Protected couplings and BPS dyons in half-maximal supersymmetric string vacua

Guillaume Bossard, Charles Cosnier-Horeau, Boris Pioline

Physics Letters B ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Protected couplings and BPS dyons in half-maximal supersymmetric string vacua

Guillaume Bossard [a], Charles Cosnier-Horeau [a,b], Boris Pioline [c,b]

[a] Centre de Physique Théorique, Ecole Polytechnique, Université Paris-Saclay, 91128 Palaiseau Cedex, France
[b] Sorbonne Universités, Laboratoire de Physique Théorique et Hautes Energies, CNRS UMR 7589, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05, France
[c] CERN, Theoretical Physics Department, 1211 Geneva 23, Switzerland

ARTICLE INFO

Article history:
Received 11 November 2016
Accepted 12 December 2016
Available online xxxx
Editor: N. Lambert

ABSTRACT

We analyze four- and six-derivative couplings in the low energy effective action of $D = 3$ string vacua with half-maximal supersymmetry. In analogy with an earlier proposal for the $(\nabla\Phi)^4$ coupling, we propose that the $\nabla^2(\nabla\Phi)^4$ coupling is given exactly by a manifestly U-duality invariant genus-two modular integral. In the limit where a circle in the internal torus decompactifies, the $\nabla^2(\nabla\Phi)^4$ coupling reduces to the $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ couplings in $D = 4$, along with an infinite series of corrections of order e^{-R}, from four-dimensional 1/4-BPS dyons whose worldline winds around the circle. Each of these contributions is weighted by a Fourier coefficient of a meromorphic Siegel modular form, explaining and extending standard results for the BPS index of 1/4-BPS dyons.

String vacua with half-maximal supersymmetry offer an interesting window into the non-perturbative regime of string theory and the quantum physics of black holes, unobstructed by intricacies present in vacua with less supersymmetry. In particular, the low-energy effective action at two-derivative order does not receive any quantum corrections, and all higher-derivative interactions are expected to be invariant under the action of an arithmetic group $G(\mathbb{Z})$, known as the U-duality group, on the moduli space G/K of massless scalars [1–4]. This infinite discrete symmetry also constrains the spectrum of BPS states, and allows to determine, for any values of the electromagnetic charges, the number of BPS black hole micro-states (counted with signs) in terms of Fourier coefficients of certain modular forms [5–7]. This property has been used to confirm the validity of the microscopic stringy description of BPS black holes at an exquisite level of precision, both for small black holes (preserving half of the supersymmetries of the background) [8,9] and for large black holes (preserving a quarter of the same) [10–18].

In this letter, we shall exploit U-duality invariance and supersymmetry Ward identities to determine certain higher-derivative couplings in the low-energy effective action of three-dimensional string vacua with 16 supercharges, for all values of the moduli. These protected couplings are analogues of the $\mathcal{R}^4$ and $\nabla^6\mathcal{R}^4$ couplings in toroidal compactifications of type II strings, which have been determined exactly in [19,20] and in many subsequent works. Our motivation for studying these protected couplings in $D = 3$ is that they are expected to encode the infinite spectrum of BPS black holes in $D = 4$, in a way consistent with the U-duality group $G_3(\mathbb{Z})$. The latter contains the four-dimensional U-duality group $G_4(\mathbb{Z})$, but is potentially far more constraining. Thus, these protected couplings provide analogues of 'black hole partition functions', which do not suffer from the usual difficulties in defining thermodynamical partition functions in theories of quantum gravity, and are manifestly automorphic [21].

The fact that solitons in $D = 4$ may induce instanton corrections to the quantum effective potential in dimension $D = 3$ is well known in the context of gauge theories with compact $U(1)$ [22]. In the context of quantum field theories with 8 rigid supersymmetries, BPS dyons in four dimensions similarly correct the moduli space metric after reduction on a circle [23,24]. In string vacua with 16 local supersymmetries, one similarly expects that 1/2-BPS dyons in $D = 4$ will contribute to four-derivative scalar couplings of the form $F_{abcd}(\Phi)\,\nabla\Phi^a\nabla\Phi^b\nabla\Phi^c\nabla\Phi^d$ in $D = 3$, while both 1/2-BPS and 1/4-BPS dyons in $D = 4$ will contribute to six-derivative scalar couplings of the form

E-mail addresses: guillaume.bossard@polytechnique.edu (G. Bossard), charles.cosnier-horeau@polytechnique.edu (C. Cosnier-Horeau), boris.pioline@cern.ch (B. Pioline).

http://dx.doi.org/10.1016/j.physletb.2016.12.035

$G_{ab,cd}(\Phi)\,\nabla(\nabla\Phi^a\nabla\Phi^b)\nabla(\nabla\Phi^c\nabla\Phi^d)$ (here, ∇ denote space–time derivatives, contracted so as to make a Lorentz scalar). In either case, the contribution of a four-dimensional BPS state with electric and magnetic charges (Q,P) is expected to be suppressed by $e^{-2\pi R\mathcal{M}(Q,P)}$, where $\mathcal{M}(Q,P)$ is the BPS mass and R the radius of the circle on which the four-dimensional theory is compactified, and weighted by a suitable BPS index $\Omega(Q,P)$ counting the number of BPS states with given charges. In addition, coupling to gravity implies additional $\mathcal{O}(e^{-R^2/\ell_P^2})$ corrections from gravitational Taub-NUT instantons, which are essential for invariance under $G_3(\mathbb{Z})$ (here, ℓ_P is the Planck length in four dimensions).

