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Mathematical analysis/Partial differential equations

Non-convex, non-local functionals converging to the total variation

Convergence de fonctionnelles non convexes et non locales vers la variation totale

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\textbf{Abstract}

We present new results concerning the approximation of the total variation, $\int_{\Omega} |\nabla u|$, of a function $u$ by non-local, non-convex functionals of the form

$$\Lambda_{\delta}(u) = \int_{\Omega} \int_{\Omega} \frac{\delta \varphi(|u(x) - u(y)|/\delta)}{|x-y|^{d+1}} \, dx \, dy,$$

as $\delta \to 0$, where $\Omega$ is a domain in $\mathbb{R}^d$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function satisfying some appropriate conditions. The mode of convergence is extremely delicate, and numerous problems remain open. The original motivation of our work comes from Image Processing.

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\textbf{Résumé}

Nous présentons des résultats nouveaux concernant l'approximation de la variation totale $\int_{\Omega} |\nabla u|$ d'une fonction $u$ par des fonctionnelles non convexes et non locales de la forme

$$\Lambda_{\delta}(u) = \int_{\Omega} \int_{\Omega} \frac{\delta \varphi(|u(x) - u(y)|/\delta)}{|x-y|^{d+1}} \, dx \, dy,$$

as $\delta \to 0$, where $\Omega$ is a domain in $\mathbb{R}^d$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function satisfying some appropriate conditions. The mode of convergence is extremely delicate, and numerous problems remain open. The original motivation of our work comes from Image Processing.

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1. Introduction

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be non-decreasing, and continuous on $[0, +\infty)$ except at a finite number of points in $(0, +\infty)$. Assume that $\varphi(0) = 0$ and that $\varphi(t) = \varphi(t-)$ for all $t > 0$. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain of $\mathbb{R}^d$. Given a measurable function $u$ on $\Omega$, and $\delta > 0$, we define the following non-local functionals:

$$\Lambda(u) := \int_{\Omega} \int_{\Omega} \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{d+1}} \, dx \, dy \leq +\infty \quad \text{and} \quad \Lambda_{\delta}(u) := \delta \Lambda(u/\delta).$$

We make the following three basic assumptions on $\varphi$:

$$\varphi(t) \leq at^2 \quad \text{in} \quad [0, 1] \quad \text{for some positive constant} \ a, \quad (1)$$

$$\varphi(t) \leq b \quad \text{in} \quad \mathbb{R}_+ \quad \text{for some positive constant} \ b, \quad (2)$$

and

$$\gamma_{d} \int_{0}^{\infty} \varphi(t) t^{-2} \, dt = 1, \quad \text{where} \quad \gamma_{d} := 2|B^{d-1}|; \quad (3)$$

here $B^{d-1}$ denotes the unit ball in $\mathbb{R}^{d-1}$ and $|B^{d-1}|$ denotes its $(d - 1)$-Hausdorff measure (with $\gamma_{d} = 2$ when $d = 1$). Condition (3) is a normalization condition prescribed in order to have (7) below with constant 1 in front of $\int_{\Omega} |\nabla u|$. Denote

$$A = \{ \varphi, \ \varphi \text{ satisfies (1)-(3)} \}. \quad (4)$$

Note that $\Lambda$ is never convex when $\varphi \in A$.

Here are three examples of functions $\varphi$ that we have in mind. They all satisfy (1) and (2). In order to achieve (3), we choose $\varphi = c_{i} \tilde{\varphi}_{i}$, where $\tilde{\varphi}_{i}$ is taken from the list below and $c_{i}$ is an appropriate constant:

$$\tilde{\varphi}_{1}(t) = \begin{cases} 0 & \text{if} \ t \leq 1 \\ 1 & \text{if} \ t > 1 \end{cases}, \quad \tilde{\varphi}_{2}(t) = \begin{cases} t^{2} & \text{if} \ t \leq 1 \\ 1 & \text{if} \ t > 1, \end{cases} \quad \text{and} \quad \tilde{\varphi}_{3}(t) = 1 - e^{-t^{2}}.$$

Example 1 is extensively studied in [3,6,10–14] (see also [5,15]). Examples 2 and 3 are motivated by Image Processing (see [8,17]).

