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FINITE ELEMENT METHOD FOR DARCY'S PROBLEM COUPLED WITH THE HEAT EQUATION

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Abstract. In this article, we study theoretically and numerically the heat equation coupled with Darcy's law by a nonlinear viscosity depending on the temperature. We establish existence and uniqueness of the exact solution by using a Galerkin method. We propose and analyze two numerical schemes based on finite element methods. An optimal *a priori* error estimate is then derived for each numerical scheme. Numerical experiments are presented that confirm the theoretical accuracy of the discretization.

Keywords. Darcy's equations; heat equation; Stampacchia's method; finite element method; *a priori* error estimates.

1. Introduction.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded simply-connected open domain in $\mathbb{R}^d$, with a Lipschitz-continuous boundary Γ. This work studies the temperature distribution of a fluid in a porous medium modelled by a convection-diffusion equation coupled with Darcy's law. The system of equations is

$$(\mathrm{P}) \begin{cases} \nu(T(\mathbf{x}))\mathbf{u}(\mathbf{x}) + \nabla p(\mathbf{x}) & = & \mathbf{f}(\mathbf{x}) & \text{in}\,\Omega, \\ (\operatorname{div}\mathbf{u})(\mathbf{x}) & = & 0 & \text{in}\,\Omega, \\ -\alpha\Delta T(\mathbf{x}) + (\mathbf{u}\cdot\nabla T)(\mathbf{x}) & = & g(\mathbf{x}) & \text{in}\,\Omega, \\ (\mathbf{u}\cdot\mathbf{n})(\mathbf{x}) & = & 0 & \text{on}\,\Gamma, \\ T(\mathbf{x}) & = & 0 & \text{on}\,\Gamma, \end{cases}$$

where $\mathbf{n}$ is the unit outward normal vector on Γ. The unknowns are the velocity $\mathbf{u}$, the pressure p and the temperature T of the fluid. The function $\mathbf{f}$ represents an external density force and g an external heat source. The viscosity ν depends on the temperature (Hooman and Gurgenci [15] or Rashad [17]) while the parameter α is a positive constant that corresponds to the diffusion coefficient.

We analyze the system (P) by setting it in an equivalent variational formulation and reducing it to a single diffusion-convection equation for the temperature where the driving velocity depends implicitly on the temperature, see (2.19)–(2.20). Existence of a solution is derived without restriction on the data by Galerkin's method and Brouwer's Fixed Point. Global uniqueness is established when the solution is slightly smoother and the data are suitably restricted. We also introduce an alternative equivalent variational formulation. Both variational formulations are discretized by finite element schemes. We derive existence, conditional uniqueness, convergence, and optimal *a priori* error estimates for the solutions of both schemes. Next, these schemes are linearized by suitable convergent successive approximation algorithms. Finally, we present some numerical experiments for a model problem that confirm the theoretical rates of convergence developed in this work.

The study of heat convection in a liquid medium whose motion is described by the Navier-Stokes equations coupled with the heat equation has been the object of many publications (see, for instance Bernardi,

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Métivet and Pernaud-Thomas [4], Deteix, Jendoubi and Yakoubi [10], or Gaultier and Lezaun [11]). A different coupling Darcy's system with the heat equation with constant viscosity but exterior force depending on the temperature has been analyzed by (Bernardi, Yacoubi and Maarouf [5] or Boussinesq [6]) and discretized with a spectral method. The present work can easily be modified to also take into account the dependency of the external force on the temperature.

This article is organized as follows:

- Section 2 is devoted to the continuous problem and the analysis of the corresponding variational formulation.
- In section 3, we introduce the discrete problems, recall their main properties, study their *a priori* errors and derive optimal estimates.
- In section 4, we introduce an iterative algorithm and prove its convergence.
- Numerical results validating the numerical analysis are presented in Section 5.

2. Analysis of the model

2.1. **Notation.** Let Ω be a bounded open domain of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz-continuous boundary Γ, and unit outward normal $\mathbf{n}$. To simplify, we define the notions below in three dimensions. We denote by $\mathcal{D}(\Omega)$ the space of functions that have compact support in Ω and have continuous derivatives of all orders in Ω. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a triple of non negative integers, set $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and define the partial derivative ∂^α by

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}. \tag{2.1}$$

Then, for any positive integer m and number $p \geq 1$, recall the classical Sobolev space (Adams [2] or Nečas [16])

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega);\, \forall\, |\alpha| \leq m,\ \partial^\alpha v \in L^p(\Omega)\}, \tag{2.2}$$

equipped with the seminorm

$$|v|_{W^{m,p}(\Omega)} = \big\{ \sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v(\mathbf{x})|^p \, d\mathbf{x} \,\big\}^{\frac{1}{p}} \tag{2.3}$$

and the norm

$$\|v\|_{W^{m,p}(\Omega)} = \big\{ \sum_{0\leq k\leq m} |v|^p_{W^{k,p}(\Omega)} \big\}^{\frac{1}{p}}. \tag{2.4}$$

When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. In particular, the scalar product of $L^2(\Omega)$ is denoted by $(.,.)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\mathbf{v}$ be a vector valued function; we set

$$\|\mathbf{v}\|_{L^p(\Omega)} = \big(\int_\Omega |\mathbf{v}(\mathbf{x})|^p \, d\mathbf{x} \big)^{\frac{1}{p}}, \tag{2.5}$$

where $|.|$ denotes the Euclidean vector norm.
For vanishing boundary values, we define

$$H^1_0(\Omega) \quad = \quad \{v \in H^1(\Omega);\, v_{|\Gamma} = 0\}. \tag{2.6}$$

We shall often use Sobolev's imbeddings: for any real number $p \geq 1$, there exists constants S_p and S^0_p such that

$$\forall\, v \in H^1(\Omega),\ \|v\|_{L^p(\Omega)} \leq S_p \|v\|_{H^1(\Omega)} \tag{2.7}$$

and

$$\forall\, v \in H^1_0(\Omega),\ \|v\|_{L^p(\Omega)} \leq S^0_p |v|_{H^1(\Omega)}. \tag{2.8}$$

When $p = 2$, (2.8) reduces to Poincaré's inequality.
We shall also use the standard spaces for Darcy's equations

$$L^2_0(\Omega) \quad = \quad \{v \in L^2(\Omega);\, \int_\Omega v(\mathbf{x}) \, d\mathbf{x} = 0\}, \tag{2.9}$$

$$H(\mathrm{div},\Omega) = \{\mathbf{v} \in (L^2(\Omega))^d;\ \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}, \tag{2.10}$$

$$H_0(\mathrm{div},\Omega) = \{\mathbf{v} \in H(\mathrm{div},\Omega);\ (\mathbf{v}\cdot\mathbf{n})|_\Gamma = 0\}, \tag{2.11}$$

equipped with the norm

$$\|\mathbf{v}\|^2_{H(\mathrm{div},\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)^d} + \|\mathrm{div}\,\mathbf{v}\|^2_{L^2(\Omega)}, \tag{2.12}$$

and also the space

$$\mathcal{V} = \{\mathbf{v} \in H_0(\mathrm{div},\Omega);\ \mathrm{div}\,\mathbf{v} = 0\}. \tag{2.13}$$

Finally, we recall the inf-sup condition between $L^2_0(\Omega)$ and $H_0(\mathrm{div},\Omega)$,

$$\inf_{q\in L^2_0(\Omega)} \sup_{\mathbf{v}\in H_0(\mathrm{div},\Omega)} \frac{\displaystyle\int_\Omega (\mathrm{div}\,\mathbf{v})(\mathbf{x})q(\mathbf{x})\,d\mathbf{x}}{\|\mathbf{v}\|_{H(\mathrm{div},\Omega)}\|q\|_{L^2(\Omega)}} \geq \beta, \tag{2.14}$$

with a constant $\beta > 0$, and the inf-sup condition between $H^1(\Omega)\cap L^2_0(\Omega)$ and $L^2(\Omega)^d$,

$$\inf_{q\in H^1(\Omega)\cap L^2_0(\Omega)} \sup_{\mathbf{v}\in L^2(\Omega)^d} \frac{\displaystyle\int_\Omega \mathbf{v}(\mathbf{x}).\nabla q(\mathbf{x})\,d\mathbf{x}}{\|\mathbf{v}\|_{L^2(\Omega)^d}|q|_{H^1(\Omega)}} \geq 1. \tag{2.15}$$

The first one follows immediately by solving a Laplace equation in Ω with a Neumann boundary condition on Γ, and the second by choosing $\mathbf{v} = \nabla q$.

2.2. **Variational formulation.** Before setting (P) in variational form, let us make precise the assumptions on the function ν

- ν is Lipschitz-continuous with Lipschitz constant λ, i.e.,

$$\forall s,t \in \mathbb{R},\quad |\nu(s) - \nu(t)| \leq \lambda|s-t|. \tag{2.16}$$

- ν is bounded and there exist two positive constants ν_1 and ν_2 such that for any $\tau \in \mathbb{R}$

$$\nu_1 \leq \nu(\tau) \leq \nu_2. \tag{2.17}$$

In many publications, the model used for the viscosity function $\nu(\cdot)$ is not necessarily bounded over $\mathbb{R}$, but then the mathematical analysis of the problem is much more complex. However, since in practical situations, $\nu(T)$ is neither infinite nor zero, we prefer to assume (2.17); this substantially simplifies the analysis. The other assumptions on the data are, $\mathbf{f} \in L^2(\Omega)^d$ and $g \in L^2(\Omega)$. With these assumptions and data, the space for Darcy's velocity and pressure $(\mathbf{u}, p)$ is $H_0(\mathrm{div},\Omega)\times L^2_0(\Omega)$ or $H_0(\mathrm{div},\Omega)\times H^1(\Omega)\cap L^2_0(\Omega)$ and for the temperature T is $H^1_0(\Omega)$. Then, whereas there is no difficulty in setting Darcy's system in variational form, a variational formulation of the temperature equation is not that obvious. Indeed, the convection term $\mathbf{u}\cdot\nabla T$ cannot be tested by an H^1 function, since it is only in $L^1(\Omega)$. Of course, it can be observed that the temperature equation implies necessarily that this product belongs to $H^{-1}(\Omega)$, meaning in fact that T belongs to the weighted space

$$H_{\mathbf{u}} = \{S \in H^1_0(\Omega)\,;\, \mathbf{u}\cdot\nabla S \in H^{-1}(\Omega)\}. \tag{2.18}$$

However, for the moment, it is simpler to set aside this space and choose instead the test functions in $H^1_0(\Omega)\cap L^\infty(\Omega)$. Thus, we propose the following variational problem:

$$(V)\begin{cases} \text{Find } (\mathbf{u}, p, T) \in H_0(\mathrm{div},\Omega)\times L^2_0(\Omega)\times H^1_0(\Omega) \text{ such that} \\ \forall\,\mathbf{v} \in H_0(\mathrm{div},\Omega),\ \displaystyle\int_\Omega \nu(T(\mathbf{x}))\mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,d\mathbf{x} - \int_\Omega p(\mathbf{x})(\mathrm{div}\,\mathbf{v})(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,d\mathbf{x}, \\ \forall\,q \in L^2_0(\Omega),\ \displaystyle\int_\Omega q(\mathbf{x})(\mathrm{div}\,\mathbf{u})(\mathbf{x})\,d\mathbf{x} = 0, \\ \forall\,S \in H^1_0(\Omega)\cap L^\infty(\Omega),\ \alpha\displaystyle\int_\Omega \nabla T(\mathbf{x})\cdot\nabla S(\mathbf{x})\,d\mathbf{x} + \int_\Omega (\mathbf{u}\cdot\nabla T)(\mathbf{x})\,S(\mathbf{x})\,d\mathbf{x} = \int_\Omega g(\mathbf{x})S(\mathbf{x})\,d\mathbf{x}. \end{cases}$$

A straightforward argument shows that this problem is equivalent to the original problem (P) in the sense that any solution $(\mathbf{u}, p, T)$ of problem (P) in $H_0(\mathrm{div},\Omega)\times L^2_0(\Omega)\times H^1_0(\Omega)$ solves (V) and conversely.

When there is no ambiguity, to simplify, from now on, we denote the scalar products by parentheses.