For simplicity we shall restrict attention to the simplest three-dimensional string vacuum with 16 supercharges, obtained by compactifying the ten-dimensional heterotic string on T^7. Our construction can however be generalized to other half-maximal supersymmetric models with reduced rank [25] with some effort [26]. The moduli space in three dimensions is the symmetric space $\mathcal{M}_3 = G_{24,8}$ [28], where $G_{p,q} = O(p,q)/O(p)\times O(q)$ denotes the orthogonal Grassmannian of q-dimensional positive planes in $\mathbb{R}^{p,q}$. In the limit where the heterotic string coupling g_3 becomes small, $\mathcal{M}_3$ decomposes as

$$G_{24,8} \to \mathbb{R}^+ \times G_{23,7} \ltimes \mathbb{R}^{30}\,, \tag{1}$$

where the first factor corresponds to g_3, the second factor to the Narain moduli space (parametrizing the metric, B-field and gauge field on T^7), and $\mathbb{R}^{30}$ to the scalars a^I dual to the gauge fields in three dimensions. At each order in g_3^2, the low-energy effective action is known to be invariant under the T-duality group $O(23,7,\mathbb{Z})$, namely the automorphism group of the even self-dual Narain lattice $\Lambda_{23,7}$ [27]. The latter leaves g_3 invariant, acts on $G_{23,7}$ by left multiplication and on the last factor in (1) by the defining representation. U-duality postulates that this symmetry is extended to $G_3(\mathbb{Z}) = O(24,8,\mathbb{Z})$, the automorphism group of the 'non-perturbative Narain lattice' $\Lambda_{24,8} = \Lambda_{23,7}\oplus\Lambda_{1,1}$, where $\Lambda_{1,1}$ is the standard even-self dual lattice of signature $(1,1)$ [29].

In the limit where the radius R of one circle of the internal torus becomes large, $\mathcal{M}_3$ instead decomposes as

$$G_{24,8} \to \mathbb{R}^+ \times \left[G_{2,1}\times G_{22,6}\right] \ltimes \mathbb{R}^{56+1}\,, \tag{2}$$

where the first factor now corresponds to $R^2/(g_4^2\ell_H^2) = R/(g_3^2\ell_H) = R^2/\ell_P^2$ (with ℓ_H being the heterotic string scale and g_4 the string coupling in $D=4$), the second correspond to the moduli space $\mathcal{M}_4$ in 4 dimensions, the third factor to the holonomies a^{1I}, a^{2I} of the 28 electric gauge fields and their magnetic duals along the circle, along with the NUT potential ψ, dual to the Kaluza–Klein gauge field. The factor $G_{2,1}\cong SL(2)/U(1)$ is parametrized by the axio-dilaton $S = S_1 + \mathrm{i}S_2 = B + \mathrm{i}/g_4^2$, while $G_{22,6}$ is the Narain moduli space of T^6, with coordinates ϕ. In the limit $R\to\infty$, the U-duality group is broken to $SL(2,\mathbb{Z})\times O(22,6,\mathbb{Z})$, where the first factor $SL(2,\mathbb{Z})$ is the famous S-duality in four dimensions [1,2].

Besides being automorphic under $G_3(\mathbb{Z})$, the couplings F_{abcd} and $G_{ab,cd}$ must satisfy supersymmetric Ward identities. To state them, we introduce the covariant derivative $\mathcal{D}_{a\hat{b}}$ on the Grassmannian $G_{p,q}$, defined by its action on the projectors $p^I_{L,a}$ and $p^I_{R,\hat{a}}$ on the time-like p-plane and its orthogonal complement (here and below, $a, b,..., \hat{a}, \hat{b}..., I, J...$ take values 1 to p, q, and $p+q$, respectively):

$$\mathcal{D}_{a\hat{b}}\, p^I_{L,c} = \tfrac{1}{2}\delta_{ac}\, p^I_{R,\hat{b}}\,, \quad \mathcal{D}_{a\hat{b}}\, p^I_{R,\hat{c}} = \tfrac{1}{2}\delta_{\hat{b}\hat{c}}\, p^I_{L,a}\,. \tag{3}$$

The trace of the operator $\mathcal{D}^2_{ef} = \mathcal{D}_{(e}{}^{\hat{g}}\mathcal{D}_{f)\hat{g}}$ is equal to (1/2 times) the Laplacian on $G_{p,q}$. On-shell linearized superspace methods indicate that F_{abcd} and $G_{ab,cd}$ have to satisfy [26]

$$\mathcal{D}^2_{ef}\, F_{abcd} = c_1\,\delta_{ef}\, F_{abcd} + c_2\,\delta_{e)(a}\, F_{bcd)(f} + c_3\,\delta_{(ab}\, F_{cd)ef}\;, \tag{4}$$

$$\begin{aligned}\mathcal{D}^2_{ef}\, G_{ab,cd} = {}& c_4\delta_{ef}\, G_{ab,cd} + c_5\left[\delta_{e)(a}G_{b)(f,cd} + \delta_{e)(c}G_{d)(f,ab}\right]\\ &+ c_6\left[\delta_{ab}\, G_{ef,cd} + \delta_{cd}\, G_{ef,ab} - 2\delta_{a)(c}\, G_{ef,d)(b}\right]\\ &+ c_7\left[F_{abk(e}\, F_{f)cd}{}^{k} - F_{c)ka(e}\, F_{f)b(d}{}^{k}\right],\end{aligned} \tag{5}$$

$$\mathcal{D}_{[e}{}^{[\hat{e}}\mathcal{D}_{f]}{}^{\hat{f}]}F_{abcd} = 0\,, \quad \mathcal{D}_{[e}{}^{[\hat{e}}\mathcal{D}_{f}{}^{\hat{f}}\mathcal{D}_{g]}{}^{\hat{g}]}G_{ab,cd} = 0\,. \tag{6}$$

The first two constraints are analogous to those derived in [30] for H^4 and $\nabla^2 H^4$ couplings in Type IIB string theory on K3. The numerical coefficients $c_1, \ldots c_7$ will be fixed below from the knowledge of perturbative contributions.

