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**Abstract**

Gurson (1977)'s famous model of the behavior of porous ductile solids, initially developed for spherical cavities, was extended by Gologanu *et al.* (1993, 1994, 1997) to spheroidal, both prolate and oblate voids. The aim of this work is to further extend it to general (non-spheroidal) ellipsoidal cavities, through approximate homogenization of some representative elementary porous cell. As a first step, we perform in the present Part I a limit-analysis of such a cell, namely an ellipsoidal volume made of some rigid-ideal-plastic von Mises material and containing a confocal ellipsoidal void, loaded arbitrarily under conditions of homogeneous boundary strain rate. This analysis provides an estimate of the overall plastic dissipation based on a family of trial incompressible velocity fields recently discovered by Leblond and Gologanu (2008), satisfying conditions of homogeneous strain rate on all ellipsoids confocal with the void and the outer boundary. The asymptotic behavior of the integrand in the expression of the global plastic dissipation is studied both far from and close to the void. The results obtained suggest approximations leading to explicit approximate expressions of the overall dissipation and yield function. These expressions contain parameters the full determination of which will be the object of Part II.

*Keywords:* Porous ductile solids; ellipsoidal voids; homogenization; limit-analysis; homogeneous boundary strain rate; plastic dissipation

## 1 Introduction

The most classical model of the overall behavior of porous ductile solids is due to Gurson (1977). This model was derived from homogenization, using a limit-analysis of a spherical cell made of some rigid-ideal-plastic von Mises material, containing a spherical void and loaded arbitrarily through conditions of homogeneous boundary strain rate (Mandel, 1964; Hill, 1967). As such, it applied to materials containing spherical voids.

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Voids actually encountered in real materials are often non-spherical, which induced Gologanu *et al.* (1993, 1994, 1997) to extend Gurson (1977)'s model to spheroidal voids; the extended model is currently known as the *GLD model*. These authors again combined homogenization and limit-analysis to derive estimates of the yield surface of spheroidal, prolate (Gologanu *et al.*, 1993) and oblate (Gologanu *et al.*, 1994) cells containing a confocal spheroidal void, and subjected to conditions of homogeneous boundary strain rate. The treatment used incompressible axisymmetric trial velocity fields satisfying conditions of homogeneous strain rate on every spheroid confocal with the boundaries of the void and the cell.[1] The model was further improved by Gologanu (1997) and Gologanu *et al.* (1997) by considering more trial velocity fields, and using some general rigorous results of Ponte-Castaneda (1991), Willis (1991) and Michel and Suquet (1992) on homogenization of nonlinear composites.

Variants and extensions of the Gurson and GLD models have been proposed by various authors. Some of these were based on use of more sophisticated trial velocity fields (Garajeu (1995): exact solution for a hollow elastic sphere loaded through conditions of homogeneous boundary strain rate ; Monchiet *et al.* (2007): exact Eshelby solution for the ellipsoidal inclusion problem in an infinite elastic matrix). Some others considered matrices obeying Hill's anisotropic yield criterion instead of that of von Mises (Keralavarma and Benzerga, 2008; Monchiet *et al.*, 2008). Benzerga and Leblond (2010)'s recent review paper provides a synthesis of these works. But more general (non-spheroidal) ellipsoidal voids have not been considered within the approach initiated by Gologanu *et al.* (1993, 1994, 1997), although such voids are common in practice, for instance in laminated plates.

The problem of general ellipsoidal voids was however attacked from another angle by Kaisalam and Ponte-Castaneda (1998) and Danas and Ponte-Castaneda (2008a,b). The first authors initially derived a model based on the concept of "linear comparison material". Unfortunately, although this model theoretically applied to porous plastic (and visco-plastic) solids containing arbitrary ellipsoidal voids and subjected to arbitrary loadings, its predictions revealed rather inaccurate for purely hydrostatic loadings. Danas and Ponte-Castaneda (2008a,b) recently proposed an improved model based on results of some "second-order homogenization method" (Ponte-Castaneda, 2002; Idiart and Ponte-Castaneda, 2005); but the accuracy of this new proposal has not yet been assessed.

The approach initiated by Gologanu *et al.* (1993, 1994, 1997) remains an interesting alternative for the derivation of models for porous plastic materials containing general ellipsoidal voids, since it offers the major advantage of yielding explicit, and basically simple approximate expressions of the overall plastic dissipation and yield function. The aim of this work is to develop such a model.

