $f_q \in F$ be the first edge in F that we meet.
  - If f_q exists and q is odd, choose inequality (r,i),
  - If f_q does not exist or f_q exists and q is even, choose inequality (l,i).

We are going to show that the sum over $i = 2, \ldots, k-1$ of the inequalities chosen according to the above rule gives inequality (8). Let us consider first any edge $1j$ $(3 \leq j \leq k-1)$ which is in $E_n \setminus E$ and show that x_{1j} vanishes in the sum. Note that x_{1j} appears only in two chosen inequalities which correspond respectively to the triangles $j-1$ and j. There are four possible cases:

- $(j-1)j$ and $j(j+1) \notin F$, hence the two chosen inequalities for the triangles $j-1$ and j are of the same type: either (l,j-1) and (l,j) or (r,j-1) and (r,j). In both cases, the signs of x_{1j} in these two inequalites are opposite.

- $(j-1)j$ is an edge $f_q \in F$ and $j(j+1) \in C \setminus F$. If q is even, then the two chosen inequalities are (a,j-1) and (r,j) in which the signs of x_{1j} are opposite. If q is odd, then the two chosen inequalities are (m,j-1) and (l,j) in which the signs of x_{1j} are also opposite.

- $(j-1)j \in C \setminus F$ and $j(j+1)$ is an edge $f_q \in F$. If q is even, then the two chosen inequalities are (l,j-1) and (a,j) in which the signs of x_{1j} are opposite. If q is odd, then the two chosen inequalities are (r,j-1) and (m,j) in which the sign of x_{1j} are also opposite.

- both $(j-1)j$ and $j(j+1)$ are in F. Let $(j-1)j = f_q \in F$. If q is even, then the two chosen inequalities are (a,j-1) and (m,j) in which the signs of x_{1j} are opposite. Similarly, if q is odd, then the two chosen inequalities are (m,j-1) and (a,j) in which the signs of x_{1j} are opposite.

In all cases, the signs of x_{1j} in the two chosen inequalities containing it are opposite, thus x_{1j} vanishes in the sum.

For any edge $e \in C$, x_e appears only in one of the chosen inequalities, the one which corresponds to the triangle having e as the middle edge. It is clear that by the choice of this inequality, the coefficient of x_e in the sum is 1 if $e \in F$ and -1 if $e \in C \setminus F$.

It remains to show that the sum of the right hand sides is $p-1$. We can see that the only chosen inequalities with non-zero right hand side are of type (a,i), i.e., the ones corresponding to the triangles having $f_q \in F$ with q even as the middle edge. There are clearly $\frac{p-1}{2}$ such inequalities with 2 as the right hand side. Hence, the sum of the right hand sides of the chosen inequalities is $p-1$.

Since the triangles created by the triangulation θ of C are in $\mathcal{T}'$, the chosen triangle inequalities are all in $\mathrm{RMETP}(K_n)$. The sum of these inequalities thus in fact produces (8) as a result of a (partial) application of the Fourier-Motzkin elimination procedure to $\mathrm{RMETP}(K_n)$. □

Thanks to the above lemma, we are now in a position to complete the proof of Theorem 1.

Proof: *of Theorem 1.* We will show the first part of Theorem 1, i.e., $\mathrm{RMETP}(K_n)_G = \mathrm{METP}(G)$. We will see that the second part will follow.

We prove first that $\mathrm{METP}(G) \subseteq \mathrm{RMETP}(K_n)_G$. This result simply follows the facts that $\mathrm{METP}(K_n) \subset \mathrm{RMETP}(K_n)$ and $\mathrm{METP}(K_n)_G = \mathrm{METP}(G)$.

Now, we prove that $\mathrm{RMETP}(K_n)_G \subseteq \mathrm{METP}(G)$. Note that we can obtain $\mathrm{RMETP}(K_n)_G$ by applying completely the Fourier-Motzkin elimination procedure (see Balas (2001), Conforti et al. (2013)) on $\mathrm{RMETP}(K_n)$ to eliminate successively the variables in $E_n \setminus E$. Lemma 1 shows that we can obtain all the inequalities of $\mathrm{METP}(G)$ by doing the projection of $\mathrm{RMETP}(K_n)$ on $\mathbb{R}^E$ by Fourier-Motzkin elimination procedure. Hence, $\mathrm{RMETP}(K_n)_G \subseteq \mathrm{METP}(G)$.