1. Exact $(\nabla\Phi)^4$ couplings in $D=3$

Based on the known one-loop contribution [31–33], it was proposed in [34] (a proposal revisited in [35]) that the four-derivative scalar coupling F_{abcd} is given exactly by the genus-one modular integral

$$F^{(24,8)}_{abcd} = \int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\, \frac{\partial^4}{(2\pi\mathrm{i})^4\partial y^a\partial y^b\partial y^c\partial y^d}\bigg|_{y=0} \frac{\Gamma_{24,8}}{\Delta} \tag{7}$$

where $\mathcal{F}_1$ is the standard fundamental domain for the action of $SL(2,\mathbb{Z})$ on the Poincaré upper half-plane, $\Delta = \eta^{24}$ is the unique cusp form of weight 12, and $\Gamma_{24,8}$ is the partition function of the non-perturbative Narain lattice,

$$\Gamma_{24,8} = \rho_2^4 \sum_{Q\in\Lambda_{24,8}} e^{\mathrm{i}\pi\, Q_L^2\rho - \mathrm{i}\pi\, Q_R^2\bar{\rho} + 2\pi\mathrm{i}\, Q_L\cdot y + \frac{\pi(y\cdot y)}{2\rho_2}} \tag{8}$$

where $Q_L \equiv p^I_L Q_I$, $Q_R \equiv p^I_R Q_I$, and $|Q|^2 = Q_L^2 - Q_R^2$ takes even values on $\Lambda_{24,8}$. It will be important that the Fourier coefficients of $1/\Delta = \sum_{m\ge -1} c(m)\, q^m$ count the number of 1/2-BPS states in the $D=4$ vacuum obtained by decompactifying a circle inside T^7. This is obvious from the fact that these states are dual to perturbative string states carrying only left-moving excitations [5,8]. It is also worth noting that the ansatz (7) is a special case of a more general class of modular integrals, which we shall denote by $F^{(q+16,q)}_{abcd}$, where the lattice $\Lambda_{24,8}$ is replaced by an even self-dual lattice $\Lambda_{q+16,q}$ and the factor ρ_2^4 by $\rho_2^{q/2}$. The integral $F^{(q+16,q)}_{abcd}$ converges for $q<6$, and is defined for $q\ge 6$ by a suitable regularization prescription. For $q\le 7$, the modular integral $F^{(q+16,q)}_{abcd}$ controls the one-loop contribution to the F^4 coupling in heterotic string compactified on T^q [31–33]. For any value of q, it can be checked that $F^{(q+16,q)}_{abcd}$ satisfies (4) and (6) with $c_1 = \frac{2-q}{4}$, $c_2 = 4-q$, $c_3 = 3$.

By construction, the ansatz (7) is a solution of the supersymmetric Ward identity, which is manifestly invariant under $G_3(\mathbb{Z})$. Its expansion at weak coupling (corresponding to the parabolic decomposition (1), such that the non-perturbative Narain lattice $\Lambda_{24,8}$ degenerates to $\Lambda_{23,7}\oplus\Lambda_{1,1}$) can be computed using the standard unfolding trick. For simplicity, we shall assume that none of the indices $abcd$ lies along $\Lambda_{1,1}$:

$$\begin{aligned}F^{(24,8)}_{\alpha\beta\gamma\delta} = {}& \frac{c_0}{16\pi\, g_3^4}\,\delta_{(\alpha\beta}\delta_{\gamma\delta)} + \frac{F^{(23,7)}_{\alpha\beta\gamma\delta}}{g_3^2} + 4\sum_{k=1}^{3}\sum_{Q\in\Lambda^\star_{23,7}} P^{(k)}_{\alpha\beta\gamma\delta}\\ &\times \bar{c}(Q)\, g_3^{2k-9}\,|\sqrt{2}Q_R|^{k-\frac{7}{2}}\, K_{k-\frac{7}{2}}\left(\frac{2\pi}{g_3^2}|\sqrt{2}Q_R|\right) e^{-2\pi\mathrm{i}a^I Q_I}\end{aligned} \tag{9}$$

where $c_0 = 24$ is the constant term in $1/\Delta$, $\Lambda^\star = \Lambda\backslash\{0\}$, $P^{(1)}_{\alpha\beta\gamma\delta}(Q) = Q_{L\alpha}Q_{L\beta}Q_{L\gamma}Q_{L\delta}$, $P^{(2)}_{\alpha\beta\gamma\delta} = -\frac{3}{2\pi}\delta_{(\alpha\beta}Q_{L\gamma}Q_{L\delta)}$, $P^{(3)}_{\alpha\beta\gamma\delta} = \frac{3}{16\pi^2}\times\delta_{(\alpha\beta}\delta_{\gamma\delta)}$, $K_\nu(z)$ is the modified Bessel function of the second kind, behaving as $\sqrt{\frac{\pi}{2z}}e^{-z}(1+\mathcal{O}(1/z))$ for large positive values of z, and

Please cite this article in press as: G. Bossard et al., Protected couplings and BPS dyons in half-maximal supersymmetric string vacua, Phys. Lett. B (2016), http://dx.doi.org/10.1016/j.physletb.2016.12.035

$$\bar{c}(Q)=\sum_{d|Q} d\, c\left(-\tfrac{|Q|^2}{2d}\right). \tag{10}$$

After rescaling from Einstein frame to string frame, the first and second terms in (9) are recognized as the tree-level and one-loop $(\nabla\Phi)^4$ coupling in perturbative heterotic string theory, while the remaining terms correspond to NS5-brane and KK5-branes wrapped on any possible T^6 inside T^7 [34].