Problem (V) can also be written as a function of the single unknown T. Indeed, for given T, the Darcy system has a unique solution $(\mathbf{u}, p)$; this is easily deduced from (2.17) and the inf-sup condition (2.14). Thus $\mathbf{u}$ and p are functions of T, $(\mathbf{u}, p) = (\mathbf{u}(T), p(T))$, and problem (V) is equivalent to the following reduced formulation: Find T in $H_0^1(\Omega)$, such that

$$\forall S \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \alpha(\nabla T, \nabla S) + \int_\Omega (\mathbf{u}(T) \cdot \nabla T)(\mathbf{x})\, S(\mathbf{x})\, d\mathbf{x} = (g, S), \tag{2.19}$$

where $\mathbf{u}(T)$ is the velocity solution of: Find $(\mathbf{u}(T), p(T)) \in H_0(\mathrm{div}, \Omega) \times L_0^2(\Omega)$, such that

$$\begin{aligned} \forall \mathbf{v} \in H_0(\mathrm{div}, \Omega), \quad (\nu(T)\mathbf{u}(T), \mathbf{v}) - (p(T), \mathrm{div}\, \mathbf{v}) &= (\mathbf{f}, \mathbf{v}), \\ \forall q \in L_0^2(\Omega), \quad (q, \mathrm{div}\, \mathbf{u}(T)) &= 0. \end{aligned} \tag{2.20}$$

By testing (2.20) with $\mathbf{v} = \mathbf{u}(T)$, we immediately derive from (2.17) and (2.14) the a priori bounds,

$$\begin{aligned} \|\mathbf{u}(T)\|_{L^2(\Omega)^d} \le \frac{1}{\nu_1} \|\mathbf{f}\|_{L^2(\Omega)^d}\ , \ \|\sqrt{\nu(T)}\mathbf{u}(T)\|_{L^2(\Omega)} \le \frac{1}{\sqrt{\nu_1}} \|\mathbf{f}\|_{L^2(\Omega)^d}, \\ \|p(T)\|_{L^2(\Omega)} \le \frac{1}{\beta} \big(\|\mathbf{f}\|_{L^2(\Omega)^d} + \nu_2 \|\mathbf{u}(T)\|_{L^2(\Omega)^d}\big). \end{aligned} \tag{2.21}$$

These bounds imply the following continuity:

Lemma 2.1. *Let ν satisfy (2.16), (2.17) and $(T_k)_{k\ge1}$ be a sequence of functions in $L^2(\Omega)$ that converges strongly to T in $L^2(\Omega)$. Then, the sequence $(\mathbf{u}(T_k), p(T_k))_{k\ge1}$ converges weakly to $(\mathbf{u}(T), p(T))$ in $H_0(\mathrm{div}, \Omega) \times L_0^2(\Omega)$ and*

$$\begin{aligned} \lim_{k\to\infty} \sqrt{\nu(T_k)}\mathbf{u}(T_k) &= \sqrt{\nu(T)}\mathbf{u}(T) \quad \text{strongly in } L^2(\Omega)^d, \\ \lim_{k\to\infty} p(T_k) &= p(T) \quad \text{strongly in } L^2(\Omega). \end{aligned} \tag{2.22}$$

Proof. The bounds (2.21) yield first the weak convergence (up to a subsequence) of $(\mathbf{u}(T_k), p(T_k))_{k\ge1}$ in $L^2(\Omega)^d \times L^2(\Omega)$ to some function $(\mathbf{u}, p)$, and next that $(\mathbf{u}, p)$ belong to $H_0(\mathrm{div}, \Omega) \times L_0^2(\Omega)$. Since T_k converges almost everywhere and $\nu(T_k) \le \nu_2$, the Lebesgue dominated convergence implies that for any real number $r > 0$

$$\lim_{k\to\infty} \nu(T_k) = \nu(T) \quad \text{strongly in } L^r(\Omega). \tag{2.23}$$

By a standard argument, this allows to pass to the limit in (2.20) with T_k instead of T and smooth test functions, thus showing that T solves (2.20). Hence $\mathbf{u} = \mathbf{u}(T)$ and $p = p(T)$.

As far as the strong convergences are concerned, first the weighted bound for the velocity in (2.21) implies that, again up to a subsequence, $(\sqrt{\nu(T_k)}\mathbf{u}(T_k))_{k\ge1}$ tends weakly to some function $\mathbf{w}$ in $L^2(\Omega)^d$ and a standard argument shows that $\mathbf{w} = \sqrt{\nu(T)}\mathbf{u}(T)$. Next, by testing (2.20) (written with T_k instead of T) with $\mathbf{v} = \mathbf{u}(T_k)$, we obtain

$$\|\sqrt{\nu(T_k)}\mathbf{u}(T_k)\|^2_{L^2(\Omega)^d} = (\mathbf{f}, \mathbf{u}(T_k)) = (\nu(T)\mathbf{u}(T), \mathbf{u}(T_k)).$$

Hence,

$$\lim_{k\to\infty} \|\sqrt{\nu(T_k)}\mathbf{u}(T_k)\|^2_{L^2(\Omega)^d} = \|\sqrt{\nu(T)}\mathbf{u}(T)\|^2_{L^2(\Omega)^d}, \tag{2.24}$$

thus implying the strong weighted convergence of the velocity. Regarding the pressure, owing to (2.14), for each k there exists a function $\mathbf{v}_k$ in $H_0(\mathrm{div}, \Omega)$ such that (see Girault and Raviart [13])

$$\mathrm{div}\, \mathbf{v}_k = p(T_k) \quad \text{and} \quad \|\mathbf{v}_k\|_{H(\mathrm{div},\Omega)} \le \frac{1}{\beta} \|p(T_k)\|_{L^2(\Omega)}. \tag{2.25}$$

The bound (2.25) yields weak convergence (up to a subsequence) of $(\mathbf{v}_k)_{k\ge1}$ in $H(\mathrm{div}, \Omega)$ to some function $\mathbf{v}$ in $H_0(\mathrm{div}, \Omega)$ with $\mathrm{div}\, \mathbf{v} = p(T)$, and by testing (2.20) (written with T_k instead of T) with $\mathbf{v} = \mathbf{v}_k$, we derive

$$\begin{aligned} \|p(T_k)\|^2_{L^2(\Omega)} = (p(T_k), \mathrm{div}\, \mathbf{v}_k) &= -(\mathbf{f}, \mathbf{v}_k) + (\nu(T_k)\mathbf{u}(T_k), \mathbf{v}_k) \\ &= (p(T), \mathrm{div}\, \mathbf{v}_k) - (\nu(T)\mathbf{u}(T), \mathbf{v}_k) + (\nu(T_k)\mathbf{u}(T_k), \mathbf{v}_k). \end{aligned}$$

For passing to the limit in the nonlinear term, we write $(\nu(T_k)\mathbf{u}(T_k), \mathbf{v}_k) = (\sqrt{\nu(T_k)}\mathbf{u}(T_k), \sqrt{\nu(T_k)}\mathbf{v}_k)$. In view of (2.17) and (2.25), the last factor is bounded in $L^2(\Omega)^d$ and hence (up to a subsequence) converges weakly to some function $\mathbf{w}$ in $L^2(\Omega)^d$. As above, an easy argument shows that $\mathbf{w} = \sqrt{\nu(T)}\mathbf{v}$. This permits to take the limit of the nonlinear term, leading to

$$\lim_{k\to\infty} \|p(T_k)\|^2_{L^2(\Omega)} = \|p(T)\|^2_{L^2(\Omega)}, \tag{2.26}$$

and to the strong convergence of $p(T_k)$. Finally, uniqueness of the solution of (2.20) implies the convergence of the whole sequence. □

2.3. **Existence.** Here, we propose to construct a solution of (2.19) by Galerkin's method. Since $H^2(\Omega)$ is separable, so is its closed subspace $H^2(\Omega) \cap H^1_0(\Omega)$; therefore, it has a countable basis $\{\theta_i\}_{i\geq 1}$. Let Θ_m be the space spanned by the first m basis functions, $\{\theta_i\}_{1\leq i\leq m}$. The reduced problem (2.19) is discretized in Θ_m by the square system of nonlinear equations: *Find* $T_m = \sum_{1\leq i\leq m} w_i\theta_i \in \Theta_m$, *solution of*

$$\forall 1 \leq i \leq m, \quad \alpha(\nabla T_m, \nabla \theta_i) + \int_\Omega (\mathbf{u}(T_m)\cdot\nabla T_m)(\mathbf{x})\,\theta_i(\mathbf{x})\,d\mathbf{x} = \ (g,\theta_i), \tag{2.27}$$

where the pair $(\mathbf{u}(T_m), p(T_m))$ solves (2.20) with $T = T_m$. Then, given T_m in Θ_m, we introduce the auxiliary problem, *find* $\Phi(T_m) \in \Theta_m$ *such that,*

$$\forall S_m \in \Theta_m, \quad (\nabla\Phi(T_m), \nabla S_m) = \alpha(\nabla T_m, \nabla S_m) + \int_\Omega (\mathbf{u}(T_m)\cdot\nabla T_m)(\mathbf{x})\,S_m(\mathbf{x})\,d\mathbf{x} - (g, S_m). \tag{2.28}$$

On one hand, (2.28) defines a mapping from Θ_m into Θ_m, and we easily derive its continuity from the finite dimension and the continuity Lemma 2.1. On the other hand, Green's formula (valid because the basis functions are smooth) gives,

$$(\nabla\Phi(T_m), \nabla T_m) = \alpha|T_m|^2_{H^1(\Omega)} - (g, T_m) \geq |T_m|_{H^1(\Omega)}\big(\alpha|T_m|_{H^1(\Omega)} - S_2^0\|g\|_{L^2(\Omega)}\big). \tag{2.29}$$

Therefore Brouwer's Fixed-Point Theorem implies immediately the next result.

Lemma 2.2. *The discrete problem* (2.27) *has at least one solution* $T_m \in \Theta_m$ *and this solution satisfies the bound*

$$|T_m|_{H^1(\Omega)} \leq \frac{S_2^0}{\alpha}\|g\|_{L^2(\Omega)}. \tag{2.30}$$

Existence of a solution of (2.19) stems from Lemmas 2.1 and 2.2.

Theorem 2.3. *Let ν satisfy (2.16) and (2.17). Then for any $\mathbf{f} \in L^2(\Omega)^d$, $g \in L^2(\Omega)$, and positive constant α, problem* (2.19) *has at least one solution $T \in H^1_0(\Omega)$ and this solution satisfies the bound* (2.30)*.*

Proof. To simplify the discussion, the proof is written when $d = 3$; it is simpler when $d = 2$. The uniform bound (2.30) implies that, up to a subsequence, $(T_m)_m$ converges weakly to some function T in $H^1_0(\Omega)$. Therefore, it converges strongly in $L^r(\Omega)$, $r < 6$, and it follows from Lemma 2.1 that $(\mathbf{u}(T_m), p(T_m))_m$ converges weakly to $(\mathbf{u}(T), p(T))$ in $H_0(\mathrm{div},\Omega)\times L^2_0(\Omega)$, $(\sqrt{\nu(T_m)}\mathbf{u}(T_m))_m$ converges strongly to $\sqrt{\nu(T)}\mathbf{u}(T)$ in $L^2(\Omega)^3$, and $(p(T_m))_m$ converges strongly to $p(T)$ in $L^2(\Omega)$. Now, let us freeze the index i in (2.27) and let m tend to infinity. To pass to the limit in the nonlinear term, by applying Green's formula (owing again to the smoothness of the basis) we write,

$$\int_\Omega (\mathbf{u}(T_m)\cdot\nabla T_m)(\mathbf{x})\theta_i(\mathbf{x})\,d\mathbf{x} = -\int_\Omega (\mathbf{u}(T_m)\cdot\nabla\theta_i)(\mathbf{x})T_m(\mathbf{x})\,d\mathbf{x}. \tag{2.31}$$

The strong convergence of $(T_m)_m$ in $L^4(\Omega)$ and the fact that $\nabla\theta_i$ belongs to $L^4(\Omega)^3$ imply that $(T_m\nabla\theta_i)_m$ converges strongly to $T\nabla\theta_i$ in $L^2(\Omega)^3$. Since $\mathbf{u}(T_m)$ converges weakly to $\mathbf{u}(T)$ in $L^2(\Omega)^3$, these two convergences imply

$$\lim_{m\to\infty}\int_\Omega (\mathbf{u}(T_m)\cdot\nabla T_m)(\mathbf{x})\theta_i(\mathbf{x})\,d\mathbf{x} = -\int_\Omega (\mathbf{u}(T)\cdot\nabla\theta_i)(\mathbf{x})T(\mathbf{x})\,d\mathbf{x}, \tag{2.32}$$

and consequently the limit functions satisfy for any $i \geq 1$,

$$\alpha(\nabla T, \nabla\theta_i) - \int_\Omega (\mathbf{u}(T)\cdot\nabla\theta_i)(\mathbf{x})T(\mathbf{x})\,d\mathbf{x} = \ (g,\theta_i). \tag{2.33}$$

