

# **Protected couplings and BPS dyons in half-maximal supersymmetric string vacua**

Guillaume Bossard, Charles Cosnier-Horeau, Boris Pioline

## **To cite this version:**

Guillaume Bossard, Charles Cosnier-Horeau, Boris Pioline. Protected couplings and BPS dyons in half-maximal supersymmetric string vacua. Modern Physics Letters B, 2016, 765, pp.377 - 381. 10.1016/j.physletb.2016.12.035 . hal-01419203

# **HAL Id: hal-01419203 <https://hal.sorbonne-universite.fr/hal-01419203v1>**

Submitted on 19 Dec 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



[Distributed under a Creative Commons Attribution 4.0 International License](http://creativecommons.org/licenses/by/4.0/)

ARTICLE IN PRESS

#### [Physics Letters B](http://dx.doi.org/10.1016/j.physletb.2016.12.035) ••• (••••) •••-•••



1 Contents lists available at [ScienceDirect](http://www.ScienceDirect.com/) 66

Physics Letters B 4 69

### $11$  Drotoctod couplings and DDS dyons in half maximal supersymmetric  $16$  $\frac{11}{12}$  Protected couplings and BPS dyons in half-maximal supersymmetric  $\frac{76}{77}$ 13 String vacua and the string vacual string vacual string vacual string vacual string vacual string vacual str 14 79

<sup>15</sup> Guillaume Bossard <sup>a</sup>, Charles Cosnier-Horeau <sup>a, b</sup>, Boris Pioline <sup>c, b</sup>  $16$  81

17 82 <sup>a</sup> *Centre de Physique Théorique, Ecole Polytechnique, Université Paris-Saclay, 91128 Palaiseau Cedex, France*

18 <sup>b</sup> Sorbonne Universités, Laboratoire de Physique Théorique et Hautes Energies, CNRS UMR 7589, Université Pierre et Marie Curie,

 $\frac{19}{2}$   $\frac{1000}{2}$   $\$ *4 place Jussieu, 75252 Paris Cedex 05, France*

20 85 <sup>c</sup> *CERN, Theoretical Physics Department, 1211 Geneva 23, Switzerland*

### <sup>22</sup> ARTICLE INFO ABSTRACT <sup>87</sup>

*Article history:* Received 11 November 2016 Accepted 12 December 2016 Available online xxxx Editor: N. Lambert

### 23 88

<sup>24</sup> Article history: **Example 20** We analyze four- and six-derivative couplings in the low energy effective action of  $D=3$  string vacua 25 90 with half-maximal supersymmetry. In analogy with an earlier proposal for the *(*∇*-)*<sup>4</sup> coupling, we 26 Accepted 12 December 2016 **by the Convert of Convertise Coupling** is given exactly by a manifestly U-duality invariant genus-two <sup>91</sup> 27 Billiance online XXXX<br>
modular integral. In the limit where a circle in the internal torus decompactifies, the  $\nabla^2(\nabla\Phi)^4$  coupling 92 28 Euron. N. Lambert **Figure 1 Figure 1 F** order *e<sup>−R</sup>*, from four-dimensional 1/4-BPS dyons whose worldline winds around the circle. Each of these sa <sub>30</sub> sontributions is weighted by a Fourier coefficient of a meromorphic Siegel modular form, explaining and s<sub>95</sub> extending standard results for the BPS index of 1/4-BPS dyons.

 $\degree$  2016 Published by Elsevier B.V. This is an open access article under the CC BY license  $\frac{97}{97}$ 33 98 [\(http://creativecommons.org/licenses/by/4.0/\)](http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP3.

> These protected couplings are analogues of the  $\mathcal{R}^4$  and  $\nabla^6 \mathcal{R}^4$ couplings in toroidal compactifications of type II strings, which have been determined exactly in [19,20] and in many subsequent works. Our motivation for studying these protected couplings in  $D = 3$  is that they are expected to encode the infinite spectrum of BPS black holes in  $D = 4$ , in a way consistent with the Uduality group  $G_3(\mathbb{Z})$ . The latter contains the four-dimensional Uduality group  $G_4(\mathbb{Z})$ , but is potentially far more constraining. Thus,

> tion functions', which do not suffer from the usual difficulties in defining thermodynamical partition functions in theories of quan-

> The fact that solitons in  $D = 4$  may induce instanton corrections to the quantum effective potential in dimension  $D = 3$ is well known in the context of gauge theories with compact

> supersymmetries, BPS dyons in four dimensions similarly correct the moduli space metric after reduction on a circle [23,

tum gravity, and are manifestly automorphic [21].

 $37$  $_{38}$  String vacua with half-maximal supersymmetry offer an inter-couplings in the low-energy effective action of three-dimensional  $_{\rm 103}$ <sub>39</sub> esting window into the non-perturbative regime of string theory string vacua with 16 supercharges, for all values of the moduli. 104 40 and the quantum physics of black holes, unobstructed by intri- These protected couplings are analogues of the  $\mathcal{R}^4$  and  $\nabla^{\text{o}}\mathcal{R}^4$  -105  $_{41}$  cacies present in vacua with less supersymmetry. In particular, couplings in toroidal compactifications of type II strings, which  $_{106}$  $42$  the low-energy effective action at two-derivative order does not have been determined exactly in [19,20] and in many subsequent  $107$  $_{43}$  receive any quantum corrections, and all higher-derivative interac- works. Our motivation for studying these protected couplings in  $_{108}$ 44 tions are expected to be invariant under the action of an arithmetic  $D=3$  is that they are expected to encode the infinite spectrum  $105$ 45 group  $G(\mathbb{Z})$ , known as the U-duality group, on the moduli space of BPS black holes in  $D=4$ , in a way consistent with the U-110 46 G/*K* of massless scalars [1–4]. This infinite discrete symmetry also duality group  $G_3(\mathbb{Z})$ . The latter contains the four-dimensional U-111 47 constrains the spectrum of BPS states, and allows to determine, and allity group  $G_4(\mathbb{Z})$ , but is potentially far more constraining. Thus, and the states, and allows to determine,  $G_4$ 48 for any values of the electromagnetic charges, the number of BPS these protected couplings provide analogues of 'black hole parti-<br>13 49 114 black hole micro-states (counted with signs) in terms of Fourier 50 coefficients of certain modular forms [5–7]. This property has been defining thermodynamical partition functions in theories of quan-51 used to confirm the validity of the microscopic stringy description tum gravity, and are manifestly automorphic [21]. The manifestive of the microscopic stringy description to the manifest of the manifest of the manifes  $52$  of BPS black holes at an exquisite level of precision, both for small  $\overline{10}$  ine fact that solitons in  $D=4$  may induce instanton cor-53 black holes (preserving half of the supersymmetries of the back-<br> $\frac{1}{2}$  rections to the quantum energy obtained in dimension  $D = 3$  and the supersymmetries of the supersymmetries of the supersymmetries of the back-54 spound) [8,9] and for large black holes (preserving a quarter of the the Samell Khown in the context or gauge theories with compact 119  $U(1)$  [22]. In the context of quantum field theories with 8 rigid 120<br>55  $\frac{1}{2}$ String vacua with half-maximal supersymmetry offer an interesting window into the non-perturbative regime of string theory for any values of the electromagnetic charges, the number of BPS same) [10–18].

56 11 In this letter, we shall exploit U-duality invariance and super- supersymmetries, BPS dyons in four dimensions similarly cor- $^{57}$  symmetry Ward identities to determine certain higher-derivative erect the moduli space metric after reduction on a circle [23,  $^{122}$ 58 **123** 123 123 123 124. In string vacua with 16 local supersymmetries, one similarly

62 boris.pioline@cern.ch (B. Pioline). The state of the form the six-derivative scalar couplings of the form the top of the form the six-derivative scalar couplings of the form the top of the form the six-derivative scalar [boris.pioline@cern.ch](mailto:boris.pioline@cern.ch) (B. Pioline).

63 128 <http://dx.doi.org/10.1016/j.physletb.2016.12.035>

64 0370-2693/© 2016 Published by Elsevier B.V. This is an open access article under the CC BY license [\(http://creativecommons.org/licenses/by/4.0/](http://creativecommons.org/licenses/by/4.0/)). Funded by SCOAP<sup>3</sup>. <sup>129</sup> 65 and the contract of the con

<sup>60</sup>  $\overline{f_{\text{p},\text{mid}}^{\text{25}}}$  and addresses: guillaume bossard@polytechnique.edu (C. Bossard) **derivative scalar couplings of the form**  $F_{abcd}(\Phi) \nabla \Phi^d \nabla \Phi^b \nabla \Phi^c \nabla \Phi^d$  125 *E-mail addresses:* [guillaume.bossard@polytechnique.edu](mailto:guillaume.bossard@polytechnique.edu) (G. Bossard),

<sup>61</sup> 126 in *D* = 3, while both 1/2-BPS and 1/4-BPS dyons in *D* = 4 [charles.cosnier-horeau@polytechnique.edu](mailto:charles.cosnier-horeau@polytechnique.edu) (C. Cosnier-Horeau),

2 *G. Bossard et al. / Physics Letters B* ••• *(*••••*)* •••*–*•••

RTICI E IN

2 67 derivatives, contracted so as to make a Lorentz scalar). In either 3 case, the contribution of a four-dimensional BPS state with elec-  $D_{ef}^cG_{ab,cd} = C_4\delta_{ef}G_{ab,cd} + C_5[\delta_{e})(aG_{b)(f,cd} + \delta_{e})(cG_{d)(f,ab}]$ 4 tric and magnetic charges  $(Q, P)$  is expected to be suppressed by  $+ C_1 \delta_{A} C_2 C_3 + \delta_{A} C_4 C_5 + 2 \delta_{A} (C_5 C_5 + \delta_{A} C_6 C_6 + \delta_{A} C_7 C_7 + \delta_{A} C_8 C_8 + \delta_{A} C_8 C_9 + \delta_{A} C_9 C_9 + \delta$  $e^{-2\pi R \mathcal{M}(Q, P)}$ , where  $\mathcal{M}(Q, P)$  is the BPS mass and *R* the radius  $e^{-2\pi R \mathcal{M}(Q, P)}$  is the BPS mass and *R* the radius  $e^{-2\pi R \mathcal{M}(Q, P)}$  is the BPS mass and *R* the radius  $e^{-2\pi R \mathcal{M}(Q, P)}$  is the BPS mass and <sup>6</sup> of the circle on which the four-dimensional theory is compacti-<br><sup>6</sup>  $+c_7|F_{abk(e)}F_{c,k(e)}F_{c,k(e)}F_{c,k(e)}|$ , 7 fied, and weighted by a suitable BPS index  $\Omega(Q, P)$  counting the  $\Omega(Q, P)$  counting the  $\Omega(Q, P)$ 8 number of BPS states with given charges. In addition, coupling  $D[e^{i\omega}D_f]^T F_{abcd} = 0$ ,  $D[e^{i\omega}D_f^T D_g]$ <sup>51</sup> $G_{ab,cd} = 0$ . (b) 73 <sup>9</sup> to gravity implies additional  $O(e^{-R^2/\ell_P^2})$  corrections from gravi-<br>
∴ The first two constraints are analogous to those derived in [30] <sup>10</sup> tational Taub-NUT instantons, which are essential for invariance for  $H^4$  and  $\nabla^2 H^4$  couplings in Type IIB string theory on K3. The <sup>75</sup>