In the large radius limit (corresponding to the parabolic decomposition (2), such that the non-perturbative Narain lattice $\Lambda_{24,8}$ degenerates to $\Lambda_{22,6}\oplus\Lambda_{2,2}$), we get instead (in units where $\ell_P=1$)

$$\begin{aligned} F^{(24,8)}_{\alpha\beta\gamma\delta} &= R^2\left(\frac{c_0}{16\pi}\widehat{E}_1(S)\,\delta_{(\alpha\beta}\delta_{\gamma\delta)}+F^{(22,6)}_{\alpha\beta\gamma\delta}\right) \\ &\quad +4\sum_{k=1}^{3}\sum_{Q'\in\Lambda^\star_{22,6}}\sum_{m,n}{}' c\left(-\tfrac{|Q'|^2}{2}\right)R^{5-k}\,P^{(k)}_{\alpha\beta\gamma\delta} \\ &\quad K_{k-\frac{7}{2}}\left(\tfrac{2\pi R|mS+n|}{\sqrt{S_2}}|\sqrt{2}Q'_R|\right)e^{-2\pi \mathrm{i}(ma^1+na^2)\cdot Q'}+\dots \end{aligned} \tag{11}$$

where $\widehat{E}_1(S)=-\frac{3}{\pi}\log S_2|\eta(S)|^4$. The first term in (11) originates from the dimensional reduction of the $\mathcal{R}^2$ and F^4 couplings in $D=4$ [33,36], after dualizing the gauge fields into scalars. The term $F^{(22,6)}_{\alpha\beta\gamma\delta}$ can also be traced to the four-derivative scalar couplings studied in [32]. The second term in (11) is of order $e^{-2\pi R\mathcal{M}(Q,P)}$, where $\mathcal{M}$ is the mass of a four-dimensional 1/2-BPS state with electromagnetic charges $(Q,P)=(mQ',nQ')$. The phase factor is the expected minimal coupling of a dyonic state to the holonomies of the electric and magnetic gauge fields along the circle. Fixing charges (Q,P) such that Q and P are collinear, the sum over (m,n) induces a measure factor

$$\mu(Q,P)=\sum_{d|(Q,P)} c\left(-\tfrac{\gcd(Q^2,P^2,Q\cdot P)}{2d^2}\right), \tag{12}$$

which is recognized as the degeneracy of 1/2-BPS states with charges (Q,P). In particular for a purely electric state ($P=0$) with primitive charge, it reduces to the well-known result $c(-|Q|^2/2)$ [5]. The dots in (11) stand for terms of order $e^{-2\pi R^2|k|+2\pi \mathrm{i}k\psi}$, characteristic of a Kaluza–Klein monopole of the form $\mathrm{TN}_k\times T^6$, where TN_k is Euclidean Taub–NUT space with charge k. These contributions will be discussed in [26].

2. Exact $\nabla^2(\nabla\Phi)^4$ couplings in $D=3$

We now turn to the six-derivative coupling $G_{ab,cd}$, which is expected to receive both 1/2-BPS and 1/4-BPS instanton contributions. Based on U-duality invariance, supersymmetric Ward identities and the known two-loop contribution [37,38], it is natural to conjecture that $G_{ab,cd}$ is given by the genus-two modular integral

$$G^{(24,8)}_{ab,cd}=\int_{\mathcal{F}_2}\frac{\mathrm{d}^3\Omega_1\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\frac{\frac{1}{2}(\varepsilon_{ik}\varepsilon_{jl}+\varepsilon_{il}\varepsilon_{jk})\partial^4}{(2\pi\mathrm{i})^4\partial y^a_i\partial y^b_j\partial y^c_k\partial y^d_l}\bigg|_{y=0}\frac{\varGamma_{24,8,2}}{\Phi_{10}}, \tag{13}$$

where $\mathcal{F}_2$ is the standard fundamental domain for the action of $Sp(4,\mathbb{Z})$ on the Siegel upper half-plane of degree two [39], $|\Omega_2|$ is the determinant of the imaginary part of $\Omega=\Omega_1+\mathrm{i}\Omega_2$, Φ_{10} is the unique cusp form of weight 10 under the Siegel modular group $Sp(4,\mathbb{Z})$ (whose inverse counts micro-states of 1/4-BPS black holes [6]), and $\varGamma_{24,8,2}$ is the genus-two partition function of the non-perturbative Narain lattice,

$$\varGamma_{24,8,2}=|\Omega_2|^4\sum_{Q^i\in\Lambda^{\otimes 2}_{24,8}} e^{\mathrm{i}\pi(Q^i_L\Omega_{ij}Q^j_L-Q^i_R\bar{\Omega}_{ij}Q^j_R+2Q^i_Ly_i)+\frac{\pi}{2}y^a_i\Omega_2^{-1ij}y_{ja}} \tag{14}$$

Acting with the y^a_i-derivatives results in the insertion of a polynomial $P_{\alpha\beta,\gamma\delta}(Q^i_L,\Omega_2^{-1})$ of degree 4 and 2 in its first and second arguments. We shall denote by $G^{(q+16,q)}_{ab,cd}$ the analogue of (14) where the lattice $\Lambda_{24,8}$ is replaced by $\Lambda_{q+16,q}$ and the power of $|\Omega_2|$ by $q/2$. The integral $G^{(q+16,q)}_{ab,cd}$ is convergent for $q<6$, and defined for $q\geq 6$ by a suitable regularization prescription [40]. For $q\leq 7$, the modular integral $G^{(q+16,q)}_{ab,cd}$ controls the two-loop contribution to the ∇^2F^4 coupling in heterotic string compactified on T^q [37,38].