From this system and the density of the basis in $H^2(\Omega)\cap H^1_0(\Omega)$, we infer in the sense of distributions,

$$-\alpha\Delta T + \operatorname{div}(\mathbf{u}(T)\,T) = g \quad \text{i.e.,} \quad -\alpha\Delta T + \mathbf{u}(T)\cdot\nabla T = g.$$

This implies in particular that $\mathbf{u}(T)\cdot\nabla T$ belongs to $H^{-1}(\Omega)$; hence by taking the duality with $S\in H^1_0(\Omega)$, we recover,

$$\forall S\in H^1_0(\Omega),\ \ \alpha(\nabla T,\nabla S) + <\mathbf{u}(T)\cdot\nabla T, S>_{H^{-1}(\Omega),H^1_0(\Omega)} = (g,S), \tag{2.34}$$

which is a slightly sharper version of (2.19). $\square$

2.4. **Uniqueness.** Before examining uniqueness of the solution, let us establish uniqueness of the solution $T\in H^1_0(\Omega)$ of (2.19) for a given divergence-free velocity $\mathbf{u}\in H_0(\operatorname{div},\Omega)$,

$$\forall S\in H^1_0(\Omega)\cap L^\infty(\Omega),\ \ \alpha(\nabla T,\nabla S) + \int_\Omega (\mathbf{u}\cdot\nabla T)(\mathbf{x})S(\mathbf{x})\,d\mathbf{x} = (g,S). \tag{2.35}$$

Existence is easily proved by a simpler version of the Galerkin technique used above and it yields a solution satisfying (2.30). But uniqueness is far from straightforward because the obvious choice of test function, $S=T$, is not available since T is not necessarily in $L^\infty(\Omega)$. To by-pass this difficulty, we shall apply a renormalizing technique in the spirit of the work of Stampacchia [19].

For a given real number $k>0$, let τ_k be the truncation function of one variable defined by

$$\forall t\in\mathbb{R},\quad \tau_k(t) = \begin{cases} t & \text{if } |t|\le k\\ k\,\operatorname{sgn}(t) & \text{if } |t|>k,\end{cases} \tag{2.36}$$

and let σ_k be its primitive:

$$\forall t\in\mathbb{R},\quad \sigma_k(t) = \int_0^t \tau_k(s)\,ds. \tag{2.37}$$

The function τ_k belongs to $W^{1,\infty}(\mathbb{R})$ and for any S in $H^1_0(\Omega)$, $\tau_k(S)$ belongs to $H^1_0(\Omega)$ and a.e. in Ω,

$$\nabla\tau_k(S) = \begin{cases}\nabla S & \text{if } |S|\le k\\ 0 & \text{if } |S|>k.\end{cases} \tag{2.38}$$

The function σ_k is Lipschitz continuous, it is piecewise $\mathcal{C}^1(\mathbb{R})$, it satisfies $\sigma_k(0)=0$, and for all S in $H^1_0(\Omega)$, $\sigma_k(S)$ belongs to $H^1_0(\Omega)$. Then, we have the following result.

Lemma 2.4. *For any $\alpha>0$, any g in $L^2(\Omega)$, and any $\mathbf{u}$ in $H_0(\operatorname{div},\Omega)$ satisfying $\operatorname{div}\mathbf{u}=0$, problem (2.35) has one and only one solution T in $H^1_0(\Omega)$; hence T is a function of $\mathbf{u}$. The solution T satisfies the bound*

$$|T|_{H^1(\Omega)} \le \frac{S_2^0}{\alpha}\|g\|_{L^2(\Omega)}. \tag{2.39}$$

Proof. As stated above, existence is an easy variant of the existence proof in Section 2.3. Regarding uniqueness, let T be any solution of (2.35); the regularity of $\tau_k(T)$ implies that we can test (2.35) with $S=\tau_k(T)$. This gives

$$\alpha\big(\nabla T,\nabla\tau_k(T)\big) + \int_\Omega(\mathbf{u}\cdot\nabla T)(\mathbf{x})\tau_k(T(\mathbf{x}))\,d\mathbf{x} = \big(g,\tau_k(T)\big). \tag{2.40}$$

First (2.38) implies

$$\big(\nabla T,\nabla\tau_k(T)\big) = \|\nabla\tau_k(T)\|^2_{L^2(\Omega)^d}. \tag{2.41}$$

Next, from (2.37), we observe that

$$\nabla\sigma_k(T) = \tau_k(T)\nabla T, \tag{2.42}$$

and hence

$$\int_\Omega(\mathbf{u}\cdot\nabla T)(\mathbf{x})\tau_k(T(\mathbf{x}))\,d\mathbf{x} = \big(\mathbf{u},\nabla\sigma_k(T)\big). \tag{2.43}$$

Therefore Green's formula and the fact that $\mathbf{u}$ is divergence-free yield

$$\int_\Omega(\mathbf{u}\cdot\nabla T)(\mathbf{x})\tau_k(T(\mathbf{x}))\,d\mathbf{x} = -\big(\operatorname{div}\mathbf{u},\sigma_k(T)\big) = 0.$$

Hence, if $T \in H_0^1(\Omega)$ is any solution of (2.35), it satisfies the equality

$$\alpha \|\nabla \tau_k(T)\|^2_{L^2(\Omega)^d} = \big(g, \tau_k(T)\big) \tag{2.44}$$

and therefore $\tau_k(T)$ satisfies the bound (2.39). The strong convergence of $\tau_k(T)$ to T in $H^1(\Omega)$ allows to derive (2.39), as k tends to infinity. Finally, since (2.35) is a linear equation in T, (2.39) for all solutions T implies uniqueness. □

This lemma has the important consequence that all solutions of (2.19) satisfy the bound (2.39). Of course all velocity and pressure solutions satisfy (2.21).

Now, we turn to uniqueness. Let $(\mathbf{u}_1, p_1, T_1)$ and $(\mathbf{u}_2, p_2, T_2)$ be two solutions of problem (V). Their difference denoted $(\mathbf{U}, p, T)$ satisfies,

$$\begin{aligned} \big(\nu(T_2)\mathbf{U}, \mathbf{v}\big) + \big((\nu(T_1) - \nu(T_2))\mathbf{u}_1, \mathbf{v}\big) = 0,\\ \alpha\big(\nabla T, \nabla S\big) + \big(\mathbf{U}\cdot\nabla T_1, S\big) + \big(\mathbf{u}_2\cdot\nabla T, S\big) = 0, \end{aligned} \tag{2.45}$$

for all divergence-free $\mathbf{v}$ in $H_0(\mathrm{div}, \Omega)$ and all S in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Clearly, uniqueness cannot be derived from (2.45) without regularity assumptions on the solution. This is the object of the next theorem. To simplify, it is stated when $d = 3$.

Theorem 2.5. *Let $d = 3$ and ν satisfy (2.16) and (2.17). In addition to the assumptions of Theorem 2.3, we suppose that problem* (2.19) *has a solution T_1 in $L^\infty(\Omega)$, that $\mathbf{u}(T_1)$ belongs to $L^3(\Omega)^3$ and that*

$$\frac{\lambda S_6^0}{\alpha\,\nu_1}\|T_1\|_{L^\infty(\Omega)}\|\mathbf{u}(T_1)\|_{L^3(\Omega)^3} < 1. \tag{2.46}$$

Then problem (2.19) *has no other solution in $H_0^1(\Omega)$.*

Proof. Set $\mathbf{u}_1 = \mathbf{u}(T_1)$ and use the notation of (2.45). From the first part of (2.45), and the above assumptions, we immediately derive,

$$\nu_1\|\mathbf{U}\|_{L^2(\Omega)^3} \le \|(\nu(T_1) - \nu(T_2))\mathbf{u}_1\|_{L^2(\Omega)^3} \le \lambda\|T\|_{L^6(\Omega)}\|\mathbf{u}_1\|_{L^3(\Omega)^3} \le \lambda S_6^0|T|_{H^1(\Omega)}\|\mathbf{u}_1\|_{L^3(\Omega)^3}. \tag{2.47}$$

To deduce a useful bound for T from the second part of (2.45), we first apply Green's formula to the second term, a valid operation since both S and T_1 belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega (\mathbf{U}\cdot\nabla T_1)(\mathbf{x})S(\mathbf{x})\,d\mathbf{x} = -\int_\Omega (\mathbf{U}\cdot\nabla S)(\mathbf{x})T_1(\mathbf{x})\,d\mathbf{x}, \tag{2.48}$$

and we test (2.45) with $S = \tau_k(T)$. Then arguing as in the proof of Lemma 2.4, we obtain

$$\int_\Omega (\mathbf{u}_2\cdot\nabla T)(\mathbf{x})\tau_k(T(\mathbf{x}))\,d\mathbf{x} = 0. \tag{2.49}$$

Hence

$$\alpha|\tau_k(T)|^2_{H^1(\Omega)} \le |\tau_k(T)|_{H^1(\Omega)}\|T_1\|_{L^\infty(\Omega)}\|\mathbf{U}\|_{L^2(\Omega)^3}, \tag{2.50}$$

implying that for all $k > 0$,

$$\alpha|\tau_k(T)|_{H^1(\Omega)} \le \|T_1\|_{L^\infty(\Omega)}\|\mathbf{U}\|_{L^2(\Omega)^3}. \tag{2.51}$$

From this bound and the strong convergence of $\tau_k(T)$ to T as k tends to infinity, we deduce

$$\alpha|T|_{H^1(\Omega)} \le \|T_1\|_{L^\infty(\Omega)}\|\mathbf{U}\|_{L^2(\Omega)^3}. \tag{2.52}$$

Then by substituting the bound (2.47) for $\mathbf{U}$, we infer

$$\alpha|T|_{H^1(\Omega)} \le \frac{\lambda S_6^0}{\nu_1}|T|_{H^1(\Omega)}\|\mathbf{u}_1\|_{L^3(\Omega)^3}\|T_1\|_{L^\infty(\Omega)}. \tag{2.53}$$

This proves uniqueness when (2.46) holds. □

The smallness condition (2.46) for uniqueness is of course restrictive, but for nonlinear problems, uniqueness is rarely guaranteed without restrictions. On the other hand, although the regularity assumptions on the solution in the statement of Theorem 2.5 are not easily inferred from the equations, they are pretty reasonable from a physical point of view.

2.5. **Alternative Variational formulation.** The variational problem (V) introduced in Section 2.2 is well adapted to locally conservative discrete schemes. However, the numerical implementation of such schemes is not so straightforward and can be simplified by eliminating the divergence from the first two equations of (V) by means of Green's formula. This leads to the following alternative:

$$(V_a)\left\{\begin{array}{c}\text{Find } (\mathbf{u},p,T)\in L^2(\Omega)^d\times(H^1(\Omega)\cap L^2_0(\Omega))\times H^1_0(\Omega)\,\text{such that}\\ \forall\,\mathbf{v}\in L^2(\Omega)^d,\ \displaystyle\int_\Omega \nu(T(\mathbf{x}))\mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,d\mathbf{x}+\int_\Omega \nabla p(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,d\mathbf{x}=\int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,d\mathbf{x}\,,\\ \forall\,q\in H^1(\Omega)\cap L^2_0(\Omega),\ \displaystyle\int_\Omega \nabla q(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,d\mathbf{x}=0,\\ \forall\,S\in H^1_0(\Omega)\cap L^\infty(\Omega),\ \alpha\displaystyle\int_\Omega \nabla T(\mathbf{x})\cdot\nabla S(\mathbf{x})\,d\mathbf{x}+\int_\Omega(\mathbf{u}\cdot\nabla T)(\mathbf{x})\,S(\mathbf{x})\,d\mathbf{x}=\int_\Omega g(\mathbf{x})S(\mathbf{x})\,d\mathbf{x}\,,\end{array}\right.$$

which is obviously equivalent to (V). It leads to numerical schemes that are more easily implemented.

