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► To cite this version:

Leticia F Cugliandolo. Critical percolation in bidimensional coarsening. *Journal of Statistical Mechanics: Theory and Experiment*, 2016, 2016 (11), pp.114001. 10.1088/1742-5468/2016/11/114001 . hal-01419911

HAL Id: hal-01419911

<https://hal.sorbonne-universite.fr/hal-01419911>

Submitted on 20 Dec 2016

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Critical percolation in bidimensional coarsening

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December 19, 2016

Abstract

I discuss a recently unveiled feature in the dynamics of two dimensional coarsening systems on the lattice with Ising symmetry: they first approach a critical percolating state via the growth of a new length scale, and only later enter the usual dynamic scaling regime. The time needed to reach the critical percolating state diverges with the system size. These observations are common to Glauber, Kawasaki, and voter dynamics in pure and weakly disordered systems. An extended version of this account appeared in *Comptes Rendus de Physique* (2016). I refer to the relevant publications for details.

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1 Introduction

Coarsening or phase ordering kinetics is the process whereby an open classical system orders locally in its equilibrium states until the maximal order compatible with thermal fluctuations, conservation laws and boundary conditions is achieved [1, 2, 3, 4, 5, 6]. This phenomenon occurs when a macroscopic system is taken across a

Bray94, Onuki02, Puri09,

continuous phase transition by changing (slowly or abruptly) one of its parameters or the environmental conditions.

The quantitative characterisation of coarsening has been largely circumscribed to the one of the space-time correlation [1, 2, 3, 4, 6] and linear response [4, 7, 8] functions, while the direct analysis of the morphology of the spatial structures has remained, in comparison, much less developed. However, such a description can have fundamental and practical interest since, from a material science point of view, not only the average grain size but also its stochastic counterpart, the size distribution function, influences many material properties.

In this article I shortly summarise recent advances in this line of research. I focus on the $2d$ ferromagnetic Ising spin model with microscopic dynamics satisfying detailed balance, and the planar voter model that goes beyond the physically constrained dynamic rules. The effects of quenched disorder will be shortly mentioned. A final discussion section closes the paper with ideas for future research.

1.1 Models

The classical *Ising model* is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j \quad (1)$$

with $J > 0$. The bimodal spin variables take values $s_i = \pm 1$ and the sum runs over first-neighbours on a lattice.

Weak quenched disorder stands for random interactions or fields that do not change the nature of the equilibrium states. This is achieved, for example, by choosing the exchanges from a probability distribution with positive support, or by using random fields with symmetric distribution around zero. These are the *random bond Ising model* and the *random field Ising model*, respectively.

The stochastic dynamics mimic the spin flips due to the coupling to a thermal bath. They can conserve the local order parameter or not. In the former case, only pairs of neighbouring anti-aligned spins are updated. In the latter case, any spin in the sample can be turned round. The moves are accepted with Monte Carlo rules that satisfy detailed balance ensuring that the dynamics can (but not necessarily do) take a finite sample to thermal equilibrium at any temperature. The flip is accepted with probability

$$1 \text{ for } \Delta E < 0, \quad e^{-\beta \Delta E} \text{ for } \Delta E > 0, \quad \text{and} \quad p \text{ for } \Delta E = 0, \quad (2)$$

and, at $T = 0$, no update with $\Delta E > 0$ is allowed. For $p = 0$ the system can get easily trapped. The behaviour for all $p > 0$ is essentially the same and typical choices are $p = 1/2$ for the heat-bath and Glauber rules and $p = 1$ for the Monte Carlo rule. The unit of Monte Carlo time is defined as N attempts to perform a spin-flip, with N the number of spins in the sample.

At a larger length-scale one follows the Ginzburg-Landau paradigm and proposes a stochastic differential equation for the time-evolution of a local field, that is to say, a coarse-grained description of the spin configuration. This is the time-dependent Ginzburg-Landau equation that can incorporate, or not, the conservation of the order parameter. The analysis of the scalar equation with non-conserved order parameter, implies that the motion of any point on a rather flat domain wall is

driven by its local curvature [9]. The conserved order parameter dynamics of Ising-like systems [10, 11] and the Potts model can also be similarly treated [12].