For any value of q, one can show that $G^{(q+16,q)}_{ab,cd}$ satisfies (5) and (6) with $c_4=\frac{3-q}{2}$, $c_5=\frac{6-q}{2}$, $c_6=\frac{1}{2}$, $c_7=-\pi$. In particular, the quadratic source term on the r.h.s. of (5) originates from the pole of $1/\Phi_{10}$ on the separating degeneration divisor, similar to the analysis in [30,40]. Thus, $G^{(24,8)}$ is a solution of the supersymmetric Ward identity, which is manifestly invariant under $G_3(\mathbb{Z})$. It remains to check that it produces the expected terms at weak coupling, when $\Lambda_{24,8}$ degenerates to $\Lambda_{23,7}\oplus\Lambda_{1,1}$. This limit can be studied using a higher-genus version of the unfolding trick [41,42]. Using results about the Fourier–Jacobi expansion of $1/\Phi_{10}$ from [16], we find

$$\begin{aligned} G^{(24,8)}_{\alpha\beta,\gamma\delta} &= \frac{G^{(23,7)}_{\alpha\beta,\gamma\delta}}{g_3^4}-\frac{\delta_{\alpha\beta}G^{(23,7)}_{\gamma\delta}+\delta_{\gamma\delta}G^{(23,7)}_{\alpha\beta}-2\delta_{\gamma)(\alpha}G^{(23,7)}_{\beta)(\delta}}{12g_3^6} \\ &\quad -\frac{1}{2\pi g_3^8}\left[\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha(\gamma}\delta_{\delta)\beta}\right]+\dots \end{aligned} \tag{15}$$

where

$$G^{(q+16,q)}_{ab}=\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\frac{\partial^2}{(2\pi\mathrm{i})^2\partial y^a\partial y^b}\bigg|_{y=0}\frac{\widehat{E}_2\,\varGamma_{q+16,q}}{\Delta}, \tag{16}$$

with $\widehat{E}_2=\frac{12}{\mathrm{i}\pi}\partial_\rho\log\eta-\frac{3}{\pi\rho_2}$ the almost holomorphic Eisenstein series of weight 2. The first and second terms in (15) corresponds to the zero and rank 1 orbits, respectively. The third term is necessary for consistency with the quadratic term on the r.h.s. of the supersymmetric Ward identity (5), although a naive unfolding procedure fails to produce it, presumably due to the singularity of the integrand in the separating degeneration limit. After rescaling to string frame, the first three terms in (15) correspond to the expected two-loop [37,38], one-loop [43] and tree-level contributions [44,45] to the $\nabla^2(\nabla\Phi)^4$ coupling in heterotic string on T^7, while the dots stand for terms of order e^{-1/g_3^2} ascribable to NS5-brane and KK5-brane instantons, which will be discussed in [26]. Note that the tree-level single trace ∇^2F^4 term in [44] proportional to $\zeta(3)$ vanishes on the Cartan subalgebra [46], and does not contribute to this coupling.

Having shown that our ansatz (13) passes all consistency conditions in $D=3$, let us now analyze its large radius limit, where $\Lambda_{24,8}$ degenerates to $\Lambda_{22,6}\oplus\Lambda_{2,2}$. Again, the unfolding trick gives

$$\begin{aligned} &G^{(24,8)}_{\alpha\beta,\gamma\delta} \\ &=R^4\Big[G^{(22,6)}_{\alpha\beta,\gamma\delta}-\frac{\widehat{E}_1(S)}{12}\left(\delta_{\alpha\beta}G^{(22,6)}_{\gamma\delta}+\delta_{\gamma\delta}G^{(22,6)}_{\alpha\beta}-2\delta_{\gamma)(\alpha}G^{(22,6)}_{\beta)(\delta}\right) \\ &\quad +g(S)(\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha(\gamma}\delta_{\delta)\beta})\Big]+G^{(1)}_{\alpha\beta,\gamma\delta}+G^{(2)}_{\alpha\beta,\gamma\delta}+G^{(\mathrm{KKM})}_{\alpha\beta,\gamma\delta} \end{aligned} \tag{17}$$

The two terms on the first line (which correspond to the constant term with respect to the parabolic decomposition (2)) originate from the reduction of the ∇^2F^4 and $\mathcal{R}^2F^2$ couplings in four dimensions. The term proportional to $g(S)$ originates presumably from the separating degeneration divisor, and is determined by

Please cite this article in press as: G. Bossard et al., Protected couplings and BPS dyons in half-maximal supersymmetric string vacua, Phys. Lett. B (2016), http://dx.doi.org/10.1016/j.physletb.2016.12.035

the differential equation (6). The terms $G^{(1)}$ and $G^{(2)}$ are independent of the NUT potential ψ, and correspond to the Abelian Fourier coefficients. They are both suppressed as $e^{-2\pi R\mathcal{M}(Q,P)}$, but $G^{(1)}$ has support on electromagnetic charges (Q,P) which Q and P collinear, hence corresponds to contributions of 1/2-BPS states winding the circle, while $G^{(2)}$ has support on generic charges, corresponding to 1/4-BPS states. The last term $G^{(\mathrm{KKM})}$ includes all terms with non-zero charge with respect to the NUT potential, corresponding to Kaluza–Klein monopole contributions.

In this letter, we focus on the contribution $G^{(2)}$ from 1/4-BPS black holes. This contribution originates from the 'Abelian rank 2 orbit', whose stabilizer is the parabolic subgroup $GL(2,\mathbb{Z})\ltimes\mathbb{Z}^3$ inside $Sp(4,\mathbb{Z})$. Thus, the integral can be unfolded onto $\mathcal{P}_2/PGL(2,\mathbb{Z})\times[0,1]^3$, where $\mathcal{P}_2$ denotes the space of positive definite 2×2 matrices Ω_2:

$$G^{(2)}_{\alpha\beta,\gamma\delta}=R^4\int_{\mathcal{P}_2}\frac{\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\int_{[0,1]^3}\mathrm{d}^3\Omega_1\,\frac{(\varepsilon_{ik}\varepsilon_{jl}+\varepsilon_{il}\varepsilon_{jk})\partial^4}{(2\pi\mathrm{i})^4\partial y^a_i\partial y^b_j\partial y^c_k\partial y^d_l}\bigg|_{y=0}$$
$$\times\frac{\langle e^{-2\pi\mathrm{i}a^{iI}A_{ij}Q^j_I}\rangle_{22,6,2}}{\Phi_{10}}\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\|A|\neq 0}}e^{-\frac{\pi R^2}{S_2}\mathrm{Tr}\left[\Omega_2^{-1}\cdot A^\intercal\cdot\left(\begin{smallmatrix}1&S_1\\S_1&|S|^2\end{smallmatrix}\right)\cdot A\right]}\tag{18}$$