For the second part of the theorem, i.e., $\mathrm{RMET}(K_n) = \mathrm{METP}(G)$, we first see that $MET(G) \subseteq \mathrm{RMET}(K_n)$. And we can show similarly as above that $\mathrm{RMET}(K_n) \subseteq \mathrm{MET}(G)$ by remarking that the result of Lemma 1 can be applied in particular for the cycle inequalities issued from C and the sets F of cardinality equal to 1. □

Recently, in Lancia and Serafini (2011), the authors express the separation problem of the cycle inequalities as a linear program to form a mixed 0/1 program with $2n^2 + m$ variables and $4nm + 2n$ constraints (trivial inequalities not included). They prove that this program is equivalent to the integer formulation of max-cut problem formed by $\mathrm{METP}(K_n)$ and integrality constraints on the variables on the original space $\mathbb{R}^E$. Note that the formulation in Lancia and Serafini (2011) involves additional inequalities other than triangle inequalities. The polytope $\mathrm{RMETP}(K_n)$ offers similar results while featuring fewer variables (by a factor of 4 actually) than in Lancia and Serafini (2011) and is based on the use of triangle inequalities only. Note that, the max-cut problem can be also formulated as a 0-1 quadratic program and different linearization methods for the latter can give linear relaxations which are more or less strong than the relaxation given by the metric polytope $\mathrm{METP}(K_n)$ (e.g., see Boros et al. (1992), Gueye and Michelon (2009)). However, to obtain a relaxation as strong as the metric polytope, these methods have to use at least $O(n^3)$ constraints.

3 Linear size formulations for $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$ in series-parallel graphs

Note that the extended formulations $\mathrm{RMET}(K_n)$ and $\mathrm{RMETP}(K_n)$ described in Section 2 respectively for $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$ have $O(nm)$ constraints and $O(n^2)$ variables. Hence, even for special sparse graphs such as planar graphs when $m = O(n)$, there are always $O(n^2)$ constraints and variables in these formulations. In this section, we show that one can obtain extended formulations of linear size, i.e., of $O(n)$ variables and constraints when G is series-parallel.

A *series-parallel graph* is a graph which can be obtained from a single edge by applying repeatedly the following operations:

- add a parallel edge to an existing edge (parallel operation).
- or subdivide an existing edge, that is replace the edge by a path of length two (series operation).

In this section, we will assume that G is series-parallel. Given an elementary path P in G, the set of nodes in P is denoted by $V(P)$, and if u and v are two distinct nodes in $V(P)$, we denote by $P(u-v)$, the subpath of P connecting u and v. An *ear decomposition* of an undirected graph G is defined as a partition of the edges of G into a sequence of *ears* P_1, P_2, …,P_k. Each ear is a path in the graph with the following properties:

- If two nodes in the path are the same, then they should be the two end-nodes of the path.
- The two end-nodes of each ear P_i, $i > 1$, appear in previous ears P_j and P'_j with $j < i$ and $j' < i$.
- No interior node (i.e., not an end-node) of P_i is in P_j for any $j < i$.

An *open ear decomposition* is one in which each ear is an elementary path. Suppose that $ED = \{P_1, P_2, \ldots, P_k\}$ is an open ear decomposition of G, we say that P_i *is nested* in P_j, denoted by $P_i \sqsubseteq P_j$, if $j < i$ and the end-nodes of P_i both appear in P_j. For such i and j, let the *nested interval* of P_i with respect to P_j be the subpath of P_j between the two end-nodes of P_i.

We recall below the notion of *nested ear decomposition* as defined in Eppstein (1992) while simultaneously introducing the concepts of *precursor*, of *"being covered"* and of *"overlap each other"* for two ears having the same precursor.

