On the strong influence of imperfections upon the quick deviation of a mode I+III crack from coplanarity

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Abstract

This work explores the possibility that quick deviations of cracks loaded in mode I+III from coplanarity may be greatly facilitated by inevitable fluctuations of the fracture toughness. The idea is that such fluctuations must induce in-plane undulations of the crack front resulting, because of the presence of the mode III load, in non-zero values of the local stress intensity factor of mode II, implying future local out-of-plane deviations of the crack which might be "unstable" in Cotterell and Rice (1980)'s sense if the local non-singular stress parallel to the direction of propagation is positive.

Exploration of this idea implies evaluation of the variations of the local stress intensity factors and non-singular stresses arising from a slight but otherwise arbitrary in-plane perturbation of a semi-infinite crack. These quantities were calculated in works of Rice (1985), Gao and Rice (1986) and Gao (1992) but the evaluation of the non-singular stresses was incomplete, and is supplemented here by using the theory of 3D weight functions (Rice, 1985; Bueckner, 1987).

Inspection of the results shows that for in-plane sinusoidal undulations of the crack front of sufficient (though still small) amplitude, the conditions of nonzero local stress intensity factor of mode II and positive local non-singular stress parallel to the direction of propagation are simultaneously met on some parts of the front, implying the possibility of future local deviations of the crack from coplanarity "unstable" in Cotterell and Rice (1980)'s sense, and thus confirming the idea investigated.

Keywords : Imperfections; deviation from coplanarity; mode I+III crack; non-singular stresses; Cotterell and Rice's directional stability criterion

1 Introduction

The propagation of cracks loaded in mixed-mode I+III has been investigated in various materials: inorganic glass (Sommer, 1969), polymeric glass (Knauss, 1970), epoxy resin

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(Hull, 1995), PMMA (Lazarus *et al.*, 2008), Homalite (Lin *et al.*, 2010), alumina (Suresh and Tschegg, 1987), steels (Hourlier and Pineau, 1979; Yates and Miller, 1989; Lazarus, 1997; Lazarus *et al.*, 2001b), rocks (Pollard *et al.*, 1982; Pollard and Aydin, 1988; Cooke and Pollard, 1996), gypsum and cheese (Goldstein and Osipenko, 2012), to name just a few experimental papers on the subject. In all cases, it was observed that the crack propagates through formation of small fracture facets which may either abruptly "tilt" or gradually "twist" about the direction of propagation.

It has been remarked by Hourlier and Pineau (1979) that two types of facets are in fact formed: "type A" ones rotating in such a way that the local stress intensity factor (SIF) of mode I increases with the distance of propagation while that of mode III decreases, and "type B" ones rotating oppositely so that the behavior of the local SIF is the reverse. Hourlier and Pineau (1979) also noted that the crack propagates preferentially along type A facets. A rationale for this observation was provided by Lazarus *et al.* (2001a,b) who showed, using theoretical estimates of the SIF after some short continuous twisting, that for a given facet length, the energy-release-rate is larger at the center of type A facets than at that of type B facets, implying that propagation of the former facets is more "energetically favored" that that of the latter ones.

Recently, Pons and Karma (2010) performed numerical simulations of crack propagation in mode I+III based on a "phase field" model developed by Karma *et al.* (2001), which included a phenomenological description of failure mechanisms in the process zone around the crack front. These simulations reproduced both the gradual deviation of the crack from its original plane through formation of an array of inclined facets, and the quicker propagation of type A facets as compared to type B ones, in a remarkable way.

Although the theoretical framework employed by Pons and Karma (2010) differed from standard linear elastic fracture mechanics (LEFM), it has been shown by Hakim and Karma (2009) that in the limit where the system size becomes much larger than the process zone size, Karma *et al.* (2001)'s phase field model in fact predicts that quasistatic crack propagation in isotropic media is governed by a combination of two classical LEFM criteria: a condition of uniform energy-release-rate along the front (Griffith (1920)'s criterion), and a condition of zero SIF of mode II (Goldstein and Salganik (1974)'s *principle of local symmetry*). This was the motivation for Leblond *et al.* (2011)'s very recent theoretical analysis, within the framework of LEFM, of the possible bifurcation from coplanar to non-coplanar propagation of cracks loaded in mode I+III. This analysis combined assumptions of constant value of the local energy-release-rate and zero value of the local mode II SIF all along the crack front, in line with Hakim and Karma (2009)'s and Pons and Karma (2010)'s findings, with technical results of Gao and Rice (1986) and Movchan *et al.* (1998) on in-plane and out-of-plane perturbations of a plane crack. A bifurcation from coplanar to non-coplanar propagation was concluded to exist for values of the ratio of the mode III to mode I SIF larger than some threshold depending on Poisson's ratio.

However, the threshold was found to be of the order of 0.5 for standard values of Poisson's ratio. The bifurcation analysis could therefore *not* explain the fact that deviations of the crack from its original plane are currently observed for much smaller values of the ratio of the mode III to mode I SIF - a threshold of the order of 0.05 was mentioned by Sommer (1969), and Ravi-Chandar (2010) has even claimed that there is no threshold at all.

The aim of this paper is to propose a possible explanation to this observation. The idea is that even for low values of the ratio of the mode III to mode I SIF, for which no bifurcation is predicted, deviations from coplanarity might occur because of a strong influence of imperfections upon the propagation path - quite in the same way as the influence of imperfections explains, for thin shells, the quick deviations from the fundamental deformed state currently observed for loads much lower than the theoretical buckling load. A typical example of inevitable imperfections consists of random fluctuations of the fracture toughness within the crack plane. Such fluctuations are bound to generate in-plane undulations of the crack front. It is intuitively obvious, and has been proved rigorously by Gao and Rice (1986), that the mode III load must generate nonzero and opposite local mode II SIF on the two sides of a local coplanar protrusion of the front; this implies that the crack will tend to extend out of its original plane in opposite directions on these two sides, thus giving birth to an incipient non-coplanar facet. If, in addition, Cotterell and Rice (1980)'s well-known "directional stability criterion" happens to be violated because of a locally positive non-singular stress in the direction of crack propagation, the deviation of this facet from the original crack plane may quickly increase as the crack propagates, even in the absence of a true bifurcation.

Investigation of this idea, in the typical case of a semi-infinite crack in some infinite body, requires that the local SIF and non-singular stresses be known for such a crack, endowed with a slightly, coplanarly perturbed front. The calculation of the SIF was carried out by Rice (1985) and Gao and Rice (1986), with definitive results. That of the non-singular stresses was carried out by Gao (1992), but with restrictive hypotheses and incomplete results, making the completion of the task an indispensable prerequisite.

The paper is organized as follows:

- Section 2 briefly recalls some elements of Rice (1985)'s and Bueckner (1987)'s theory of 3D weight functions, which serve as a basis in the analysis to follow.
- From there, Section 3 presents the calculation of the first-order variation of the stresses on the crack plane resulting from some small but otherwise arbitrary in-plane perturbation of the crack front.
- We then derive from there, in Section 4, the first-order variations of the non-singular stresses under similar conditions.
- Section 5 briefly recalls Leblond (1999)'s 3D extension of Cotterell and Rice (1980)'s original 2D directional stability analysis of a propagating crack, indispensable for the application of the preceding results to crack propagation in mode I+III.
- Finally Section 6 considers the case of a sinusoidal in-plane perturbation of the crack front, and examines whether Cotterell and Rice (1980)'s directional stability criterion (as extended to the 3D case) is met or not, distinguishing between those parts of the undulated front about to give birth to type A and type B facets.

2 Elements of Rice (1985)'s and Bueckner (1987)'s 3D weight function theory

Consider an arbitrary body Ω made of some linear elastic isotropic material, and containing an arbitrary planar crack (Figure 1). Assume that prescribed displacements are imposed on the portion $\partial\Omega_u$ of the boundary of this body, while prescribed tractions are imposed on the complementary portion $\partial\Omega_T$. This loading generates a distribution of SIF $K_I^0(s)$, $K_{II}^0(s)$, $K_{III}^0(s)$, where s denotes a curvilinear abscissa along the crack front $\mathcal{C}$, on this front.