<sup>12</sup> For simplicity we shall restrict attention to the simplest three- edge of perturbative contributions. <sup>13</sup> dimensional string vacuum with 16 supercharges, obtained by **18 and 18 and 18** <sup>14</sup> compactifying the ten-dimensional heterotic string on  $T^7$ . Our con-**1. Exact**  $(\nabla \Phi)^4$  **couplings in**  $D=3$   $T^8$ <sup>15</sup> struction can however be generalized to other half-maximal su-<sup>16</sup> persymmetric models with reduced rank [25] with some effort Based on the known one-loop contribution [31–33], it was pro-<sup>17</sup> [26]. The moduli space in three dimensions is the symmetric space [15] posed in [34] (a proposal revisited in [35]) that the four-derivative  $\frac{82}{3}$ <sup>18</sup>  $\mathcal{M}_3 = G_{24.8}$  [28], where  $G_{p,q} = O(p,q)/O(p) \times O(q)$  denotes the scalar coupling  $F_{abcd}$  is given exactly by the genus-one modular <sup>83</sup> 19 orthogonal Grassmannian of *q*-dimensional positive planes in  $\mathbb{R}^{p,q}$ . Integral and the state of <sup>20</sup> In the limit where the heterotic string coupling  $g_3$  becomes small,  $\frac{24.8}{\pi}$  do.do.  $\frac{34}{4}$  |  $\frac{1}{24.8}$ persymmetric models with reduced rank [25] with some effort [26]. The moduli space in three dimensions is the symmetric space  $\mathcal{M}_3 = G_{24,8}$  [28], where  $G_{p,q} = O(p,q)/O(p) \times O(q)$  denotes the

$$
G_{24,8} \to \mathbb{R}^+ \times G_{23,7} \times \mathbb{R}^{30}, \tag{1}
$$

 $_{25}$  where the first factor corresponds to g<sub>3</sub>, the second factor to the *SL*(2, Z) on the Poincaré upper half-plane,  $Δ = η<sup>24</sup>$  is the unique so <sub>26</sub> Narain moduli space (parametrizing the metric, B-held and gauge cusp form of weight 12, and *Γ*<sub>24,8</sub> is the partition function of the 91  $\frac{27}{27}$  field on  $T^7$ ), and  $\mathbb{R}^{30}$  to the scalars  $a^I$  dual to the gauge fields non-perturbative Narain lattice, 28 in three dimensions. At each order in  $g_3^2$ , the low-energy effec-<br>33 <sub>29</sub> tive action is known to be invariant under the T-duality group  $\Gamma_{24,8} = \rho_2^4$   $\sum e^{i\alpha} \epsilon_1^{\mu} e^{i\alpha} \epsilon_1^{\mu} e^{i\alpha} \epsilon_1^{\mu} e^{i\alpha} \epsilon_1^{\mu} e^{i\alpha}$  $\overline{a_3}$   $O(23, 7, \mathbb{Z})$ , namely the automorphism group of the even self-dual  $\overline{Q \in A_{24,8}}$ 31 96  $\sigma_{23}$   $\sigma_{23,7}$  by left multiplication and on the last factor in (1) by the where  $Q_L \equiv p_L^I Q_I$ ,  $Q_R \equiv p_R^I Q_I$ , and  $|Q|^2 = Q_L^2 - Q_R^2$  takes even by  $Q_L$  $\frac{33}{24}$  extended to  $G_3(\mathbb{Z}) = 0$  (24, 8,  $\mathbb{Z}$ ), the automorphism group of the  $1/\Delta = \sum_{m \geq -1} c(m) q^m$  count the number of 1/2-BPS states in the  $\frac{98}{90}$ extended to  $G_3(\mathbb{Z}) = O(24, 8, \mathbb{Z})$ , the automorphism group of the  $1/\Delta = \sum_{m \geq -1} c(m) q^m$  count the number of 1/2-BPS states in the 35 'non-perturbative Narain lattice'  $\Lambda_{24,8} = \Lambda_{23,7} \oplus \Lambda_{1,1}$ , where  $\Lambda_{1,1}$   $D=4$  vacuum obtained by decompactifying a circle inside T'. This <sub>100</sub>  $_{36}$  is the standard even-self dual lattice of signature  $(1,1)$  [29]. Is obvious from the fact that these states are dual to perturbative  $_{101}$ where the first factor corresponds to *g*3, the second factor to the Narain moduli space (parametrizing the metric, *B*-field and gauge Narain lattice  $\Lambda_{23.7}$  [27]. The latter leaves  $g_3$  invariant, acts on *G*<sub>23</sub><sup>7</sup> by left multiplication and on the last factor in (1) by the defining representation. U-duality postulates that this symmetry is

 $_{37}$  In the limit where the radius *R* of one circle of the internal string states carrying only left-moving excitations [5,8]. It is also  $_{102}$ torus becomes large,  $\mathcal{M}_3$  instead decomposes as

$$
G_{24,8} \to \mathbb{R}^+ \times [G_{2,1} \times G_{22,6}] \times \mathbb{R}^{56+1},
$$
 (2)

 $\frac{42}{100}$   $\frac{1}{100}$   $\frac{1}{100}$   $\frac{1}{100}$   $\frac{1}{100}$   $\frac{1}{100}$   $\frac{1}{100}$   $\frac{1}{100}$   $\frac{1}{100}$   $\frac{1}{100}$  or  $q < b$ , and is defined for  $q \ge 6$  by a suitable regularization  $\frac{107}{107}$  $A^2$   $R^2/\ell_p^2$  (with  $\ell_H$  being the heterotic string scale and  $g_4$  the string<br>coupling in  $D = 4$ ), the second correspond to the moduli space<br>generally prescription. For  $q \le 7$ , the modular integral  $F_{abcd}^{(q+16,q)}$  co  $\frac{44}{10}$  one-loop contribution to the  $F^4$  coupling in heterotic string com-<br>A *l*, in A dimensions the third factor to the holonomies all a<sup>21</sup> one-loop contribution to the  $F^4$  coupling in heterotic string com-45 of the 28 electric grups fields and their magnetic duals along the pactified on  $T^q$  [31–33]. For any value of *q*, it can be checked that <sup>110</sup> 46 111 <sup>46</sup> of the 26 electric gauge fields and their magnetic duals along the<br>
<sup>47</sup> circle, along with the NUT potential  $\psi$ , dual to the Kaluza–Klein  $F_{abcd}^{(q+16,q)}$  satisfies (4) and (6) with  $c_1 = \frac{2-q}{4}$ ,  $c_2 = 4-q$ ,  $c_3 =$ 48 sange heral the factor  $\sigma_{2,1} = 52(27)$  (1) to parametric by the settic Ward identity, which is manifestly invariant under  $G_3(\mathbb{Z})$ . <sup>113</sup> axio-dilaton  $S = S_1 + iS_2 = B + i/g^2$ , while  $G_{2,6}$  is the Narain metric Ward  $\frac{49}{100}$  and the set of  $T^6$  with coordinates  $\frac{4}{5}$ . In the limit  $R \rightarrow \infty$  the set of the parabolic spansion at weak coupling (corresponding to the parabolic spansion  $\frac{1}{50}$  in the number of the number  $\frac{1}{50}$  in the number of  $\frac{1}{50}$  and  $\frac{1}{50}$  decomposition (1), such that the non-perturbative Narain lattice  $\frac{115}{50}$ 51 5 duality group is broken to  $5E(z, \omega) \times 6$  ( $\epsilon z$ ,  $\sigma$ ,  $\omega$ ), where the first  $\Lambda_{24,8}$  degenerates to  $\Lambda_{23,7} \oplus \Lambda_{1,1}$ ) can be computed using the <sup>116</sup> factor SL(2 %) is the famous S-duality in four dimensions where the first factor now corresponds to  $R^2/(g_4^2 \ell_H^2) = R/(g_3^2 \ell_H) =$  $R^2/\ell_P^2$  (with  $\ell_H$  being the heterotic string scale and  $g_4$  the string coupling in  $D = 4$ ), the second correspond to the moduli space  $\mathcal{M}_4$  in 4 dimensions, the third factor to the holonomies  $a^{11}, a^{21}$ of the 28 electric gauge fields and their magnetic duals along the circle, along with the NUT potential *ψ*, dual to the Kaluza–Klein gauge field. The factor  $G_{2,1} \cong SL(2)/U(1)$  is parametrized by the axio-dilaton  $S = S_1 + iS_2 = B + i/g_4^2$ , while  $G_{22,6}$  is the Narain moduli space of  $T^6$ , with coordinates  $\phi$ . In the limit  $R \to \infty$ , the U-duality group is broken to  $SL(2,\mathbb{Z}) \times O(22,6,\mathbb{Z})$ , where the first factor  $SL(2, \mathbb{Z})$  is the famous S-duality in four dimensions [1,2].