3. Discretization

From now on, we assume that Ω is a polygon when $d=2$ or polyhedron when $d=3$, so it can be completely meshed. Now, we describe the discretization space. A regular (see Ciarlet [8]) family of triangulations $(\mathcal{T}_h)_h$ of Ω, is a set of closed non degenerate triangles or tetrahedra, called elements, satisfying,

- for each h, $\bar\Omega$ is the union of all elements of $\mathcal{T}_h$;
- the intersection of two distinct elements of $\mathcal{T}_h$ is either empty, a common vertex, or an entire common edge or face;
- the ratio of the diameter of an element K in $\mathcal{T}_h$ to the diameter of its inscribed circle or ball is bounded by a constant independent of h.

As usual, h denotes the maximal diameter of all elements of $\mathcal{T}_h$. For each K in $\mathcal{T}_h$, we denote by $\mathbb{P}_1(K)$ the space of restrictions to K of polynomials in d variables and total degree at most one.

In what follows, $c, c', C, C', c_1, \dots$ stand for generic constants which may vary from line to line but are always independent of h. For a given triangulation $\mathcal{T}_h$, we define the following finite dimensional spaces:

$$Z_h=\{S_h\in\mathcal{C}^0(\bar\Omega);\ \forall\,\kappa\in\mathcal{T}_h,\ S_h|_K\in\mathbb{P}_1(K)\}\quad\text{and}\quad X_h=Z_h\cap H^1_0(\Omega).\tag{3.1}$$

There exists an approximation operator (when $d=2$, see Bernardi and Girault [3] or Clément [9]; when $d=2$ or $d=3$, see Scott and Zhang [20]), $R_h\in\mathcal{L}(W^{1,p}(\Omega);Z_h)$ and $R_h\in\mathcal{L}(W^{1,p}(\Omega)\cap H^1_0(\Omega);X_h)$ such that for all K in $\mathcal{T}_h$, $m=0,1$, $l=0,1$, and all $p\geq 2$,

$$\forall\,S\in W^{l+1,p}(\Omega),\ |S-R_hS|_{W^{m,p}(K)}\leq C(p,m,l)\,h^{l+1-m}|S|_{W^{l+1,p}(\Delta_K)},\tag{3.2}$$

where Δ_K is the macro element containing the values of S used in defining $R_h(S)$.

3.1. **First discrete scheme.** The velocity and pressure are discretized by RT_0 elements. More precisely, the discrete spaces $(\mathcal{W}_{h,1}, M_{h,1})$ are defined as follows:

$$\begin{aligned}\mathcal{W}_h=&\{\mathbf{v}_h\in H(\mathrm{div},\Omega);\ \mathbf{v}_h(\mathbf{x})|_K=a_K\mathbf{x}+\mathbf{b}_K, a_K\in\mathbb{R},\mathbf{b}_K\in\mathbb{R}^d,\ \forall\,K\in\mathcal{T}_h\},\\ \mathcal{W}_{h,1}=&\mathcal{W}_h\cap H_0(\mathrm{div},\Omega),\end{aligned}\tag{3.3}$$

$$M_h=\{q_h\in L^2(\Omega);\ \forall\,K\in\mathcal{T}_h,\ q_h|_K\text{ is constant}\}\quad\text{and}\quad M_{h,1}=M_h\cap L^2_0(\Omega).\tag{3.4}$$

The kernel of the divergence in $\mathcal{W}_{h,1}$ is denoted by $\mathcal{V}_{h,1}$,

$$\mathcal{V}_{h,1}=\{\mathbf{v}_h\in\mathcal{W}_{h,1};\ \mathrm{div}\,\mathbf{v}_h=0\ \text{in}\ \Omega\}.\tag{3.5}$$

There exists an approximation operator $\xi^1_h\in\mathcal{L}(H^1(\Omega);\mathcal{W}_h)$ and $\xi^1_h\in\mathcal{L}(H^1(\Omega)\cap H_0(\mathrm{div},\Omega);\mathcal{W}_{h,1})$ such that for all K in $\mathcal{T}_h$ (Roberts and Thomas [18]):

$$\forall\,\mathbf{v}\in H^1(\Omega)^d,\ \|\mathbf{v}-\xi^1_h(\mathbf{v})\|_{L^2(K)^d}\leq C_1\,h|\mathbf{v}|_{H^1(K)^d},\tag{3.6}$$

and

$$\forall\,\mathbf{v}\in H^1(\Omega)^d\,\text{with}\,\mathrm{div}\,\mathbf{v}\in H^1(\Omega),\ \|\mathrm{div}(\mathbf{v}-\xi^1_h(\mathbf{v}))\|_{L^2(K)}\leq C_2\,h|\mathrm{div}\,\mathbf{v}|_{H^1(K)}.\tag{3.7}$$

Furthermore, if $\operatorname{div} \mathbf{u} = 0$ then $\operatorname{div}(\xi_h^1(\mathbf{u})) = 0$. In addition, we shall use the operator $\rho_h \in \mathcal{L}(L^2(\Omega); M_h)$ and $\rho_h \in \mathcal{L}(L_0^2(\Omega); M_{h,1})$ defined by

$$\rho_h(q)|_K = \frac{1}{|K|} \int_K q(\mathbf{x})\, d\mathbf{x}\,,\ \forall\, K \in \mathcal{T}_h; \tag{3.8}$$

it satisfies

$$\forall q \in H^1(\Omega), \quad \|q - \rho_h(q)\|_{L^2(K)} \le c\, h\, |q|_{H^1(K)}. \tag{3.9}$$

The following discrete inf-sup condition holds (see Roberts and Thomas [18]):

$$\forall\, q_h \in M_{h,1}, \quad \sup_{\mathbf{v}_h \in \mathcal{W}_{h,1}} \frac{\displaystyle\int_\Omega q_h(\mathbf{x})(\operatorname{div} \mathbf{v}_h)(\mathbf{x})\, d\mathbf{x}}{\|\mathbf{v}_h\|_{H(\operatorname{div},\Omega)}} \ge \beta_1 \|q_h\|_{L^2(\Omega)}, \tag{3.10}$$

with a constant $\beta_1 > 0$ independent of h. We then consider the straightforward discretization of Problem (V):

$$(V_{h,1}) \begin{cases} \text{Find } (\mathbf{u}_h, p_h, T_h) \in \mathcal{W}_{h,1} \times M_{h,1} \times X_h \text{ such that} \\ \forall\, \mathbf{v}_h \in \mathcal{W}_{h,1},\ \displaystyle\int_\Omega \nu(T_h(\mathbf{x}))\mathbf{u}_h(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})\, d\mathbf{x} - \int_\Omega p_h(\mathbf{x})(\operatorname{div} \mathbf{v}_h)(\mathbf{x})\, d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})\, d\mathbf{x}\,, \\ \forall\, \mathbf{q}_h \in M_{h,1},\ \displaystyle\int_\Omega q_h(\mathbf{x})(\operatorname{div} \mathbf{u}_h)(\mathbf{x})\, d\mathbf{x} = 0, \\ \forall\, S_h \in X_h,\ \alpha \displaystyle\int_\Omega \nabla T_h(\mathbf{x}) \cdot \nabla S_h(\mathbf{x})\, d\mathbf{x} + \int_\Omega (\mathbf{u}_h \cdot \nabla T_h)(\mathbf{x}) S_h(\mathbf{x})\, d\mathbf{x} = \int_\Omega g(\mathbf{x}) S_h(\mathbf{x})\, d\mathbf{x}\,. \end{cases}$$

It is easy to see that the second equation above implies that $\operatorname{div} \mathbf{u}_h = 0$ in Ω. Hence this scheme exactly preserves the zero divergence condition.

3.1.1. *First scheme: Existence, convergence, and uniqueness.* Existence of a solution of $(V_{h,1})$ is derived by duplicating the steps of Section 2.3. First $(V_{h,1})$ is split as in (2.19)–(2.20), i.e., find T_h in X_h, such that

$$\forall\, S_h \in X_h,\ \ \alpha(\nabla T_h, \nabla S_h) + (\mathbf{u}_h(T_h) \cdot \nabla T_h, S_h) = (g, S_h), \tag{3.11}$$

where $\mathbf{u}_h(T_h)$ is the velocity solution of: Find $(\mathbf{u}_h(T_h), p_h(T_h)) \in \mathcal{W}_{h,1} \times M_{h,1}$, such that

$$\begin{aligned} \forall\, \mathbf{v}_h \in \mathcal{W}_{h,1},\ \ (\nu(T_h)\mathbf{u}_h(T_h), \mathbf{v}_h) - (p_h(T_h), \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \\ \forall\, q_h \in M_{h,1},\ \ (q_h, \operatorname{div} \mathbf{u}_h(T_h)) = 0. \end{aligned} \tag{3.12}$$

Indeed, since the approximation is conforming and (3.10) holds, an easy argument shows that, for given $T_h \in X_h$, (3.12) (which is a square linear system in finite dimension) has a unique solution $(\mathbf{u}_h(T_h), p_h(T_h))$, and this solution satisfies the same bounds as (2.21), uniform in h,

$$\begin{aligned} \|\mathbf{u}_h(T_h)\|_{L^2(\Omega)^d} \le \frac{1}{\nu_1} \|\mathbf{f}\|_{L^2(\Omega)^d}\ ,\ \ \|\sqrt{\nu(T_h)}\mathbf{u}_h(T_h)\|_{L^2(\Omega)} \le \frac{1}{\sqrt{\nu_1}} \|\mathbf{f}\|_{L^2(\Omega)^d}, \\ \|p_h(T_h)\|_{L^2(\Omega)} \le \frac{1}{\beta_1}\big(\|\mathbf{f}\|_{L^2(\Omega)^d} + \nu_2 \|\mathbf{u}_h(T_h)\|_{L^2(\Omega)^d}\big) \le \frac{1}{\beta_1} \|\mathbf{f}\|_{L^2(\Omega)^d}\big(1 + \frac{\nu_2}{\nu_1}\big). \end{aligned} \tag{3.13}$$

Moreover, in view of the regularity of functions of X_h, we immediately derive that every solution of (3.11)–(3.12) satisfies the a priori bound, uniform in h,

$$|T_h|_{H^1(\Omega)} \le \frac{S_2^0}{\alpha} \|g\|_{L^2(\Omega)}. \tag{3.14}$$

As a consequence, the argument of the existence lemma 2.2 can be applied to (3.11), thus establishing that (3.11) has at least one solution. Similarly, the convergence proof of Theorem 2.3 carries over to (3.11), considering the approximation properties of the operators R_h, ξ_h^1 and ρ_h. Finally, uniqueness follows easily from Green's formula, since $\mathbf{u}_h$ is in $L^\infty(\Omega)^d$ and T_h in $W^{1,\infty}(\Omega)$. This is summed up in the following existence, convergence and uniqueness theorems. To simplify, the uniqueness theorem is stated when $d = 3$.

Theorem 3.1. *Let ν satisfy (2.17). Then for any data $(\mathbf{f},g) \in L^2(\Omega)^d \times L^2(\Omega)$, $(V_{h,1})$ has at least a solution $(\mathbf{u}_h, p_h, T_h) \in \mathcal{W}_{h,1} \times M_{h,1} \times X_h$. Moreover, every solution of $(\mathcal{V}_{h,1})$ satisfies the bounds* (3.13) *and* (3.14).

Theorem 3.2. *Let ν satisfy (2.16), (2.17) and $(\mathbf{u}_h, p_h, T_h)$ be any solution of the discrete problem $(V_{h,1})$. We can extract a subsequence, still denoted $(\mathbf{u}_h, p_h, T_h)$ such that*

$$\begin{aligned} \lim_{h\to 0} T_h = T \quad &\textit{weakly in } H^1(\Omega),\\ \lim_{h\to 0} \mathbf{u}_h = \mathbf{u} \quad &\textit{weakly in } H(\mathrm{div},\Omega),\\ \lim_{h\to 0} \sqrt{\nu(T_h)}\mathbf{u}_h = \sqrt{\nu(T)}\mathbf{u} \quad &\textit{strongly in } L^2(\Omega)^d,\\ \lim_{h\to 0} p_h = p \ &\textit{strongly in } L^2(\Omega), \end{aligned} \tag{3.15}$$

where $(\mathbf{u}, p, T)$ solves problem (V).

