Mathematical analysis/Partial differential equations

Non-convex, non-local functionals converging to the total variation

Convergence de fonctionnelles non convexes et non locales vers la variation totale

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\textbf{ABSTRACT}

We present new results concerning the approximation of the total variation, $\int_{\Omega} |\nabla u|$, of a function $u$ by non-local, non-convex functionals of the form

$$
\Lambda_{\delta}(u) = \int_{\Omega} \int_{\Omega} \frac{\delta \varphi(|u(x) - u(y)|/\delta)}{|x - y|^{d+1}} \, dx \, dy,
$$

as $\delta \to 0$, where $\Omega$ is a domain in $\mathbb{R}^d$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function satisfying some appropriate conditions. The mode of convergence is extremely delicate, and numerous problems remain open. The original motivation of our work comes from Image Processing.

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\textbf{RÉSUMÉ}

Nous présentons des résultats nouveaux concernant l'approximation de la variation totale $\int_{\Omega} |\nabla u|$ d'une fonction $u$ par des fonctionnelles non convexes et non locales de la forme

$$
\Lambda_{\delta}(u) = \int_{\Omega} \int_{\Omega} \frac{\delta \varphi(|u(x) - u(y)|/\delta)}{|x - y|^{d+1}} \, dx \, dy,
$$

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1. Introduction

Let \( \varphi : [0, +\infty) \to [0, +\infty) \) be non-decreasing, and continuous on \( [0, +\infty) \) except at a finite number of points in \((0, +\infty)\). Assume that \( \varphi(0) = 0 \) and that \( \varphi(t) = \varphi(t^-) \) for all \( t > 0 \). Let \( \Omega \subset \mathbb{R}^d \) be a smooth bounded domain of \( \mathbb{R}^d \).

Given a measurable function \( u \) on \( \Omega \), and \( \delta > 0 \), we define the following non-local functionals:

\[
\Lambda(u) := \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x-y|^{d+1}} \, dx \, dy \leq +\infty \quad \text{and} \quad \Lambda_\delta(u) := \delta \Lambda(u/\delta).
\]

We make the following three basic assumptions on \( \varphi \):

\[
\varphi(t) \leq at^2 \text{ in } [0, 1] \quad \text{for some positive constant } a, \quad \text{(1)}
\]

\[
\varphi(t) \leq b \text{ in } \mathbb{R}_+ \quad \text{for some positive constant } b, \quad \text{(2)}
\]

and

\[
\gamma_d \int_0^\infty \varphi(t)t^{-2} \, dt = 1, \text{ where } \gamma_d := 2|B^{d-1}|; \quad \text{(3)}
\]

here \( B^{d-1} \) denotes the unit ball in \( \mathbb{R}^{d-1} \) and \( |B^{d-1}| \) denotes its \((d-1)\)-Hausdorff measure (with \( \gamma_d = 2 \) when \( d = 1 \)). Condition (3) is a normalization condition prescribed in order to have (7) below with constant 1 in front of \( \int_\Omega |\nabla u| \). Denote

\[
A = \{ \varphi; \ \varphi \text{ satisfies (1)--(3)} \}.
\]

Note that \( \Lambda \) is never convex when \( \varphi \in A \).

Here are three examples of functions \( \varphi \) that we have in mind. They all satisfy (1) and (2). In order to achieve (3), we choose \( \varphi = c_1 \tilde{\varphi}_1 \), where \( \tilde{\varphi}_1 \) is taken from the list below and \( c_1 \) is an appropriate constant:

\[
\tilde{\varphi}_1(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ 1 & \text{if } t > 1, \end{cases} \quad \text{and} \quad \tilde{\varphi}_2(t) = \begin{cases} t^2 & \text{if } t \leq 1 \\ 1 & \text{if } t > 1, \end{cases} \quad \text{and} \quad \tilde{\varphi}_3(t) = 1 - e^{-t^2}.
\]

Example 1 is extensively studied in [3,6,10–14] (see also [5,15]). Examples 2 and 3 are motivated by Image Processing (see [8,17]).