Another interesting case is the *voter model* [13] with no energy function and dynamic rules that mimic the spreading of opinions in a society with finite-range interactions. At each microscopic time-step a spin with two possible orientations is chosen at random and it is aligned with one of its nearest-neighbours, also chosen at random. If the spins were already in agreement no change is induced in this time-step. Instead, the chosen spin flips. In $2d$ and for finite system size, this essentially out of equilibrium evolution approaches an absorbing state with full consensus of one of the two kinds. The dynamics are driven by interface noise without bulk noise, as no spin surrounded by neighbours of the same kind can fluctuate.

The density of spins of each type has strong effects on the dynamics. For rules that do not conserve the order parameter, a finite imbalance drives the system to equilibrium very quickly. For rules that do conserve the density of all species, the dynamics of slightly unbalanced initial states is still interesting.

There are, basically, two types of interesting initial states:

- with only short-range correlations as obtained, for example, by sampling equilibrium configurations in the disordered phase.
- with long-range correlations as, for example, in equilibrium configurations at a critical point.

This paper concerns the dynamics at relatively short time scales when the disordered configuration builds short-range correlations and approach a critical percolation state. To discuss this phenomenon we recall, in the next Section, the percolation problem and some of its geometric properties.

1.2 Geometry

The *site percolation problem* is very simply defined. A lattice with typically free (FBC) or periodic (PBC) boundary conditions is chosen. (It is also possible to consider regular or random graphs but we will not discuss these in this note.) The sites of the lattice are occupied or not with probability p and $1 - p$, respectively. A *cluster* is the ensemble of neighbouring sites occupied by particles. p controls the sizes of the clusters and the possible presence of a spanning or wrapping cluster. For $p \rightarrow 0$ all configurations are void while for $p \rightarrow 1$ the full lattice is always occupied by a single cluster. In the limit $L \rightarrow \infty$ a sharp threshold at a finite p_c separates a phase with no spanning cluster (or no wrapping cluster for PBC) from one with one spanning cluster (or a wrapping cluster for PBC) of occupied sites for FBC. For finite L there is only a rounded changeover around p_c . The critical parameter p_c depends on the geometry of the lattice and some values are given in Table 1. The interest is set on the full characterisation of the number and structure of these clusters, as a function of their size, as a function of p .

The percolation problem is completely geometric. Still, the transition at p_c is a critical phenomenon characterised by a set of universal critical exponents that describe, for example, the fractal dimension of the percolating cluster and the size distribution of the non-percolating ones independently of the microscopic details. $p = 1/2$ is below threshold for all the lattices in Table 1 but for the triangular one.

Geometric clusters can also be defined in spins models by joining together neighbouring spins with the same orientation. (These are not, however, the relevant structures that describe the critical phenomenon of the spin models [16, 17].) The

Lattice	n	p_c	z_p
Honeycomb	3	0.70	∞
Kagomé	4	0.65	1/2
Square	4	0.59	1/2
Bowtie-a	5	0.55	0.38(5)
Triangular	6	1/2	1/3

Table 1: Connectivities, n , critical thresholds, p_c , and exponents for the approach to critical percolation, z_p , after a $T = 0$ quench of the Ising model on several $2d$ lattices evolved with non-conserved order parameter dynamics. In the case of the bow-tie lattice, $n = 5$ is the average between the connectivities of the kinds of sites with connectivities 4 and 6. (Regularly odd-coordinated lattices are not suited for our study because the evolution rapidly freezes due to metastable droplets. The honeycomb lattice is one such case [14] and this is the reason why an exponential in L time-scale is found [15].)

table:uno

interfaces are the *domain walls* between clusters of different kind. In models with bimodal variables they must be closed loops for PBCs or start and end at the system's border for FBCs. The *hull* is the external border of a domain, constructed by joining sites on the dual lattice with links that cut broken bonds on the original lattice. Hull-enclosed areas do not have holes while domains can have. In various papers we focused on the hull and the area enclosed by it [18, 19, 20].

At a critical point, be it thermodynamic or geometric, the critical objects are characterised by fractal exponents,

$$V \sim R^{D_V} \quad A \simeq R^{D_A} \quad \ell \simeq R^{D_\ell} \quad (3)$$

with R the radius of the smallest sphere that contains the loop and $\ell \gg l_0$, the microscopic length-scale. A hull-enclosed area in $d = 2$ is compact, and $D_A = 2$. Instead, a domain and its interface at the critical $2d$ Ising point are not compact and their area and length satisfy $A \simeq \ell^{D_A/D_\ell}$.