 $\frac{118}{53}$  118<br>and C<sub>1</sub>, must satisfy supersymmetric Ward identities To state of the indices abcd lies along  $\Lambda_{1,1}$ :  $_{54}$  and  $G_{ab,cd}$  must satisfy supersymmetric Ward identities. To state  $\frac{G_{ab,cd}}{G_{ab,cd}}$  and  $G_{ab,cd}$  must satisfy supersymmetric Ward identities. To state 55 them, we introduce the covariant derivative  $\mathcal{D}_{a\hat{b}}$  on the Grassman-<br>  $\mathcal{D}_{a\hat{b}}$  on the Grassman-<br>  $\mathcal{D}_{a\hat{b}}$  on the Grassman-56 nian  $G_{p,q}$ , defined by its action on the projectors  $p_{L,q}^I$  and  $p_{R,\hat{a}}^I$   $F_{\alpha\beta\gamma\delta}^{(\alpha4,0)} = \frac{g_{\alpha\beta\delta}^I}{16\pi\sigma^4}\delta_{(\alpha\beta}\delta_{\gamma\delta)} + \frac{g_{\beta\gamma\delta}^I}{\sigma^2} + 4\sum_{\alpha\beta}^I \sum_{\alpha\beta\gamma\delta}^{} P_{\alpha\beta\gamma\delta}^{(\alpha)}$ 57 122 on the time-like *p*-plane and its orthogonal complement (here and 58 below,  $a, b,..., \hat{a}, \hat{b}...$ , I, J... take values 1 to p, q, and  $p + q$ , respec-<br>58 below,  $a, b,..., \hat{a}, \hat{b}...$ , I, J... take values 1 to p, q, and  $p + q$ , respec-59 tively):  $\times c(Q) g_2^{2\kappa-3} |\sqrt{2}Q_R|^{\kappa-2} K_{1/2} |\sqrt{2}Q_R|^{1/2} \leq K_{1/2} |\sqrt{2}Q_R| |\sqrt{2}Q_R|^{1/2}$ Besides being automorphic under  $G_3(\mathbb{Z})$ , the couplings  $F_{abcd}$ tively):

$$
\mathcal{D}_{ab} \ p_{L,c}^I = \frac{1}{2} \delta_{ac} \ p_{R,\hat{b}}^I \ , \quad \mathcal{D}_{ab} \ p_{R,\hat{c}}^I = \frac{1}{2} \delta_{\hat{b}\hat{c}} \ p_{L,a}^I \ . \tag{3}
$$
 where  $c_0 = 24$  is the constant term in  $1/\Delta$ ,  $\Lambda^* = \Lambda \setminus \{0\}$ ,  $P_{\alpha\beta\gamma\delta}^{(1)}(Q)$ 

64 the Laplacian on  $G_{p,q}$ . On-shell linearized superspace methods in-<br> $\frac{\partial (\alpha \beta \sigma \gamma \delta)}{\partial \alpha}$ ,  $\frac{\partial$ The trace of the operator  $\mathcal{D}^2_{ef} = \mathcal{D}_{(e}{}^{\hat{g}} \mathcal{D}_{f)\hat{g}}$  is equal to (1/2 times) dicate that *Fabcd* and *Gab,cd* have to satisfy [26]

$$
G_{ab,cd}(\Phi) \nabla (\nabla \Phi^a \nabla \Phi^b) \nabla (\nabla \Phi^c \nabla \Phi^d) \quad \text{(here, } \nabla \text{ denote space-time} \qquad \mathcal{D}_{ef}^2 F_{abcd} = c_1 \delta_{ef} F_{abcd} + c_2 \delta_{e)(a} F_{bcd}(f) + c_3 \delta_{(ab} F_{cd)ef} \tag{4}
$$

$$
\mathcal{D}_{ef}^{2} G_{ab,cd} = c_{4} \delta_{ef} G_{ab,cd} + c_{5} \left[ \delta_{e}(a G_{b)(f,cd} + \delta_{e})(c G_{d)(f,ab}) \right] \n+ c_{6} \left[ \delta_{ab} G_{ef,cd} + \delta_{cd} G_{ef,ab} - 2 \delta_{a}(c G_{ef,d)(b)} \right]
$$
\n(5)

$$
+ c_7 \left[ F_{abk(e} F_{f)cd}^{\ \ k} - F_{c)ka(e} F_{f)bd}^{\ \ k} \right],
$$

$$
\mathcal{D}_{[e}{}^{[\hat{e}}\mathcal{D}_{f]}{}^{\hat{f}]}F_{abcd} = 0 \,, \quad \mathcal{D}_{[e}{}^{[\hat{e}}\mathcal{D}_{f}{}^{\hat{f}}\mathcal{D}_{g]}{}^{\hat{g}]}G_{ab,cd} = 0 \,. \tag{6}
$$

<sup>11</sup> under  $G_3(\mathbb{Z})$  (here,  $\ell_P$  is the Planck length in four dimensions). In numerical coefficients  $c_1,...c_7$  will be fixed below from the knowl-The first two constraints are analogous to those derived in [30] for  $H^4$  and  $\nabla^2 H^4$  couplings in Type IIB string theory on K3. The numerical coefficients  $c_1$ , ... $c_7$  will be fixed below from the knowledge of perturbative contributions.

### **1. Exact**  $(\nabla \Phi)^4$  **couplings** in  $D = 3$

integral

In the limit where the heterotic string coupling 
$$
g_3
$$
 becomes small,  
\n
$$
F_{abcd}^{(24,8)} = \int_{\mathcal{F}_1} \frac{d\rho_1 d\rho_2}{\rho_2^2} \frac{\partial^4}{(2\pi i)^4 \partial y^a \partial y^b \partial y^c \partial y^d} \Big|_{y=0} \frac{\Gamma_{24,8}}{\Delta}
$$
\n(7)

24 where  $\mathcal{F}_1$  is the standard fundamental domain for the action of 89 non-perturbative Narain lattice,

$$
\Gamma_{24,8} = \rho_2^4 \sum_{Q \in \Lambda_{24,8}} e^{i\pi Q_L^2 \rho - i\pi Q_R^2 \bar{\rho} + 2\pi i Q_L \cdot y + \frac{\pi (y \cdot y)}{2\rho_2}}
$$
(8)

 $38$  103 becomes large,  $\sqrt{13}$  instead decomposes as  $38$  and  $30$  and  $30$ 39 104 eral class of modular integrals, which we shall denote by *<sup>F</sup> (q*+16*,q) abcd* ,  $\mu_{40}$   $G_{24,8} \rightarrow \mathbb{R}^+ \times [G_{2,1} \times G_{22,6}] \times \mathbb{R}^{30+1}$ ,  $(2)$  where the lattice  $\Lambda_{24,8}$  is replaced by an even self-dual lattice  $\Lambda_{105}$ 41 where the first factor new corresponds to  $R^2/(\sigma^2 \ell^2)$   $R^2/\langle \sigma^2 \ell^2 \rangle$   $R^2/\langle \sigma^2 \ell^2 \rangle$   $R^2/\langle \sigma^2 \ell^2 \rangle$   $R^2$  and the factor  $\rho_2^2$  by  $\rho_2$  . The integral  $F_{abcd}$  converges the values on  $\Lambda_{24,8}$ . It will be important that the Fourier coefficients of  $D = 4$  vacuum obtained by decompactifying a circle inside  $T<sup>7</sup>$ . This is obvious from the fact that these states are dual to perturbative string states carrying only left-moving excitations  $[5,8]$ . It is also worth noting that the ansatz  $(7)$  is a special case of a more gen- $\Lambda_{q+16,q}$  and the factor  $\rho_2^4$  by  $\rho_2^{q/2}$ . The integral  $F_{abcd}^{(q+16,q)}$  converges for  $q < 6$ , and is defined for  $q \ge 6$  by a suitable regularization

 $\frac{52}{15}$  1 according to the sum of the sum of the southpast  $\frac{1}{2}$ . Standard unfolding trick. For simplicity, we shall assume that none 117

$$
F_{\alpha\beta\gamma\delta}^{(24,8)} = \frac{c_0}{16\pi g_3^4} \delta_{(\alpha\beta}\delta_{\gamma\delta)} + \frac{F_{\alpha\beta\gamma\delta}^{(23,7)}}{g_3^2} + 4 \sum_{k=1}^3 \sum_{\mathcal{Q} \in \Lambda_{23,7}^{\star}} P_{\alpha\beta\gamma\delta}^{(k)}
$$
(9)

58 below, *a*, *b*,..., *a*, *b*..., *h*, *J*... take values 1 to *p*, *q*, and *p* + *q*, respectively):  
\n59 (a) 
$$
g_3^{2k-9} |\sqrt{2}Q_R|^{k-\frac{7}{2}} K_{k-\frac{7}{2}} \left(\frac{2\pi}{g_3^2} |\sqrt{2}Q_R|\right) e^{-2\pi i a^T Q_I}
$$

62  $\sqrt{2}$  127 <sup>62</sup><br>63 The trace of the operator  $\mathcal{D}^2_{\alpha f} = \mathcal{D}_{\alpha} e^{\hat{g}} \mathcal{D}_{f \hat{g}}$  is equal to (1/2 times)  $= Q_{L\alpha} Q_{L\beta} Q_{L\gamma} Q_{L\delta}$ ,  $P^{(2)}_{\alpha\beta\gamma\delta} = -\frac{3}{2\pi} \delta_{(\alpha\beta} Q_{L\gamma} Q_{L\delta)}$ ,  $P^{(3)}_{\alpha\beta\gamma\delta} = \frac{3}{16\pi^2} \times 128$ 65 dicate that  $F_{abcd}$  and  $G_{ab,cd}$  have to satisfy [26] behaving as  $\sqrt{\frac{\pi}{2z}}e^{-z}(1+\mathcal{O}(1/z))$  for large positive values of *z*, and 130 *δ*<sub>( $\alpha \beta \delta$ γ*δ*),  $K$ *ν*</sub>(*z*) is the modified Bessel function of the second kind,

Please cite this article in press as: G. Bossard et al., Protected couplings and BPS dyons in half-maximal supersymmetric string vacua, Phys. Lett. B (2016), http://dx.doi.org/10.1016/j.physletb.2016.12.035

*G. Bossard et al. / Physics Letters B* ••• *(*••••*)* •••*–*••• 3

$$
\bar{c}(Q) = \sum_{d|Q} dc\left(-\frac{|Q|^2}{2d}\right). \tag{10}
$$

After rescaling from Einstein frame to string frame, the first and second terms in (9) are recognized as the tree-level and oneloop  $(∇Φ)^4$  coupling in perturbative heterotic string theory, while the remaining terms correspond to NS5-brane and KK5-branes wrapped on any possible  $T^6$  inside  $T^7$  [34].