In the present Part I, we begin by extending Gologanu *et al.* (1993)'s and Gologanu *et al.* (1994)'s first works on spheroidal voids, by performing a limit-analysis of some general ellipsoidal cell made of some rigid-ideal-plastic von Mises material, containing a confocal ellipsoidal void and loaded arbitrarily through conditions of homogeneous boundary strain rate.

[1] A variant using a velocity field orthogonal to each such spheroid was proposed by Garajeu (1995) and Garajeu *et al.* (2000).

The paper is organized as follows:

- As a geometric preliminary, Section 2 is devoted to general ellipsoidal coordinates.
- Section 3 then recalls the expression of a family of incompressible velocity fields recently discovered by Leblond and Gologanu (2008). These fields satisfy conditions of homogeneous strain rate on an arbitrary family of confocal ellipsoids, and thus appear as generalizations of those used by Gologanu *et al.* (1993) and Gologanu *et al.* (1994), which satisfied such conditions on confocal spheroids.
- Section 4 presents the principle of the application of this family of trial velocity fields to some limit-analysis of the ellipsoidal cell considered.
- Section 5 then expounds a thorough study of the asymptotic behavior of the integrand in the integral expression of the estimated overall plastic dissipation, both far from and near the void.
- Finally Section 6 shows how to deduce from this study reasonable approximations allowing for an explicit calculation of this dissipation and the associated, Gurson-like approximate overall yield function.

The parameters appearing in the approximate expression of the overall yield function are not fully determined yet at this stage. Part II will be devoted to their determination.

## 2 Geometric preliminaries

We first define general ellipsoidal coordinates, following Morse and Feshbach (1953). To each triplet $(a, b, c)$ of numbers such that

$$a > b > c > 0 \tag{1}$$

is attached a set of curvilinear coordinates $(\lambda, \mu, \nu)$ such that

$$\lambda > -c^2 > \mu > -b^2 > \nu > -a^2; \tag{2}$$

the relations connecting these coordinates to ordinary Cartesian ones $(x, y, z)$ are

$$\begin{cases} x = \pm \left( \dfrac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)} \right)^{1/2} \\ y = \pm \left( \dfrac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-c^2)(b^2-a^2)} \right)^{1/2} \\ z = \pm \left( \dfrac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)} \right)^{1/2} \end{cases} \Leftrightarrow \begin{cases} \dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} + \dfrac{z^2}{c^2+\lambda} = 1 \\ \dfrac{x^2}{a^2+\mu} + \dfrac{y^2}{b^2+\mu} + \dfrac{z^2}{c^2+\mu} = 1 \\ \dfrac{x^2}{a^2+\nu} + \dfrac{y^2}{b^2+\nu} + \dfrac{z^2}{c^2+\nu} = 1. \end{cases} \tag{3}$$

(Equations $(3)_{4,5,6}$ implicitly define $\lambda$, $\mu$, $\nu$ in terms of $x$, $y$, $z$ through polynomial equations of the third degree). Note that equations $(3)_{1,2,3}$ leave the signs of $x$, $y$, $z$ unspecified so that there are in fact 8 possible triplets $(x, y, z)$ for each triplet $(\lambda, \mu, \nu)$. Also, equations $(3)_{4,5,6}$, combined with inequalities (2), show that surfaces of constant $\lambda$ are confocal ellipsoids, denoted $\mathcal{E}_\lambda$ in the sequel, of semi-axes $\sqrt{a^2+\lambda}$, $\sqrt{b^2+\lambda}$, $\sqrt{c^2+\lambda}$, whereas surfaces of constant $\mu$ and $\nu$ are hyperboloids of one and two sheets, respectively.

Define the function

$$v(\rho) \equiv \sqrt{|a^2+\rho|\,|b^2+\rho|\,|c^2+\rho|}\ . \tag{4}$$

This function appears in the expression of the infinitesimal volume element:

$$d\Omega = -\frac{(\lambda-\mu)\,(\mu-\nu)\,(\nu-\lambda)}{8v(\lambda)v(\mu)v(\nu)}\,d\lambda d\mu d\nu \tag{5}$$

and in other instances, as will appear below. Note that up to a factor of $4\pi/3$, $v(\lambda)$ represents the volume enclosed within the ellipsoidal surface $\mathcal{E}_\lambda$. (The geometrical interpretations of $v(\mu)$ and $v(\nu)$ are less straightforward and will not be needed). In the sequel, the short notation $v$ will often be used to represent $v(\lambda)$, without any ambiguity.