In this note, we are concerned with modes of convergence of $\Lambda_{\delta}$ to the total variation as $\delta \to 0$. The convergence to the total variation of a sequence of convex non-local functionals $J_{\varepsilon}$, defined by

$$J_{\varepsilon}(u) = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|}{|x - y|} \rho_{\varepsilon}(|x - y|) \, dx \, dy, \quad (5)$$

where $\rho_{\varepsilon}$ is a sequence of radial mollifiers, was originally analyzed by J. Bourgain, H. Brezis and P. Mironescu and thoroughly investigated in [1,2,4,9].

The asymptotic analysis of $\Lambda_{\delta}$ is much more delicate than the one of $J_{\varepsilon}$, because two basic properties satisfied by $J_{\varepsilon}$ (which played an important role in [1]) are not fulfilled by $\Lambda_{\delta}$:

i) there is no constant $C$ such that

$$\Lambda_{\delta}(u) \leq C \int_{\Omega} |\nabla u| \quad \forall \ u \in C^{1} (\bar{\Omega}), \ \forall \ \delta > 0, \quad (6)$$

ii) $\Lambda_{\delta}(u)$ is not a convex functional.
2. Statement of the main results

Concerning the pointwise limit of $\Lambda_\delta$ as $\delta \to 0$, i.e. the convergence of $\Lambda_\delta(u)$ for fixed $u$, we prove that, for every $\varphi \in A$, $\Lambda_\delta(u)$ converges, as $\delta \to 0$, to $TV(u) = \int_\Omega |\nabla u| \ \forall u \in \bigcup_{p>1} W^{1,p}(\Omega)$. (7)

If $u \in W^{1,1}(\Omega)$, we can only assert that, for every $\varphi \in A$,

$$\liminf_{\delta \to 0} \Lambda_\delta(u) \geq \int_\Omega |\nabla u|.$$  

Surprisingly, for every $d \geq 1$ and for every $\varphi \in A$, one can construct a function $u \in W^{1,1}(\Omega)$ such that

$$\lim_{\delta \to 0} \Lambda_\delta(u) = +\infty.$$  

This kind of “pathology” was originally discovered by A. Ponce and presented in [10] for $\varphi = c_1 \tilde{\varphi}_1$ (for another example, see [7]). One may also construct (see [7]) functions $u \in W^{1,1}(\Omega)$ such that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u| \text{ and } \limsup_{\delta \to 0} \Lambda_\delta(u) = +\infty.$$  

When dealing with functions $u \in BV(\Omega)$, the situation becomes even more intricate. It may happen, for some $\varphi \in A$ and some $u \in BV(\Omega)$, that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) < \int_\Omega |\nabla u|.$$  

All these facts suggest that the mode of convergence of $\Lambda_\delta$ to $TV$ as $\delta \to 0$ is delicate and that a theory of pointwise convergence is out of reach. It turns out that $\Gamma$-convergence (in the sense of E. De Giorgi) is the appropriate framework to analyze the asymptotic behavior of $\Lambda_\delta$ as $\delta \to 0$.

Our main result is the following.

**Theorem 1.** For every $\varphi \in A$, there exists a constant $K = K(\varphi) \in (0, 1]$, which is independent of $\Omega$, such that, as $\delta \to 0$, $\Lambda_\delta$ $\Gamma$-converges to $\Lambda_0$ in $L^1(\Omega)$, where $\Lambda_0$ is defined on $L^1(\Omega)$ by

$$\Lambda_0(u) = K \int_\Omega |\nabla u| \text{ for } u \in BV(\Omega), \text{ and } +\infty \text{ otherwise.}$$  

The proof of **Theorem 1** is extremely involved and it would be desirable to simplify it. When $\varphi = c_1 \tilde{\varphi}_1$ and $\Omega = \mathbb{R}^d$, **Theorem 1** is originally due to H.-M. Nguyen [11,13]. One of the key ingredients was the following earlier result, basically due to J. Bourgain and H.-M. Nguyen [3, Lemma 2].