We say that an open ear decomposition $ED = \{P_1, P_2, \ldots, P_k\}$ is *nested* if the following conditions hold:

- For each $i > 1$ there is some $j < i$ such that P_i is nested in P_j. Let j_0 denote the minimum index value in the set $\{j \ : \ P_i \sqsubseteq P_j\}$ then the ear P_{j_0} is called *the precursor* of P_i. Figure 3 gives an example where P_1 is the precursor of P_2, P_3, P_4 and P_6, P_2 is the precursor of P_5 and P_6 is the precursor of P_7.
- If two ears P_i and $P_{i'}$ have the same precursor, then exactly one of the following situations arises for their nested intervals with respect to the common precursor:
  - **(a)** the nested intervals of P_i and $P_{i'}$ coincide. We say that P_i and $P_{i'}$ *overlap* each other. An example is illustrated in Figure 3 where P_3 and P_4 overlap each other;
  - **(b)** the nested interval of P_i strictly contains the one of $P_{i'}$. We say that $P_{i'}$ *is covered* by P_i which will be denoted by $P_{i'} \propto P_i$. For example in Figure 3, P_2, P_3 and P_4 are covered by P_6;
  - **(c)** the nested interval of $P_{i'}$ strictly contains the one of P_i, i.e., $P_i \propto P_{i'}$;

(d) the two nested intervals are disjoint. This is the case for P_2 and P_3 in the example of Figure 3.

Note that the relation $\propto$ is only defined for two ears in ED having the same precursor.
A directed two terminal graph is a directed graph with

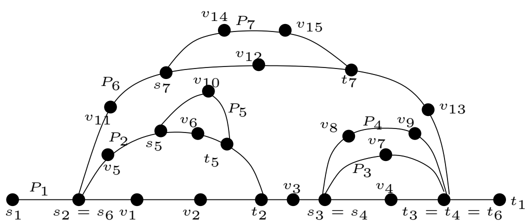

Figure 3: An open nested ear decomposition.

two specific vertices s and t such that there is a path from s to any vertex and from any vertex to t. An undirected graph is *two terminal series parallel* with terminals s and t if for some orientation of its edges it forms a directed two terminal series parallel graph with respect to these terminals. Moreover, an undirected graph is series parallel if for some two vertices s and t it is two terminal series parallel with those terminals. In Eppstein (1992), the author shows the following,

Theorem 2 *Eppstein (1992) Any simple undirected two terminal series parallel graph has an open nested ear decomposition starting with a path between the terminals. And any simple undirected graph with an open nested ear decomposition is two terminal series parallel with its terminals being the end-nodes of the first ear.*

A *biconnected graph* is either a 2-connected graph or a single edge. If G is a biconnected simple series parallel graph, one can find an open nested ear decomposition of G in logarithmic time (see Maon et al. (1986)). If G is not biconnected, it is easy to determine two nodes s and t in V such that the addition of the edge st into G makes G biconnected simple series parallel with s and t as the terminals (see Eppstein (1992)).
Let G be a simple biconnected series parallel graph and let $ED = \{P_1, \ldots, P_k\}$ be an open nested ear decomposition of G found by using for example the algorithm in Maon et al. (1986). As the definition of an open ear decomposition imposes only conditions on the indices of the ears regarding to the relation $\sqsubset$ but not to the relation $\propto$, without loss of generality, we can in addition impose an order on the indices of the ears regarding to the relation $\propto$ as follows.

> For any two ears P_i and $P_{i'}$ having the same precursor P_j, if $P_{i'} \propto P_i$ then $i' < i$ and for all j' such that $P_{j'} \sqsubseteq P_{i'}$, we have also $j' < i$. As an example, in Figure 3, we can take $i = 6$, $i' = 2$, $j = 1$ and $j' = 5$.