The expressions of the derivatives $\partial\lambda/\partial x$, $\partial\lambda/\partial y$, $\partial\lambda/\partial z$ will be needed; they are readily found by differentiating equation $(3)_4$ at constant $(y,z)$, $(z,x)$ and $(x,y)$ respectively:

$$\begin{cases} \dfrac{\partial\lambda}{\partial x} = \dfrac{2x}{(a^2+\lambda)\,T} \\ \dfrac{\partial\lambda}{\partial y} = \dfrac{2y}{(b^2+\lambda)\,T} \\ \dfrac{\partial\lambda}{\partial z} = \dfrac{2z}{(c^2+\lambda)\,T} \end{cases} ,\quad T \equiv \frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2}. \tag{6}$$

## 3 Incompressible velocity fields satisfying conditions of homogeneous strain rate on an arbitrary family of confocal ellipsoids

The velocity fields mentioned in the title were recently discovered by Leblond and Gologanu (2008), who established the existence, uniqueness and explicit expression of a field of such a type for any possible value of the overall strain rate tensor imposed on the outermost ellipsoid. We concentrate here on their results, postponing a short overview of their treatment to Appendix A for completeness.

Leblond and Gologanu (2008)'s conclusions are as follows: *the velocity fields considered are, in ellipsoidal coordinates, of the form*

$$\mathbf{v}(\mathbf{r}) \equiv \mathbf{D}(\lambda).\mathbf{r} \quad \Leftrightarrow \quad \begin{cases} v_x(\mathbf{r}) \equiv D_{xx}(\lambda)x + D_{xy}(\lambda)y + D_{xz}(\lambda)z \\ v_y(\mathbf{r}) \equiv D_{yx}(\lambda)x + D_{yy}(\lambda)y + D_{yz}(\lambda)z \\ v_z(\mathbf{r}) \equiv D_{zx}(\lambda)x + D_{zy}(\lambda)y + D_{zz}(\lambda)z, \end{cases} \tag{7}$$

$\mathbf{r} \equiv x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ *being the current position vector and* $\mathbf{D}(\lambda)$ *a symmetric second-rank tensor depending on* $\lambda$ *of the form*

$$\mathbf{D}(\lambda) \equiv \mathcal{A}\mathbf{D}^0(\lambda) + \mathbf{\Delta} \tag{8}$$

*where $\mathcal{A}$ is an arbitrary number, $\mathbf{\Delta}$ an arbitrary constant traceless symmetric second-rank tensor, and $\mathbf{D}^0(\lambda)$ the symmetric second-rank tensor depending on $\lambda$ given by*

$$\begin{cases} D^0_{xx}(\lambda) \equiv \displaystyle\int_\lambda^{+\infty} \frac{d\rho}{2\,(a^2+\rho)\,v(\rho)} \\ D^0_{yy}(\lambda) \equiv \displaystyle\int_\lambda^{+\infty} \frac{d\rho}{2\,(b^2+\rho)\,v(\rho)} \\ D^0_{zz}(\lambda) \equiv \displaystyle\int_\lambda^{+\infty} \frac{d\rho}{2\,(c^2+\rho)\,v(\rho)} \\ D^0_{xy}(\lambda) \equiv D^0_{yz}(\lambda) \equiv D^0_{zx}(\lambda) \equiv 0. \end{cases} \tag{9}$$

The integrals defining the diagonal components of $\mathbf{D}^0(\lambda)$ are of elliptic type. However it is important to note that the trace of this tensor has a simple expression:

$$\operatorname{tr}\mathbf{D}^0(\lambda) = \int_\lambda^{+\infty}\left(\frac{1}{a^2+\rho}+\frac{1}{b^2+\rho}+\frac{1}{c^2+\rho}\right)\frac{d\rho}{2v(\rho)} = -\int_\lambda^{+\infty}\frac{d}{d\rho}\left(\frac{1}{v(\rho)}\right)d\rho = \frac{1}{v(\lambda)}. \tag{10}$$

Some remarks pertaining to the velocity field

$$\mathbf{v}^0(\mathbf{r}) \equiv \mathbf{D}^0(\lambda).\mathbf{r} \tag{11}$$

are in order:

- In the spherical case ($a = b = c$), this velocity field is just the radial field inversely proportional to the square of the distance to the origin.
- In the circular cylindrical case ($a = +\infty, b = c$), it is the planar radial field inversely proportional to the distance to the axis of rotational symmetry.
- In the spheroidal, prolate ($a > b = c$) or oblate ($a = b > c$) cases, it coincides with the trial field used by Gologanu *et al.* (1993, 1994) to describe the expansion of the void.