Theorem 3.3. *Let $d = 3$ and ν satisfy (2.16) and (2.17). Suppose that problem* (3.11) *has a solution $T_h \in X_h$ such that*

$$\frac{\lambda S_6^0}{\alpha\,\nu_1}\|T_h\|_{L^\infty(\Omega)}\|\mathbf{u}_h(T_h)\|_{L^3(\Omega)^3} < 1. \tag{3.16}$$

Then problem (3.11) *has no other solution $T_h \in X_h$.*

3.1.2. *First discrete scheme. A priori error estimates.* A priori error estimates are obtained when the exact solution satisfies a slightly stronger smoothness and smallness condition than the uniqueness condition (2.46) of Theorem 2.5.

Theorem 3.4. *Let $d = 3$ and ν satisfy (2.16) and (2.17). We suppose that problem* (2.19) *has a solution T in $W^{1,3}(\Omega)$, that $\mathbf{u} = \mathbf{u}(T)$ belongs to $L^3(\Omega)^3$, and that*

$$\lambda\,(S_6^0)^2\,\|\mathbf{u}\|_{L^3(\Omega)^3}|T|_{W^{1,3}(\Omega)} < \alpha\,\nu_1. \tag{3.17}$$

Then the following error inequalities hold:

$$\begin{aligned} \Big(1 - \frac{\lambda (S_6^0)^2}{\alpha\,\nu_1}\|\mathbf{u}\|_{L^3(\Omega)^3}|T|_{W^{1,3}(\Omega)}\Big)|T - T_h|_{H^1(\Omega)} \le 2|T - R_h(T)|_{H^1(\Omega)}\\ + \frac{S_6^0}{\alpha\,\nu_1}\|\mathbf{f}\|_{L^2(\Omega)^3}|T - R_h(T)|_{W^{1,3}(\Omega)} + \frac{S_6^0}{\alpha}\big(1 + \frac{\nu_2}{\nu_1}\big)|T|_{W^{1,3}(\Omega)} \inf_{\mathbf{w}_h \in \mathcal{V}_{h,1}} \|\mathbf{u} - \mathbf{w}_h\|_{L^2(\Omega)^3}, \end{aligned} \tag{3.18}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)^3} \le \big(1 + \frac{\nu_2}{\nu_1}\big) \inf_{\mathbf{w}_h \in \mathcal{V}_{h,1}} \|\mathbf{u} - \mathbf{w}_h\|_{L^2(\Omega)^3} + \frac{\lambda S_6^0}{\nu_1}\|\mathbf{u}\|_{L^3(\Omega)^3}|T - T_h|_{H^1(\Omega)}, \tag{3.19}$$

$$\|p - p_h\|_{L^2(\Omega)} \le 2\,\|p - \rho_h(p)\|_{L^2(\Omega)} + \frac{1}{\beta_1}\Big(\nu_2\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)^3} + \lambda S_6^0\|\mathbf{u}\|_{L^3(\Omega)^3}|T - T_h|_{H^1(\Omega)}\Big). \tag{3.20}$$

Proof. Let $(\mathbf{u}, p, T)$ and $(\mathbf{u}_h, p_h, T_h)$ solve respectively (V) and $(V_{h,1})$. We shall prove first (3.19), next (3.20), and finally (3.18).

1) Let us estimate the velocity error in terms of the temperature error. By taking the difference between the second equations of (V) and $(V_{h,1})$ and testing with $\mathbf{v} = \mathbf{v}_h \in \mathcal{V}_{h,1}$, we obtain

$$(\nu(T)\mathbf{u}, \mathbf{v}_h) = (\nu(T_h)\mathbf{u}_h, \mathbf{v}_h). \tag{3.21}$$

Then by inserting $\nu(T_h)$ and an arbitrary $\mathbf{w}_h \in \mathcal{V}_{h,1}$, and testing with $\mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h$ that belongs indeed to $\mathcal{V}_{h,1}$, we easily derive

$$\|(\nu(T_h))^{1/2}(\mathbf{u}_h - \mathbf{w}_h)\|^2_{L^2(\Omega)^3} = ((\nu(T) - \nu(T_h))\mathbf{u}, \mathbf{u}_h - \mathbf{w}_h) + (\nu(T_h)(\mathbf{u} - \mathbf{w}_h), \mathbf{u}_h - \mathbf{w}_h). \tag{3.22}$$

Hence (2.17) and the Lipschitz continuity of ν yield

$$\nu_1\|\mathbf{u}_h - \mathbf{w}_h\|_{L^2(\Omega)^3} \le \nu_2\|\mathbf{u} - \mathbf{w}_h\|_{L^2(\Omega)^3} + \lambda\|\mathbf{u}\|_{L^3(\Omega)^3}\|T - T_h\|_{L^6(\Omega)^3} \tag{3.23}$$

and (3.19) follows immediately from Sobolev's imbedding and the triangle inequality.

2) The proof of the error estimate for the pressure follows the same lines. By taking the difference between the second equations of (V) and $(V_{h,1})$, inserting $\rho_h(p)$, and testing with $\mathbf{v}_h$ in $\mathcal{W}_{h,1}$, we obtain

$$(\rho_h(p) - p_h, \operatorname{div} \mathbf{v}_h) = (\rho_h(p) - p, \operatorname{div} \mathbf{v}_h) + (\nu(T)\mathbf{u} - \nu(T_h)\mathbf{u}_h, \mathbf{v}_h). \tag{3.24}$$

It follows from the inf-sup condition (3.10) (see for instance Girault and Raviart [13]) that there exists $\mathbf{v}_h$ in $\mathcal{W}_{h,1}$ such that

$$\operatorname{div} \mathbf{v}_h = \rho_h(p) - p_h \quad \text{and} \quad \|\mathbf{v}_h\|_{H(\operatorname{div},\Omega)} \leq \frac{1}{\beta_1}\|\rho_h(p) - p_h\|_{L^2(\Omega)}. \tag{3.25}$$

With this $\mathbf{v}_h$, (3.24) implies

$$\|\rho_h(p) - p_h\|_{L^2(\Omega)} \leq \|\rho_h(p) - p\|_{L^2(\Omega)} + \frac{1}{\beta_1}\|\nu(T)\mathbf{u} - \nu(T_h)\mathbf{u}_h\|_{L^2(\Omega)^3}. \tag{3.26}$$

By treating the last term as above, we recover (3.20).

3) By taking the difference between the first equation of (V) and $(V_{h,1})$, tested with S_h, and inserting $R_h(T)$, we obtain

$$\begin{aligned}\alpha\big(\nabla(R_h(T) - T_h), \nabla S_h\big) = {} & \alpha\big(\nabla(R_h(T) - T), \nabla S_h\big) + \big(\mathbf{u}_h \cdot \nabla(T_h - R_h(T)), S_h\big) \\ & + \big(\mathbf{u}_h \cdot \nabla(R_h(T) - T), S_h\big) + \big((\mathbf{u}_h - \mathbf{u}) \cdot \nabla T, S_h\big).\end{aligned}$$

Then the choice $S_h = R_h(T) - T_h$ and the antisymmetric property of the transport term yield

$$\begin{aligned}\alpha|R_h(T) - T_h|^2_{H^1(\Omega)} = {} & \alpha(\nabla(R_h(T) - T), \nabla(R_h(T) - T_h)) + ((\mathbf{u}_h - \mathbf{u}) \cdot \nabla T, R_h(T) - T_h) \\ & + (\mathbf{u}_h \cdot \nabla(R_h(T) - T), R_h(T) - T_h).\end{aligned}$$

With Hölder's inequality, this becomes

$$\begin{aligned}\alpha|R_h(T) - T_h|^2_{H^1(\Omega)} \leq {} & \alpha\,|R_h(T) - T|_{H^1(\Omega)}|R_h(T) - T_h|_{H^1(\Omega)} \\ & + \Big(\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)^3}|T|_{W^{1,3}(\Omega)} + \|\mathbf{u}_h\|_{L^2(\Omega)^3}|R_h(T) - T|_{W^{1,3}(\Omega)}\Big)\|R_h(T) - T_h\|_{L^6(\Omega)}.\end{aligned}$$

Then Sobolev's imbedding implies

$$\begin{aligned}|R_h(T) - T_h|_{H^1(\Omega)} \leq {} & |R_h(T) - T|_{H^1(\Omega)} \\ & + \frac{S_6^0}{\alpha}\Big(\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)^3}|T|_{W^{1,3}(\Omega)} + \|\mathbf{u}_h\|_{L^2(\Omega)^3}|R_h(T) - T|_{W^{1,3}(\Omega)}\Big).\end{aligned}$$

By substituting (3.19) and the first part of (3.13) into this inequality and using the triangle inequality, we derive

$$\begin{aligned}|T - T_h|_{H^1(\Omega)} \leq {} & 2\,|T - R_h(T)|_{H^1(\Omega)} + \frac{S_6^0}{\alpha\,\nu_1}\|\mathbf{f}\|_{L^2(\Omega)^3}|T - R_h(T)|_{W^{1,3}(\Omega)} \\ & + \frac{S_6^0}{\alpha}|T|_{W^{1,3}(\Omega)}\Big(\big(1 + \frac{\nu_2}{\nu_1}\big)\inf_{\mathbf{w}_h \in \mathcal{V}_{h,1}}\|\mathbf{u} - \mathbf{w}_h\|_{L^2(\Omega)^3} \\ & + \frac{\lambda\,S_6^0}{\nu_1}\|\mathbf{u}\|_{L^3(\Omega)^3}|T - T_h|_{H^1(\Omega)}\Big).\end{aligned} \tag{3.27}$$

Then (3.18) follows by collecting terms in (3.27) and applying the assumption (3.17). □

Remark 3.5. Under the assumptions of Theorem 3.4, the solution of the scheme $(V_{h,1})$ converges strongly to the solution of (V) when h tends to zero. Indeed, for $\mathbf{u} \in L^3(\Omega)^3$ and $T \in W^{1,3}(\Omega)$, the right-hand sides of the three error inequalities (3.18), (3.19) and (3.20) tend to zero as h tends to zero. □

Remark 3.6. When the exact solution $(\mathbf{u}, p, T) \in H^1(\Omega)^3 \times H^1(\Omega) \times W^{2,3}(\Omega)$, (3.18), (3.19) and (3.20) yield a specific rate of convergence,

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{div},\Omega)} + \|p - p_h\|_{L^2(\Omega)} + |T - T_h|_{H^1(\Omega)} \leq C\,h\,\big(|\mathbf{u}|_{H^1(\Omega)^3} + |p|_{H^1(\Omega)} + |T|_{W^{2,3}(\Omega)}\big). \tag{3.28}$$

□

3.2. **Second discrete scheme.** Let K be an element of $\mathcal{T}_h$ with vertices a_i, $1 \le i \le d+1$, and corresponding barycentric coordinates λ_i. We denote by $b_K \in \mathbb{P}_{d+1}(K)$ the basic bubble function

$$b_K(\mathbf{x}) = \lambda_1(\mathbf{x})...\lambda_{d+1}(\mathbf{x}). \tag{3.29}$$

We observe that $b_K(\mathbf{x}) = 0$ on ∂K and that $b_K(\mathbf{x}) > 0$ in the interior of K.
Let $(\mathcal{W}_{h,2}, M_{h,2})$ be a pair of discrete spaces approximating $L^2(\Omega)^d \times \big(H^1(\Omega) \cap L^2_0(\Omega)\big)$ defined by

$$\mathcal{W}_{h,2} = \{\mathbf{v}_h \in (\mathcal{C}^0(\bar\Omega))^d;\ \forall K \in \mathcal{T}_h,\ \mathbf{v}_h|_K \in \mathcal{P}(K)^d\}, \tag{3.30}$$

$$\tilde M_h = \{q_h \in \mathcal{C}^0(\bar\Omega);\ \forall K \in \mathcal{T}_h,\ q_h|_K \in \mathbb{P}_1(K)\} \quad \text{and} \quad M_{h,2} = \tilde M_h \cap L^2_0(\Omega), \tag{3.31}$$

where

$$\mathcal{P}(K) = \mathbb{P}_1(K) \oplus \mathrm{Vect}\{b_K\}.$$

Let $\mathcal{V}_{h,2}$ be the kernel of the divergence in $\mathcal{W}_{h,2}$,

$$\mathcal{V}_{h,2} = \{\mathbf{v}_h \in \mathcal{W}_{h,2};\ \forall q_h \in M_{h,2},\ (\mathrm{div}\,\mathbf{v}_h, q_h) = 0\}. \tag{3.32}$$