In a dynamic situation the reference linear scale R is a growing function of time, $\xi_d(t)$. The observed length and areas should then be scaled with the growing length and growing areas (including, possibly, fractal exponents), to describe the area-length relations and also the time-dependent number densities discuss below.

Take a geometric object with a given property (area, length, etc.) generically called x that takes values X . Close to criticality, its probability distribution per lattice site or number density, $n_x(X)$, takes the form

$$n_x(X) \simeq X^{-\alpha_x} f_x(X^{\sigma_x}(p - p_c)) + n_{fs}(X/L^{D_x}) . \quad (4)$$

We called p the control parameter that could be p for percolation or T for a statistical physics problem. $m_x = (p - p_c)^{1/\sigma_x}$ is sometimes called the ‘mass’ and α_x the ‘Fisher exponent’. The fractal dimension D_x and α_x are linked by $D_x = d/(\alpha_x - 1)$ with d the dimension of space. The scaling function, f_x , at criticality, takes a constant value $f_x(0) = c_x$ that depends on the quantity x . The first contribution is the only one that survives in the infinite size limit while the second one controls the finite size corrections. In the infinite system size limit, the number density of hull-enclosed and domain areas at critical percolation is

$$n_{h,d}(A) \simeq 2c_{h,d}/A^{\alpha_{h,d}} . \quad (5)$$

eq:nhA

The value of c_h is known exactly $c_h = 1/(16\sqrt{3}\pi)$ [21] and $\alpha_h = 2$. At the critical Ising point the distribution is the same without the factor 2. The values of the exponent α_d and consequently the fractal dimensions have been elucidated in the 80s. The values of c_d are known only numerically and some approximate relations to c_h have been discussed in [19].

At the percolation threshold, the probability of there being (i) a cluster that percolates in the two Cartesian directions, (ii) one or more clusters that percolate in the horizontal or the vertical direction, (iii) one or more clusters that percolate diagonally are known exactly (for different boundary conditions) [22]. These observables are very useful to decide whether a dynamic configuration has come close to critical percolation or not.

In the continuum limit the percolation hulls are conformally invariant curves described by a stochastic Loewner evolution SLE_κ . The parameter κ can be determined numerically by computing the averaged variance of the winding angle, $\langle \theta^2(x) \rangle$, for two points chosen at random at a distance x along the curve. For critical systems in two dimensions [23, 24]:

$$\langle \theta^2(x) \rangle = \text{cst} + \frac{4\kappa}{8 + \kappa} \ln x. \quad (6)$$

For critical percolation hulls $\kappa = 6$ and for the critical Ising model $\kappa = 3$.

2 Dynamics

After a quench below T_c the systems tend to order, the number of domains steadily decreases and the average linear size of the remaining ones steadily increases. Indeed, from the simple visual inspection of the time-evolving configurations, one observes that there is a growing length $\xi_d(t)$ that, *grosso modo*, corresponds to the average linear size of the *finite-size ordered patches*. The *dynamic scaling hypothesis* states that this is the unique growing length that controls the scaling of all correlation functions and observables in these systems.

The time-dependence of $\xi_d(t)$ serves to classify different dynamic universality classes determined by the dimension of the order parameter and the conservation laws. In clean systems (i.e., with no quench disorder)

$$\xi_d(t) \simeq t^{1/z_d} \quad (7)$$

with $z_d = 2$ for curvature driven grain growth and interface controlled Ostwald ripening, and $z_d = 3$ for phase separation and diffusion controlled Ostwald ripening. These values have been derived analytically for the continuous description of coarsening with non-conserved [9] and conserved [10, 11] scalar order parameter dynamics, respectively. For the voter model in $d = 2$, $\xi_d(t) \simeq t^{1/z_d}$ with $z_d = 2$ [13, 25] but there are logarithmic violations to dynamic scaling. In cases with weak disorder $\xi_d(t)$ is expected to grow logarithmically [26].