These elements suggest that the field $\mathbf{v}^0(\mathbf{r})$ may plausibly be used to represent the expansion of an arbitrary ellipsoidal void.

## 4 Limit analysis of an ellipsoidal cell containing a confocal ellipsoidal void

### *4.1 Presentation of the cell - Notations*

We shall now consider an ellipsoidal cell containing a confocal ellipsoidal void, and loaded arbitrarily through conditions of homogeneous boundary strain rate (Mandel, 1964; Hill, 1967). Although this shape is admittedly somewhat arbitrary, it is commonly considered as an acceptable approximation of the actual shape of some representative elementary cell in a porous material. [2] The boundary conditions imposed on it are also commonly

[2] Danas and Ponte-Castaneda (2008a) have argued that for the small porosities of practical interest, elementary cells of *any* shape are acceptable, provided that they respect the given

accepted as representative, at least prior to the onset of strain localization phenomena which lie outside of the scope of the present work.

The semi-axes of the inner ellipsoid (the boundary of the void) and the outer one (the boundary of the cell) are denoted $a, b, c$ ($a > b > c$) and $A, B, C$ ($A > B > C$) respectively; they are related through the confocality conditions $A^2 - a^2 = B^2 - b^2 = C^2 - c^2$. The volumes of these ellipsoids are $\frac{4\pi}{3}\omega$ and $\frac{4\pi}{3}\Omega$ where

$$\omega \equiv abc \quad ; \quad \Omega \equiv ABC, \tag{12}$$

and the porosity (void volume fraction) is

$$f \equiv \frac{\omega}{\Omega}. \tag{13}$$

We shall use the ellipsoidal coordinates $(\lambda, \mu, \nu)$ associated to the triplet $(a, b, c)$, as defined in Section 2. The values of $\lambda$ on the inner and outer ellipsoids are $\lambda \equiv 0$ and $\lambda \equiv \Lambda$ respectively, so that these ellipsoids may be identified, with the notation introduced above, to $\mathcal{E}_0$ and $\mathcal{E}_\Lambda$, on which the values of $v(\lambda)$ are $v(0) = \omega$ and $v(\Lambda) = \Omega$ respectively. The semi-axes $A, B, C$ of $\mathcal{E}_\Lambda$ are related to those, $a, b, c$, of $\mathcal{E}_0$ plus the parameter $\Lambda$ through the relations

$$A \equiv \sqrt{a^2 + \Lambda} \quad ; \quad B \equiv \sqrt{b^2 + \Lambda} \quad ; \quad C \equiv \sqrt{c^2 + \Lambda}. \tag{14}$$

It follows from this equation, the definition (13) of the porosity and the expression (12) of $\omega$ and $\Omega$, that the parameter $\Lambda$ is determined in terms of the semi-axes of the void and the porosity by the following third-degree polynomial equation:

$$\left(a^2 + \Lambda\right)\left(b^2 + \Lambda\right)\left(c^2 + \Lambda\right) - \frac{a^2 b^2 c^2}{f^2} = 0. \tag{15}$$

The completely flat ellipsoid confocal with $\mathcal{E}_0$ and $\mathcal{E}_\Lambda$ will play a fundamental role in the sequel. The semi-axes $\bar{a}, \bar{b}, \bar{c}$ of this ellipsoid are given by

$$\bar{a} \equiv \sqrt{a^2 - c^2} \quad ; \quad \bar{b} \equiv \sqrt{b^2 - c^2} \quad ; \quad \bar{c} \equiv 0. \tag{16}$$

The family of confocal ellipsoids $\mathcal{E}_\lambda$ may be characterized by the single dimensionless parameter

$$k \equiv \frac{\bar{b}}{\bar{a}} = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}} \tag{17}$$

or the related one

$$k' \equiv \sqrt{1 - k^2} = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}. \tag{18}$$

In general, $k$ and $k'$ are in the interval $(0, 1)$. The special values ($k = 0, k' = 1$) and ($k = 1, k' = 0$) correspond to spheroidal, prolate ($a > b = c$) and oblate ($a = b > c$) voids respectively. For a spherical void ($a = b = c$), $k$ and $k'$ are indeterminate.