10 75 degenerates to 22*,*<sup>6</sup> ⊕ 2*,*2), we get instead (in units where 11 degenerates to  $\Lambda_{2,2}$ ,  $\sigma$   $\Lambda_{2,2}$ , we get instead (in times where  $\sigma$  For any value of *q*, one can show that  $G_{ab,cd}^{(q+16,q)}$  satisfies (5)  $\sigma$ <sub>77</sub> In the large radius limit (corresponding to the parabolic decomposition  $(2)$ , such that the non-perturbative Narain lattice  $\Lambda_{24,8}$  $\ell_p = 1$ 

$$
F_{\alpha\beta\gamma\delta}^{(24,8)} = R^2 \left( \frac{c_0}{16\pi} \widehat{E}_1(S) \, \delta_{(\alpha\beta} \delta_{\gamma\delta)} + F_{\alpha\beta\gamma\delta}^{(22,6)} \right) \tag{11}
$$
\n
$$
+ 4 \sum_{k=1}^3 \sum_{Q' \in \Lambda_{22,6}^*} \sum_{m,n}^{\prime} c \left( -\frac{|Q'|^2}{2} \right) R^{5-k} P_{\alpha\beta\gamma\delta}^{(k)}
$$
\n
$$
K_{k-\frac{7}{2}} \left( \frac{2\pi R|mS+n|}{\sqrt{5_2}} |\sqrt{2}Q'_R| \right) e^{-2\pi i (ma^1 + na^2) \cdot Q'} + \dots
$$

22 nates from the dimensional reduction of the  $\mathcal{R}^2$  and  $F^4$  cou-<br>22 nates from the dimensional reduction of the  $\mathcal{R}^2$  and  $F^4$  cou-23 plings in  $D = 4$  [33,36], after dualizing the gauge fields into scalars.  $G^{(23,7)}_{\alpha\beta}$   $\delta_{\alpha\beta} G^{(23,7)}_{\alpha\beta} + \delta_{\gamma\delta} G^{(23,7)}_{\alpha\beta} - 2\delta_{\gamma\delta} G^{(23,7)}_{\alpha\beta}$ 24 The term  $F^{(22,6)}_{\alpha\beta\gamma\delta}$  can also be traced to the four-derivative scalar  $G^{(24,8)}_{\alpha\beta,\gamma\delta} = \frac{(\alpha\beta\gamma\gamma\delta}{g_A^4} - \frac{(\alpha\beta\gamma\gamma\delta)^2}{g_A^4} - \frac{(\alpha\beta\gamma\delta)^2}{g_A^4} - \frac{(\alpha\beta\gamma\delta)^2}{g_A^4} - \frac{(\alpha\beta\gamma\delta)^2}{g_A^4}$ 25  $\alpha \rho \gamma_0$  90  $26 \t 37 \t 77 \t 84(0, 8)$ 27  $\alpha(\gamma \delta) \beta + \cdots$  (15)  $\alpha$  is the mass of a four anneholder. 28  $^{28}$  The phase factor is the expected minimal coupling of a dyonic state  $29$  and the holonomies of the electric and magnetic gauge fields along where  $29$  $\frac{30}{10}$   $\frac{1}{10}$   $\frac{30}{10}$   $\frac{30$ the circle. Fixing charges  $(Q, P)$  such that  $Q$  and  $P$  are collinear,<br>31 couplings studied in [32]. The second term in (11) is of or-der *e*−2*<sup>π</sup> <sup>R</sup>*M*(<sup>Q</sup> ,P)* , where M is the mass of a four-dimensional 1/2-BPS state with electromagnetic charges  $(Q, P) = (mQ', nQ'),$ 

$$
\mu(Q, P) = \sum_{d|(Q, P)} c\left(-\frac{\gcd(Q^2, P^2, Q \cdot P)}{2d^2}\right),
$$
\n
$$
\text{with } \widehat{E}_2 = \frac{12}{\pi} \partial_\rho \log \eta - \frac{3}{\pi \omega} \text{ the almost holomorphic Eisenstein sequence.}
$$
\n
$$
\mu(Q, P) = \sum_{d|(Q, P)} c\left(-\frac{\gcd(Q^2, P^2, Q \cdot P)}{2d^2}\right),
$$
\n
$$
\text{with } \widehat{E}_2 = \frac{12}{\pi} \partial_\rho \log \eta - \frac{3}{\pi \omega} \text{ the almost holomorphic Eisenstein sequence.}
$$

tions will be discussed in [26].

### **2. Exact**  $\nabla^2(\nabla\Phi)^4$  couplings in  $D=3$

 $_{49}$  ties and the known two-loop contribution [37,38], it is natural to tribute to this coupling. expected to receive both 1/2-BPS and 1/4-BPS instanton contributions. Based on U-duality invariance, supersymmetric Ward identiconjecture that *Gab,cd* is given by the genus-two modular integral

$$
G_{ab,cd}^{(24,8)} = \int_{\mathcal{F}_2} \frac{d^3 \Omega_1 d^3 \Omega_2}{|\Omega_2|^3} \frac{\frac{1}{2} (\varepsilon_{ik} \varepsilon_{jl} + \varepsilon_{il} \varepsilon_{jk}) \partial^4}{(2\pi i)^4 \partial y_i^a \partial y_j^b \partial y_k^c \partial y_l^d}\bigg|_{y=0} \frac{\Gamma_{24,8,2}}{\Phi_{10}}, (13) \qquad \begin{array}{c} \text{and this in } D = 3, \text{ let us now analyze its large radius limit, where} \\ \Lambda_{24,8} \text{ degenerates to } \Lambda_{22,6} \oplus \Lambda_{2,2}. \text{ Again, the unfolding trick gives} \\ G_{ab,cd}^{(24,8)} = \int_{\mathcal{F}_2} \frac{d^3 \Omega_1 d^3 \Omega_2}{(2\pi i)^4 \partial y_i^a \partial y_j^b \partial y_k^c \partial y_l^d}\bigg|_{y=0} \frac{\Gamma_{24,8,2}}{\Phi_{10}}, (13) \qquad \begin{array}{c} \text{and } \\ \Lambda_{24,8} \text{ degenerates to } \Lambda_{22,6} \oplus \Lambda_{2,2}. \text{ Again, the unfolding trick gives} \end{array} \end{array}
$$

 $\begin{bmatrix} 6 \end{bmatrix}$ , and  $\Gamma_{24,8,2}$  is the genus-two partition function of the non-<br>60 perturbative Narain lattice,

$$
\Gamma_{24,8,2} = |\Omega_2|^4 \sum_{Q^i \in \Lambda_{24,8}^{\otimes 2}} e^{i\pi (Q^i_L \Omega_{ij} Q^j_L - Q^i_R \bar{\Omega}_{ij} Q^j_R + 2Q^i_L y_i) + \frac{\pi}{2} y^a_i \Omega_2^{-1ij} y_{ja}}
$$
\n(14)

 $\bar{c}(Q) = \sum d c \left(-\frac{|Q|^2}{2d}\right)$ . (10) Acting with the  $y_i^a$ -derivatives results in the insertion of a poly- 66  $\frac{d}{d}$   $\frac{d}{d}$  **h**  $\frac{d}{d}$  **c**  $\frac{d}{d}$  *c*  $\frac{d}{d$ 3 68 After rescaling from Einstein frame to string frame, the first and ond arguments. We shall denote by  $G_{ab,cd}^{(q+16,q)}$  the analogue of (14) 68  $5 - \frac{1}{10}$   $\frac{1}{10}$   $\frac{1}{10}$  $\frac{1}{6}$  loop  $(\nabla \Phi)^4$  coupling in perturbative heterotic string theory, while  $\frac{1}{2}$  by  $q/2$ . The integral  $G_{ab,cd}^{(q+16,q)}$  is convergent for  $q < 6$ , and de-<br>the remaining terms correspond to NSE hange and *WE* ha  $7 \frac{\text{m}}{\text{m}}$  is the set of  $\frac{1}{2}$  of  $\frac{1}{2}$  is the state and the states inner for  $q \ge 0$  by a suitable regularization prescription  $\frac{140}{10}$ . For  $\frac{72}{2}$  $\alpha$  8  $\alpha$  8  $\alpha$  8  $\alpha$  8  $\alpha$  8  $\alpha$  8  $\alpha$ 8 9  $\mu$  and  $\mu$ where the lattice  $\Lambda_{24,8}$  is replaced by  $\Lambda_{q+16,q}$  and the power of fined for  $q \ge 6$  by a suitable regularization prescription [40]. For *q*  $\leq$  7, the modular integral  $G_{ab,cd}^{(q+16,q)}$  controls the two-loop contri-[37,38].