In systems with an energy function, the growing length is usually measured from the excess energy with respect to the ground state one. In a simulation, the extracted function $\ell_G(t)$ does not quickly reach the expected stable power law t^{1/z_d} for a clean system or $(\ln t)^{1/\psi}$ in a weak disordered one, but instead it is described by a time and/or temperature dependent effective exponent $t^{1/z_{\text{eff}}}$ [15, 27, 28, 29, 30]. This is a fact to be taken into account in the dynamic scaling analysis.

This discussion assumes that there is a single growing length $\xi_d(t)$, controlling the full dynamics of the system. However, as we will show, after the quench another process takes place, one of aggregation in which these systems approach critical percolation, with another growing length $\xi_p(t)$ with a different time-dependence and that has to be taken into account in all scaling arguments.

An initial condition taken from equilibrium in the disorder phase has a vanishing order parameter and short-range correlations unless $T_0 \simeq T_c$. There is no spanning cluster of aligned spins in it. The fate of this initial state under the short-time evolution is the subject of the rest of this section.

2.1 Approach to critical percolation

Evidence for critical percolation influencing the scaling regime and asymptotic states reached by the $2d$ square-lattice Ising model after a quench from high to low temperature was first given in [18, 19]. These papers show that after a very short time span, a few Monte Carlo (MC) steps for the simulated cells, the configurations look like the ones at critical percolation. More precisely, the morphological and statistical properties of structures such as the areas of the domains or the lengths of the interfaces, that are larger than the average ones, are the ones of percolation at its threshold. As the occupation probability for up and down spins in a high-temperature equilibrium configuration on a square lattice is smaller than the one at critical percolation, this fact suggests that the system must have reached critical percolation at some point of its evolution while it was building correlations.

A detailed analysis of the time needed to reach the critical percolation state after quenches from infinite to $T = 0$ Glauber dynamics (non-conserved order parameter) in the $2d$ Ising model on different lattices was performed in [31, 15]. Several criteria lead to the predictions in the third column in Table I for the exponent characterising the algebraic divergence of this time-scale with the linear size of the system

$$t_p \simeq L^{1/z_p} \quad (8)$$

independently of the boundary conditions, and with a prefactor of the order of one. We do not know whether the fractions are exact results, we only measure numerical values that are very close to them. This law was found from: (1) The study of the overlap between two copies of the system, $Q(t, t_w)$, made at a time t_w scaled with L in such a way to find a non-zero asymptotic value. (2) The scaling of the size of the largest geometric cluster area, its hull-enclosed area and their interfaces. (3) The distribution of the largest geometric cluster and hull-enclosed areas. (4) The behaviour of the spanning or wrapping probabilities, depending on the boundary conditions. (5) The average of the square winding angle of various interfaces. In [31] we conjectured, on the basis of the study of the four last lattices in Table I, that

$$z_p = z_d/n \quad (9)$$

or at least that the exponent $1/z_p$ should increase for faster dynamics (smaller z_d) and for increasing coordination of the lattice. (The honeycomb lattice is special as it allows for the existence of blocked domains with finite size at zero temperature [14].)

The fact that the conserved order parameter dynamics of an Ising model, à la Kawasaki, also approaches critical percolation was already observed in [20, 32]. A similar study on segregating binary Bose Einstein condensates arrived at the same

conclusion [33]. This study suggests that in systems with such a slow dynamics the dynamic growing length itself should be used to characterise $\xi_p(t)$. More precisely,

$$\xi_p(t) = \xi_d^n(t) . \quad (10)$$

We have not seen any important influence of temperature on the time-scale t_p of the Ising model [15]. At the other end of the evolution, temperature fluctuations quite naturally shorten the life-time of striped metastable.

The approach to critical percolation is not exclusive of physical dynamics respecting detailed balance. In the voter model this phenomenon is achieved in a much longer time scale; on the square lattice [34]

$$t_p \simeq L^{1.66} . \quad (11)$$

One can be tempted to propose that in this case the exponent is $z_p = 3/5$. The connectivity of the lattice should not be a relevant parameter for this problem since there is no energy function and the update of a spin is decided by a single of its neighbours, independently of the geometry of the lattice (see [15] for a check). In this sense, $n = 1$ for all lattices and an exponent close to 2 in (11) seems natural.

The effects of weak quenched disorder are under study [35].