In this note, we are concerned with modes of convergence of \( \Lambda_\delta \) to the total variation as \( \delta \to 0 \). The convergence to the total variation of a sequence of convex non-local functionals \( J_\varepsilon \), defined by

\[
J_\varepsilon(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x-y|} \rho_\varepsilon(|x-y|) \, dx \, dy,
\]

where \( \rho_\varepsilon \) is a sequence of radial mollifiers, was originally analyzed by J. Bourgain, H. Brezis and P. Mironescu and thoroughly investigated in [1,2,4,9].

The asymptotic analysis of \( \Lambda_\delta \) is much more delicate than the one of \( J_\varepsilon \), because two basic properties satisfied by \( J_\varepsilon \) (which played an important role in [1]) are not fulfilled by \( \Lambda_\delta \):

i) there is no constant \( C \) such that

\[
\Lambda_\delta(u) \leq C \int_\Omega |\nabla u|, \quad \forall u \in C^1(\overline{\Omega}), \quad \forall \delta > 0,
\]

ii) \( \Lambda_\delta(u) \) is not a convex functional.
2. Statement of the main results

Concerning the pointwise limit of $\Lambda_\delta$ as $\delta \to 0$, i.e.
the convergence of $\Lambda_\delta(u)$ for fixed $u$, we prove that,
for every $\varphi \in A$,
$$
\Lambda_\delta(u) \text{ converges, as } \delta \to 0, \text{ to } TV(u) = \int_{\Omega} |\nabla u|, \quad \forall u \in \bigcup_{p>1} W^{1,p}(\Omega). (7)
$$

If $u \in W^{1,1}(\Omega)$, we can only assert that, for every $\varphi \in A$,
$$
\liminf_{\delta \to 0} \Lambda_\delta(u) \geq \int_{\Omega} |\nabla u|.
$$

Surprisingly, for every $d \geq 1$ and for every $\varphi \in A$, one can construct a function $u \in W^{1,1}(\Omega)$ such that
$$
\lim_{\delta \to 0} \Lambda_\delta(u) = +\infty.
$$

This kind of "pathology" was originally discovered by A. Ponce and presented in [10] for $\varphi = c_1 \tilde{\varphi}_1$ (for another example, see [7]). One may also construct (see [7]) functions $u \in W^{1,1}(\Omega)$ such that
$$
\liminf_{\delta \to 0} \Lambda_\delta(u) = \int_{\Omega} |\nabla u| \quad \text{and} \quad \limsup_{\delta \to 0} \Lambda_\delta(u) = +\infty.
$$

When dealing with functions $u \in BV(\Omega)$, the situation becomes even more intricate. It may happen, for some $\varphi \in A$ and some $u \in BV(\Omega)$, that
$$
\liminf_{\delta \to 0} \Lambda_\delta(u) < \int_{\Omega} |\nabla u|.
$$

All these facts suggest that the mode of convergence of $\Lambda_\delta$ to $TV$ as $\delta \to 0$ is delicate and that a theory of pointwise convergence is out of reach. It turns out that $\Gamma$-convergence (in the sense of E. De Giorgi) is the appropriate framework to analyze the asymptotic behavior of $\Lambda_\delta$ as $\delta \to 0$.

Our main result is the following.

**Theorem 1.** For every $\varphi \in A$, there exists a constant $K = K(\varphi) \in (0, 1]$, which is independent of $\Omega$, such that, as $\delta \to 0$,
$$
\Lambda_\delta \quad \Gamma\text{-converges to } \Lambda_0 \text{ in } L^1(\Omega),
$$
where $\Lambda_0$ is defined on $L^1(\Omega)$ by
$$
\Lambda_0(u) = K \int_{\Omega} |\nabla u| \text{ for } u \in BV(\Omega), \text{ and } +\infty \text{ otherwise.}
$$

The proof of Theorem 1 is extremely involved and it would be desirable to simplify it. When $\varphi = c_1 \tilde{\varphi}_1$ and $\Omega = \mathbb{R}^d$, Theorem 1 is originally due to H.-M. Nguyen [11,13]. One of the key ingredients was the following earlier result, basically due to J. Bourgain and H.-M. Nguyen [3, Lemma 2].