**Lemma 1.** Let $\Omega = (0, 1)$, $\varphi = c_1 \tilde{\varphi}_1$. There exists a constant $k > 0$ such that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) \geq k|u(t_2) - u(t_1)|,$$

for every $u \in L^1(\Omega)$, and for all Lebesgue points $t_1, t_2 \in (0, 1)$ of $u$.

Furthermore, one can show that

$$\inf_{\varphi \in A} K(\varphi) > 0.$$  

One of the most intriguing remaining questions is

**Open Problem 1.** Is it true that for every $\varphi \in A$, $K(\varphi) < 1$ in **Theorem 1**?
It has been proved in [11] (see also [7]) that \( K(c_1\hat{\varphi}_1) < 1 \). However, the answer to Open Problem 1 is not known for \( \varphi = c_2\hat{\varphi}_2 \) and \( \varphi = c_3\hat{\varphi}_3 \), even when \( d = 1 \).

Motivated by questions arising in Image Processing (see, e.g., [7,8,16,17]), we consider the problem

\[
\inf_{u \in L^1(\Omega)} E_\delta(u),
\]

where

\[
E_\delta(u) = \lambda \int_\Omega |u - f|^q + \Lambda_\delta(u),
\]

\( q \geq 1, f \in L^q(\Omega) \) is given, and \( \lambda \) is a fixed positive constant. Our goal is twofold: investigate the existence of minimizers for \( E_\delta \) (for fixed \( \delta \)) and analyze their behavior as \( \delta \to 0 \). The existence of a minimizer in (9) is not obvious since \( \Lambda_\delta \) is not convex and one cannot invoke the standard tools of Functional Analysis. Our main result in this direction is the following.

**Theorem 2.** Assume that \( \varphi \in A \) and \( \psi(t) > 0 \) for all \( t > 0 \). Let \( q \geq 1 \) and \( f \in L^q(\Omega) \). For each \( \delta > 0 \), there exists a minimizer \( u_\delta \) of (9). Moreover, \( u_\delta \to u_0 \) in \( L^q(\Omega) \) as \( \delta \to 0 \), where \( u_0 \) is the unique minimizer of the functional \( E_0 \) defined on \( L^q(\Omega) \cap BV(\Omega) \) by

\[
E_0(u) := \lambda \int_\Omega |u - f|^q + K \int_\Omega |\nabla u|,
\]

and \( 0 < K \leq 1 \) is the constant coming from Theorem 1.

Note that the minimizers \( u_\delta \) of (9) need not be unique, but the convergence assertion in **Theorem 2** holds for any choice of minimizers. The proof of the existence of a minimizer for (9) relies on the following compactness lemma for fixed \( \delta \), e.g., with \( \delta = 1 \).

**Lemma 2.** Let \( \varphi \in A \) be such that \( \varphi(t) > 0 \) for all \( t > 0 \), and let \((u_n)\) be a bounded sequence in \( L^1(\Omega) \) such that

\[
\sup_n \Lambda(u_n) < +\infty.
\]

There exists a subsequence \((u_{n_k})\) of \((u_n)\) and \( u \in L^1(\Omega) \) such that \((u_{n_k})\) converges to \( u \) in \( L^1(\Omega) \).

The proof of the convergence as \( \delta \to 0 \) in **Theorem 2** relies heavily on the \( \Gamma \)-convergence of \( \Lambda_\delta \) (Theorem 1), and also on the following compactness lemma (with roots in H.-M. Nguyen [14]).

**Lemma 3.** Let \( \varphi \in A, (\delta_n) \to 0 \), and let \((u_n)\) be a bounded sequence in \( L^1(\Omega) \) such that

\[
\sup_n \Lambda_{\delta_n}(u_n) < +\infty.
\]

There exists a subsequence \((u_{n_k})\) of \((u_n)\) and \( u \in L^1(\Omega) \) such that \((u_{n_k})\) converges to \( u \) in \( L^1(\Omega) \).

The proofs of the results announced in this note are given in [7].

**References**