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value of the porosity and the shape of the voids; indeed the influence of the distribution function of the centers of the voids is of second order in the porosity.

The hollow cell will be assumed to be made of some rigid-ideal-plastic material with yield stress $\sigma_0$ in simple tension, obeying von Mises's criterion and the associated flow rule.

### *4.2 Limit-analysis*

The cell is loaded through conditions of homogeneous boundary strain rate:

$$\mathbf{v}(\mathbf{r}) = \mathbf{D}.\mathbf{r} \quad , \quad \forall \mathbf{r} \in \mathcal{E}_\Lambda \tag{19}$$

where $\mathbf{D}$ is the macroscopic strain rate tensor. There is a unique incompressible velocity field of the type discussed in Section 3 obeying these boundary conditions, that is, the equation

$$\mathcal{A}\mathbf{D}^0(\Lambda) + \mathbf{\Delta} = \mathbf{D} \tag{20}$$

has a unique solution in $(\mathcal{A}, \mathbf{\Delta})$; indeed taking the trace of both sides, one gets by equation (10) plus the fact that $\mathbf{\Delta}$ must be traceless:

$$\mathcal{A} \ \mathrm{tr}\, \mathbf{D}^0(\Lambda) = \frac{\mathcal{A}}{v(\Lambda)} = \frac{\mathcal{A}}{\Omega} = \mathrm{tr}\, \mathbf{D} \quad \Rightarrow \quad \mathcal{A} = \Omega \ \mathrm{tr}\, \mathbf{D}, \tag{21}$$

and it follows that

$$\mathbf{\Delta} = \mathbf{D} - \mathcal{A}\mathbf{D}^0(\Lambda) = \mathbf{D} - \Omega\,(\mathrm{tr}\, \mathbf{D})\, \mathbf{D}^0(\Lambda). \tag{22}$$

The thus unambiguously defined velocity field may be used in a limit-analysis of the domain considered. This analysis provides an upper estimate $\Pi^+(\mathbf{D})$ of the overall plastic dissipation, identical to the average value of the local plastic dissipation corresponding to the trial velocity field considered over the ellipsoidal domain:

$$\Pi^+(\mathbf{D}) \equiv \frac{1}{\frac{4\pi}{3}\Omega} \int_{\lambda=0}^{\lambda=\Lambda} \int_{\mu=-b^2}^{\mu=-c^2} \int_{\nu=-a^2}^{\nu=-b^2} \left( \sum_{i=1}^{8} \sigma_0 \, d_{eq}^{(i)}(\lambda, \mu, \nu) \right) d\Omega. \tag{23}$$

In this equation the symbols $d_{eq}^{(i)}(\lambda, \mu, \nu)$ $(i = 1, ..., 8)$ denote the values of the local von Mises equivalent strain rate

$$d_{eq}(\mathbf{r}) \equiv \left( \frac{2}{3} \mathbf{d}(\mathbf{r}) : \mathbf{d}(\mathbf{r}) \right)^{1/2} \tag{24}$$

(where $\mathbf{d}(\mathbf{r})$ is the local strain rate tensor) for the 8 triplets $(x, y, z)$ corresponding to the triplet $(\lambda, \mu, \nu)$, and the elementary volume element $d\Omega$ is given by equation (5).

It is unfortunately impossible to provide an exact explicit expression of the integral in equation (23), even in the simplest case of a spherical void. We therefore introduce some approximation here. First, integrating over successive confocal ellipsoids $\mathcal{E}_\lambda$, we re-express $\Pi^+(\mathbf{D})$ as

$$\Pi^+(\mathbf{D}) = \frac{\sigma_0}{\Omega} \int_{\lambda=0}^{\lambda=\Lambda} \langle d_{eq}(\mathbf{r}) \rangle_{\mathcal{E}_\lambda} \, d(v(\lambda)). \tag{25}$$