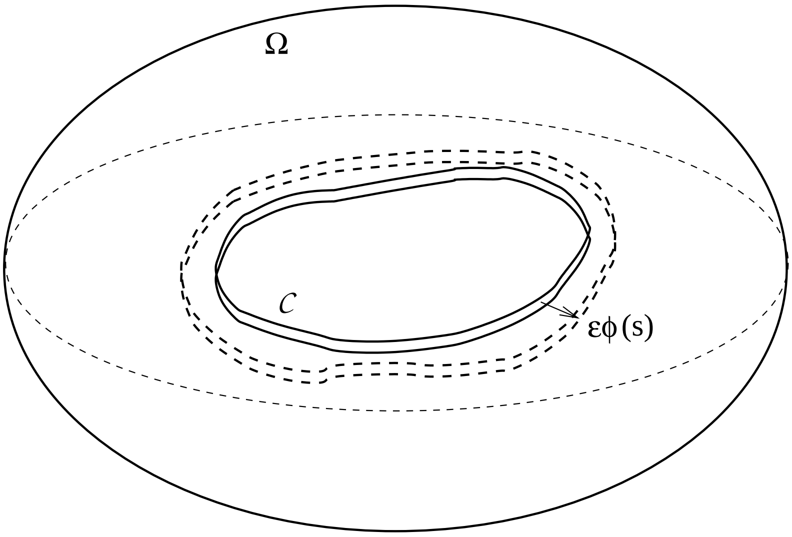

Fig. 1. Slight in-plane perturbation of a plane crack in an arbitrary body

Now slightly perturb $\mathcal{C}$ within the crack plane, while keeping the loading applied on $\partial\Omega_u$ and $\partial\Omega_T$ unchanged. Let $\epsilon\phi(s)$, where ϵ is a small parameter and $\phi(s)$ a given, fixed function, denote the distance from the original front to the perturbed one, as measured perpendicularly to the former front. The components $\delta u_i(\mathbf{r})$, in an arbitrary orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, of the resulting variation of displacement $\delta\mathbf{u}$ at the point $\mathbf{r}$ of the body, are given to first order by Rice (1985)'s formula:

$$\delta u_i(\mathbf{r}) = \int_{\mathcal{C}} 2\Lambda_{\alpha\beta}\, h_{i\alpha}(\mathbf{r}, s) K_\beta^0(s)\, \epsilon\phi(s) ds. \tag{1}$$

In this expression:

- the indices α and β take the values I, II and III and Einstein's implicit summation convention is used for them;
- the coefficients $\Lambda_{\alpha\beta}$ are those appearing in the quadratic form of the SIF defining the energy-release-rate, given by [1]

$$\Lambda_{I,I} = \Lambda_{II,II} = \frac{1-\nu^2}{E} \quad ; \quad \Lambda_{III,III} = \frac{1+\nu}{E} \quad ; \quad \text{other } \Lambda_{\alpha\beta} = 0 \tag{2}$$

where E and ν denote Young's modulus and Poisson's ratio;

[1] For the sake of clarity, commas separating the indices are exceptionally introduced into this equation.

- finally the $h_{i\alpha}(\mathbf{r}, s)$ are the *3D weight functions* of the cracked geometry considered (Rice, 1985; Bueckner, 1987); $h_{i\alpha}(\mathbf{r}, s)$ represents the α-th SIF generated at the point s of the crack front by a unit point force applied in the direction $\mathbf{e}_i$ at the point $\mathbf{r}$ of the body, zero displacements being simultaneously prescribed on $\partial\Omega_u$ and zero tractions on $\partial\Omega_T$.

Now, f being an arbitrary function of position and $\mathbf{r}$ an arbitrary point on the crack plane lying inside the crack contour, let

$$\langle f\rangle(\mathbf{r}) \equiv \frac{1}{2}\left[f(\mathbf{r}^+) + f(\mathbf{r}^-)\right] \tag{3}$$

denote the average of the values of this function at the points $\mathbf{r}^+$, $\mathbf{r}^-$ of the upper ($^+$) and lower ($^-$) faces of the crack. With this notation, application of equation (1) on the crack faces yields

$$\langle \delta u_i\rangle(\mathbf{r}) = \int_{\mathcal{C}} 2\Lambda_{\alpha\beta}\,\langle h_{i\alpha}\rangle(\mathbf{r}, s)K^0_\beta(s)\,\epsilon\phi(s)ds \tag{4}$$

where $\langle h_{i\alpha}\rangle(\mathbf{r}, s)$, a *crack-face weight function* (CFWF), now represents the α-th SIF generated at the point s of the crack front by two half-unit point forces applied in the direction $\mathbf{e}_i$ at the points $\mathbf{r}^+$, $\mathbf{r}^-$ of the crack faces, zero displacements being simultaneously prescribed on $\partial\Omega_u$ and zero tractions on $\partial\Omega_T$.

3 First-order variation of the stresses on the crack faces

3.1 Generalities

Consider now, more specifically, a semi-infinite crack located in some infinite body subjected to prescribed forces only (Figure 2). Following the usual convention, define a Cartesian frame (O, x, y, z) with O on the unperturbed crack front, x in the direction of propagation, y in the direction of the normal to the crack plane, and z in the direction of the crack front. Also, characterize the position of the unperturbed front Oz through its distance a to some fixed "reference line" parallel to it in the crack plane.

Equation (4) then takes the special form, with obvious notations:

$$\langle \delta u_i\rangle(x, z) = \int_{-\infty}^{+\infty} 2\Lambda_{\alpha\beta}\,\langle h_{i\alpha}\rangle(x, z; z')K^0_\beta(z')\,\epsilon\phi(z')dz'. \tag{5}$$

The aim of this section is to derive from there the expressions of the average variations $\langle\delta\sigma_{xx}\rangle$, $\langle\delta\sigma_{zz}\rangle$, $\langle\delta\sigma_{xz}\rangle$ of the in-plane stresses on the crack faces. When equation (5) is used for this sole purpose, some simplifications arise:

- it is enough to know the in-plane components $\langle\delta u_x\rangle$, $\langle\delta u_z\rangle$ of the average variation of displacement $\langle\delta\mathbf{u}\rangle$, the out-of-plane component $\langle\delta u_y\rangle$ is not needed;
- the expressions of these in-plane components involve the CFWF $\langle h_{x\alpha}\rangle$ and $\langle h_{z\alpha}\rangle$ which are nonzero only for $\alpha = I$, since the loadings implied, consisting of half-unit point

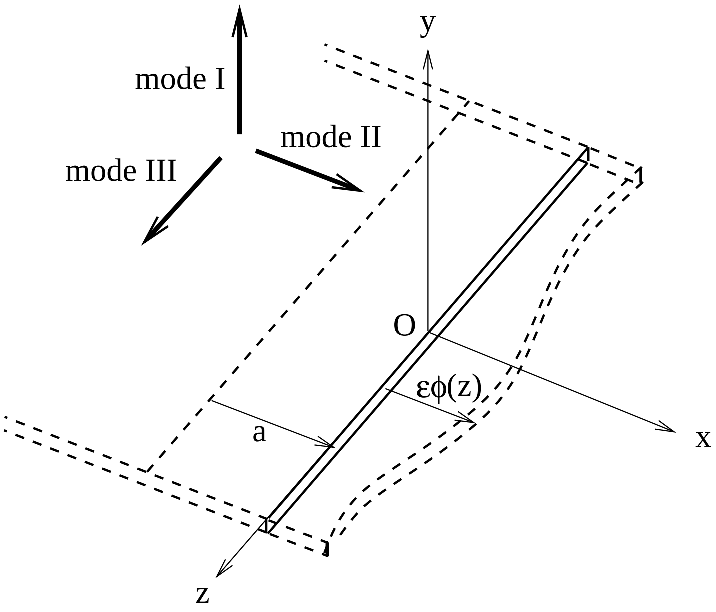

Fig. 2. Slight in-plane perturbation of a semi-infinite crack loaded arbitrarily

forces applied on the crack faces in the directions x and z, are symmetric with respect to the crack plane and therefore do not generate any mode II or III.