where $\langle f(Q)\rangle_{22,6,2}$ denotes the partition function $\Gamma_{22,6,2}$ with an insertion of $f(Q)$ in the sum. The integral over Ω_1 at fixed Ω_2 extracts the Fourier coefficient $C\left[\left(\begin{smallmatrix}-\frac12|Q|^2&-Q\cdot P\\-Q\cdot P&-\frac12|P|^2\end{smallmatrix}\right);\Omega_2\right]$ of $1/\Phi_{10}$. Due to the zeros of Φ_{10}, the latter is a locally constant function of Ω_2, discontinuous across certain real codimension 1 walls in $\mathcal{P}_2$ [47,48]. For large R however, the remaining integral over Ω_2 is dominated by a saddle point $\Omega_2^\star$ (see (24) below), so to all orders in $1/R$ around the saddle point, we can replace the above Fourier coefficient by its value at $\Omega_2^\star$. The remaining integral over Ω_2 can be computed using

$$\int_{\mathcal{P}_2}\mathrm{d}^3S\,|S|^{\delta-\frac32}\,e^{-\pi\,\mathrm{Tr}\,(SA+S^{-1}B)}=2\left(\frac{|B|}{|A|}\right)^{\delta/2}\widetilde{B}_\delta(AB)\,,\tag{19}$$

where $\widetilde{B}_\delta(Z)$ is a matrix-variate generalization of the modified Bessel function [49],[1]

$$\widetilde{B}_\delta(Z)=\int_0^\infty\frac{\mathrm{d}t}{t^{3/2}}\,e^{-\pi t-\frac{\pi\mathrm{Tr}Z}{t}}\,K_\delta\left(\frac{2\pi\sqrt{|Z|}}{t}\right)\,.\tag{20}$$

In the limit where all entries in Z are large, one has

$$\widetilde{B}_\delta(Z)\sim\frac12\left[|Z|\left(\mathrm{Tr}Z+2\sqrt{|Z|}\right)\right]^{-\frac14}e^{-2\pi\sqrt{\mathrm{Tr}Z+2\sqrt{|Z|}}}\,.\tag{21}$$

Further relabelling $\left(\begin{smallmatrix}Q\\P\end{smallmatrix}\right)=A\left(\begin{smallmatrix}Q_1\\Q_2\end{smallmatrix}\right)$, we find

$$G^{(2)}_{\alpha\beta,\gamma\delta}=2R^7\sum_{Q,P\in\Lambda^\star_{22,6}}e^{-2\pi\mathrm{i}(a^1Q+a^2P)}\frac{\mu(Q,P)}{|2P_R\wedge Q_R|^{\frac32}}$$
$$\times P_{\alpha\beta,\gamma\delta}\left(\tfrac{1}{\sqrt{S_2}}\begin{pmatrix}1&S_1\\0&S_2\end{pmatrix}\begin{pmatrix}Q_L\\P_L\end{pmatrix},-\tfrac{1}{\pi R^2}\tfrac{\partial}{\partial Y}\right)\cdot$$
$$\left(|Y|^{\frac34}\widetilde{B}_{\frac32}\left[Y\tfrac{2R^2}{S_2}\begin{pmatrix}1&S_1\\0&S_2\end{pmatrix}\begin{pmatrix}|Q_R|^2&P_R\cdot Q_R\\P_R\cdot Q_R&|P_R|^2\end{pmatrix}\begin{pmatrix}1&0\\S_1&S_2\end{pmatrix}\right]\right)\bigg|_{Y=\mathbb{1}}\tag{22}$$

where $|P_R\wedge Q_R|=\sqrt{(P_R^2)(Q_R^2)-(P_R\cdot Q_R)^2}$, $P_{\alpha\beta,\gamma\delta}$ is the polynomial defined below (14),

$$\mu(Q,P)=\sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\A^{-1}\left(\begin{smallmatrix}Q\\P\end{smallmatrix}\right)\in\Lambda^{\otimes2}_{22,6}}}|A|\,C\left[A^{-1}\begin{pmatrix}-\frac12|Q|^2&-Q\cdot P\\-Q\cdot P&-\frac12|P|^2\end{pmatrix}A^{-\intercal};\Omega_2^\star\right]\tag{23}$$

and $\Omega_2^\star$ is the location of the afore-mentioned saddle point,

$$\Omega_2^\star=\frac{R}{\mathcal{M}(Q,P)}A^\intercal\left[\tfrac{1}{S_2}\begin{pmatrix}1&S_1\\S_1&|S|^2\end{pmatrix}+\tfrac{1}{|P_R\wedge Q_R|}\begin{pmatrix}|P_R|^2&-P_R\cdot Q_R\\-P_R\cdot Q_R&|Q_R|^2\end{pmatrix}\right]A\,.\tag{24}$$