12 77 13  $F^{(24,8)} = P^2 \left( \frac{C_0}{F_1(S) \delta_{(1,0)} \delta_{(1,0)}} + F^{(22,0)} \right)$  (11) in the set of the set o  $\frac{14}{10\pi}$  (10 $\pi$   $\frac{10\pi}{79}$ ) and the quantity scale term on the finite of  $\frac{1}{10}$ , originates from the  $\frac{1}{79}$ 15 Bott  $\sum_{15}^{3}$   $\sum_{c}^{f}$   $\left(\frac{|Q'|^2}{2}\right)$   $\sum_{p=0}^{5-k} p(k)$  the analysis in [30,40]. Thus,  $G^{(24,8)}$  is a solution of the supersym-16  $+4\sum_{k=1}^{\infty}\sum_{\alpha' \neq 1}^{\infty} \sum_{\substack{n,m,n \ k \text{ odd}}}^{\infty} c\left(-\frac{|\alpha|}{2}\right) R^{\alpha-k} P^{\alpha\beta}_{\alpha\beta\gamma\delta}$  are the dialysis in [50, 0]. Thus, and is a solution of the supersymmum at the supersymmum of the supersymmum of the supersymmu  $k=1$   $Q' \in \Lambda_{22,6}^{\star}$  m,n<br>the counterpart of the expected terms at weak cou-18 and the contract of the con 19  $K_{12}$   $\left(\frac{2\pi R |mS+n|}{\sqrt{2}Q_{p}'}\right) e^{-2\pi i (ma^{1}+na^{2}) \cdot Q'} + \dots$  pling, when  $\Lambda_{24,8}$  degenerates to  $\Lambda_{23,7} \oplus \Lambda_{1,1}$ . This limit can be added to  $\Lambda_{24,8}$  $z_0$   $k-\frac{1}{2}$   $\sqrt{5_2}$   $\sqrt{5_2}$   $\sqrt{5_3}$   $\sqrt{5_4}$   $\sqrt{5_5}$   $\sqrt{5_6}$   $\sqrt{5_7}$   $\sqrt{5_8}$   $\sqrt{5_8}$   $\sqrt{5_8}$   $\sqrt{5_9}$   $\sqrt{$ 21 where  $\widehat{E}_1(S) = -\frac{3}{\pi} \log S_2 |\eta(S)|^4$ . The first term in (11) origi-<br> $\frac{16!}{16!}$  we find and (6) with  $c_4 = \frac{3-q}{2}$ ,  $c_5 = \frac{6-q}{2}$ ,  $c_6 = \frac{1}{2}$ ,  $c_7 = -\pi$ . In particular, the quadratic source term on the r.h.s. of (5) originates from the pole of  $1/\Phi_{10}$  on the separating degeneration divisor, similar to Using results about the Fourier–Jacobi expansion of  $1/\Phi_{10}$  from [16], we find

$$
G_{\alpha\beta,\gamma\delta}^{(24,8)} = \frac{G_{\alpha\beta,\gamma\delta}^{(23,7)}}{g_3^4} - \frac{\delta_{\alpha\beta}G_{\gamma\delta}^{(23,7)} + \delta_{\gamma\delta}G_{\alpha\beta}^{(23,7)} - 2\delta_{\gamma\gamma}G_{\beta\gamma\delta}^{(23,7)}}{12g_3^6} - \frac{1}{2\pi g_3^8} \left[ \delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha(\gamma}\delta_{\delta)\beta} \right] + \dots \tag{15}
$$

where

the circle. Fixing charges (Q, P) such that Q and P are collinear,  
\nthe sum over (m, n) induces a measure factor  
\n
$$
G_{ab}^{(q+16,q)} = \int_{\mathcal{F}_1} \frac{d\rho_1 d\rho_2}{\rho_2^2} \frac{\partial^2}{(2\pi i)^2 \partial y^a \partial y^b} \Big|_{y=0} \frac{\widehat{E}_2 F_{q+16,q}}{\Delta},
$$
\n(16)

 $d|(\overline{Q}, P)$  100  $\overline{E}_2 = \frac{12}{1\pi} \partial_\rho \log \eta - \frac{3}{\pi \rho_2}$  the almost holomorphic Eisenstein se-<br>35  $_{36}$  which is recognized as the degeneracy of 1/2-BPS states with the of weight 2. The first and second terms in (15) corresponds  $_{101}$  $_{37}$  charges  $(Q, P)$ . In particular for a purely electric state (*P* = 0) with to the zero and rank 1 orbits, respectively. The third term is nec-38 primitive charge, it reduces to the well-known result  $c(-|Q|^2/2)$  essary for consistency with the quadratic term on the r.h.s. of the  $\frac{103}{103}$ 39 [5]. The dots in (11) stand for terms of order  $e^{-2π R^2 |k|+2πikψ}$ , char-<br>39 [5] although a naive unfolding pro- $_{40}$  acteristic of a Kaluza–Klein monopole of the form TN<sub>k</sub>  $\times$  T<sup>6</sup>, where cedure fails to produce it, presumably due to the singularity of <sub>105</sub> 41 106 TN*<sup>k</sup>* is Euclidean Taub–NUT space with charge *k*. These contribu- $\frac{42}{42}$  1000s will be discussed in [26].<br>
pected two-loop [37,38], one-loop [43] and tree-level contributions 43 **2. Exact**  $\nabla^2(\nabla \Phi)^4$  couplings in  $D = 3$   $[44,45]$  to the  $\nabla^2(\nabla \Phi)^4$  coupling in heterotic string on  $T^7$ , while 44 **2. Exact v** (v v) couplings in  $D = 3$ <br>the dots stand for terms of order  $e^{-1/g^2}$  ascribable to NS5-brane  $\frac{110}{110}$ <sup>45</sup><br><sub>46</sub> We now turn to the six-derivative coupling  $G_{ab,cd}$ , which is and KK5-brane instantons, which will be discussed in [26]. Note  $\frac{110}{111}$ 46 We now turn to the six-derivative coupling  $G_{ab,cd}$ , which is and KK5-brane instantons, which will be discussed in [26]. Note  $\frac{111}{111}$  $_{47}$  expected to receive both 1/2-BPS and 1/4-BPS instanton contribu-<br>that the tree-level single trace  $\nabla^2 F^4$  term in [44] proportional to <sub>48</sub> tions. Based on U-duality invariance, supersymmetric Ward identi- *ζ*(3) vanishes on the Cartan subalgebra [46], and does not conries of weight 2. The first and second terms in (15) corresponds to the zero and rank 1 orbits, respectively. The third term is necessary for consistency with the quadratic term on the r.h.s. of the supersymmetric Ward identity (5), although a naive unfolding procedure fails to produce it, presumably due to the singularity of the integrand in the separating degeneration limit. After rescaling to string frame, the first three terms in (15) correspond to the extribute to this coupling.

<sub>50</sub> conjecture that G<sub>ab,cd</sub> is given by the genus-two modular integral **Frankling** shown that our ansatz (13) passes all consistency con- $\frac{1}{2}$   $\frac{1}{8}$   $\frac{1}{8}$ 

where 
$$
F_2
$$
 is the standard fundamental domain for the action of  
\n $Sp(4,\mathbb{Z})$  on the Siegel upper half-plane of degree two [39],  $|\Omega_2|$  is  
\nthe determinant of the imaginary part of  $\Omega = \Omega_1 + i\Omega_2$ ,  $\Phi_{10}$  is the  
\nunique cusp form of weight 10 under the Siegel modular group  
\n $Sp(4,\mathbb{Z})$  (whose inverse counts micro-states of 1/4-BPS black holes  
\n[6]), and  $\Gamma_{24.8.2}$  is the genus-two partition function of the non-

61 126 The two terms on the first line (which correspond to the constant  $\epsilon^2$   $\Gamma_{24.8,2} = |\Omega_2|^4$   $\sum e^{i\pi (Q_L^1 \Omega_{ij} Q_L^j - Q_R^1 \Omega_{ij} Q_R^j + 2Q_L^1 y_i) + \frac{\pi}{2} y_i^a \Omega_2^{-i\eta} y_{ja}}$  term with respect to the parabolic decomposition (2)) originate 127  $63$   $128$   $128$   $160$   $160$   $128$   $128$   $129$   $160$   $128$   $129$  64 129 mensions. The term proportional to *g(S)* originates presumably <sup>65</sup> 130 to 130 to 14 130 from the separating degeneration divisor, and is determined by

Please cite this article in press as: G. Bossard et al., Protected couplings and BPS dyons in half-maximal supersymmetric string vacua, Phys. Lett. B (2016), http://dx.doi.org/10.1016/j.physletb.2016.12.035

4 *G. Bossard et al. / Physics Letters B* ••• *(*••••*)* •••*–*•••

ARTICLE IN PRE:

3 Fourier coefficients. They are both suppressed as  $e^{-2\pi R M(Q, P)}$ , but  $\frac{1}{2}$  and defined server (1.1), 7 **7** *P P* 8 73 terms with non-zero charge with respect to the NUT potential, cor-

10 In this letter, we focus on the contribution  $G^{(2)}$  from  $1/4$ -BPS due  $\frac{1}{2}$  is the location of the alore-inentioned saddle point, <sup>11</sup> black holes. This contribution originates from the 'Abelian rank  $\overline{R}$   $\overline{R}$ 12 2 orbit', whose stabilizer is the parabolic subgroup  $GL(2,\mathbb{Z})\ltimes\mathbb{Z}^3$   $\Omega_2^{\star} = \frac{R}{L(\mathcal{L}_2 - R)}A^{\top} \left( \frac{1}{S_1} \left( \frac{1}{S_1} \sum_{i=1}^{N_1} \frac{1}{(P_{i-1} \Omega_{i})} \left( \frac{|P_{i}|^2}{(P_{i-1} \Omega_{i})} - \frac{|P_{i}|^2}{(P_{i-1} \Omega_{i})} \right) \right) A$ 13 inside  $Sp(4, \mathbb{Z})$ . Thus, the integral can be unfolded onto  $\mathcal{P}_2$ / $\mathcal{M}(Q, P)$   $\mathbb{L}^{32} \setminus ^{31}$   $\mathbb{L}^{52} \setminus ^{31}$   $\mathbb{L}^{57}$   $\mathbb{L}^{78} \setminus ^{10}$   $\mathbb{R}^{12}$   $\mathbb{L}^{78}$ 14  $PGL(2, \mathbb{Z}) \times [0, 1]^3$ , where  $\mathcal{P}_2$  denotes the space of positive def-<br>12  $\frac{1}{2}$   $\frac{1}{2}$  15 inite  $2 \times 2$  matrices  $\Omega$ <sub>2</sub>:  $\Omega$  and  $\Omega$  a inite  $2 \times 2$  matrices  $\Omega_2$ :

$$
G_{\alpha\beta,\gamma\delta}^{(2)} = R^4 \int_{P_2} \frac{d^3 \Omega_2}{|\Omega_2|^3} \int_{[0,1]^3} d^3 \Omega_1 \frac{(\varepsilon_{ik}\varepsilon_{jl} + \varepsilon_{il}\varepsilon_{jk})\partial^4}{(2\pi i)^4 \partial y_i^a \partial y_j^b \partial y_k^c \partial y_l^d} \Big|_{y=0}
$$
  