In this expression the symbol $\langle f(\mathbf{r}) \rangle_{\mathcal{E}_\lambda}$ denotes the average value of an arbitrary function $f(\mathbf{r})$ over the ellipsoidal surface $\mathcal{E}_\lambda$, with a weight equal to the infinitesimal volume element between the surfaces $\mathcal{E}_\lambda$ and $\mathcal{E}_{\lambda+d\lambda}$:

$$\begin{aligned}\langle f(\mathbf{r})\rangle_{\mathcal{E}_\lambda} &\equiv \frac{1}{\frac{4\pi}{3}d(v(\lambda))}\int_{\mu=-b^2}^{\mu=-c^2}\int_{\nu=-a^2}^{\nu=-b^2}\left(\sum_{i=1}^{8} f^{(i)}(\lambda,\mu,\nu)\right)d\Omega \\ &= -\frac{1}{\frac{4\pi}{3}v(\lambda)\frac{dv}{d\lambda}(\lambda)}\int_{\mu=-b^2}^{\mu=-c^2}\int_{\nu=-a^2}^{\nu=-b^2}\left(\sum_{i=1}^{8} f^{(i)}(\lambda,\mu,\nu)\right)\frac{(\lambda-\mu)\,(\mu-\nu)\,(\nu-\lambda)}{8v(\mu)v(\nu)}\,d\mu d\nu\end{aligned} \tag{26}$$

where equation (5) has been used; the $f^{(i)}(\lambda,\mu,\nu)$ here again represent the values of $f(\mathbf{r})$ for the 8 triplets $(x,y,z)$ corresponding to the triplet $(\lambda,\mu,\nu)$. It then follows from the classical inequality $\langle f(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \leq \sqrt{\langle f^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}}$ (applicable whatever the weighting function chosen to define the average value) that

$$\Pi^+(\mathbf{D}) \leq \Pi^{++}(\mathbf{D}) \equiv \frac{\sigma_0}{\Omega}\int_{\lambda=0}^{\lambda=\Lambda}\sqrt{\langle d_{eq}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}}\;d(v(\lambda)), \tag{27}$$

and the following approximation is introduced:

$\mathcal{A}_1$: *The quantity* $\Pi^{++}(\mathbf{D})$ *represents an acceptable approximation of* $\Pi^+(\mathbf{D})$.

This is exactly the hypothesis which must be made to derive Gurson's criterion for spherical voids (see e.g. Leblond (2003), Chapter 8) and the GLD criterion for spheroidal ones (Gologanu *et al.*, 1993, 1994, 1997). Since it was found to be quite acceptable in these cases, it may be hoped to again yield satisfactory results for general ellipsoidal voids.

The calculation of the integral in equation (27) is still a difficult task which will require further approximations detailed in the sequel. Once this task is completed, an external estimate of the overall yield surface of the cell may be obtained from the equation

$$\mathbf{\Sigma} = \frac{\partial\Pi^{++}}{\partial\mathbf{D}}(\mathbf{D}) \tag{28}$$

where the components of $\mathbf{D}$ act as parameters. (This equation defines a 5D surface in the 6D space of overall stresses $\mathbf{\Sigma}$ since the 6 components of $\mathbf{\Sigma}$ depend only on the ratios of 5 components of $\mathbf{D}$ to the last one, the function $\frac{\partial\Pi^{++}}{\partial\mathbf{D}}(\mathbf{D})$ being positively homogeneous of degree 0).

## 5 Asymptotic study of the integrand in the expression of the overall plastic dissipation

### *5.1 Generalities*

The aim of this Section is to study the asymptotic behavior of the integrand in equation $(27)_2$ defining $\Pi^{++}(\mathbf{D})$, both far from and close to the origin. The results found will play a fundamental role in the search for a reasonable approximate expression of this integrand allowing for an explicit calculation of the overall plastic dissipation and the associated yield criterion.