**Lemma 1.** Let $\Omega = (0, 1)$, $\varphi = c_1 \tilde{\varphi}_1$. There exists a constant $k > 0$ such that
$$
\liminf_{\delta \to 0} \Lambda_\delta(u) \geq k|u(t_2) - u(t_1)|,
$$
for every $u \in L^1(\Omega)$, and for all Lebesgue points $t_1, t_2 \in (0, 1)$ of $u$.

Furthermore, one can show that
$$
\inf_{\varphi \in A} K(\varphi) > 0.
$$

One of the most intriguing remaining questions is

**Open Problem 1.** Is it true that for every $\varphi \in A$, $K(\varphi) < 1$ in Theorem 1?
It has been proved in [11] (see also [7]) that $K(c_1 \psi_1) < 1$. However, the answer to Open Problem 1 is not known for $\varphi = c_2 \bar{\psi}_2$ and $\varphi = c_3 \bar{\psi}_3$, even when $d = 1$.

Motivated by questions arising in Image Processing (see, e.g., [7,8,16,17]), we consider the problem

$$\inf_{u \in \mathcal{E}(\Omega)} E_\delta(u),$$

where

$$E_\delta(u) = \lambda \int_\Omega |u - f|^q + \Lambda_\delta(u),$$  \hspace{1cm} (9)

$q \geq 1$, $f \in L^q(\Omega)$ is given, and $\lambda$ is a fixed positive constant. Our goal is twofold: investigate the existence of minimizers for $E_\delta$ (for fixed $\delta$) and analyze their behavior as $\delta \to 0$. The existence of a minimizer in (9) is not obvious since $\Lambda_\delta$ is not convex and one cannot invoke the standard tools of Functional Analysis. Our main result in this direction is the following.

**Theorem 2.** Assume that $\varphi \in \mathcal{A}$ and $\varphi(t) > 0$ for all $t > 0$. Let $q \geq 1$ and $f \in L^q(\Omega)$. For each $\delta > 0$, there exists a minimizer $u_\delta$ of (9). Moreover, $u_\delta \to u_0$ in $L^p(\Omega)$ as $\delta \to 0$, where $u_0$ is the unique minimizer of the functional $E_0$ defined on $L^q(\Omega) \cap BV(\Omega)$ by

$$E_0(u) := \lambda \int_\Omega |u - f|^q + K \int_\Omega |\nabla u|,$$

and $0 < K \leq 1$ is the constant coming from Theorem 1.

Note that the minimizers $u_\delta$ of (9) need not be unique, but the convergence assertion in Theorem 2 holds for any choice of minimizers. The proof of the existence of a minimizer for (9) relies on the following compactness lemma for fixed $\delta$, e.g., with $\delta = 1$.

**Lemma 2.** Let $\varphi \in \mathcal{A}$ be such that $\varphi(t) > 0$ for all $t > 0$, and let $(u_n)$ be a bounded sequence in $L^1(\Omega)$ such that

$$\sup_n \Lambda(u_n) < +\infty. \hspace{1cm} (11)$$

There exists a subsequence $(u_{n_k})$ of $(u_n)$ and $u \in L^1(\Omega)$ such that $(u_{n_k})$ converges to $u$ in $L^1(\Omega)$.

The proof of the convergence as $\delta \to 0$ in Theorem 2 relies heavily on the $\Gamma$-convergence of $\Lambda_\delta$ (Theorem 1), and also on the following compactness lemma (with roots in H.-M. Nguyen [14]).

**Lemma 3.** Let $\varphi \in \mathcal{A}$, $(\delta_n) \to 0$, and let $(u_n)$ be a bounded sequence in $L^1(\Omega)$ such that

$$\sup_n \Lambda_{\delta_n}(u_n) < +\infty. \hspace{1cm} (12)$$

There exists a subsequence $(u_{n_k})$ of $(u_n)$ and $u \in L^1(\Omega)$ such that $(u_{n_k})$ converges to $u$ in $L^1(\Omega)$.

The proofs of the results announced in this note are given in [7].

**References**