Using (21), we see that these contributions behave as $e^{-2\pi R\mathcal{M}(Q,P)}$ in the limit $R\to\infty$, where

$$\mathcal{M}(Q,P)=\sqrt{2\frac{|Q_R+SP_R|^2}{S_2}+4\sqrt{\begin{vmatrix}|Q_R|^2&Q_R\cdot P_R\\Q_R\cdot P_R&|P_R|^2\end{vmatrix}}}\tag{25}$$

is recognized as the mass of a 1/4-BPS dyon with electromagnetic charges (Q,P) [50,51]. Moreover, in cases where only $A=\mathbb{1}$ contributes to (23), the instanton measure $\mu(Q,P)$ agrees with the BPS index $\Omega(Q,P;S,\phi)$ in the corresponding chamber of the moduli space $\mathcal{M}_4$ in $D=4$, computed with the contour prescription in [52]. Our result (23) generalizes this prescription to arbitrary electromagnetic charges (Q,P) and recovers the results of [53–55] for dyons with torsion, fixing a subtlety in the choice of chamber. Additional (exponentially suppressed) contributions to $G^{(2)}$ arise from the difference between $C\left[\left(\begin{smallmatrix}-\frac12|Q|^2&-Q\cdot P\\-Q\cdot P&-\frac12|P|^2\end{smallmatrix}\right);\Omega_2\right]$ and its value at the saddle point. The relation between the jumps of these Fourier coefficients and the possible splittings of a 1/4-BPS bound state into two 1/2-BPS constituents [47] is crucial for consistency with the quadratic source term in the supersymmetric Ward identity (5). These contributions, along with the terms $G^{(2)}$ and $G^{(\mathrm{KKM})}$ which we have ignored here, will be discussed in [26].

3. Discussion

In this work, we have conjectured the exact form of the $(\nabla\Phi)^4$ and $\nabla^2(\nabla\Phi)^4$ couplings in the low energy effective action of $D=3$ string vacua with half-maximal supersymmetry, focussing on the simplest model, heterotic string compactified on T^7. Our ansätze (7) and (13) are manifestly U-duality invariant, satisfy the requisite supersymmetric Ward identities, reproduce the known perturbative contributions at weak heterotic coupling and the known F^4, $\mathcal{R}^2$, D^2F^4 and $\mathcal{R}^2F^2$ couplings in $D=4$ in the limit where the radius of one circle inside T^7 becomes large. While we do not yet have a rigorous proof that these constraints uniquely determine the functions F_{abcd} and $G_{ab,cd}$, we expect that additional contributions from cusp forms are ruled out by the supersymmetric Ward identities (4) and (5), by the same type of arguments which apply for the $\mathcal{R}^4$ and $\nabla^6\mathcal{R}^4$ couplings.

In the limit where the radius of one circle inside T^7 becomes large, we find, in addition to the afore-mentioned power-like terms, an infinite series of corrections of order $e^{-2\pi R\mathcal{M}(Q,P)}$ which are interpreted as Euclidean counterparts of four-dimensional BPS states with mass $\mathcal{M}(Q,P)$, whose worldline winds around the circle. Rather remarkably, the contribution from a 1/4-BPS dyon is weighted by the BPS index $\Omega(Q,P;S,\phi)$, extracted from the Siegel modular form $1/\Phi_{10}$ using the very same contour prescription as in [52]. Indeed, it was suggested in [56] (see also [57,58]) to represent 1/4-BPS dyons as heterotic strings wrapped on a genus-two

[1] Our $\widetilde{B}_\delta(Z)$ is related to $B_\delta(Z)$ in [49] via $\widetilde{B}_\delta(Z/\pi^2)=\frac12(\det Z)^{\delta/2}B_\delta(Z)$.

curve holomorphically embedded in a T^4 inside T^7. This picture was further used in [59] to justify the contour prescription of [52]. Our analysis of the $\nabla^2(\nabla\Phi)^4$ coupling in $D=3$ gives a concrete basis to these heuristic ideas, and explains why 1/4-BPS dyons in $D=4$ are counted by a Siegel modular form of genus two. We emphasize that the introduction of the Siegel modular form $1/\Phi_{10}$ in the conjectured formula (13) is necessary to match the perturbative 2-loop amplitude, where it appears explicitly [37,38]. A more detailed analysis of the weak coupling and large radius expansions of the $\nabla^2(\nabla\Phi)^4$ coupling will appear in [26], with particular emphasis on the consequences of wall-crossing for three-dimensional couplings.

Acknowledgements

We are grateful to Eric d'Hoker, Ioannis Florakis and Rodolfo Russo for valuable discussions on genus-two modular integrals, and to Sameer Murthy for discussions on wall-crossing in $\mathcal{N}=4$ string vacua. G.B. and C.C.H. thank CERN for its hospitality.