\n
$$
\times \frac{\left(e^{-2\pi i a^{il} A_{ij} Q_j^j}\right)_{22,6,2}}{\Phi_{10}} \times \frac{\left(e^{-2\pi i a^{il} A_{ij} Q_j^j}\right)_{22,6,2}}{\Phi_{10}}
$$
\n
$$
G_{\alpha\beta,\gamma\delta}^{(2)} = R^4 \int_{[0,1]^3} \frac{d^3 \Omega_1 \frac{(\varepsilon_{ik}\varepsilon_{jl} + \varepsilon_{il}\varepsilon_{jk})\partial^4}{(2\pi i)^4 \partial y_i^a \partial y_j^b \partial y_k^c \partial y_l^d} \Big|_{y=0}
$$
\n
$$
G_{\alpha\beta,\gamma\delta}^{(2)} = R^4 \int_{[0,1]^3} \frac{d^3 \Omega_1 \frac{(\varepsilon_{ik}\varepsilon_{jl} + \varepsilon_{il}\varepsilon_{jk})\partial^4}{(2\pi i)^4 \partial y_i^a \partial y_j^b \partial y_k^c \partial y_l^d} \Big|_{y=0}
$$
\n
$$
G_{\alpha\beta,\gamma\delta}^{(2)} = R^4 \int_{[0,1]^3} \frac{d^3 \Omega_1 \frac{(\varepsilon_{ik}\varepsilon_{jl} + \varepsilon_{il}\varepsilon_{jk})\partial^4}{(2\pi i)^4 \partial y_i^a \partial y_j^b \partial y_k^c \partial y_l^d} \Big|_{y=0}
$$
\n
$$
G_{\alpha\beta,\gamma\delta}^{(2)} = R^4 \int_{[0,1]^3} \frac{d^3 \Omega_1 \frac{(\varepsilon_{ik}\varepsilon_{jl} + \varepsilon_{il}\varepsilon_{jk})\partial^4}{(2\pi i)^4 \partial y_i^a \partial y_j^b \partial y_k^c \partial y_l^d} \Big|_{y=0}
$$
\n
$$
G_{\alpha\beta,\gamma\delta}^{(2)} = R^4 \int_{[0,1]^3} \
$$

insertion of  $f(Q)$  in the sum. The integral over  $\Omega_1$  at fixed  $\Omega_2$  ex- $-Q \cdot P = \frac{1}{2} |P|^2$  $\Big), \Omega_2$  of  $1/\Phi_{10}$ . Due to the zeros of  $\Phi_{10}$ , the latter is a locally constant function of  $\Omega_2$ , discontinuous across certain real codimension 1 walls in  $\mathcal{P}_2$ [47,48]. For large *R* however, the remaining integral over  $\Omega_2$  is dominated by a saddle point  $\Omega_2^{\star}$  (see (24) below), so to all orders in 1*/R* around the saddle point, we can replace the above Fourier be computed using

$$
\int_{39}^{37} d^3S |S|^{\delta-\frac{3}{2}} e^{-\pi \text{Tr}(SA+S^{-1}B)} = 2 \left( \frac{|B|}{|A|} \right)^{\delta/2} \widetilde{B}_{\delta}(AB),
$$
\n(19) 
$$
\int_{39}^{1 \text{det}(I + S)} \text{ which we have ignored here, will be discussed in [26].}
$$
\n
$$
3. Discussion
$$

41 where  $B_\delta(Z)$  is a matrix-variate generalization of the modified  $106$ Bessel function [49],<sup>1</sup>

$$
\widetilde{B}_{\delta}(Z) = \int_{0}^{\infty} \frac{dt}{t^{3/2}} e^{-\pi t - \frac{\pi \text{Tr}Z}{t}} K_{\delta}\left(\frac{2\pi \sqrt{|Z|}}{t}\right).
$$
 (20)

In the limit where all entries in *Z* are large, one has

$$
\widetilde{B}_{\delta}(Z) \sim \frac{1}{2} \left[ |Z| \left( \text{Tr} Z + 2\sqrt{|Z|} \right) \right]^{-\frac{1}{4}} e^{-2\pi \sqrt{\text{Tr} Z + 2\sqrt{|Z|}}}.
$$
 (21)

Further relabelling  $({}^{\mathbb{Q}}_P) = A({}^{\mathbb{Q}_1}_{Q_2})$ , we find

$$
G_{\alpha\beta,\gamma\delta}^{(2)} = 2R^7 \sum_{Q,P \in \Lambda_{22,6}^{\star}} e^{-2\pi i (a^1 Q + a^2 P)} \frac{\mu(Q,P)}{|\rho_R \wedge Q_R|^{\frac{3}{2}}}
$$
  
\n
$$
\times P_{\alpha\beta,\gamma\delta} \left(\frac{1}{\sqrt{5_2}} \left(\frac{1}{0} S_1\right) \left(\frac{Q_L}{P_L}\right), -\frac{1}{\pi R^2} \frac{\partial}{\partial Y}\right).
$$
  
\n
$$
\times P_{\alpha\beta,\gamma\delta} \left(\frac{1}{\sqrt{5_2}} \left(\frac{1}{0} S_2\right) \left(\frac{Q_L}{P_L}\right), -\frac{1}{\pi R^2} \frac{\partial}{\partial Y}\right).
$$
  
\nfor the  $\mathcal{R}^4$  and  $\nabla^6 \mathcal{R}^4$  couplings.  
\nIn the limit where the radius of one circle inside  $T^7$  becomes  
\nterms, an infinite series of corrections of order  $e^{-2\pi R M(Q,P)}$  which  
\nare interpreted as Euclidean counterparts of four-dimensional BPS  
\nstate with mass  $M(Q,P)$ , whose worldline winds around the cir-  
\nthe BPS index  $\Omega(Q,P; S, \phi)$ , extracted from the Siegel  
\n
$$
\times P_{\alpha\beta,\gamma\delta} \left(\frac{1}{\sqrt{5_2}} \left(\frac{1}{0} S_2\right) \left(\frac{Q_L}{P_R \cdot Q_R} - \frac{|P_R \cdot Q_R|}{|P_R|^2}\right) \left(\frac{1}{S_1} S_2\right)\right)\Big|_{Y=1}
$$
  
\n
$$
\times P_{\alpha\beta,\gamma\delta} \left(\frac{1}{\sqrt{5_2}} \left(\frac{1}{0} S_2\right) \left(\frac{|Q_R|^2}{P_R \cdot Q_R} - \frac{|P_R \cdot Q_R|}{|P_R|^2}\right) \left(\frac{1}{S_1} S_2\right)\right)\Big|_{Y=1}
$$
  
\n
$$
\times P_{\alpha\beta,\gamma\delta} \left(\frac{1}{\sqrt{5_2}} \left(\frac{1}{0} S_2\right) \left(\frac{|Q_R|^2}{P_R \cdot Q_R} - \frac{|P_R \cdot Q_R|}{|P_R|^2}\right) \left(\frac{1}{S_1} S_2\right)\right)\Big|_{Y=1}
$$
  
\

1 the differential equation (6). The terms  $G^{(1)}$  and  $G^{(2)}$  are inde-<br>where  $|P_R \wedge Q_R| = \sqrt{(P_R^2)(Q_R^2) - (P_R \cdot Q_R)^2}$ ,  $P_{\alpha\beta,\gamma\delta}$  is the polyno-2 pendent of the NUT potential  $\psi$ , and correspond to the Abelian  $\mu$  mial defined below (14) where  $|P_R \wedge Q_R| = \sqrt{(P_R^2)(Q_R^2) - (P_R \cdot Q_R)^2}$ ,  $P_{\alpha\beta,\gamma\delta}$  is the polynomial defined below (14),

$$
G^{(1)}
$$
 has support on electromagnetic charges  $(Q, P)$  which Q and  
\n $P$  collinear, hence corresponds to contributions of 1/2-BPS states  
\n $P$  collinear, hence corresponds to contributions of 1/2-BPS states  
\n $P$  is a support on generic charges, cor-  
\n $A \in M_2(\mathbb{Z})/GL(2,\mathbb{Z})$   
\n $A^{-1}(\rho^0) \in \Lambda_{22,6}^{\infty/2}$   
\n $A^{-1}(\rho^0) \in \Lambda_{22,6}^{\infty/2}$   
\n $A^{-1}(\rho^0) \in \Lambda_{22,6}^{\infty/2}$   
\n $A^{-1}(\rho^0) \in \Lambda_{22,6}^{\infty/2}$ 

(23)

9 74 responding to Kaluza–Klein monopole contributions. and  $\Omega_2^*$  is the location of the afore-mentioned saddle point,

$$
\Omega_2^{\star} = \frac{R}{\mathcal{M}(Q, P)} A^{\mathsf{T}} \left[ \frac{1}{S_2} {1 \choose S_1 |S|^2} + \frac{1}{|P_R \wedge Q_R|} \left( \frac{|P_R|^2}{-P_R \cdot Q_R} - \frac{P_R \cdot Q_R}{|Q_R|^2} \right) \right] A. \tag{24}
$$

16 81 Using (21), we see that these contributions behave as *e*−2*<sup>π</sup> <sup>R</sup>*M*(<sup>Q</sup> ,P)* in the limit  $R \rightarrow \infty$ , where

$$
\mathcal{M}(Q, P) = \sqrt{2 \frac{|Q_R + SP_R|^2}{S_2} + 4 \sqrt{\left| \frac{|Q_R|^2}{Q_R \cdot P_R} - \frac{Q_R \cdot P_R}{|P_R|^2} \right|}}
$$
(25)