The labels s_i and t_i for the end-nodes of P_i where $1 < i \leq k$ are supposed to be assigned according to the following rule: let P_j be the precursor of P_i, if one follows the path P_j from s_j then one should meet s_i before t_i (see Figure 3).
An ear P_i such that there is no P_j which overlaps P_i, will be called *distinct*. When several ears mutually overlap, only the ear with smallest index will be called *distinct*. For instance, in Figure 3, P_6 is distinct and as P_3 and P_4 overlap each other, P_3 is distinct while P_4 is not.
For each ear P_i where $1 < i \leq k$ and P_j the (unique) precursor of P_i, we define *the base* of P_i as $B(P_i) = V(P_j(s_i - t_i)) \setminus \{s_i, t_i\}$, that is the set of the nodes in the nested interval of P_i with respect to P_j except the two end-nodes s_i and t_i. For exemple, let us consider the ear P_6 in Figure 3, its precursor is P_1 and $V(P_1(s_6 - t_6)) = \{s_6, v_1, v_2, t_2, v_3, s_3, v_4, t_6\}$. Hence, $B(P_6) = \{v_1, v_2, t_2, v_3, s_3, v_4\}$.

Remark 1 *Given $1 < h < i \leq k$, if P_h is covered by P_i, i.e., $P_h \propto P_i$, then, $B(P_h) \subset B(P_i)$.*

We also define the *unshared subbase* of P_i, $USB(P_i) = B(P_i) \setminus \bigcup_{P_h \propto P_i} B(P_h)$ which is the set of the nodes in $B(P_i)$ which do not belong to any other base $B(P_h)$ of some ear P_h covered by P_i. For example, in Figure 3, the unshared subbase of P_6, $USB(P_6) = \{t_2, v_3, s_3\}$ as v_1 and v_2 also belong to the base of P_2 and v_4 also belongs to the base of P_3. We can see that ears that overlap each other have the same unshared subbase. For example in Figure 3, $USB(P_3) = USB(P_4) = \{v_4\}$. Given two nodes

u and v belonging to $USB(P_i)$, we say that they are *consecutive* in $USB(P_i)$ if when we follow the sub-path $P_j(s_i - t_i)$ starting from s_i and count only the nodes in $USB(P_i)$, we meet u and v consecutively. For example, in Figure 3, t_2 and v_3 are consecutive in $USB(P_6)$.

Lemma 2 *Given any $1 \leq j < k$ and $v \in V(P_j)$, there is at most one unshared subbase $USB(P_i)$ that contains v, such that $j < i \leq k$ and P_i distinct.*

Proof: Suppose that there are two ears P_i and P_h with $1 \leq j < i < h \leq k$ that do not overlap (i.e., $USB(P_i) \neq USB(P_h)$) and $v \in USB(P_i) \cap USB(P_h)$. As $USB(P_i) \cap USB(P_h) \neq \emptyset$, we have $B(P_i) \cap B(P_h) \neq \emptyset$. This implies that the nested intervals of P_i and P_h with respect to theirs precursors are not disjoint. By the definition of nested ear decomposition above, P_i should be covered by P_h and consequently $B(P_i) \subset B(P_h)$. By the definition of unshared subbase, we have $USB(P_i) \cap USB(P_h) = \emptyset$ which is a contradiction. □

Let us build *the augmented graph* $G' = (V, E')$ of G where $E' = E \cup E^0$ contains the edges in E plus some additional edges constructed as follows.

At initialization $E^0 \leftarrow \emptyset$. For each ear P_i where $i > 1$, for each $v \in (V(P_i) \setminus \{s_i\})$, let us add $s_i v$ to E^0. If P_i is distinct then for each $v \in USB(P_i)$, let us add $s_i v$ to E^0.

Set $E' = E \cup E^0$.