2.2 Dynamic scaling

In the asymptotic time domain, when $\xi_d(t)$ has grown much larger than any microscopic length and remains smaller than the system size, $\ell_0 \ll \xi_d(t) \ll L$, a *dynamic scaling symmetry* sets in, similarly to the usual scaling symmetry observed in equilibrium critical phenomena. According to this hypothesis, the growth of $\xi_d(t)$ is the only relevant process and the whole time-dependence enters only through $\xi_d(t)$. This implies that the statistical properties should not depend on time provided that distances are measured in units of the dominant length $\xi_d(t)$. Due to this fact, correlations such as $C(r, t) = \langle s_i(t) s_j(t) \rangle$, where $r = |\vec{r}_i - \vec{r}_j|$, take the scaling form

$$C(r, t) \simeq f(r/\xi_d(t)) \quad (12)$$

where $f(x)$ is a scaling function (different for $T_0 > T_c$ and $T_0 = T_c$) expressing the fact that there is a unique relevant length in the system.

The existence of a different growing correlation, $\xi_p(t) \simeq t^{1/z_p}$, suggests to extend the scaling of the space-time correlation function as [31]

$$C(r, t; L) \simeq f(r/\xi_d(t), \xi_p(t)/L) . \quad (13)$$

The numerical data for the 2d Ising model with non-conserved order parameter are much better scaled when this extended form is used. The study of the relevance of $\xi_p(t)$ in the scaling of the linear response functions should still be evaluated.

The analysis of the scaling properties of the correlations in other systems (conserved order parameter, Potts model, weak-disordered systems) is under way.

The *hyper-universality* hypothesis states that the scaling functions of the correlation functions should be independent of weak disorder, its strength, distribution, etc., as long as it does not change the nature of the equilibrium phases [26, 1], once the correct growing length is used in the scaling variables. The idea behind this proposal is that the length at which the effects of weak disorder are important

should be shorter than the growing length $\xi_d(t)$ and, therefore, the mechanism for the dynamics of the large structures should remain the one of the pure system. Favourable numerical evidence for this feature appeared in [36, 37, 38, 27] although some recent studies tend to falsify the claim [39]. In our view, a more detailed analysis is needed to conclude, taking into account the new growing length $\xi_p(t)$ that will be affected by weak quench disorder. This study is also under way [35].

2.3 Number density of finite areas and lengths

A statistical analysis of the morphology of the system at any instant can be achieved by counting the clusters with a given area or by counting the interfaces with a given length, for example. In this way one constructs number densities (using the total fixed area of the system to normalise) that give quite a detailed information on the dynamic internal structure. From the analysis of these quantities one confirms the existence of the two dynamic regimes controlled by $\xi_p(t)$ and $\xi_d(t)$:

- $t < t_p$, approach to percolation.

An educated guess based on the results of the regime $t \gg t_p$ let us look at the behaviour of $n_h(A, t)$ in the form $A^{\alpha_A} n_h(A, t)$ against $A/\xi_p^{D_A}(t)$ in a variety of models. At short time scales the good data scaling confirms that $\xi_p(t)$ is the dominant growing length until t_p is reached. the scaling function is constant at small arguments and it has a weak variation at large arguments.

- $t \gg t_p$, percolation established.

Thanks to the dynamic scaling hypothesis, the distribution of various geometrical objects relative to their average is expected to reach a *universal function* independent of the initial state, as long as it is one with short-range correlations, since the long-distance properties are controlled by the infinite temperature ‘fixed point’. A critical initial state, with long-range correlations, shares the same growing length as the disordered ones but leads to different scaling functions [1].