The formulae required therefore simply read, by equation (2):

$$\begin{cases} \langle \delta u_x \rangle(x,z) = 2\dfrac{1-\nu^2}{E} \displaystyle\int_{-\infty}^{+\infty} \langle h_{xI} \rangle(x,z;z') K_I^0(z')\, \epsilon\phi(z') dz' \\ \langle \delta u_z \rangle(x,z) = 2\dfrac{1-\nu^2}{E} \displaystyle\int_{-\infty}^{+\infty} \langle h_{zI} \rangle(x,z;z') K_I^0(z')\, \epsilon\phi(z') dz'. \end{cases} \tag{6}$$

3.2 Variations of the displacement components and their spatial derivatives

The CFWF $\langle h_{xI} \rangle$, $\langle h_{zI} \rangle$ for a semi-infinite crack have been calculated by Bueckner (1987), Kuo (1993) and Movchan *et al.* (1998). The simplest way of expressing the results is as follows (Kuo, 1993): assume that point forces of intensities F_x and F_z are simultaneously applied in the directions x and z on the points $(x, y = 0^+, z)$ and $(x, y = 0^-, z)$ of the upper and lower crack faces; these forces together generate a mode I SIF k_I at the point z' of the crack front given by

$$\begin{aligned} k_I(z') &\equiv 2F_x \langle h_{xI} \rangle(x,z;z') + 2F_z \langle h_{zI} \rangle(x,z;z') \\ &= \frac{1}{4\sqrt{\pi}} \frac{1-2\nu}{1-\nu} \mathrm{Re} \left\{ \frac{F_x + iF_z}{[-x + i(z'-z)]^{3/2}} \right\} \end{aligned} \tag{7}$$

where the cut of the complex power function is along the half-straight line of non-positive reals. Since this compact formula "couples" the CFWF $\langle h_{xI} \rangle$ and $\langle h_{zI} \rangle$, it seems appropriate, when inserting it into the expressions (6) of $\langle \delta u_x \rangle$ and $\langle \delta u_z \rangle$, to introduce arbitrary real parameters α, β and consider the single quantity $\langle \alpha\delta u_x + \beta\delta u_z \rangle$ rather than $\langle \delta u_x \rangle$ and $\langle \delta u_z \rangle$ individually. One thus gets

$$
\begin{aligned}
\langle \alpha \delta u_x + \beta \delta u_z \rangle (x,z) &= 2\frac{1-\nu^2}{E}\int_{-\infty}^{+\infty} \langle \alpha h_{xI} + \beta h_{zI} \rangle (x,z;z') K_I^0(z')\, \epsilon\phi(z') dz' \\
&= \frac{(1+\nu)(1-2\nu)}{4\sqrt{\pi}\, E} \mathrm{Re}\left\{ \int_{-\infty}^{+\infty} \frac{\alpha + i\beta}{[-x+i(z'-z)]^{3/2}}\, K_I^0(z')\, \epsilon\phi(z') dz' \right\}
\end{aligned}
$$

or equivalently, after integration by parts,

$$
\langle \alpha \delta u_x + \beta \delta u_z \rangle (x,z) = \frac{(1+\nu)(1-2\nu)}{2\sqrt{\pi}\, E} \mathrm{Re}\left\{ \int_{-\infty}^{+\infty} \frac{\beta - i\alpha}{[-x+i(z'-z)]^{1/2}} \left(K_I^0\, \epsilon\phi \right)' (z') dz' \right\}. \tag{8}
$$

Differentiating this equation with respect to x and z, and then ascribing the values $(1,0)$ and $(0,1)$ to the pair (α,β), one gets the average spatial derivatives of the components of the variation of displacement:

$$
\begin{cases}
\langle \dfrac{\partial\, \delta u_x}{\partial x} \rangle (x,z) = -\langle \dfrac{\partial\, \delta u_z}{\partial z} \rangle (x,z) = \dfrac{(1+\nu)(1-2\nu)}{4\sqrt{\pi}\, E}\, \mathrm{Im}\left\{ \displaystyle\int_{-\infty}^{+\infty} \frac{\left(K_I^0\, \epsilon\phi \right)'(z')}{[-x+i(z'-z)]^{3/2}}\, dz' \right\} \\
\langle \dfrac{\partial\, \delta u_x}{\partial z} \rangle (x,z) = \langle \dfrac{\partial\, \delta u_z}{\partial x} \rangle (x,z) \quad = \dfrac{(1+\nu)(1-2\nu)}{4\sqrt{\pi}\, E}\, \mathrm{Re}\left\{ \displaystyle\int_{-\infty}^{+\infty} \frac{\left(K_I^0\, \epsilon\phi \right)'(z')}{[-x+i(z'-z)]^{3/2}}\, dz' \right\}.
\end{cases} \tag{9}
$$

3.3 Variations of the stress components

The stress components σ_{yx}, σ_{yy} and σ_{yz} being zero on the crack faces, the average variations $\langle \delta\sigma_{xx} \rangle$, $\langle \delta\sigma_{zz} \rangle$, $\langle \delta\sigma_{xz} \rangle$ of the in-plane stresses on these faces may be obtained from expressions (9) through application of the plane stress elastic stiffness tensor:

$$
\begin{cases}
\langle \delta\sigma_{xx} \rangle (x,z) = -\langle \delta\sigma_{zz} \rangle (x,z) = \dfrac{1-2\nu}{4\sqrt{\pi}}\, \mathrm{Im}\left\{ \displaystyle\int_{-\infty}^{+\infty} \frac{\left(K_I^0\, \epsilon\phi \right)'(z')}{[-x+i(z'-z)]^{3/2}}\, dz' \right\} \\
\langle \delta\sigma_{xz} \rangle (x,z) \qquad\qquad\qquad\quad = \dfrac{1-2\nu}{4\sqrt{\pi}}\, \mathrm{Re}\left\{ \displaystyle\int_{-\infty}^{+\infty} \frac{\left(K_I^0\, \epsilon\phi \right)'(z')}{[-x+i(z'-z)]^{3/2}}\, dz' \right\}.
\end{cases} \tag{10}
$$

Equation $(10)_1$ implies in particular that the average variation $\langle \delta\sigma_{xx} + \delta\sigma_{zz} \rangle$ is zero whatever the perturbation of the crack front; this means that *at a given point of the broken region of the crack plane, the average 2D trace of the stress tensor, $\langle \sigma_{xx} + \sigma_{zz} \rangle$, is independent of the position and shape of the crack front.* This remarkable property is noted here in the special case of a semi-infinite crack, but was shown by Gao (1992) to in fact hold for *any* planar crack of arbitrary contour in some infinite body.