Example 1. Let us consider the set E^0 built for the graph of Figure 3. For ear P_2, the edges s_2v_5, s_2s_5, s_2v_6, s_2t_5 and s_2t_2 are added to E^0. As P_2 is distinct and $USB(P_2) = \{v_1, v_2\}$, the edges s_2v_1 and s_2v_2 are added to E^0. For ear P_3, the edges s_3v_7 and s_3t_3 are added to E^0. AS $P3$ is distinct and $USB(P_3) = \{v_4\}$, the edge s_3v_4 is added to E^0. For ear P_4, the edges s_4v_8, s_4v_9 are added to E^0 (not s_4t_4 since it is the same edge as s_3t_3 added previously to E^0). As P_4 is not dinstinct, no edge s_4v with $v \in USB(P_4)$ is added to E^0. For ear P_5, the edges s_5v_{10} and s_5t_5 are added to E^0. As P_5 is distinct and $USB(P_5) = \{v_6\}$, the edge s_5v_6 is added to E^0. For ear P_6, the edges s_6v_{11}, s_6s_7, s_6v_{12}, s_6t_7, s_6v_13 and s_6t_6 are added to E^0. As P_6 is distinct and $USB(P_6) = \{t_2, v_3, s_3\}$, the edges s_6t_2, s_6v_3 and s_6s_3 are added to E^0. For ear P_7, the edges s_7v_14, s_7v_15 and s_7t_7 are added to E^0. As P_7 is distinct and $USB(P_7) = \{v_{12}\}$, the edge s_7v_{12} is added to E^0. Notice that E and E_0 are not disjoint sets.

Remark 2 *The number of additional edges $|E^0|$ is at most $2n$ and $|E'| \in O(n)$.*

Proof: The first part of the remark straightforwardly follows from the fact that each node $v \in V$ belong to exactly one ear and to at most one unshared base associated with a distinct ear. The second part is derived from the first part and the fact that $|E'| = |E| + |E^0|$ and $|E| \leq 3n - 6$ as G is planar. □

Remark 3 *The augmented graph G' remains series-parallel.*

Proof: For each P_i with $1 < i \leq k$ and P_j its precursor, one can consider the additional edges $s_i v$ where $v \in V(P_i)$ or $v \in V(P_i) \cup USB(P_i)$ if P_i is distinct as additional ears that one can easily insert in ED. More precisely, if P_i is distinct, we insert before P_i the edges $s_i v$ for all $v \in USB(P_i)$ with respect to the order of increasing distance (in terms of number of edges) from s_i to v in P_j. Then we insert after P_i in the sequence ED the edges $s_i v$ for all $v \in V(P_i)$ with respect to the order of increasing distance from s_i to v in P_i . The final obtained sequence represents an open nested ear decomposition for G'. Hence, by Theorem 2, G' is a series-parallel graph. □

Lemma 3 *Given $1 < i \leq k$ and two nodes v and w belonging to $USB(P_i)$ then there is an edge vw in E' if and only if v and w are consecutive in $USB(P_i)$.*

Proof: $\Leftarrow$ Suppose that v and w are consecutive in $USB(P_i)$ and let P_j be the precursor of P_i. By the definition of $USB(P_i)$, there are two possible cases.

- v and w are also consecutive when going from s_i to t_i through $P_j(s_i - t_i)$. This implies that the edge vw belongs to E and also to E'.

- v and w are not consecutive when going from s_i to t_i through $P_j(s_i - t_i)$. By the definition of $USB(P_i)$, v and w should be the end-nodes of some P_h covered by P_i. We can see that in this case there is an edge vw in E^0. Thus there is an edge vw in E'.

$\Rightarrow$ Suppose that there is an edge $vw \in E'$ and v and w are not consecutive in $USB(P_i)$. By the definition of $USB(P_i)$, v and w are not consecutive in $B(P_i)$, i.e., there is no edge vw in E.
Suppose that there is an edge $vw \in E^0$, then v should be a node s_h with $1 < h < i \leq k$ of some ear P_h covered by P_i. As $w \in USB(P_i)$, by Lemma 2, we have $w \notin USB(P_h)$. Hence, the only case for an edge vw to exist in E^0 is $w = t_h$. But in this case v and w are consecutive in $USB(P_i)$, contradicting the assumption. □
Let $\mathcal{T}''$ be the set of triples $u, v, w \in V$ such that there exists some $1 < i \leq k$ such that $u = s_i$ and the nodes v and w satisfy one of the following conditions.