Since $\mathcal{W}_{h,2}$ contains the polynomials of degree one in each K, we can construct a variant π_h of R_h (cf. Girault and Lions [12] or Scott and Zhang [20]) in $\mathcal{L}(L^2(\Omega)^d; Z_h)$ that is quasi-locally stable in $L^2(\Omega)$, i.e., for all K in $\mathcal{T}_h$

$$\forall \mathbf{v} \in L^2(\Omega)^d, \quad \|\pi_h(\mathbf{v})\|_{L^2(K)^d} \le C\|\mathbf{v}\|_{L^2(\Delta_K)^d}, \tag{3.33}$$

and has the same quasi-local approximation properties as R_h for all K in $\mathcal{T}_h$, for $m = 0, 1$ and $1 \le l \le 2$,

$$\forall\, \mathbf{v} \in H^l(\Omega)^d,\ |\mathbf{v} - \pi_h(\mathbf{v})|_{H^m(K)^d} \le C\, h^{l-m}|\mathbf{v}|_{H^l(\Delta_K)^d}. \tag{3.34}$$

Regarding the pressure, since Z_h coincides with $\tilde M_h$, an easy modification of R_h yields an operator $r_h \in \mathcal{L}(H^1(\Omega); \tilde M_h)$ and $r_h \in \mathcal{L}(H^1(\Omega) \cap L^2_0(\Omega); M_{h,2})$ (see for instance Abboud, Girault and Sayah [1]), satisfying (3.2). We approximate problem (V_a) by the following discrete scheme:

$$(V_{h,2}) \begin{cases} \text{Find } (\mathbf{u}_h, p_h, T_h) \in \mathcal{W}_{h,2} \times M_{h,2} \times X_h \,\text{such as} \\ \forall\, \mathbf{v}_h \in \mathcal{W}_{h,2},\ \displaystyle\int_\Omega \nu(T_h(\mathbf{x}))\mathbf{u}_h(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})\, d\mathbf{x} + \int_\Omega \nabla p_h(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})\, d\mathbf{x} \,= \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})\, d\mathbf{x}\,, \\ \forall\, \mathbf{q}_h \in M_{h,2},\ \displaystyle\int_\Omega \nabla q_h(\mathbf{x}) \cdot \mathbf{u}_h(\mathbf{x})\, d\mathbf{x} \,= 0, \\ \forall\, S_h \in X_h,\ \alpha \displaystyle\int_\Omega \nabla T_h(\mathbf{x}) \cdot \nabla S_h(\mathbf{x})\, d\mathbf{x} + \int_\Omega (\mathbf{u}_h \cdot \nabla T_h)(\mathbf{x}) S_h(\mathbf{x})\, d\mathbf{x} \\ \qquad + \dfrac{1}{2}\displaystyle\int_\Omega (\mathrm{div}\,\mathbf{u}_h)(\mathbf{x}) T_h(\mathbf{x}) S_h(\mathbf{x})\, d\mathbf{x} \,= \int_\Omega g(\mathbf{x}) S_h(\mathbf{x})\, d\mathbf{x}\,, \end{cases}$$

where as usual, the second nonlinear term in the last equation is added to compensate for the fact that $\mathrm{div}\,\mathbf{u}_h \ne 0$. It is well-known that Green's formula and the functions regularity imply that

$$\big(\mathbf{u}_h \cdot \nabla\, T_h, S_h\big) + \frac{1}{2}\big((\mathrm{div}\,\mathbf{u}_h) T_h, S_h\big) = \frac{1}{2}\Big(\big(\mathbf{u}_h \cdot \nabla\, T_h, S_h\big) - \big(\mathbf{u}_h \cdot \nabla\, S_h, T_h\big)\Big), \tag{3.35}$$

so that the nonlinear term is antisymmetric. One of the key points for studying $(V_{h,2})$ is the discrete inf-sup condition satisfied by the pair of spaces $(\mathcal{W}_{h,2}, M_{h,2})$. Its proof consists in using the continuous inf-sup condition and Fortin's lemma (see for instance Girault and Raviart [13]) based on the operator

$$\mathcal{F}_h(\mathbf{v}) = \pi_h(\mathbf{v}) + \sum_{K \in \mathcal{T}_h} \alpha_K(\mathbf{v}) b_K,$$

where

$$\alpha_K(\mathbf{v}) = \frac{1}{\displaystyle\int_K b_K(\mathbf{x})\, d\mathbf{x}} \int_K (\mathbf{v} - \pi_h(\mathbf{v}))(\mathbf{x})\, d\mathbf{x}\,.$$

Fortin's lemma holds with this operator and leads to the following discrete inf-sup condition:

$$\forall\, q_h \in M_{h,2},\ \sup_{\mathbf{v}_h \in \mathcal{W}_{h,2}} \frac{\displaystyle\int_\Omega \nabla q_h(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})\, d\mathbf{x}}{\|\mathbf{v}_h\|_{L^2(\Omega)^d}} \ge \beta_2\, |q_h|_{H^1(\Omega)}, \tag{3.36}$$

with a constant $\beta_2 > 0$ independent of h. We also have the following bound in each element K,

$$\forall\, \mathbf{v} \in H^1(\Omega)^d,\ \ \|\mathbf{v} - \mathcal{F}_h(\mathbf{v})\|_{L^2(K)^d} \le C\, h |\mathbf{v}|_{H^1(\Delta_K)^d}. \tag{3.37}$$

Owing to this inf-sup condition, $(V_{h,2})$ has the same splitting as $(V_{h,1})$, i.e., find T_h in X_h, such that

$$\forall S_h \in X_h,\ \ \alpha(\nabla T_h, \nabla S_h) + (\mathbf{u}_h(T_h)\cdot \nabla T_h, S_h) + \frac{1}{2}((\operatorname{div} \mathbf{u}_h(T_h))T_h, S_h) = (g, S_h), \tag{3.38}$$

where $\mathbf{u}_h(T_h)$ is the velocity solution of (3.12) stated in $\mathcal{W}_{h,2} \times M_{h,2}$. Of course, $\mathbf{u}_h(T_h)$ and $p_h(T_h)$ satisfy the bounds (3.13) with β_2 instead of β_1. Moreover, as all functions involved are smooth enough, Green's formula implies the bound (3.14) for T_h. Hence we have the analogue of Theorem 3.1 with the same proof.

Theorem 3.7. *Let ν satisfy (2.17). Then for any data $(\mathbf{f}, g) \in L^2(\Omega)^d \times L^2(\Omega)$, problem $(V_{h,2})$ has at least a solution $(\mathbf{u}_h, p_h, T_h) \in \mathcal{W}_{h,2} \times M_{h,2} \times X_h$ and every solution of $(\mathcal{V}_{h,2})$ satisfies the bounds* (3.13) *and* (3.14).

Because the divergence of the discrete velocity does not vanish, the sufficient condition for uniqueness is more restrictive.

Theorem 3.8. *Let $d = 3$ and ν satisfy (2.16) and (2.17). Suppose that problem* (3.38) *has a solution $T_h \in X_h$ such that*

$$\frac{\lambda S_6^0}{2\,\alpha\,\nu_1}\|\mathbf{u}_h(T_h)\|_{L^3(\Omega)^3}\big(\|T_h\|_{L^\infty(\Omega)} + S_6^0 |T_h|_{W^{1,3}(\Omega)}\big) < 1. \tag{3.39}$$

Then problem (3.38) *has no other solution $T_h \in X_h$.*

Proof. Here again, we consider two solutions of problem (3.38) and denote the difference in velocity and in temperature by $\mathbf{U}_h$ and T_h. On one hand, since the velocity equation is the same for both discretizations, $\mathbf{U}_h$ satisfies the analogue of (2.47),

$$\nu_1 \|\mathbf{U}_h\|_{L^2(\Omega)^3} \le \lambda S_6^0 |T_h|_{H^1(\Omega)} \|\mathbf{u}_{h,1}\|_{L^3(\Omega)^3}. \tag{3.40}$$

On the other hand, using (3.35), the difference in the temperature equation reads with $S_h = T_h$,

$$\alpha |T_h|^2_{H^1(\Omega)} + \frac{1}{2}\Big(\big(\mathbf{U}_h, \cdot \nabla\, T_{h,1}, T_h\big) - \big(\mathbf{U}_h \cdot \nabla\, T_h, T_{h,1}\big)\Big) = 0. \tag{3.41}$$

Then the above estimate for $\|\mathbf{U}_h\|_{L^2(\Omega)^3}$ and condition (3.39) imply uniqueness. $\square$

We have the same convergence of a discrete to an exact solution, but the proof is slightly more involved, again due to the non zero divergence.

Theorem 3.9. *Let ν satisfy (2.16), (2.17) and $(\mathbf{u}_h, p_h, T_h)$ be any solution of the discrete problem $(V_{h,2})$. We can extract a subsequence, still denoted $(\mathbf{u}_h, p_h, T_h)$ such that*

$$\begin{aligned}
\lim_{h\to 0} T_h &= T \quad \text{weakly in } H^1(\Omega),\\
\lim_{h\to 0} \mathbf{u}_h &= \mathbf{u} \quad \text{weakly in } H(\operatorname{div}, \Omega),\\
\lim_{h\to 0} \sqrt{\nu(T_h)}\mathbf{u}_h &= \sqrt{\nu(T)}\mathbf{u} \quad \text{strongly in } L^2(\Omega)^d,\\
\lim_{h\to 0} p_h &= p \quad \text{weakly in } H^1(\Omega) \ \text{ and strongly in } L^2(\Omega),
\end{aligned} \tag{3.42}$$

where $(\mathbf{u}, p, T)$ solves problem (V).

Proof. The convergences are the same since the solutions satisfy the same bounds, but passing to the limit in (3.38) is slightly different. Let us use the expression (3.35) with the choice $S_h = R_h(S)$ for a smooth function S. The convergence of $\big(\mathbf{u}_h \cdot \nabla S_h, T_h\big)$ is done as in Theorem 2.3. For $\big(\mathbf{u}_h \cdot \nabla T_h, S_h\big)$ we use the strong convergence of $\sqrt{\nu(T_h)}\mathbf{u}_h$. Indeed, we write

$$\big(\mathbf{u}_h \cdot \nabla T_h, S_h\big) = \big(\sqrt{\nu(T_h)}\mathbf{u}_h \cdot \nabla T_h, \frac{1}{\sqrt{\nu(T_h)}} S_h\big), \tag{3.43}$$

which is the sum of terms of the form

$$(\sqrt{\nu(T_h)}u_{h,i}, \frac{1}{\sqrt{\nu(T_h)}} S_h \frac{\partial T_h}{\partial x_i}), \tag{3.44}$$

where $u_{h,i}$ denotes the i-th component of $\mathbf{u}_h$. The first factor converges strongly to $\sqrt{\nu(T)}u_i$ in $L^2(\Omega)$, while the second factor is bounded in $L^2(\Omega)$; therefore, again up to a subsequence, it converges weakly in $L^2(\Omega)$, and a standard argument shows that its limit is

$$\frac{1}{\sqrt{\nu(T)}} S \frac{\partial T}{\partial x_i}. \tag{3.45}$$

Thus, we conclude that $(\mathbf{u}, p, T)$ solves problem (V_a). □

3.2.1. *A priori error estimates for the second scheme.* As the equations satisfied by $\mathbf{u}_h(T_h)$ and $p_h(T_h)$ are the same for the two schemes, the error estimates for the discrete velocity and pressure in terms of the temperature error are the same with an additional term $|p - r_h(p)|_{H^1(\Omega)}$ in the velocity error,

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)^3} \le \big(1+\frac{\nu_2}{\nu_1}\big) \inf_{\mathbf{w}_h\in\mathcal{V}_{h,2}} \|\mathbf{u}-\mathbf{w}_h\|_{L^2(\Omega)^3} + \frac{\lambda S_6^0}{\nu_1}\|\mathbf{u}\|_{L^3(\Omega)^3}|T-T_h|_{H^1(\Omega)} + \frac{1}{\nu_1}|p-r_h(p)|_{H^1(\Omega)}, \tag{3.46}$$

and ρ_h replaced by r_h in the pressure error. Therefore, we only need to establish an error estimate for the temperature. It is stated under the same regularity condition on the data, but under a slightly more restrictive smallness condition, again due to the stabilizing term.