The variations of the non-singular stresses will be deduced from the asymptotic behavior of $\langle \delta\sigma_{xx} \rangle$, $\langle \delta\sigma_{zz} \rangle$, $\langle \delta\sigma_{xz} \rangle$ near the crack front, that is in the limit $x \to 0^-$. This makes it necessary to evaluate the limit

$$
\lim_{x \to 0^-} \int_{-\infty}^{+\infty} \frac{\left(K_I^0\, \epsilon\phi \right)'(z')}{[-x+i(z'-z)]^{3/2}}\, dz'.
$$

In order to do so, rewrite the integral as

$$\int_{-\infty}^{+\infty} \frac{\left(K_I^0\,\epsilon\phi\right)'(z')}{\left[-x+i(z'-z)\right]^{3/2}}\,dz' = \int_{-\infty}^{+\infty} \frac{\left(K_I^0\,\epsilon\phi\right)'(z') - \left(K_I^0\,\epsilon\phi\right)'(z)}{\left[-x+i(z'-z)\right]^{3/2}}\,dz' + \left(K_I^0\,\epsilon\phi\right)'(z)\int_{-\infty}^{+\infty} \frac{dz'}{\left[-x+i(z'-z)\right]^{3/2}}.$$

The second integral in the right-hand side is obviously zero, and the first one goes to the limit

$$\int_{-\infty}^{+\infty} \frac{\left(K_I^0\,\epsilon\phi\right)'(z') - \left(K_I^0\,\epsilon\phi\right)'(z)}{\left[i(z'-z)\right]^{3/2}}\,dz'$$

for $x \to 0^-$. (Note that this integral is convergent at the point $z' = z$ since the integrand behaves like $|z'-z|^{-1/2}$ near it). Evaluating $\left[i(z'-z)\right]^{3/2}$ using the definition of the complex power function and distinguishing between the cases $z' < z$ and $z' > z$, one concludes that

$$\lim_{x\to 0^-}\int_{-\infty}^{+\infty} \frac{\left(K_I^0\,\epsilon\phi\right)'(z')}{\left[-x+i(z'-z)\right]^{3/2}}\,dz' = -\frac{1}{\sqrt{2}}\int_{-\infty}^{+\infty} \frac{\left(K_I^0\,\epsilon\phi\right)'(z') - \left(K_I^0\,\epsilon\phi\right)'(z)}{|z'-z|^{3/2}}\,dz' - \frac{i}{\sqrt{2}}\int_{-\infty}^{+\infty} \operatorname{sgn}(z'-z)\frac{\left(K_I^0\,\epsilon\phi\right)'(z') - \left(K_I^0\,\epsilon\phi\right)'(z)}{|z'-z|^{3/2}}\,dz' \tag{11}$$

where $\operatorname{sgn}(x)$ denotes the sign of x.

Inserting the result (11) into the expressions (10) of $\langle\delta\sigma_{xx}\rangle$, $\langle\delta\sigma_{zz}\rangle$, $\langle\delta\sigma_{xz}\rangle$, one finally gets the limits looked for:

$$\left\{\begin{aligned} \langle\delta\sigma_{xx}\rangle(0^-,z) &= -\langle\delta\sigma_{zz}\rangle(0^-,z) \\ &= -\frac{1-2\nu}{4\sqrt{2\pi}}\int_{-\infty}^{+\infty} \operatorname{sgn}(z'-z)\frac{\left(K_I^0\,\epsilon\phi\right)'(z') - \left(K_I^0\,\epsilon\phi\right)'(z)}{|z'-z|^{3/2}}\,dz' \\ \langle\delta\sigma_{xz}\rangle(0^-,z) &= -\frac{1-2\nu}{4\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\left(K_I^0\,\epsilon\phi\right)'(z') - \left(K_I^0\,\epsilon\phi\right)'(z)}{|z'-z|^{3/2}}\,dz'. \end{aligned}\right. \tag{12}$$

4 First-order variations of the non-singular stresses

4.1 Special case of an immobile point of the crack front

In a first step, we wish to derive the variations of the non-singular stresses for an *immobile* point of the crack front, having $\phi(z) = 0$. In such a case the point of observation of the in-plane stresses to be used to define the non-singular stresses, located just behind the crack front, does not move when this front is perturbed; hence the local variations of the non-singular stresses are simply related to the local average variations of the in-plane stresses.

More specifically, define a Cartesian frame $(P, x_1, x_2 \equiv y, x_3)$ "adapted" to the perturbed crack front at the immobile point P considered (Figure 3). To first order in the perturbation, the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ corresponding to the coordinates x_1, x_2, x_3 are related to those, $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ corresponding to the coordinates x, y, z adapted to the unperturbed front, through the relations

$$\mathbf{e}_1 = \mathbf{e}_x - \epsilon\phi'(z)\,\mathbf{e}_z \quad ; \quad \mathbf{e}_2 = \mathbf{e}_y \quad ; \quad \mathbf{e}_3 = \mathbf{e}_z + \epsilon\phi'(z)\,\mathbf{e}_x. \tag{13}$$

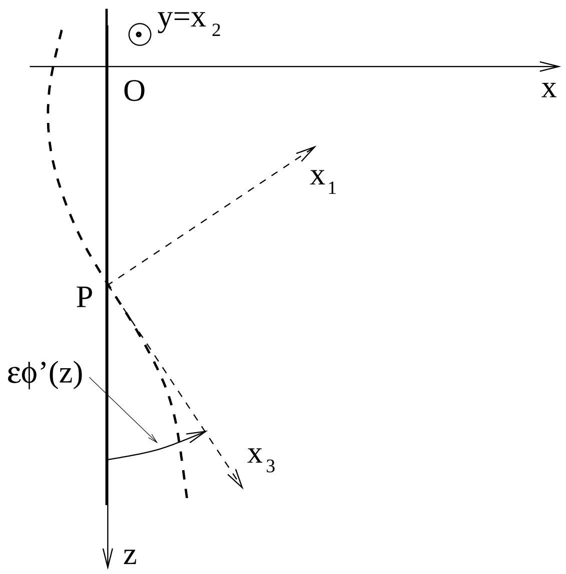

Fig. 3. Definition of local axes for the perturbed front

In the local frame, the average in-plane stress components $\langle\sigma_{11}\rangle$, $\langle\sigma_{33}\rangle$, $\langle\sigma_{13}\rangle$:

- are zero for the first, singular term of the Williams expansion of the stresses;
- are equal to the non-singular stresses T_{11}, T_{33}, T_{13} for the second term;
- vanish close to the crack front for the next terms.

Hence T_{11}, T_{33}, T_{13} may be identified to the limits of $\langle\sigma_{11}\rangle$, $\langle\sigma_{33}\rangle$, $\langle\sigma_{13}\rangle$ when the point of observation of these quantities gets infinitely close to the crack front. Therefore, unperturbed values and variations of quantities being denoted with symbols ^{0} and δ respectively, the average perturbed stress tensor $\langle\boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma}\rangle$ on the crack faces close to the front may be expressed as

$$\begin{aligned}
\langle\boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma}\rangle(0^-, z) &= \left[T^0_{11}(z) + \delta T_{11}(z)\right] \mathbf{e}_1 \otimes \mathbf{e}_1 + \left[T^0_{33}(z) + \delta T_{33}(z)\right] \mathbf{e}_3 \otimes \mathbf{e}_3 \\
&\quad + [T^0_{13}(z) + \delta T_{13}(z)]\,(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1) \\
&= \left[T^0_{11}(z) + \delta T_{11}(z)\right] \mathbf{e}_x \otimes \mathbf{e}_x + \left[T^0_{33}(z) + \delta T_{33}(z)\right] \mathbf{e}_z \otimes \mathbf{e}_z \\
&\quad + [T^0_{13}(z) + \delta T_{13}(z)]\,(\mathbf{e}_x \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_x) \\
&\quad + 2T^0_{13}(z)\,\epsilon\phi'(z)\,\mathbf{e}_x \otimes \mathbf{e}_x - 2T^0_{13}(z)\,\epsilon\phi'(z)\,\mathbf{e}_z \otimes \mathbf{e}_z \\
&\quad + [T^0_{33}(z) - T^0_{11}(z))]\,\epsilon\phi'(z)\,(\mathbf{e}_x \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_x)
\end{aligned}$$

where equation (13) has been used.

But on the other hand this average perturbed stress tensor may also be expressed as

$$\langle \boldsymbol{\sigma}^0 + \delta\boldsymbol{\sigma} \rangle(0^-, z) = \langle \sigma^0_{xx} + \delta\sigma_{xx} \rangle(0^-, z)\, \mathbf{e}_x \otimes \mathbf{e}_x + \langle \sigma^0_{zz} + \delta\sigma_{zz} \rangle(0^-, z)\, \mathbf{e}_z \otimes \mathbf{e}_z$$
$$+ \langle \sigma^0_{xz} + \delta\sigma_{xz} \rangle(0^-, z)\, (\mathbf{e}_x \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_x)\,.$$

Comparison of these two formulae and identification of the first-order terms yields the following expressions of the variations of the non-singular stresses:

$$\begin{cases} \delta T_{11}(z) = -2T^0_{13}(z)\, \epsilon\phi'(z) + \langle \delta\sigma_{xx} \rangle(0^-, z) \\ \delta T_{33}(z) = 2T^0_{13}(z)\, \epsilon\phi'(z) + \langle \delta\sigma_{zz} \rangle(0^-, z) \\ \delta T_{13}(z) = \left[T^0_{11}(z) - T^0_{33}(z)\right] \epsilon\phi'(z) + \langle \delta\sigma_{xz} \rangle(0^-, z) \end{cases} \tag{14}$$

where the average variations of the stresses are given by equations (12).