- vw is an edge of P_i (triple of Type 1).
- v and w are consecutive in $USB(P_i)$ and P_i is distinct (triple of Type 2). Note that by Lemma 3, there exists an edge vw in E'.
- v is the end-node s_j of some distinct ear P_j where $j > i$ such that $s_j \in V(P_i)$ and $w \in V(P_i) \cap USB(P_j)$ (triple Type 3).

Example 2. Let us consider the case $i = 6$ in Figure 3, then $u = s_6$. If we set $v = s_7$ and $w = v_{12}$ then the triple s_6, s_7, v_{12} is both of Type 1 and Type 3 in $\mathcal{T}''$. If we set $v = t_2$ and $w = v_3$ then the triple s_6, t_2, v_3 is of Type 2 in $\mathcal{T}''$.

Lemma 4 *The triples in $\mathcal{T}''$ form all the triangles in G'.*

Proof: Given $T = (u, v, w)$ any triangle in G', by construction, at least one node in T should be the node s_i for some $i > 1$. Suppose that $u = s_i$. We have the two following possible cases.

- $u = s_i$ is the unique s-node in T (we call *s-node*, a node s_i with $1 \leq i \leq k$) . In this case, since every edge in E^0 should have at least one s-node as end-node, vw should be an edge of E.
  - If vw is an edge of P_i then, T is a triple of Type 1 in $\mathcal{T}''$.
  - If vw is an edge of the precursor P_j of P_i and $v, w \in USB(P_i)$ then, v and w should be consecutive in $USB(P_i)$. As the edges sv and sw exist, P_i should be distinct. Thus, T is a triple of Type 2 in $\mathcal{T}''$.
- There are at least two s-nodes in T, $u = s_i$ and $v = s_j$. Suppose without loss of generality, that $1 \leq i < j \leq k$. As the edge $s_i s_j$ exists in E', either $s_j \in V(P_i)$ or $s_j \in USB(P_i)$. But this is impossible since $j > i$. Thus $s_j \in V(P_i)$.
  - If $w \in V(P_i)$, then as the edge $s_j w$ exists, w should be in $USB(P_j)$ and P_j is distinct. The node w indeed belongs to $V(P_i) \cap USB(P_j)$. Hence T is a triple of Type 3 in $\mathcal{T}''$.
  - If $w \in USB(P_i)$, then as the edge $s_j w$ exists, we have
    * either $w \in USB(P_j)$ and from Lemma 2, we obtain that $USB(P_i) = USB(P_j)$ and $s_i = s_j$ which contradicts the fact that T is a triangle.
    * or $w \in V(P_j)$, which implies that P_j is the precursor of P_i and $j < i$. But this contradicts the fact that $j > i$.

□

Let us define $\mathrm{SPMETP}(G')$ as the polytope defined by the following system,

$$\begin{aligned}
&x_{uv} + x_{uw} + x_{vw} \leq 2 \text{ for all } u, v, w \in \mathcal{T}'',\\
&x_{uv} - x_{uw} - x_{vw} \leq 0,\\
&x_{uv} - x_{uw} - x_{vw} \leq 0,\\
&x_{uv} - x_{uw} - x_{vw} \leq 0 \text{ for all } u, v, w \in \mathcal{T}'',
\end{aligned}$$

together with the nonnegativity and trivial inequalities (2) for the edges that do not belong to any triangle in $\mathcal{T}''$. We define the cone $\mathrm{SPMET}(G')$ as the one defined by homogeneous inequalities in $\mathrm{SPMETP}(G')$, the nonnegativity for the edges that do not belong to any triangle in $\mathcal{T}''$ and the trivial inequalities (2) for all the edges in G'.

Remark 4 *The number of variables and the number of inequalities in* $\mathrm{SPMETP}(G')$ *(respectively* $\mathrm{SPMET}(G')$*) is in* $O(n)$.