Theorem 3.10. *We retain the setting and assumptions of Theorem 3.4 and in addition, we suppose that* $T \in L^\infty(\Omega)$ *and*

$$\lambda\, S_6^0\, \|\mathbf{u}\|_{L^3(\Omega)^3}\big(S_6^0\, |T|_{W^{1,3}(\Omega)} + \|T\|_{L^\infty(\Omega)}\big) < 2\,\alpha\,\nu_1. \tag{3.47}$$

Then $\mathbf{u}_h - \mathbf{u}$ *satisfies* (3.46), $p_h - p$ *satisfies* (3.20) *with* r_h *instead of* ρ_h *and* β_2 *instead of* β_1, *and* $T_h - T$ *satisfies*

$$\begin{aligned}\Big(1 - \frac{\lambda\, S_6^0}{2\,\alpha\,\nu_1}\|\mathbf{u}\|_{L^3(\Omega)^3}\big(S_6^0\, |T|_{W^{1,3}(\Omega)} + \|T\|_{L^\infty(\Omega)}\big)\Big)|T - T_h|_{H^1(\Omega)} \le 2|T - R_h(T)|_{H^1(\Omega)}\\ + \frac{1}{2\,\alpha\,\nu_1}\|\mathbf{f}\|_{L^2(\Omega)^3}\big(S_6^0|T - R_h(T)|_{W^{1,3}(\Omega)} + \|T - R_h(T)\|_{L^\infty(\Omega)}\big)\\ + \frac{1}{2\,\alpha}\Big(\big(1 + \frac{\nu_2}{\nu_1}\big) \inf_{\mathbf{w}_h\in\mathcal{V}_{h,2}} \|\mathbf{u} - \mathbf{w}_h\|_{L^2(\Omega)^3} + \frac{1}{\nu_1}|p - r_h(p)|_{H^1(\Omega)}\Big)\big(S_6^0\, |T|_{W^{1,3}(\Omega)} + \|T\|_{L^\infty(\Omega)}\big).\end{aligned} \tag{3.48}$$

Proof. As stated above, the velocity error is given by (3.46) and the pressure error is unchanged; it remains to establish the temperature error. Again, we use the expression (3.35); then for any function S_h in X_h, the temperature's error equation is,

$$\begin{aligned}\alpha\big(\nabla(R_h(T) - T_h), \nabla S_h\big) &= \alpha\big(\nabla(R_h(T) - T), \nabla S_h\big)\\ &+ \frac{1}{2}\Big(\big(\mathbf{u}_h\cdot\nabla(T_h - R_h(T)), S_h\big) - \big(\mathbf{u}_h\cdot\nabla S_h, T_h - R_h(T)\big)\Big)\\ &+ \frac{1}{2}\Big(\big(\mathbf{u}_h\cdot\nabla(R_h(T) - T), S_h\big) - \big(\mathbf{u}_h\cdot\nabla S_h, R_h(T) - T\big)\Big)\\ &+ \frac{1}{2}\Big(\big((\mathbf{u}_h - \mathbf{u})\cdot\nabla T, S_h\big) - \big((\mathbf{u}_h - \mathbf{u})\cdot\nabla S_h, T\big)\Big).\end{aligned}$$

Up to the factor $\frac{1}{2}$, the terms in the last two lines of the right-hand side are bounded by

$$\begin{aligned}&\|\mathbf{u}_h\|_{L^2(\Omega)^3}\Big(|R_h(T) - T|_{W^{1,3}(\Omega)}\|S_h\|_{L^6(\Omega)} + \|R_h(T) - T\|_{L^\infty(\Omega)}|S_h|_{H^1(\Omega)}\Big)\\ &+ \|\mathbf{u}_h - \mathbf{u}\|_{L^2(\Omega)^3}\Big(|T|_{W^{1,3}(\Omega)}\|S_h\|_{L^6(\Omega)} + \|T\|_{L^\infty(\Omega)}|S_h|_{H^1(\Omega)}\Big).\end{aligned}$$

Then the choice $S_h = R_h(T) - T_h$, the antisymmetric property of the transport term, and Sobolev's imbedding yield

$$\begin{aligned}|R_h(T) - T_h|_{H^1(\Omega)} \leq & |R_h(T) - T|_{H^1(\Omega)} \\ & + \frac{1}{2\,\alpha}\|\mathbf{u}_h\|_{L^2(\Omega)^3}\big(S_6^0|R_h(T) - T|_{W^{1,3}(\Omega)} + \|R_h(T) - T\|_{L^\infty(\Omega)}\big) \\ & + \frac{1}{2\,\alpha}\|\mathbf{u}_h - \mathbf{u}\|_{L^2(\Omega)^3}\big(S_6^0|T|_{W^{1,3}(\Omega)} + \|T\|_{L^\infty(\Omega)}\big).\end{aligned}$$

By substituting (3.46) into this inequality and using the triangle inequality, we derive

$$\begin{aligned}|T - T_h|_{H^1(\Omega)} \leq & 2\,|T - R_h(T)|_{H^1(\Omega)} \\ & + \frac{1}{2\,\alpha}\|\mathbf{u}_h\|_{L^2(\Omega)^3}\big(S_6^0|T - R_h(T)|_{W^{1,3}(\Omega)} + \|T - R_h(T)\|_{L^\infty(\Omega)}\big) \\ & + \frac{1}{2\,\alpha}\big(S_6^0|T|_{W^{1,3}(\Omega)} + \|T\|_{L^\infty(\Omega)}\big) \\ & \times \Big(\big(1 + \frac{\nu_2}{\nu_1}\big)\inf_{\mathbf{w}_h \in \mathcal{V}_{h,2}}\|\mathbf{u} - \mathbf{w}_h\|_{L^2(\Omega)^3} + |p - r_h(p)|_{H^1(\Omega)} \\ & + \frac{\lambda\, S_6^0}{\nu_1}\|\mathbf{u}\|_{L^3(\Omega)^3}|T - T_h|_{H^1(\Omega)}\Big).\end{aligned} \tag{3.49}$$

Then (3.48) follows by collecting terms in (3.49), using the first part of (3.13), and applying the assumption (3.47). □

Remark 3.11. In addition to the assumptions of Theorem 3.10, we suppose that T belongs to $W^{1,s}(\Omega)$ with $s > 3$. Then the error of the scheme $(V_{h,2})$ tends to zero as h tends to zero since, for $\mathbf{u} \in L^3(\Omega)^3$ and $T \in W^{1,s}(\Omega)$ the right-hand sides of the error inequalities tend to zero as h tends to zero. □

Remark 3.12. When the exact solution $(\mathbf{u}, p, T)$ is in $H^1(\Omega)^3 \times H^2(\Omega) \times W^{2,3}(\Omega) \cap W^{1,\infty}(\Omega)$, we can prove a specific rate of convergence,:

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)^3}+|p-p_h|_{H^1(\Omega)}+|T-T_h|_{H^1(\Omega)} \leq C\,h\,\big(|\mathbf{u}|_{H^1(\Omega)^3}+|p|_{H^2(\Omega)}+|T|_{W^{2,3}(\Omega)}+|T|_{W^{1,\infty}(\Omega)}\big). \tag{3.50}$$

□

4. Successive approximations

In order to solve the discrete system, we propose in this section a straightforward successive approximation algorithm that linearizes the discrete problem at each step and converges to the exact solution under the sufficient conditions of the error theorems in the preceding section. The same algorithm is applied to the two schemes, and for the sake of conciseness, we only discuss the first scheme; the analysis of the algorithm for the second scheme being exactly the same.

The algorithm proceeds as follows: Given a first guess T_h^0 in X_h, for $i \geq 0$, find $(\mathbf{u}_h^{i+1}, p_h^{i+1}, T_h^{i+1}) \in \mathcal{W}_{h,1} \times M_{h,1} \times X_h$ such that

$$\begin{aligned}\forall\,\mathbf{v}_h \in \mathcal{W}_{h,1}, \quad \int_\Omega \nu(T_h^i(\mathbf{x}))\mathbf{u}_h^{i+1}(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})\,d\mathbf{x} - \int_\Omega p_h^{i+1}(\mathbf{x})(\operatorname{div}\mathbf{v}_h)(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})\,d\mathbf{x}\,, \\ \forall\,q_h \in M_{h,1}, \quad \int_\Omega q_h(\mathbf{x})(\operatorname{div}\mathbf{u}_h^{i+1})(\mathbf{x})\,d\mathbf{x} = 0,\end{aligned} \tag{4.1}$$

$$\forall\,S_h \in X_h, \quad \alpha\int_\Omega \nabla T_h^{i+1}(\mathbf{x}) \cdot \nabla S_h(\mathbf{x})\,d\mathbf{x} + \int_\Omega (\mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1})(\mathbf{x})S_h(\mathbf{x})\,d\mathbf{x} = \int_\Omega g(\mathbf{x})S_h(\mathbf{x})\,d\mathbf{x}, \tag{4.2}$$

which in reduced form is equivalent to finding $T_h^i \in X_h$ such that, for all $S_h \in X_h$,

$$\alpha\int_\Omega \nabla\,T_h^{i+1}(\mathbf{x}) \cdot \nabla S_h(\mathbf{x})\,d\mathbf{x} + \int_\Omega (\mathbf{u}_h(T_h^i) \cdot \nabla T_h^{i+1})(\mathbf{x})S_h(\mathbf{x})\,d\mathbf{x} = \int_\Omega g(\mathbf{x})S_h(\mathbf{x})\,d\mathbf{x}. \tag{4.3}$$

It follows from the material of Section 3 that for each initial guess T_h^0, this algorithm generates a unique sequence $(\mathbf{u}_h^i, p_h^i, T_h^i)_{i\geq 1}$, and each sequence satisfies the bounds (3.13)–(3.14), for $i \geq 1$, that are independent of T_h^0, of i and of h. Regarding convergence, and reverting to the setting and proof of Theorem

3.4, it is easy to check that the first two components satisfy the following error bounds for all $i \geq 0$:

$$\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} \leq \big(1+\frac{\nu_2}{\nu_1}\big) \inf_{\mathbf{w}_h \in \mathcal{V}_{h,1}} \|\mathbf{u}-\mathbf{w}_h\|_{L^2(\Omega)^3} + \frac{\lambda S_6^0}{\nu_1} \|\mathbf{u}\|_{L^3(\Omega)^3} |T-T_h^i|_{H^1(\Omega)}, \tag{4.4}$$

$$\|p-p_h^{i+1}\|_{L^2(\Omega)} \leq 2\,\|p-\rho_h(p)\|_{L^2(\Omega)} + \frac{1}{\beta_1}\Big(\nu_2\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} + \lambda S_6^0\|\mathbf{u}\|_{L^3(\Omega)^3}|T-T_h^i|_{H^1(\Omega)}\Big). \tag{4.5}$$

An error bound for $T-T_h^{i+1}$ is a little more complex. The argument of the proof of Theorem 3.4 yields the analogue of (3.27),

$$\begin{aligned}|T-T_h^{i+1}|_{H^1(\Omega)} \leq & 2\,|T-R_h(T)|_{H^1(\Omega)} + \frac{S_6^0}{\alpha\,\nu_1}\|\mathbf{f}\|_{L^2(\Omega)^3}|T-R_h(T)|_{W^{1,3}(\Omega)} \\ & + \frac{S_6^0}{\alpha}|T|_{W^{1,3}(\Omega)}\Big(\big(1+\frac{\nu_2}{\nu_1}\big)\inf_{\mathbf{w}_h \in \mathcal{V}_{h,1}}\|\mathbf{u}-\mathbf{w}_h\|_{L^2(\Omega)^3} \\ & + \frac{\lambda\,S_6^0}{\nu_1}\|\mathbf{u}\|_{L^3(\Omega)^3}|T-T_h^i|_{H^1(\Omega)}\Big).\end{aligned} \tag{4.6}$$

Now, either there is an index $i_0 \geq 0$ such that

$$|T-T_h^{i_0}|_{H^1(\Omega)} \leq |T-T_h^{i_0+1}|_{H^1(\Omega)},$$

or there is none. In the first case, we have

$$\sup_{i\geq i_0}|T-T_h^i|_{H^1(\Omega)} = \max\Big(|T-T_h^{i_0}|_{H^1(\Omega)}, \sup_{i\geq i_0+1}|T-T_h^i|_{H^1(\Omega)}\Big) = \sup_{i\geq i_0+1}|T-T_h^i|_{H^1(\Omega)}.$$