4.2 General case

In order to now evaluate the variations of the non-singular stresses in the general case where $\phi(z) \neq 0$, we use the same trick as in the works of Rice (1985) and Gao and Rice (1986) on the variations of the SIF: we decompose the perturbation $\epsilon\phi$ in the form

$$\epsilon\phi(z') = \epsilon\phi(z) + \epsilon\overline{\phi}(z') \quad \text{where} \quad \overline{\phi}(z') \equiv \phi(z') - \phi(z). \tag{15}$$

- The first term of the decomposition represents a translatory motion of the unperturbed crack front by the distance $\epsilon\phi(z)$. This motion generates a variation of the non-singular stress T_{ij} equal to $\frac{\partial T^0_{ij}}{\partial a}(z)\, \epsilon\phi(z)$, where $\frac{\partial T^0_{ij}}{\partial a}(z)$ is the derivative of the unperturbed non-singular stress T^0_{ij} with respect to the position a of the straight crack front, see Figure 2. [2]
- The second term represents a motion in which the point z remains immobile. Therefore equations (14) may be used, with $\overline{\phi}(z')$ instead of $\phi(z')$, to evaluate the resulting variations of the non-singular stresses.

Adding the contributions of the two terms, and using equations (12) for the average variations of the stresses and the definition $(15)_2$ of the function $\overline{\phi}$, one finally gets the following formulae for the variations of the non-singular stresses in the general case:

$$\begin{aligned} \delta T_{11}(z) = {} & \frac{\partial T^0_{11}}{\partial a}(z)\, \epsilon\phi(z) - 2T^0_{13}(z)\, \epsilon\phi'(z) \\ & - \frac{1-2\nu}{4\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Big\{ \left(K^0_I \epsilon\phi'\right)(z') - \left(K^0_I \epsilon\phi'\right)(z) \\ & \qquad + \left(K^0_I\right)'(z') \left[\epsilon\phi(z') - \epsilon\phi(z)\right] \Big\} \frac{\operatorname{sgn}(z'-z)}{|z'-z|^{3/2}}\, dz'; \end{aligned} \tag{16}$$

[2] The additional dependence of this quantity and other ones upon the argument a is omitted to alleviate the notation.

$$
\begin{aligned}
\delta T_{33}(z) = {} & \frac{\partial T_{33}^0}{\partial a}(z)\,\epsilon\phi(z) + 2T_{13}^0(z)\,\epsilon\phi'(z) \\
& + \frac{1-2\nu}{4\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Big\{ \left(K_I^0 \epsilon\phi'\right)(z') - \left(K_I^0 \epsilon\phi'\right)(z) \\
& \qquad + \left(K_I^0\right)'(z')\,[\epsilon\phi(z') - \epsilon\phi(z)] \Big\} \frac{\mathrm{sgn}(z'-z)}{|z'-z|^{3/2}}\,dz';
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\delta T_{13}(z) = {} & \frac{\partial T_{13}^0}{\partial a}(z)\,\epsilon\phi(z) + \left[T_{11}^0(z) - T_{33}^0(z)\right]\epsilon\phi'(z) \\
& - \frac{1-2\nu}{4\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Big\{ \left(K_I^0 \epsilon\phi'\right)(z') - \left(K_I^0 \epsilon\phi'\right)(z) \\
& \qquad + \left(K_I^0\right)'(z')\,[\epsilon\phi(z') - \epsilon\phi(z)] \Big\} \frac{dz'}{|z'-z|^{3/2}}.
\end{aligned}
\tag{18}
$$

It is worth noting that *unlike the average variation $\langle \delta\sigma_{xx} + \delta\sigma_{zz} \rangle$, the variation $\delta T_{11} + \delta T_{33}$ is nonzero in general.* The effect arises solely from the terms $\frac{\partial T_{11}^0}{\partial a}(z)\,\epsilon\phi(z)$ and $\frac{\partial T_{33}^0}{\partial a}(z)\,\epsilon\phi(z)$ in the right-hand sides of equations (16) and (17), the sum of which has no reason to be zero. In physical terms, even though $\langle \sigma_{xx} + \sigma_{zz} \rangle$, *at a given, fixed point* of the broken region of the crack plane, is independent of the position and shape of the crack front, evaluating the variation of $T_{11} + T_{33}$ implies *following the point of observation* of $\langle \sigma_{xx} + \sigma_{zz} \rangle$ as the front moves, which inevitably entails a variation of this quantity.

4.3 Comparison with the work of Gao (1992)

Although Gao (1992) did not consider the sole case of an initially straight crack front like in the present work but also that of a circular one, he introduced a number of restrictive hypotheses not made here:

- A pure more I loading was assumed. However our general expressions (16), (17), (18) of δT_{11}, δT_{33}, δT_{13} show that considering more general mixed-mode loadings would not in fact have changed anything, since the unperturbed SIF K_{II}^0, K_{III}^0 of modes II and III do not appear in them.
- The unperturbed SIF K_I^0 of mode I was assumed to be independent of the position z along the crack front. The possible variation of K_I^0 along this front does modify the expressions (16), (17), (18) of δT_{11}, δT_{33}, δT_{13} through the term $(K_I^0)'(z')\,[\epsilon\phi(z') - \epsilon\phi(z)]$ appearing in the integrand in each of them.
- The perturbation of the crack front was assumed to be sinusoidal. This was in fact equivalent to providing the expressions of δT_{11}, δT_{33}, δT_{13} in Fourier's space. However our expressions (16), (17), (18) in the physical space are interesting in themselves, and getting them from those of Gao (1992) through inverse Fourier transform is not a completely straightforward operation.

With these hypotheses, Gao (1992) obtained expressions of δT_{11}, δT_{33}, δT_{13} coinciding exactly with the integral terms in the right-hand sides of our equations (16), (17), (18), as evaluated for a sinusoidal perturbation. However all additional terms proportional to the $\frac{\partial T_{ij}^0}{\partial a}$ and T_{ij}^0 were absent. These terms were apparently really missing in the sense that the hypotheses made did not seem to permit to discard them.

5 Cotterell and Rice's directional stability criterion for 3D cracks

In this section, we briefly recall, as a prerequisite to the next one, Leblond (1999)'s 3D extension of Cotterell and Rice (1980)'s classical 2D analysis of directional stability of a propagating crack.

We therefore consider, within an arbitrary 3D body, an initially planar crack of arbitrary contour, and denote s some curvilinear abscissa along this contour. This crack is loaded through some system of prescribed forces and/or displacements generating distributions of SIF $K_I(s)$, $K_{II}(s)$, $K_{III}(s)$ and non-singular stresses $T_{11}(s)$, $T_{33}(s)$, $T_{13}(s)$ along its front. These distributions are arbitrary except that the mode II SIF $K_{II}(s)$ is assumed to be everywhere small.

Because of the presence of mode II, the crack propagates in a slightly non-coplanar way. More specifically, at each point $P(s)$ of the original crack front, propagation of the crack results in the creation of some small, slightly kinked and curved extension; the length of this extension is $\eta\psi(s)$ where η is a small parameter and ψ a given function, and its equation reads, in the local "adapted" frame $(P(s), x_1, x_2, x_3)$ defined like in Figure 3:

$$x_2 = \theta(s)x_1 + a(s)x_1^{3/2} + O(x_1^2), \tag{19}$$

where $\theta(s)$ ($\ll 1$) is the local "kink angle" and $a(s)$ a local "curvature parameter" (Figure 4, where $K_{II}(s)$ is assumed to be negative in order for $\theta(s)$ to be positive, see equation (21) below). The peculiar shape of the curve defined by equation (19), resulting from the term proportional to $x_1^{3/2}$ instead of simply x_1^2, will be seen to be necessary for the propagation criterion to be satisfied.