Proof: The number of variables in $\mathrm{SPMETP}(G')$ (respectively $\mathrm{SPMET}(G')$) is equal to $|E'|$ and hence by Remark 2, it is in $O(n)$. The number of triples of Type 1 in $\mathcal{T}''$ is at most the number of edges in E and hence, it is in $O(n)$. As for each edge vw in E, there is at most one distinct P_i such that v, w are consecutive in $USB(P_i)$, the number of triples of Type 2 in $\mathcal{T}''$ is also at most $|E|$ and hence, it is in $O(n)$. As every node $w \in V$ belongs to at most one unshared base associated with a distinct ear, the number of triples of Type 3 in $\mathcal{T}''$ is at most n. Hence the total number of triples in $\mathcal{T}''$ is in $O(n)$. Consequently, the number of inequalities in $\mathrm{SPMETP}(G')$ (respectively $\mathrm{SPMET}(G')$) is in $O(n)$. □

Let $\mathrm{SPMETP}(G')_G$, (respectively, $\mathrm{SPMET}(G')_G$) be the projection of $\mathrm{SPMETP}(G')$, (respectively, $\mathrm{SPMET}(G')$) on $\mathbb{R}^E$.

Theorem 3 *Let G be a series-parallel graph and G' the corresponding augmented graph. Then* $\mathrm{SPMETP}(G')_G = \mathrm{METP}(G)$*, (respectively,* $\mathrm{SPMET}(G')_G = MET(G)$*).*

Proof: Let $\mathrm{METP}(G')$ be the metric polytope defined on $\mathbb{R}^{E'}$ and $\mathrm{METP}(G')_G$ its projection on $\mathbb{R}^E$.

Note that $\mathrm{METP}(G')_G = \mathrm{METP}(G)$ follows from results in Barahona (1993). Hence to prove the theorem, we need to show that $\mathrm{METP}(G') = \mathrm{SPMETP}(G')$.

We have obviously $\mathrm{METP}(G') \subseteq \mathrm{SPMETP}(G')$. To show $\mathrm{SPMETP}(G') \subseteq \mathrm{METP}(G')$, given any $x' \in \mathrm{SPMETP}(G')$, we shall show that the only chordless elementary cycles in G' are the triangles in $\mathcal{T}''$ which will imply that $x' \in \mathrm{METP}(G')$. We will prove this by recurrence on the number k of the ears of G.

Let us consider first the case when $k = 2$, i.e., $ED = \{P_1, P_2\}$. In this case, the only cycle in G is the cycle C formed by P_2 and the subpath $P_1(s_2 - t_2)$. The added edges in E^0 make a pointed triangulation of C at the node s_2. Hence, we can see that the only chordless cycles in G' are the triangles in $\mathcal{T}''$.

Now suppose that we have $\mathrm{METP}(G') = \mathrm{SPMETP}(G')$ for the graphs G having an open nested ear decomposition of cardinality $k-1$ with $k \geq 3$. Let us show that for the graphs G having an open nested ear decomposition $ED = \{P_1, \ldots, P_k\}$ of cardinality k, the only chordless elementary cycles in the augmented graph G' are the triangles in $\mathcal{T}''$.

Let us consider P_k. Let $E_k^0 \subset E^0$ be the subset of the edges in E^0 added to G due to P_k. We can see that E_k^0 contains the edges $s_k v$ for all $v \in V(P_k) \cup USB(P_k)$.

Let G^{k-1} denote the subgraph of G induced by the first $k-1$ ears $P_1, \ldots, P_{k-1}$. Obviously, G^{k-1} is a series-parallel graph having an open nested ear decomposition of cardinality $k-1$. Hence, any elementary cycle $C' \notin \mathcal{T}''$ in G' containing no edge in $E_k^0 \cup P_k$ is a cycle in the augmented graph of G^{k-1} (which is a subgraph of G'). By induction hypothesis, C' has a chord.

Suppose now that $C' \notin \mathcal{T}''$ is any elementary cycle in G' that contains some edge in $E_k^0 \cup P_k$. We will show that C' has a chord. As $C' \notin \mathcal{T}''$ and by Lemma 4, C' should be of length at least 4. As there is no ear P_i having the base in P_k, if C' contains some edge in P_k then C' should go through the node s_k. Hence, in all the cases, the cycle C' should contain s_k.