22  $\Phi_{10}$   $\Phi_{2}$   $\Phi_{10}$   $\Phi_{2}$   $\Phi_{2}$   $\Phi_{3}$   $\Phi_{4}$   $\Phi_{5}$   $\Phi_{7}$   $\Phi_{8}$  is recognized as the mass of a 1/4-BPS dyon with electromag- 87  $A \in M_2(\mathbb{Z})/GL(2,\mathbb{Z})$ <br>  $|A| \neq 0$  as a netic charges  $(Q, P)$  [50,51]. Moreover, in cases where only  $A = \mathbb{1}$  as 24 89 contributes to (23), the instanton measure *μ(Q , P)* agrees with  $(10)$  the BPS index  $\Omega(Q, P; S, \phi)$  in the corresponding chamber of the 90 26 where  $\langle f(Q) \rangle_{22,6,2}$  denotes the partition function  $\Gamma_{22,6,2}$  with an moduli space  $\mathcal{M}_4$  in  $D=4$ , computed with the contour pre-27 Insertion of  $f(Q)$  in the sum. The integral over  $\Omega_1$  at fixed  $\Omega_2$  ex-stription in [52]. Our result (23) generalizes this prescription to set 28 etracts the Fourier coefficient  $\epsilon$  |  $\ell$  =  $\frac{1}{2}|\mathcal{Q}|^2$  =  $\mathcal{Q} \cdot \ell$  |  $\Omega_{\rm g}$  | of 1/ $\Phi_{\rm g}$  = arbitrary electromagnetic charges  $(\mathcal{Q}, P)$  and recovers the results = 93 28 tracts the Fourier coefficient *C*  $\left[ \begin{pmatrix} -\frac{1}{2} |Q|^2 & -Q \cdot P \\ -Q \cdot P & -\frac{1}{2} |P|^2 \end{pmatrix}$ ; Ω<sub>2</sub> of 1/Φ<sub>10</sub>. arbitrary electromagnetic charges (*Q*, *P*) and recovers the results 93<br>29 of [53–55] for dyons with torsio <sub>30</sub> Due to the zeros of  $\Phi_{10}$ , the latter is a locally constant function  $\Box$  of chamber. Additional (exponentially suppressed) contributions to s 31 of sz, discontinuous actoss certain real counnension 1 wans in  $P_2 = C(2)$  axics from the difference between  $C\left[\left(1-\frac{1}{2}\left|\mathbb{Q}\right|^2\right]-Q\cdot P\right]$ ,  $C\left[\left(1-\frac{1}{2}\right)\right]$ 31 of Ω<sub>2</sub>, discontinuous across certain real codimension 1 walls in  $\mathcal{P}_2$ <br>32 [47,48]. For large R however, the remaining integral over Ω<sub>2</sub> is G<sup>(2)</sup> arise from the difference between  $C\left[\begin{pmatrix} -\frac{1}{2}|Q|^2 & -Q \cdot P \\ -Q$ 33 dominated by a saddle point  $\Omega_2^{\star}$  (see (24) below), so to all orders and its value at the saddle point. The relation between the jumps setween the same setween the jumps setween the jumps setween the jumps set of  $_{34}$  in 1/R around the saddle point, we can replace the above Fourier of these Fourier coefficients and the possible splittings of a 1/4-BPS  $_{99}$ coefficient by its value at  $\Omega_2^*$ . The remaining integral over  $\Omega_2$  can bound state into two 1/2-BPS constituents [47] is crucial for consis-<br>100 <sub>36</sub> be computed using the supersymmetric Ward the supersymmetric Ward to the supersymm  $\frac{\delta}{2}$  identity (5). These contributions, along with the terms  $G^{(2)}$  and  $\frac{\delta}{2}$ moduli space  $\mathcal{M}_4$  in  $D = 4$ , computed with the contour prearbitrary electromagnetic charges *(Q , P)* and recovers the results of [53–55] for dyons with torsion, fixing a subtlety in the choice  $-Q \cdot P = \frac{1}{2} |P|^2$  $\bigg)$ ;  $\Omega_2$ bound state into two 1/2-BPS constituents [47] is crucial for consis-

#### **3. Discussion**

42 Bessel function  $[49]$ <sup>1</sup>, to a set of the *C*  $\Phi$ <sup>4</sup> 107 and this work, we have conjectured the exact form of the  $(\nabla \Phi)^4$  107 and  $∇<sup>2</sup>(∇Φ)<sup>4</sup>$  couplings in the low energy effective action of *D* = 3 and  $\frac{44}{\epsilon}$   $\sim$   $\int dt$   $\frac{\pi}{12}$   $\frac{7\pi}{2}$   $\sqrt{2\pi}$   $\sqrt{17}$  $45$   $B_{\delta}(Z) = \int \frac{1}{63/2} e^{-\lambda (L-\epsilon)}$   $K_{\delta}(\frac{2\lambda (L+\epsilon)}{L})$ . (20) simplest model, heterotic string compactified on  $T^7$ . Our ansätze 110 46  $\frac{6}{11}$  and  $\frac{1}{11}$  and  $\frac{1}{11}$  are manifestly U-duality invariant, satisfy the req- 111 <sup>47</sup> In the limit where all entries in 7 are large one has **under the supersymmetric Ward identities**, reproduce the known per-<br> 48 113 turbative contributions at weak heterotic coupling and the known 49  $F^4, \mathcal{R}^2, D^2F^4$  and  $\mathcal{R}^2F^2$  couplings in  $D = 4$  in the limit where the 114  $50 \quad B_{\delta}(Z) \sim \frac{1}{2} |Z| (Tr Z + 2\sqrt{|Z|})$   $e^{-\frac{2\pi i}{\sqrt{2}}}$   $e^{-\frac{2\pi i}{\sqrt{2}}}$   $e^{-\frac{2\pi i}{\sqrt{2}}}$   $e^{-\frac{2\pi i}{\sqrt{2}}}$  radius of one circle inside  $T^7$  becomes large. While we do not yet 51 **116 116 have a rigorous proof that these constraints uniquely determine** 116 52 Further relabelling ( $\bar{p}$ ) =  $A(\tilde{Q}_2^t)$ , we find  $\frac{1}{2}$  tions from cusp forms are ruled out by the supersymmetric Ward  $\frac{118}{2}$  $54 \t G_{\alpha\beta}^{(2)}$ ,  $\epsilon = 2R^7$   $\sum e^{-2\pi i (a^2(1+a^2P))} \frac{1}{2}$  e<sup>-2 $\pi$ </sup>  $\epsilon$  identities (4) and (5), by the same type of arguments which apply

 $\frac{1}{256}$  121  $\frac{1}{256}$  1  $57 \rightarrow p$ ,  $\left(1 \left(1 \right) \left(1 \right) \left(2 \right) \right)$   $\left(22 \right)$  and  $\left(1 \right)$  and  $\left(3 \right)$  and  $\left(4 \right)$  and  $\left(5 \right)$  and  $\left(6 \right)$  and  $\left(7 \right)$  and  $\left(8 \right)$  and  $\left(9 \right)$  and  $\left(1 \right)$  and  $\left(1 \right)$  and  $\left(1 \right)$  and  $\left(1 \right)$  an 58  $\lambda + \alpha \beta, \gamma \delta \sqrt{\sqrt{52}} \left(0, S_2\right) \left(P_L\right), \frac{-\pi R^2}{\pi R^2} \frac{\partial Y}{\partial Y}$  terms, an infinite series of corrections of order  $e^{-2\pi R \mathcal{M}(\mathbb{Q}, P)}$  which 123  $\frac{59}{24}$   $\frac{1}{24}$   $\frac{59}{24}$   $\frac{1}{24}$   $\frac{1}{24}$   $\frac{1}{24}$   $\frac{1}{24}$   $\frac{1}{24}$  are interpreted as Euclidean counterparts of four-dimensional BPS 124 60  $\left(\frac{|Y|^{\frac{1}{4}}B_{\frac{3}{2}}}{Y\frac{2K}{2}}\right)\left(\frac{2K}{D_R}\right)\left(\frac{2K}{P_R}\right)^{\frac{1}{2}}\left(\frac{K}{D_R}\right)^{\frac{1}{2}}\left(\frac{2K}{2}\right)^{\frac{1}{2}}\left(\frac{2K}{2}\right)^{\frac{1}{2}}$  states with mass  $\mathcal{M}(Q, P)$ , whose worldline winds around the cir-61 126 cle. Rather remarkably, the contribution from a 1/4-BPS dyon is 62 **127** 127 **127 127** 63 128 modular form 1*/-*<sup>10</sup> using the very same contour prescription as 64 129 in [52]. Indeed, it was suggested in [56] (see also [57,58]) to rep- $^{65}$   $^{1}$  Our  $\widetilde{B}_\delta(Z)$  is related to  $B_\delta(Z)$  in [49] via  $\widetilde{B}_\delta(Z/\pi^2) = \frac{1}{2}$ (det  $Z)^{\delta/2}B_\delta(Z)$ . resent 1/4-BPS dyons as heterotic strings wrapped on a genus-two 130

Please cite this article in press as: G. Bossard et al., Protected couplings and BPS dyons in half-maximal supersymmetric string vacua, Phys. Lett. B (2016), http://dx.doi.org/10.1016/j.physletb.2016.12.035

<sup>&</sup>lt;sup>1</sup> Our  $\widetilde{B}_\delta(Z)$  is related to  $B_\delta(Z)$  in [49] via  $\widetilde{B}_\delta(Z/\pi^2) = \frac{1}{2} (\det Z)^{\delta/2} B_\delta(Z)$ .

## ARTICLE IN PRESS

<span id="page-5-0"></span>48 and the contract of the con  $49$ 50 115  $51$  $52$ 53 118 54 119 55 120 56 121 57 122 58 123 59 124 60 125 61 126 62 and the contract of the con 63 128

<sup>1</sup> curve holomorphically embedded in a  $T^4$  inside  $T^7$ . This picture [22] A.M. Polyakov, Nucl. Phys. B 120 (1977) 429. 2 67 was further used in [59] to justify the contour prescription of [52]. 4 basis to these heuristic ideas, and explains why  $1/4$ -BPS dyons in  $\frac{u_1x_1u_2u_3u_4u_5u_6u_7}{25}$ . Using Phys. Boy 1st, 75 (1995) 2364 <sup>6</sup> phasize that the introduction of the Siegel modular form  $1/\Phi_{10}$  in [26] G. Bossard, C. Cosnier-Horeau, B. Pioline, in press.  $\frac{1}{2}$ <sup>7</sup> the conjectured formula  $(13)$  is necessary to match the perturba-  $[27]$  K.S. Narain, Phys. Lett. B 169 (1986) 41. 8 tive 2-loop amplitude, where it appears explicitly [37,38]. A more [28] N. Marcus, J.H. Schwarz, Nucl. Phys. B 228 (1983) 145. The state of th 9 detailed analysis of the weak coupling and large radius expansions  $^{[29]}_R$  and the set that we have the set of the set <sup>11</sup> phasis on the consequences of wall-crossing for three-dimensional  $_{[31]}$  w. Lerche, B. Nilsson, A. Schellekens, N. Warner, Nucl. Phys. B 299 (1988) 91. <sup>76</sup> 12 77 [32] I. Antoniadis, B. Pioline, T.R. Taylor, Nucl. Phys. B 512 (1998) 61, arXiv:hep-th/ 13 and the contract of the con couplings.