Therefore, by taking first the supremum over i for $i \geq i_0$ of the right-hand side of (4.6) and next the supremum of the left-hand side of the resulting inequality, we deduce

$$\begin{aligned}\Big(1-\frac{\lambda(S_6^0)^2}{\alpha\,\nu_1}\|\mathbf{u}\|_{L^3(\Omega)^3}|T|_{W^{1,3}(\Omega)}\Big)\sup_{i>i_0}|T-T_h^i|_{H^1(\Omega)} \leq 2|T-R_h(T)|_{H^1(\Omega)} \\ + \frac{S_6^0}{\alpha\,\nu_1}\|\mathbf{f}\|_{L^2(\Omega)^3}|T-R_h(T)|_{W^{1,3}(\Omega)} + \frac{S_6^0}{\alpha}\big(1+\frac{\nu_2}{\nu_1}\big)|T|_{W^{1,3}(\Omega)}\inf_{\mathbf{w}_h \in \mathcal{V}_{h,1}}\|\mathbf{u}-\mathbf{w}_h\|_{L^2(\Omega)^3}.\end{aligned} \tag{4.7}$$

In the second case, we have for all $i \geq 0$,

$$|T-T_h^i|_{H^1(\Omega)} > |T-T_h^{i+1}|_{H^1(\Omega)},$$

in which case the sequence of positive numbers $\big(|T-T_h^i|_{H^1(\Omega)}\big)_{i\geq 0}$ decreases monotonically and hence converges to some nonnegative limit. Since the sequence converges, we can pass to the limit in (4.6), thus obtaining

$$\begin{aligned}\Big(1-\frac{\lambda(S_6^0)^2}{\alpha\,\nu_1}\|\mathbf{u}\|_{L^3(\Omega)^3}|T|_{W^{1,3}(\Omega)}\Big)\lim_{i\to\infty}|T-T_h^i|_{H^1(\Omega)} \leq 2|T-R_h(T)|_{H^1(\Omega)} \\ + \frac{S_6^0}{\alpha\,\nu_1}\|\mathbf{f}\|_{L^2(\Omega)^3}|T-R_h(T)|_{W^{1,3}(\Omega)} + \frac{S_6^0}{\alpha}\big(1+\frac{\nu_2}{\nu_1}\big)|T|_{W^{1,3}(\Omega)}\inf_{\mathbf{w}_h \in \mathcal{V}_{h,1}}\|\mathbf{u}-\mathbf{w}_h\|_{L^2(\Omega)^3}.\end{aligned} \tag{4.8}$$

Since, for $\mathbf{u}$ in $L^3(\Omega)^3$ and T in $W^{1,3}(\Omega)$ the right-hand sides of both (4.7) and (4.8) tend to zero as h tends to zero, we deduce the following convergence:

Theorem 4.1. *We retain the assumptions of Theorem 3.4. Then the sequence* $(T_h^i)_{i\geq 0}$ *generated by* (4.3) *either satisfies* (4.7) *in which case for some* $i_0 \geq 0$,

$$\lim_{h\to 0}\sup_{i>i_0}|T-T_h^i|_{H^1(\Omega)} = 0,$$

or it satisfies (4.8)*, in which case*

$$\lim_{h\to 0}\lim_{i\to\infty}|T-T_h^i|_{H^1(\Omega)} = 0.$$

Remark 4.2. When the exact solution is sufficiently smooth and the mesh is quasi uniform so that global inverse inequalities hold, by restricting further the size of the data, we can prove a specific rate of convergence of the algorithm. □

5. Numerical results

To validate the theoretical results, we perform several numerical simulations using Freefem++ (see [14]). We consider a square domain $\Omega =]0,3[^2$. Each edge is divided into N equal segments so that Ω is divided into $2N^2$ triangles (see Figure 1).

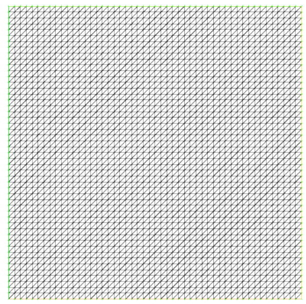

Figure 1. Geometry of the domain

We choose for exact solution $(\mathbf{u}, p, T) = (\mathbf{curl}\,\psi, p, T)$ where ψ, p and T are defined by

$$\psi(x,y) = e^{-\beta((x-1)^2+(y-1)^2)}, \tag{5.1}$$

$$p(x,y) = \cos(\frac{\pi}{3}x)\cos(\frac{\pi}{3}y), \tag{5.2}$$

and

$$T(x,y) = x^2(x-3)^2y^2(y-3)^2. \tag{5.3}$$

We henceforth take $\alpha = 3$, $\beta = 5$ and $N = 100$.
In Figures 2 and 3, we compare the numerical and the exact pressure, temperature and velocity for $\nu(T) = T + 1$ when the numerical solution is computed by using the first discrete scheme.

Figure 4 plots the global error curves versus h in logarithmic scales, global in the sense that they depict the sum of the velocity, pressure and temperature errors. The algorithm is tested as the number of segments increase from 30 to 120. The slope of the error's curve for the first discrete scheme is equal to 1.0036 for $\nu(T) = T + 1$, 0.9938 for $\nu(T) = e^{-T} + \dfrac{1}{10}$ and finally 0.9956 for $\nu(T) = \sin(T) + 2$. For the second discrete scheme, the slope is equal to 1.0122 for $\nu(T) = T + 1$, 0.9994 for $\nu(T) = e^{-T} + \dfrac{1}{10}$ and finally 1.0091 for $\nu(T) = \sin(T) + 2$.

Remark 5.1. Note that the error curves are consistent with the theoretical results in Section 3. $\square$

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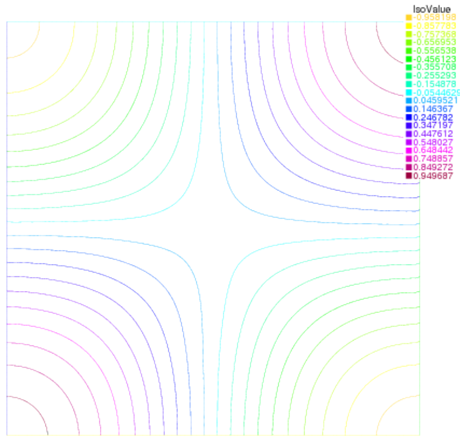
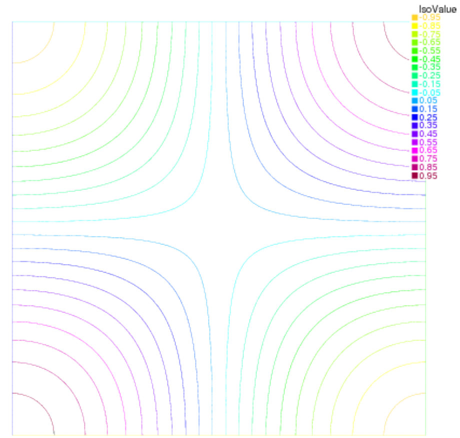
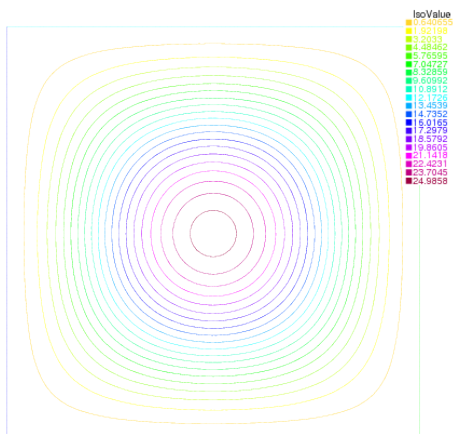


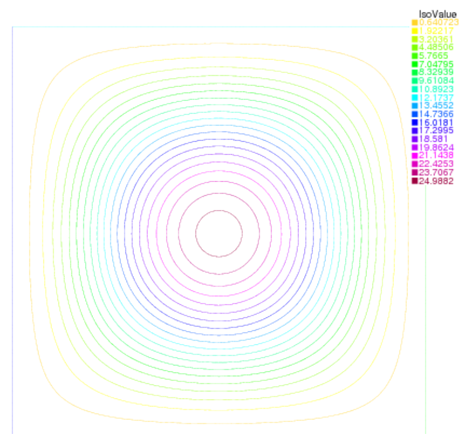
(A) Numerical pressure



(B) Exact pressure

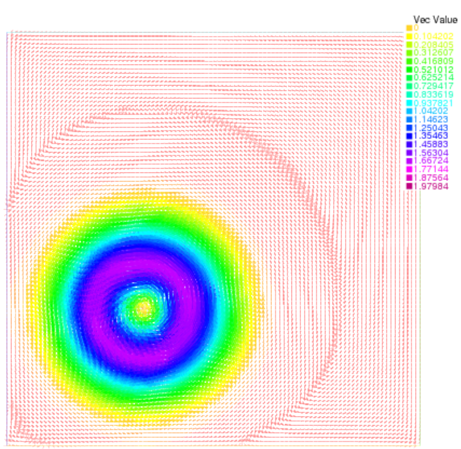


(C) Numerical temperature

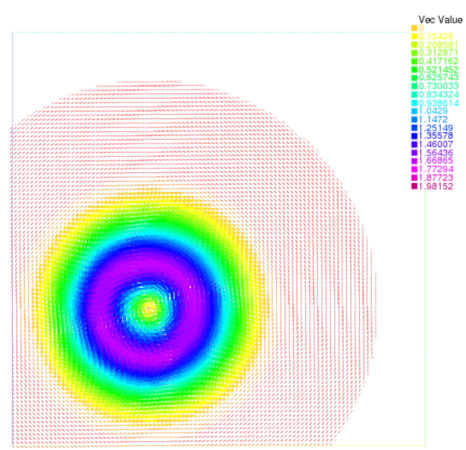


(D) Exact temperature

FIGURE 2. Comparison of numerical and exact solutions for $\nu(T) = T + 1$ for the first discrete scheme.

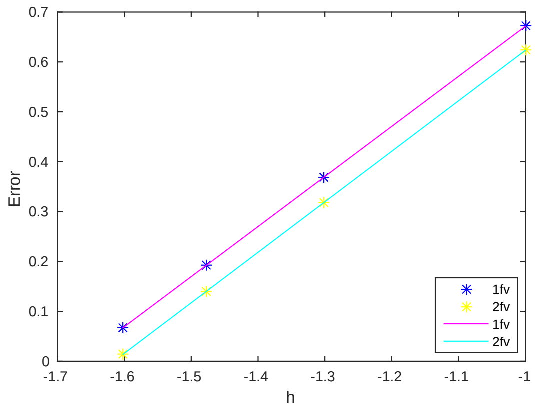


(A) Numerical velocity for $\nu(T) = T + 1$

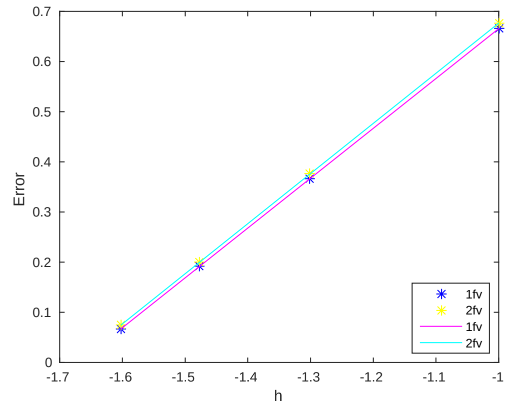


(B) Exact velocity for $\nu(T) = T + 1$

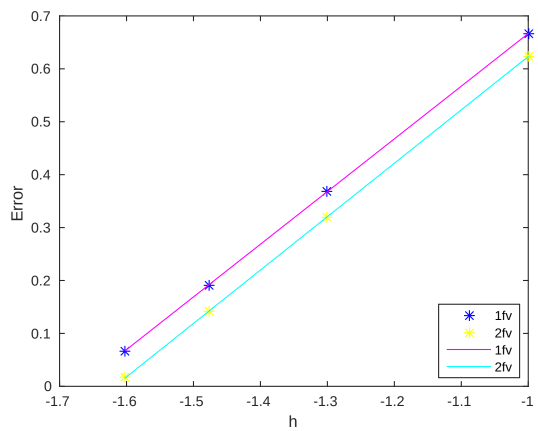
FIGURE 3. Comparison of numerical and exact velocity for $\nu(T) = T + 1$ for the first discrete scheme.



(A) $\nu(T) = T + 1$



(B) $\nu(T) = e^{-T} + \dfrac{1}{10}$



(C) $\nu(T) = \sin(T) + 2$

FIGURE 4. Error curve for different $\nu(T)$.

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