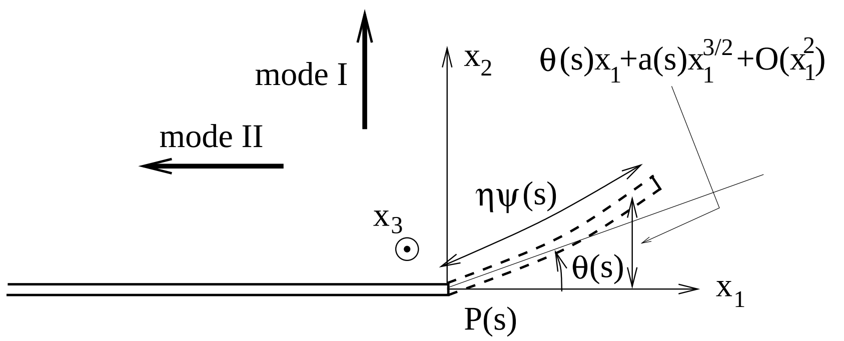

Fig. 4. Geometric hypotheses and notations for Cotterell and Rice's directional stability analysis of a propagating crack

Leblond (1999) and Leblond *et al.* (1999) have derived, under such conditions, the expansions of the SIF $K_I(s;\eta)$, $K_{II}(s;\eta)$, $K_{III}(s;\eta)$ along the extended front in powers of η. At order $\eta^{1/2}$, these "extended SIF" depend upon the geometrical and mechanical parameters only through their *local* values at that point where they are evaluated,[3] and are given by

[3] This does not remain true at order $\eta^1 = \eta$: the expressions of the extended SIF at a given point depend at that order upon the whole distribution of geometrical and mechanical parameters on the crack front, see Leblond *et al.* (1999).

formulae exactly similar to those of Cotterell and Rice (1980) for the 2D case, except for the extra dependence of all quantities on s. In particular the expansion of $K_{II}(s;\eta)$ reads

$$K_{II}(s;\eta) = K_{II}(s) + \frac{\theta(s)}{2} K_I(s) + \left[-2\sqrt{\frac{2}{\pi}}\,\theta(s) T_{11}(s) + \frac{3}{4}\,a(s) K_I(s) \right] \sqrt{\eta\psi(s)} + O(\eta). \tag{20}$$

(Note that $\lim_{\eta \to 0^+} K_{II}(s;\eta)$ differs from $K_{II}(s)$ because of the kink). This equation may be used to predict the values of the local kink angle $\theta(s)$ and curvature parameter $a(s)$, assuming the shape of the propagating crack to be governed by Goldstein and Salganik (1974)'s principle of local symmetry which demands that $K_{II}(s;\eta)$ be constantly zero after the initial kink. This condition yields:

- at order $\eta^0 = 1$:

$$\theta(s) = -2\frac{K_{II}(s)}{K_I(s)}\,; \tag{21}$$

- at order $\eta^{1/2}$:

$$a(s) = \frac{8}{3}\sqrt{\frac{2}{\pi}}\,\frac{T_{11}(s)}{K_I(s)}\,\theta(s). \tag{22}$$

Equation (22) permits to discuss the local directional stability of crack propagation. Cotterell and Rice (1980) indeed consider that directional stability prevails if the effect of the curvature parameter $a(s)$ tends to counterbalance that of the kink angle $\theta(s)$ and bring the crack back to its original plane, that is if these quantities are of opposite signs. [4] This leads to the following criterion (since necessarily $K_I(s) > 0$):

$$\text{directional stability} \quad \Leftrightarrow \quad T_{11}(s) < 0. \tag{23}$$

6 Application to deviation of a mode I+III crack from coplanarity

6.1 Position of the problem

We now consider a semi-infinite crack loaded in mode I+III in an infinite body (Figure 5). The unperturbed SIF K_I^0, K_{III}^0 on the straight configuration of the crack front are assumed to be uniform along this front. Inevitable fluctuations of the fracture toughness within the crack plane are assumed to generate small in-plane undulations of the front depicted by the typically sinusoidal perturbation

$$\epsilon\phi(z) \equiv \epsilon\cos(kz) \quad (k > 0). \tag{24}$$

[4] Though reasonable, this condition does not result from some fundamental stability theory, but simply from some *ad hoc* postulate; this is why such prudent expressions as "directional stability in the sense of Cotterell and Rice (1980)", "Cotterell and Rice (1980)'s directional stability criterion" are used in this paper.

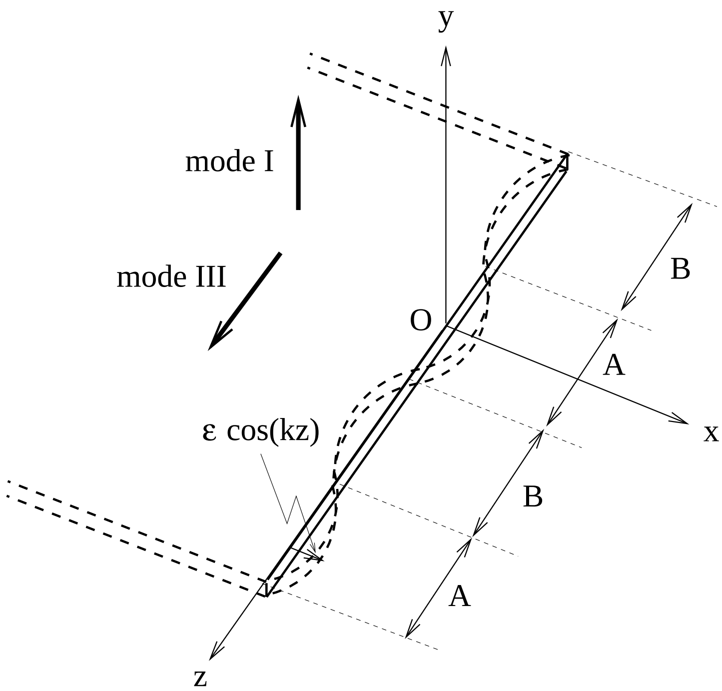

Fig. 5. In-plane sinusoidal perturbation of a semi-infinite crack loaded in mode I+III

In Figure 5, the zones of the perturbed crack front having $\cos(kz) > 0$ and < 0 are indicated with symbols A and B respectively, meaning that they are anticipated to generate future non-coplanar facets of these types. Indeed the former, more advanced zones will generate facets lying ahead of the mean position of the front, which is a typical property of type A ones, see the Introduction; conversely the latter, less advanced zones will generate facets lying behind the same position, which is typical of type B ones.

Making use of the results of the preceding sections, we wish to study the deviation from coplanarity and directional stability of the incipient facets on the two types of zones. This first requires to determine the distributions of the second SIF K_{II} and first non-singular stress T_{11} along the coplanarly perturbed crack front.