### **Acknowledgements**

16 We are grateful to Eric d'Hoker, Ioannis Florakis and Rodolfo 0006088. <sup>82</sup> Russo for valuable discussions on genus-two modular integrals, [35] S. Kachru, N.M. Paquette, R. Volpato, arXiv:1603.07330, 2016. <sup>18</sup> and to Sameer Murthy for discussions on wall-crossing in  $\mathcal{N} = 4$  [36] J.A. Harvey, G.W. Moore, Phys. Rev. D 57 (1998) 2323, arXiv:hep-th/9610237. <sup>83</sup> 19 string vacua. G.B. and C.C.H. thank CERN for its hospitality. The state of the Phong, Nucl. Phys. B 715 (2005) 3, arXiv:hep-th/0501197. The B4

#### **References**

- 23 88 [1] A. Font, L.E. Ibanez, D. Lust, F. Quevedo, Phys. Lett. B 249 (1990) 35.
- 24 89 [2] A. Sen, Int. J. Mod. Phys. A 9 (1994) 3707, arXiv:hep-th/9402002.
	- [3] C.M. Hull, P.K. Townsend, Nucl. Phys. B 438 (1995) 109, arXiv:hep-th/9410167.
	- [4] E. Witten, Nucl. Phys. B 443 (1995) 85, arXiv:hep-th/9503124.
	- [5] A. Dabholkar, J.A. Harvey, Phys. Rev. Lett. 63 (1989) 478.
- 27 92 [6] R. Dijkgraaf, H.L. Verlinde, E.P. Verlinde, Nucl. Phys. B 484 (1997) 543, arXiv: 28 1999 http://www.biz.com/individual/commond, P.J. Heslop, P.S. Howe, S.F. Kerstan, J. High Energy Phys. 08 23 hep-th/9607026.
- 29 94 [7] R. Dijkgraaf, G.W. Moore, E.P. Verlinde, H.L. Verlinde, Commun. Math. Phys. 185 (1997) 197, arXiv:hep-th/9608096.
	- [8] A. Dabholkar, Phys. Rev. Lett. 94 (2005) 241301, arXiv:hep-th/0409148.
- 32 97 [49] C.S. Herz, Ann. Math. (2) 61 (1955) 474. 096, arXiv:hep-th/0507014.
- 33 98 [10] A. Strominger, C. Vafa, Phys. Lett. B 379 (1996) 99, arXiv:hep-th/9601029.
- $_{34}$  [11] D. Shih, A. Strominger, X. Yin, J. High Energy Phys. 10 (2006) 087, arXiv:hep-th/ [51] M.J. Dutf, J.I. Ltu, J. Kahmield, Nucl. Phys. B 459 (1996) 125, arXiv:hep-  $_{99}$ 0505094.
	- [12] J.R. David, A. Sen, J. High Energy Phys. 11 (2006) 072, arXiv:hep-th/0605210.
	- [13] A. Sen, Gen. Relativ. Gravit. 40 (2008) 2249, arXiv:0708.1270.
- 37 102 [14] N. Banerjee, D.P. Jatkar, A. Sen, J. High Energy Phys. 05 (2009) 121, arXiv: 0810.3472.
- 39 104 [15] S. Banerjee, R.K. Gupta, I. Mandal, A. Sen, J. High Energy Phys. 11 (2011) 143, arXiv:1106.0080.
	-
- 41 [17] A. Dabholkar, J. Gomes, S. Murthy, J. High Energy Phys. 03 (2015) 074, arXiv: [56] D. Gaiotto, arXiv:hep-th/0506249, 2005. 42 107 [57] A. Dabholkar, D. Gaiotto, J. High Energy Phys. 12 (2007) 087, arXiv:hep-th/ 1404.0033.
	-
	- [19] M.B. Green, M. Gutperle, Nucl. Phys. B 498 (1997) 195, arXiv:hep-th/9701093. [20] M.B. Green, P. Vanhove, J. High Energy Phys. 0601 (2006) 093, arXiv:hep-th/
- 45 110 [59] S. Banerjee, A. Sen, Y.K. Srivastava, J. High Energy Phys. 03 (2009) 151, arXiv: 0510027.
- 47 112 arXiv:hep-th/0512296.
- [22] A.M. Polyakov, Nucl. Phys. B 120 (1977) 429.
- [23] N. Seiberg, E. Witten, arXiv:hep-th/9607163, 1996.
- 3 Our analysis of the  $\nabla^2(\nabla\Phi)^4$  coupling in  $D=3$  gives a concrete  $[24]$  D. Gaiotton 4722. Moore, A. Neitzke, Commun. Math. Phys. 299 (2010) 163, 68 [24] D. Gaiotto, G.W. Moore, A. Neitzke, Commun. Math. Phys. 299 (2010) 163, arXiv:0807.4723.
- $5\qquad D=4$  are counted by a Siegel modular form of genus two. We em-<br> $\frac{125J(5.10)(1000)(1000)(1000)}{375}$ [25] S. Chaudhuri, G. Hockney, J.D. Lykken, Phys. Rev. Lett. 75 (1995) 2264, arXiv:hep-th/9505054.
	- [26] G. Bossard, C. Cosnier-Horeau, B. Pioline, in press.
	- [27] K.S. Narain, Phys. Lett. B 169 (1986) 41.
	- [28] N. Marcus, J.H. Schwarz, Nucl. Phys. B 228 (1983) 145.
	- [29] A. Sen, Nucl. Phys. B 434 (1995) 179, arXiv:hep-th/9408083.
- 10 of the  $\nabla^2(\nabla\Phi)^4$  coupling will appear in [26], with particular em-<br><sup>10</sup> arXiv:1508.07305 [30] Y.-H. Lin, S.-H. Shao, Y. Wang, X. Yin, J. High Energy Phys. 12 (2015) 142, arXiv:1508.07305.
	- [31] W. Lerche, B. Nilsson, A. Schellekens, N. Warner, Nucl. Phys. B 299 (1988) 91.
	- 9707222.
- 14 **Acknowledgements 1988** Theory of Equation 133 E. Kiritsis, N.A. Obers, B. Pioline, J. High Energy Phys. 01 (2000) 029, arXiv:hepth/0001083.
- 15 80 [34] N.A. Obers, B. Pioline, J. High Energy Phys. 07 (2000) 003, arXiv:hep-th/ 0006088.
	- [35] S. Kachru, N.M. Paquette, R. Volpato, arXiv:1603.07330, 2016.
	- [36] J.A. Harvey, G.W. Moore, Phys. Rev. D 57 (1998) 2323, arXiv:hep-th/9610237.
	- [37] E. D'Hoker, D.H. Phong, Nucl. Phys. B 715 (2005) 3, arXiv:hep-th/0501197.
- 20 85 [38] E. D'Hoker, M. Gutperle, D.H. Phong, Nucl. Phys. B 722 (2005) 81, arXiv:hep- $^{21}$  References  $^{86}$ th/0503180.
	- [39] E. Gottschling, Math. Ann. 138 (1959) 103.
- 87<br>[40] B. Pioline, R. Russo, J. High Energy Phys. 12 (2015) 102, arXiv:1510.02409.<br>مصاحب المستقلة المست
	- [41] B. Pioline, Proc. Symp. Pure Math. 88 (2014) 119, arXiv:1401.4265.
	- [42] I. Florakis, B. Pioline, arXiv:1602.00308, 2016.
- 25 90 [43] N. Sakai, Y. Tanii, Nucl. Phys. B 287 (1987) 457.
- 26 91 [44] D.J. Gross, J.H. Sloan, Nucl. Phys. B 291 (1987) 41.
	- [45] E.A. Bergshoeff, M. de Roo, Nucl. Phys. B 328 (1989) 439.
	- (2003) 016, arXiv:hep-th/0305202.
	- [47] A. Sen, J. High Energy Phys. 05 (2007) 039, arXiv:hep-th/0702141.
- 30 95 [48] A. Dabholkar, D. Gaiotto, S. Nampuri, J. High Energy Phys. 01 (2008) 023,  $^{31}$  96 and Dabholkar, F. Denef, G.W. Moore, B. Pioline, J. High Energy Phys. 10 (2005) arXiv:hep-th/0702150. arXiv:hep-th/0702150.
	-
	- [50] M. Cvetic, D. Youm, Phys. Rev. D 53 (1996) 584, arXiv:hep-th/9507090.
	- [51] M.J. Duff, J.T. Liu, J. Rahmfeld, Nucl. Phys. B 459 (1996) 125, arXiv:hepth/9508094.
- <sup>100</sup> 12 E. Savid, A. Sen, J. High Energy Phys. 11 (2006) 072, arXiv:hep-th/0605210. [52] M.C.N. Cheng, E. Verlinde, J. High Energy Phys. 09 (2007) 070, arXiv:0706.2363. <sup>100</sup>
- 36 13] A. Sen, Gen. Relativ. Gravit. 40 (2008) 2249, arXiv:0708.1270. [53] S. Banerjee, A. Sen, Y.K. Srivastava, J. High Energy Phys. 05 (2008) 101, arXiv: <sup>101</sup> 0802.0544.
- 38 103 [54] S. Banerjee, A. Sen, Y.K. Srivastava, J. High Energy Phys. 05 (2008) 098, arXiv:0802.1556.
- 40 105 [16] A. Dabholkar, S. Murthy, D. Zagier, arXiv:1208.4074, 2012. [55] A. Dabholkar, J. Gomes, S. Murthy, J. High Energy Phys. 05 (2011) 059, arXiv: 0803.2692.
	- [56] D. Gaiotto, arXiv:hep-th/0506249, 2005.
- 43 [18] S. Murthy, V. Reys, J. High Energy Phys. 04 (2016) 052, arXiv:1512.01553. 0612011. 0612011.
- 44 1.9 M.B. Green, M. Gutperie, Nucl. Phys. B 498 (1997) 195, arXiv:hep-th/9701093. [58] A. Dabholkar, S. Nampuri, J. High Energy Phys. 11 (2007) 077, arXiv:hep-th/ 10g 0603066.
- 46 111 [21] M. Günaydin, A. Neitzke, B. Pioline, A. Waldron, Phys. Rev. D 73 (2006) 084019, 0808.1746.

64 129 65 and the contract of the con