- Suppose that C' contains some edge $s_k v$ with $v \in V(P_k) \setminus \{t_k\}$. In this case, from a node $v \in V(P_k) \setminus \{t_k\}$, C' can only go to another node $w \in V(P_k)$ and C' has a chord which is the edge $s_k w$.

- Suppose that C' contains no edge $s_k v$ with $v \in V(P_k) \setminus \{t_k\}$. In this case, P_k should be distinct and

  - either C' contains two edges $s_k v$ and $s_k w$ with v and w both belong to $USB(P_k) \cup \{t_k\}$. Let P_j be the precursor of P_k, then $v, w \in V(P_j)$. As C' is of length at least 4, C' does not contain the edge vw if the latter exists. If C' contains some other node $u \in (USB(P_k) \setminus \{v, w\}) \cup \{t_k\}$, then C' has a chord which is the edge $s_k u$. Thus suppose that C' contains no node $u \in USB(P_k) \setminus \{v, w\}$, we can see that the cycle C obtained by replacing in C' the two edges $s_k v$ and $s_k w$ by the subpath $P_j(v-w)$ is an elementary cycle in the augmented graph of G^{k-1}. Let $P_{v,w}$ be the other half of C which forms C with $P_j(v-w)$. By induction hypothesis, C should have a chord. If both v and w are different from t_k, by Lemma 3, this chord cannot be a chord in $P_j(v-w)$. If one of these nodes is equal to t_k, say w, it is easy to see by the definition of $USB(P_k)$ that the subpath $P_j(v-t_k) = P_j(v-w)$ should be chordless. Hence the chord in C should be a chord in $P_{v,w}$. As C' is composed by the two edges $s_k v$ and $s_k w$ and $P_{v,w}$, we conclude that C' has a chord.

- or C' contains exactly one edge $s_k v$ such that $v \in USB(P_k) \cup \{t_k\}$. This case can be handled similarly to the previous case by considering the sequence $v_1, \ldots, v_h, v$ of consecutive nodes in $USB(P_k)$ counted from s_k and by remarking that the path $s_k v_1 \ldots v_h v$ is a chordless path in the augmented graph of G^{k-1}.

Thus the only chordless cycles in G' are the triangles in $\mathcal{T}''$. Hence, $\mathrm{METP}(G') = \mathrm{SPMETP}(G')$.
From $\mathrm{METP}(G')_G = \mathrm{METP}(G)$ and $\mathrm{METP}(G') = \mathrm{SPMETP}(G')$, we conclude that $\mathrm{SPMETP}(G')_G = \mathrm{METP}(G)$. The proof for $\mathrm{SPMET}(G')_G = MET(G)$ is similar. □

4 Conclusions

In this paper, improved compact formulations featuring $O(n^2)$ variables and $O(nm)$ constraints for metric and cut polyhedra for general undirected graphs of n nodes and m edges have been proposed. This is particularly interesting in the case of sparse graphs where $m = O(n)$ leading to quadratic size formulations with $O(n^2)$ variables *and* constraints in contrast with the $O(n^2)$ variables and $O(n^3)$ constraints of standard compact formulations. Our technique of proof has also been shown to open the way to further possible improvements when considering special subclasses of sparse graphs. As a first step in this direction, we have investigated the case of series-parallel graphs for which the max-cut problem is known to be polynomial-time solvable. For the slightly more general subclass of graphs exhibited in Barahona (1986) for which max-cut is solvable in linear time, an interesting open research question raised by our result in Section 3 would be to investigate whether a linear-size representation of the metric polyhedra is still possible. Moreover, since series-parallel graphs form a subclass of planar graphs, our result in section 3 raises the question of exhibiting more special cases of planar graphs admitting linear-size representable metric polyedra. This is left for future research.
As we have mentioned in the Introduction section, $\mathrm{MET}(G)$ can be used as relaxation for graph partitioning. In Nguyen et al. (2016), we have shown that the reduction proposed in Section 2 is applied to graph partitioning problems with some generic additional constraints. For the latter, it will be interesting to see if we can have a reduction of linear size when G is series-parallel, i.e., a similar result as in Section 3.

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