The number density of *finite-size hull-enclosed areas* after a zero temperature quench in the $2d$ non-conserved order scalar parameter universality class has been derived using two previously known results: (1) The exact distribution of these objects at the critical percolation and critical Ising points [21], see Eq. (5). (2) The Allen-Cahn evolution [9] of each area:

$$dA/dt = -\lambda \quad (14)$$

with λ a non-universal parameter that depends on the model and/or material (and temperature). The functional form is [18, 19]

$$n_h(A, t) \simeq \frac{(2)c_h}{(A + \xi_d^2(t))^2} \quad (15)$$

with the factor 2 present for $T_0 > T_c$ and absent for $T_0 = T_c$. Indeed, the first result is obtained assuming that at t_p the system reached critical percolation and using the corresponding distribution in Eq. (5) as the initial one for the evolution. This result is one of the very few proofs of dynamic scaling for a nontrivial quantity and, as far as I know, the only one for a finite (not one) dimensional model with short-range interactions. It shows that the short-time dynamics have a long-lasting effect

on the morphology of the system, with small and large structures having geometric and statistical properties proper of the target and short-time state, respectively. We later confirmed this observation on a liquid crystal [40] and recent experiments using phase separating glasses (though in $3d$) observe a similar phenomenology [41].

A morphological analysis along these lines was later performed on the same $2d$ model with conserved order parameter modelling phase separation [20], see also [42, 33]. The effect of weak quenched disorder was considered in [43]. We are currently revisiting these two problems [32, 35] with the aim of computing the time-scale t_p in these cases.

In Ising models the areas and hulls or external interfaces are in one-to-one relation, from the number density of areas we have derived the number densities of interface lengths in the various cases discussed above.

2.4 Area-to-length relations

Finally, we analysed the evolution of the area against perimeters, a relation that generalises $A \simeq \ell^{D_A/D_\ell}$ out of equilibrium. For instance, for sub-critical quenches, after the time-scale t_p this relation also scales in the sense that $y = A/\xi_d^2(t)$ against $x = \ell/\xi_d(t)$ yields a master curve. This master curve has double algebraic form, with a crossover from $y = x^2$ for $x \ll 1$ to $y = x^{D_A^{(p)}/D_\ell^{(p)}}$ for $y \gg 1$ with $D_A^{(p)}$ and $D_\ell^{(p)}$ the fractal dimensions of the areas and length involved at critical percolation [19, 20].

(Special care needs to be taken in the scaling analysis of double critical quenches as the initial and final structures are fractal though with different dimensions [44].)

2.5 Metastability

The zero-temperature one dimensional Ising model always converges to complete order in one of the two ferromagnetic states. In contrast, this does not necessarily happen in $d \geq 2$. This is easily illustrated with an example. On a square lattice, any configuration with a straight interface is stable at zero temperature. More precisely, the finite-size kinetic $2d$ Ising model with non-conserved order-parameter does not reach its ground state after an instantaneous quench from infinite to zero temperature in, approximately, one third of the occasions [45, 46]. At low but finite temperature the metastable states have extremely long but finite relaxation time and equilibrium is reached in all runs after a time-scale that is longer than L^{z_d} . After our work [18, 19] the probability of appearance of each kind of blocked state was put in quantitative contact with the ones of the different kinds of spanning clusters in critical percolation [47, 22]. In particular, for a square lattice with PBC the probability of reaching a stripe state is 0.356 and for FBC it is 0.339 [48]. (Quenches from the critical Ising point to zero temperature were considered in [49] using different lattice geometries.) The topology of the asymptotic state was shown to be decided by the one of the initial critical condition by taking advantage of analytic results for various crossing probabilities (Cardy formula for percolation [50] and extensions for critical Ising [51]) and comparing them to the long-time limit of the zero temperature evolved data.

Figure 1: A sketch of the time-scales and growing length-scales after a quench from a disordered initial condition, with $t_p \simeq L^{z_p}$, $t_L \simeq L^{z_d}$ and $t_{eq} \simeq L^{z_{eq}}$, and $z_p < z_d \leq z_{eq}$.

fig:uno

3 Conclusions

We finish with a short summary of the scenario suggested by the results presented in the main text and a discussion of lines for future research.

Different stochastic dynamics of various $2d$ models with bimodal variables initially distributed at random with equal densities, and quenched below their critical point, evolve in the three time-regimes sketched in Fig. 1. These features are shared by the voter model.

Initially, the systems evolve *via* a kind of aggregation process, characterised by a growing length $\xi_p(t) \simeq t^{1/z_p}$, in which critical percolation is approached. This regime ends at a time determined by $\xi_p(t_p) \simeq L$. Interestingly enough, the models ‘self-organise’ into a critical situation, the one of critical percolation.