References

[1] A. Font, L.E. Ibanez, D. Lust, F. Quevedo, Phys. Lett. B 249 (1990) 35.
[2] A. Sen, Int. J. Mod. Phys. A 9 (1994) 3707, arXiv:hep-th/9402002.
[3] C.M. Hull, P.K. Townsend, Nucl. Phys. B 438 (1995) 109, arXiv:hep-th/9410167.
[4] E. Witten, Nucl. Phys. B 443 (1995) 85, arXiv:hep-th/9503124.
[5] A. Dabholkar, J.A. Harvey, Phys. Rev. Lett. 63 (1989) 478.
[6] R. Dijkgraaf, H.L. Verlinde, E.P. Verlinde, Nucl. Phys. B 484 (1997) 543, arXiv:hep-th/9607026.
[7] R. Dijkgraaf, G.W. Moore, E.P. Verlinde, H.L. Verlinde, Commun. Math. Phys. 185 (1997) 197, arXiv:hep-th/9608096.
[8] A. Dabholkar, Phys. Rev. Lett. 94 (2005) 241301, arXiv:hep-th/0409148.
[9] A. Dabholkar, F. Denef, G.W. Moore, B. Pioline, J. High Energy Phys. 10 (2005) 096, arXiv:hep-th/0507014.
[10] A. Strominger, C. Vafa, Phys. Lett. B 379 (1996) 99, arXiv:hep-th/9601029.
[11] D. Shih, A. Strominger, X. Yin, J. High Energy Phys. 10 (2006) 087, arXiv:hep-th/0505094.
[12] J.R. David, A. Sen, J. High Energy Phys. 11 (2006) 072, arXiv:hep-th/0605210.
[13] A. Sen, Gen. Relativ. Gravit. 40 (2008) 2249, arXiv:0708.1270.
[14] N. Banerjee, D.P. Jatkar, A. Sen, J. High Energy Phys. 05 (2009) 121, arXiv:0810.3472.
[15] S. Banerjee, R.K. Gupta, I. Mandal, A. Sen, J. High Energy Phys. 11 (2011) 143, arXiv:1106.0080.
[16] A. Dabholkar, S. Murthy, D. Zagier, arXiv:1208.4074, 2012.
[17] A. Dabholkar, J. Gomes, S. Murthy, J. High Energy Phys. 03 (2015) 074, arXiv:1404.0033.
[18] S. Murthy, V. Reys, J. High Energy Phys. 04 (2016) 052, arXiv:1512.01553.
[19] M.B. Green, M. Gutperle, Nucl. Phys. B 498 (1997) 195, arXiv:hep-th/9701093.
[20] M.B. Green, P. Vanhove, J. High Energy Phys. 0601 (2006) 093, arXiv:hep-th/0510027.
[21] M. Günaydin, A. Neitzke, B. Pioline, A. Waldron, Phys. Rev. D 73 (2006) 084019, arXiv:hep-th/0512296.
[22] A.M. Polyakov, Nucl. Phys. B 120 (1977) 429.
[23] N. Seiberg, E. Witten, arXiv:hep-th/9607163, 1996.
[24] D. Gaiotto, G.W. Moore, A. Neitzke, Commun. Math. Phys. 299 (2010) 163, arXiv:0807.4723.
[25] S. Chaudhuri, G. Hockney, J.D. Lykken, Phys. Rev. Lett. 75 (1995) 2264, arXiv:hep-th/9505054.
[26] G. Bossard, C. Cosnier-Horeau, B. Pioline, in press.
[27] K.S. Narain, Phys. Lett. B 169 (1986) 41.
[28] N. Marcus, J.H. Schwarz, Nucl. Phys. B 228 (1983) 145.
[29] A. Sen, Nucl. Phys. B 434 (1995) 179, arXiv:hep-th/9408083.
[30] Y.-H. Lin, S.-H. Shao, Y. Wang, X. Yin, J. High Energy Phys. 12 (2015) 142, arXiv:1508.07305.
[31] W. Lerche, B. Nilsson, A. Schellekens, N. Warner, Nucl. Phys. B 299 (1988) 91.
[32] I. Antoniadis, B. Pioline, T.R. Taylor, Nucl. Phys. B 512 (1998) 61, arXiv:hep-th/9707222.
[33] E. Kiritsis, N.A. Obers, B. Pioline, J. High Energy Phys. 01 (2000) 029, arXiv:hep-th/0001083.
[34] N.A. Obers, B. Pioline, J. High Energy Phys. 07 (2000) 003, arXiv:hep-th/0006088.
[35] S. Kachru, N.M. Paquette, R. Volpato, arXiv:1603.07330, 2016.
[36] J.A. Harvey, G.W. Moore, Phys. Rev. D 57 (1998) 2323, arXiv:hep-th/9610237.
[37] E. D'Hoker, D.H. Phong, Nucl. Phys. B 715 (2005) 3, arXiv:hep-th/0501197.
[38] E. D'Hoker, M. Gutperle, D.H. Phong, Nucl. Phys. B 722 (2005) 81, arXiv:hep-th/0503180.
[39] E. Gottschling, Math. Ann. 138 (1959) 103.
[40] B. Pioline, R. Russo, J. High Energy Phys. 12 (2015) 102, arXiv:1510.02409.
[41] B. Pioline, Proc. Symp. Pure Math. 88 (2014) 119, arXiv:1401.4265.
[42] I. Florakis, B. Pioline, arXiv:1602.00308, 2016.
[43] N. Sakai, Y. Tanii, Nucl. Phys. B 287 (1987) 457.
[44] D.J. Gross, J.H. Sloan, Nucl. Phys. B 291 (1987) 41.
[45] E.A. Bergshoeff, M. de Roo, Nucl. Phys. B 328 (1989) 439.
[46] J.M. Drummond, P.J. Heslop, P.S. Howe, S.F. Kerstan, J. High Energy Phys. 08 (2003) 016, arXiv:hep-th/0305202.
[47] A. Sen, J. High Energy Phys. 05 (2007) 039, arXiv:hep-th/0702141.
[48] A. Dabholkar, D. Gaiotto, S. Nampuri, J. High Energy Phys. 01 (2008) 023, arXiv:hep-th/0702150.
[49] C.S. Herz, Ann. Math. (2) 61 (1955) 474.
[50] M. Cvetic, D. Youm, Phys. Rev. D 53 (1996) 584, arXiv:hep-th/9507090.
[51] M.J. Duff, J.T. Liu, J. Rahmfeld, Nucl. Phys. B 459 (1996) 125, arXiv:hep-th/9508094.
[52] M.C.N. Cheng, E. Verlinde, J. High Energy Phys. 09 (2007) 070, arXiv:0706.2363.
[53] S. Banerjee, A. Sen, Y.K. Srivastava, J. High Energy Phys. 05 (2008) 101, arXiv:0802.0544.
[54] S. Banerjee, A. Sen, Y.K. Srivastava, J. High Energy Phys. 05 (2008) 098, arXiv:0802.1556.
[55] A. Dabholkar, J. Gomes, S. Murthy, J. High Energy Phys. 05 (2011) 059, arXiv:0803.2692.
[56] D. Gaiotto, arXiv:hep-th/0506249, 2005.
[57] A. Dabholkar, D. Gaiotto, J. High Energy Phys. 12 (2007) 087, arXiv:hep-th/0612011.
[58] A. Dabholkar, S. Nampuri, J. High Energy Phys. 11 (2007) 077, arXiv:hep-th/0603066.
[59] S. Banerjee, A. Sen, Y.K. Srivastava, J. High Energy Phys. 03 (2009) 151, arXiv:0808.1746.