6.2 Expression of the perturbed mode II stress intensity factor

Gao and Rice (1986) have calculated, for the semi-infinite crack considered, the variations of the SIF resulting from some arbitrary coplanar perturbation of the front; their result for the mode II SIF reads

$$\begin{aligned}\delta K_{II}(z) = & \frac{\partial K_{II}^0}{\partial a}(z)\,\epsilon\phi(z) - \frac{2}{2-\nu}\,K_{III}^0(z)\,\epsilon\phi'(z) \\ & + \frac{1}{2\pi}\,\frac{2-3\nu}{2-\nu}\,PV\int_{-\infty}^{+\infty} K_{II}^0(z')\frac{\epsilon\phi(z')-\epsilon\phi(z)}{(z'-z)^2}\,dz'\end{aligned} \quad (25)$$

where $\frac{\partial K_{II}^0}{\partial a}(z)$ is the derivative of the unperturbed mode II SIF K_{II}^0 with respect to the position a of the straight crack front, see Figure 2, and the symbol $PV \int$ denotes the Cauchy principal value of an integral. In the special case considered here, where K_{II}^0 and

K_{III}^0 are respectively zero and uniform along the unperturbed front and the perturbation $\epsilon\phi$ is of the form (24), this yields for the mode II SIF K_{II} along the perturbed front:

$$K_{II}(z) = \delta K_{II}(z) = \frac{2}{2-\nu}\, K_{III}^0\, k\epsilon \sin(kz). \tag{26}$$

6.3 *Expression of the perturbed first non-singular stress*

In order to now evaluate the non-singular stress T_{11} along the coplanarly perturbed crack front, we introduce the following hypotheses:

- The unperturbed SIF K_I^0 and K_{III}^0 are comparable in magnitude. Also, the unperturbed non-singular stresses T_{ij}^0 and their derivatives $\frac{\partial T_{ij}^0}{\partial a}$ with respect to the position of the straight crack front are of the order of $K_I^0 L^{-1/2}$ and $K_I^0 L^{-3/2}$ respectively, where L is the characteristic length defined by the loading (in the absence of any characteristic lengthscale defined by the geometry itself).
- The characteristic length L is much larger than the typical distance of fluctuation of the fracture toughness, and therefore than the wavelength $\lambda \equiv 2\pi/k$ of the perturbation resulting from the non-uniformity of this toughness.

The perturbation $\epsilon\phi$ and its derivative $\epsilon\phi'$ being of the order of ϵ and $k\epsilon$ respectively, it follows from the first hypothesis that the first, second and third terms in the right-hand side of the expression (16) of δT_{11} are of the order of $K_I^0 L^{-3/2}\,\epsilon$, $K_I^0 L^{-1/2}\,k\epsilon$ and $K_I^0\,k^{3/2}\epsilon$ respectively. Since the second hypothesis implies that $kL \gg 1$, the first and second terms are negligible compared to the third one. It follows that for the sinusoidal perturbation considered,

$$\begin{aligned}
\delta T_{11}(z) &\simeq -\frac{1-2\nu}{4\sqrt{2\pi}} \int_{-\infty}^{+\infty} K_I^0 \left[-k\epsilon \sin(kz') + k\epsilon \sin(kz)\right] \frac{\mathrm{sgn}(z'-z)}{|z'-z|^{3/2}}\, dz' \\
&= \frac{1-2\nu}{4\sqrt{2\pi}}\, K_I^0\, k\epsilon\, \mathrm{Im}\left[\int_{-\infty}^{+\infty} \left(e^{ikz'} - e^{ikz}\right) \frac{\mathrm{sgn}(z'-z)}{|z'-z|^{3/2}}\, dz'\right] \\
&= \frac{1-2\nu}{4\sqrt{2\pi}}\, K_I^0\, k\epsilon\, \mathrm{Im}\left[e^{ikz} \int_{-\infty}^{+\infty} \left(e^{ik\zeta} - 1\right) \frac{\mathrm{sgn}(\zeta)}{|\zeta|^{3/2}}\, d\zeta\right] \\
&= \frac{1-2\nu}{2\sqrt{2\pi}}\, K_I^0\, k\epsilon\, \mathrm{Im}\left[ie^{ikz} \int_{0}^{+\infty} \sin(k\zeta) \frac{\mathrm{sgn}(\zeta)}{\zeta^{3/2}}\, d\zeta\right]
\end{aligned}$$

where use has been made of the change of variable $\zeta \equiv z' - z$ and parity properties. Calculation of the last integral using Gradshteyn and Ryzhik (1980)'s formulae (3.7614) and (8.338.3) then yields

$$\delta T_{11}(z) \simeq \left(\frac{1}{2} - \nu\right) K_I^0\, k^{3/2} \epsilon \cos(kz). \tag{27}$$

This expression happens to exactly coincide with Gao (1992)'s similar formula (44). This means that the terms disregarded by Gao without proper justification, see Subsection 4.3 above, are indeed negligible with the hypotheses introduced above.

Equation (27) confirms that δT_{11} is of order $K_I^0 k^{3/2}\epsilon$, as anticipated, whereas T_{11}^0 is of order $K_I^0 L^{-1/2}$. We therefore introduce the following final hypothesis:

- The characteristic length L defined by the loading, and the fluctuations of toughness generating the in-plane undulations of the crack front, are sufficiently large for the dimensionless quantity $k^{3/2}\epsilon L^{1/2} = k\epsilon\sqrt{kL}$ to be much larger than unity. (Note that this hypothesis does not contradict that of smallness of the perturbation: indeed one may have $|\epsilon\phi'| \sim k\epsilon \ll 1$ but $k\epsilon\sqrt{kL} \gg 1$, since kL is assumed to be much larger than unity).

Then the unperturbed non-singular stress T_{11}^0 is negligible compared to its variation δT_{11} so that

$$T_{11}(z) \simeq \delta T_{11}(z) \simeq \left(\frac{1}{2} - \nu\right) K_I^0 k^{3/2}\epsilon\cos(kz). \tag{28}$$

6.4 Analysis of deviation from coplanarity and directional stability

As remarked by Gao and Rice (1986), equation (26) implies that the mode II SIF $K_{II}(z)$ takes nonzero and opposite values on the two sides of a local bump or hollow of the coplanarly perturbed crack front. It then follows from equation (21) (with $K_I(z)$, $K_{II}(z)$ instead of $K_I(s)$, $K_{II}(s)$) that the subsequent kink angle $\theta(z)$ will also take nonzero and opposite values on these two sides, implying formation of an incipient non-coplanar facet gradually rotating about the direction of propagation of the crack.

To pursue the analysis, assume for instance K_{III}^0 to be positive and consider a zone of the coplanarly perturbed crack front having $\cos(kz) > 0$, thus lying ahead of its mean position. Over such a zone, the function $\sin(kz)$ increases so that, by equations (21) and (26), the kink angle $\theta(z)$ of the future non-coplanar facet is a decreasing function of z. Looking at Figure 5 and imagining a facet with this property, one easily sees that it tends to rotate in time in the *positive* direction about the axis Ox, and that this rotation tends to gradually enhance the local value of the mode I SIF and lower that of the mode III SIF. Such evolutions of the SIF are, as explained in the Introduction, typical of type A facets. This confirms the anticipated property that a facet formed from a locally more advanced zone of the crack front must be of this type.

Conversely, over a zone having $\cos(kz) < 0$, thus lying behind the mean position of the front, $\sin(kz)$ decreases, $\theta(z)$ increases, the facet rotates in the negative direction about the axis Ox, so that the mode I SIF decreases in time whereas that of mode III increases. Such evolutions are typical of type B facets. This again confirms the anticipated property that facets formed from less advanced zones of the crack front must be of that other type.

The preceding analysis however says nothing about the directional stability of the facets. To examine this question, consider the expression (28) of the non-singular stress $T_{11}(z)$ on the coplanarly perturbed crack front. On a zone having $\cos(kz) > 0$, about to generate a type A facet, this non-singular stress is positive so that Cotterell and Rice (1980)'s directional stability criterion (23) is violated, implying directional instability; conversely, on a zone having $\cos(kz) < 0$, about to generate a type B facet, Cotterell and Rice

(1980)'s criterion is met, implying directional stability. This suggests that in addition to the tendency of type A facets to propagate ahead of type B ones, there is *a tendency of the former facets to deviate more and more in time from the original crack plane, versus of the latter ones to come back to it.* These elements provide some theoretical ground to the intuitive idea that "the crack would ideally like to develop exclusively along non-coplanar facets of type A, type B ones being present only because they are geometrically necessary to join them".