This ‘approach-to-percolation’ dynamics is accompanied by the usual dynamic mechanism, be it curvature-driven or another one determined by the conservation laws. The local growth of order of the competing equilibrium states is characterised by the usual growing length $\xi_d(t) \simeq t^{1/z_d}$.

This ordering process finishes when $\xi_d(t)$ reaches L , at $t_L \simeq L^{z_d}$, a time-scale usually associated to the equilibration one. However, at low temperatures the approach-to-percolation leaves configurations with stripes that span the systems horizontally, vertically or diagonally, with probabilities dictated by the ones of critical percolation (and the choice of boundary conditions and lattice geometry). At $T = 0$ these striped states are stable (on several lattices) and inhibit the system from reaching a fully ordered state. On some lattices or at low but finite temperature a new time-scale exists such that these metastable states decay and the system completely orders in one of the equilibrium or absorbing states.

The existence of two growing lengths over a certain period of time forces the use of both in the dynamic scaling of space-time correlation functions and possibly also linear response functions.

The current challenge is to give an analytical proof of the approach to critical percolation. Concomitantly, it would be very nice to compute the characteristic time t_p and its associated growing length $\xi_p(t)$ analytically.

From a theoretical perspective, similar issues can be addressed in different settings with their own physical relevance. We mention a few here.

Much attention is currently being paid to the analysis of the density of topological defects left in a system after a slow quench across its phase transition [52]. How does a slow cooling affect the approach to critical percolation is a question that needs a detailed study. We are currently working on this problem [53].

Other systems in which similar effects should arise are $2d$ kinetically constrained particle systems such as the spiral model [54, 55, 56], or frustrated magnets modelled with microscopic descriptions [57] or with the simpler the sixteen vertex model [58].

Phase ordering kinetics in stochastic dynamic models that are not necessarily

motivated by physics are also of interest. We studied the morphology of clusters in the voter model [34]. Godrèche & Pleimling [59] recently studied the asymmetric dynamics of an Ising model. The analysis of the possible approach to percolation and the morphology of clusters is being performed by this group [60].

I close the theoretical discussion with a promising connection between our results and aggregation and gelation [61, 62, 63], a problem linked to glassy dynamics as well. Aggregation is the process whereby particles (or objects) randomly distributed in space attach together upon encounter (with probability one or smaller). In some cases the links thus formed cannot be broken while in others they can, at some rate. The gelation phenomenon consists in the formation of a network that percolates across the system. The gelation time, t_G , is the time needed to construct the gel and it should depend on the density, the dynamic rules, and other details of the model but, as far as I know, no study of its possible system size dependence has been performed. The fractal properties of the percolation cluster and the size distribution of the clusters were also studied in many cases. There is some consensus that for certain models, at concentration $\phi = 0.5$, the percolating network has the properties of the percolating cluster at critical percolation [64]. Connections between the Smoluchowski mean-field model of gelation and critical percolation on the Cayley tree were also reckoned [65]. How general these features are, and how could they be put in contact with our results on simpler lattice spin models, will constitute an interesting theme of future research. Moreover, the relation to glassy physics suggests to exploit the detailed knowledge of the mesoscopic configuration in $2d$ coarsening with approaches to fluctuations in glassy dynamics [66].

Our results are amenable to experimental checks using visualisation techniques. One such study was carried out in electric field driven deracemisation in achiral bent-core liquid crystals through the formation and coarsening of chiral domains [67, 40, 68]. Other $2d$ systems should also be relatively simple to observe. Just to give another example, Castro et al [69] used atomic force microscopy to observe the surface of silicon targets eroded by an ion plasma. Any visualisation technique, giving access the areas covered by the different phases in the model with enough accuracy will allow for a statistical analysis of them along the lines shortly explained in this text.

Acknowledgements. The results here summarised were obtained in collaboration with J. J. Arenzon, A. J. Bray, F. Corberi, M. Kobayashi and M. Picco, and the master and PhD students T. Blanchard, M. P. Loureiro, F. Insalata, H. Ricateau, A. Sicilia, Y. Sarrazin, and A. Tartaglia. I warmly thank all of them for this very fruitful collaboration. We benefitted from financial support from Capes-Cofecub (collaboration with Porto Alegre, Brazil) and IRSES SoftActive (collaboration with Kyoto University, Japan). L. F. C. is a member of Institut Universitaire de France.

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