It is important to note that the tendencies just mentioned are completely independent of the value of the ratio K^0_{III}/K^0_I (as long as K^0_{III} is nonzero, which is a necessary condition for out-of-plane deviations of the crack to appear). Thus increasing deviations from coplanarity of type A facets generated by in-plane fluctuations of the fracture toughness are probable, because of a Cotterell and Rice (1980)-type instability, even for low values of this ratio. *An instability of this type is therefore a good candidate to the explanation of increasing deviations of mode I+III cracks from coplanarity observed experimentally for values of K^0_{III}/K^0_I smaller, or even much smaller than Leblond et al. (2011)'s theoretical threshold for occurrence of a bifurcation.*[5]

One may analyze in more detail the importance of Cotterell and Rice (1980)'s instability as a function of the wavelength $\lambda = 2\pi/k$ of the coplanar crack front perturbation. This requires comparing the values of the curvature parameter $a(z)$ of the incipient facets for various initial in-plane sinusoidal perturbations of the crack front, having different wavelengths. Combination of equations (21) and (22) (with $K_I(z)$, $K_{II}(z)$, $T_{11}(z)$ instead of $K_I(s)$, $K_{II}(s)$, $T_{11}(s)$), (26) and (28) yields

$$a(z) = -\frac{16}{3}\sqrt{\frac{2}{\pi}}\,\frac{1-2\nu}{2-\nu}\,\frac{K^0_{III}}{K^0_I}\,k^{5/2}\epsilon^2 \sin(kz)\cos(kz). \tag{29}$$

This expression makes it clear that when comparing the curvature parameters $a(z)$ of various perturbations, one should fix their amplitude ϵ in some way, otherwise the obvious effect of this amplitude will mask that of the wavenumber k. A logical way of doing so is to consider "homothetic" perturbations identical in shape but not in size, that is sinusoidal fronts having the same maximal "slope" $k\epsilon$ in the crack plane but different wavenumbers k. Equation (29) shows that $a(z)$ is proportional to $(k\epsilon)^2\sqrt{k}$, that is to $\sqrt{k}$ or $1/\sqrt{\lambda}$ for a given value of $k\epsilon$. Therefore, the smaller the value of λ, the larger that of $a(z)$ in absolute value; in other words, *the smaller the wavelength of the initial in-plane perturbation, the more directionally unstable the incipient facets it generates, if of type A, and the more directionally stable these facets, if of type B.* This theoretical conclusion finds some experimental support in the fact that incipient facets actually observed are generally of tiny initial wavelength, apparently limited in smallness only by the microstructure of the material. This is illustrated in Figure 6, which shows a photograph taken by Lazarus of a mode I+III fracture surface generated in a glass specimen by Buchholz, where the initial crack is located at the very top and propagates toward the bottom. (The photograph also shows that the wavelength of the facets does *not* remain small when their length

[5] It cannot be the sole explanation, however, since such deviations *could* be observed in Pons and Karma (2010)'s numerical simulations of crack propagation in mode I+III based on Karma *et al.* (2001)'s phase field model, though they did *not* include fluctuations of the fracture toughness.

increases, due to the gradual "coarsening" resulting from their progressive coalescence; such a phenomenon of course completely eludes the present analysis limited to *incipient* facets).

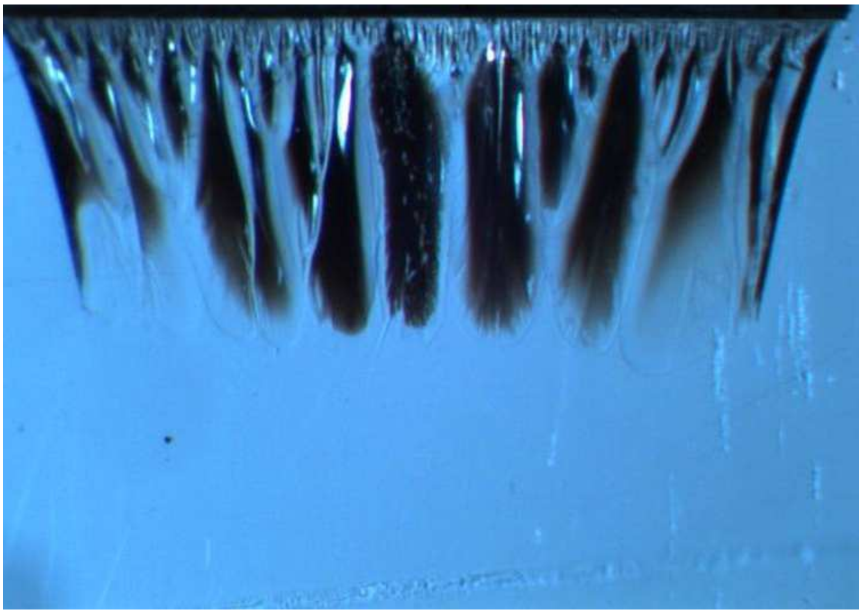

Fig. 6. Fracture facets in mode I+III - Experiment by Buchholz, photograph by Lazarus

7 Conclusion

This paper was devoted to the investigation of the idea that quick deviations of cracks loaded in mode I+III from their original plane might be made much easier by a strong influence of imperfections. Such an influence could stand as a plausible explanation to the fact that, as reported notably by Sommer (1969) and Ravi-Chandar (2010), such deviations are frequently observed experimentally for small values of the ratio of the mode III to mode I unperturbed SIF, lying below or even much below the theoretical threshold for occurrence of a bifurcation calculated by Leblond *et al.* (2011).

A typical example of inevitable imperfections consists of small fluctuations of the fracture toughness within the initial crack plane, which generate small in-plane undulations of the crack front. Rice (1985) and Gao and Rice (1986) calculated the distributions of the SIF of the three modes resulting from a slight but otherwise arbitrary in-plane perturbation of the front of a semi-infinite crack loaded arbitrarily; in particular Gao and Rice (1986) showed that the local mode II SIF takes nonzero and opposite values on the two sides of a local protrusion of this front, implying future local deviations of the propagating crack from coplanarity of opposite signs on these two sides, giving birth to an incipient "fracture facet" gradually rotating about the direction of propagation.

Gao and Rice (1986) however left open the question of the "directional stability" of these facets. It was precisely the main purpose of the present paper to complement their work by analyzing this question, using Cotterell and Rice (1980)'s well-known "directional stability criterion" (duly extended to the 3D case by Leblond (1999)).

Since Cotterell and Rice (1980)'s stability condition is on the sign of the non-singular stress parallel to the direction of propagation, its application to the analysis of directional stability of the facets requires knowledge of the distributions of the non-singular stresses for a crack with a slightly, coplanarly perturbed front. The necessary calculations were performed by Gao (1992), but with some restrictive hypotheses and unjustifiably omitting some terms, which made it necessary to revisit the problem. This has been done here by using the theory of 3D weight functions (Rice, 1985; Bueckner, 1987). Fully general formulae have been obtained for the variations of the non-singular stresses resulting from some slight but otherwise arbitrary in-plane perturbation of a semi-infinite crack, confirming and completing Gao (1992)'s partial results.

The formula obtained for the variation of the non-singular stress parallel to the direction of crack propagation was then applied to a typically sinusoidal coplanar perturbation of the front. A distinction was made between the more advanced zones of the front, about to generate an incipient facet of "type A" rotating about the direction of propagation so as to raise the local proportion of mode I versus mode III, and the less advanced ones, about to generate a facet of "type B" rotating oppositely so as to lower this proportion.

It has been found that provided that the lengthscale defined by the loading and the fluctuations of the fracture toughness are large enough, the non-singular stress parallel to the direction of propagation is positive on the former zones, implying directional instability, and negative on the latter ones, implying directional stability. This shows that even for low values of the ratio of the mode III to mode I unperturbed SIF, crack propagation in mode I+III may occur through preferential formation of non-coplanar facets of type A, because of a local Cotterell and Rice (1980)-type instability - which confirms the idea investigated.

It has also been found that the smaller the wavelength of the initial in-plane perturbation, the stronger Cotterell and Rice (1980)'s instability on the resulting incipient facets of type A. This implies preferential development of type A facets of *small* initial wavelength, in agreement with experimental observations.

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