We first note that by equations (7) and (8),

$$\mathbf{d}(\mathbf{r}) = \mathcal{A}\mathbf{d}^0(\mathbf{r}) + \boldsymbol{\Delta} \tag{29}$$

where $\mathbf{d}^0(\mathbf{r})$ is the local strain rate tensor corresponding to the velocity field $\mathbf{v}^0(\mathbf{r})$ defined by equation (11); the components of this tensor are readily deduced from equations (6) and (9):

$$\begin{cases} d^0_{xx}(\mathbf{r}) &= D^0_{xx}(\lambda) - \dfrac{x^2}{(a^2+\lambda)^2\,Tv(\lambda)} \\ d^0_{yy}(\mathbf{r}) &= D^0_{yy}(\lambda) - \dfrac{y^2}{(b^2+\lambda)^2\,Tv(\lambda)} \\ d^0_{zz}(\mathbf{r}) &= D^0_{zz}(\lambda) - \dfrac{z^2}{(c^2+\lambda)^2\,Tv(\lambda)} \end{cases} \quad ; \quad \begin{cases} d^0_{xy}(\mathbf{r}) &= -\dfrac{xy}{(a^2+\lambda)\,(b^2+\lambda)\,Tv(\lambda)} \\ d^0_{yz}(\mathbf{r}) &= -\dfrac{yz}{(b^2+\lambda)\,(c^2+\lambda)\,Tv(\lambda)} \\ d^0_{zx}(\mathbf{r}) &= -\dfrac{zx}{(c^2+\lambda)\,(a^2+\lambda)\,Tv(\lambda)}. \end{cases} \tag{30}$$

It then follows from equation (24) that

$$d^2_{eq}(\mathbf{r}) = \mathcal{A}^2 {d^0_{eq}}^2(\mathbf{r}) + \Delta^2_{eq} + \frac{4}{3}\mathcal{A}\,\mathbf{d}^0(\mathbf{r}) : \boldsymbol{\Delta} \tag{31}$$

where

$$\Delta_{eq} \equiv \left(\frac{2}{3}\boldsymbol{\Delta} : \boldsymbol{\Delta}\right)^{1/2}, \tag{32}$$

so that

$$\langle d^2_{eq}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = \mathcal{A}^2 \langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} + \Delta^2_{eq} + \frac{4}{3}\mathcal{A}\,\langle \mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda} : \boldsymbol{\Delta}. \tag{33}$$

One may immediately note that the off-diagonal components of the average value $\langle \mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ here are zero since, by equations $(30)_{4,5,6}$, the off-diagonal components of $\mathbf{d}^0(\mathbf{r})$ change sign from one of the 8 points $(x, y, z)$ corresponding to a given triplet $(\lambda, \mu, \nu)$ to another.

The asymptotic study of the scalar $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ and the tensor $\langle \mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ near infinity is easy. Indeed in this limit the ellipsoids $\mathcal{E}_\lambda$ become spheres of large radius $r \sim \sqrt{\lambda}$ (by equation $(3)_4$) and large volume $\frac{4\pi}{3}v(\lambda)$, $v(\lambda) \equiv v \sim \lambda^{3/2} \sim r^3$ (by equation (4)), the tensor $\mathbf{D}^0(\lambda)$ becomes identical to $\frac{1}{3v}\mathbf{1} \sim \frac{1}{3r^3}\mathbf{1}$ (by equation (10)), and the velocity field $\mathbf{v}^0(\mathbf{r})$ becomes identical to the radial incompressible field $\frac{1}{3r^3}\mathbf{r}$ (by equation (11)). It follows that the equivalent strain rate $d_{eq}(\mathbf{r})$ becomes uniform and equal to $\frac{2}{3r^3} \sim \frac{2}{3v}$ on the surface $\mathcal{E}_\lambda$ so that

$$\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \sim \frac{4}{9r^6} \sim \frac{4}{9v^2} \qquad \text{for } v \to +\infty. \tag{34}$$

Also, the diagonal components of the tensor $\langle \mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, of *a priori* order $O(1/r^3) = O(1/v)$, become asymptotically equal for reasons of isotropy, and their sum is zero since $\mathbf{d}^0(\mathbf{r})$ is traceless; hence they are zero at dominant order, that is,

$$\begin{cases} \langle d^0_{xx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} &= o\left(\frac{1}{r^3}\right) = o\left(\frac{1}{v}\right) \\ \langle d^0_{yy}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} &= o\left(\frac{1}{r^3}\right) = o\left(\frac{1}{v}\right) \\ \langle d^0_{zz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} &= o\left(\frac{1}{r^3}\right) = o\left(\frac{1}{v}\right) \end{cases} \qquad \text{for } v \to +\infty. \tag{35}$$

The asymptotic study of $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ and $\langle \mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ in the other limit, that is for surfaces $\mathcal{E}_\lambda$ close to the completely flat confocal ellipsoid of semi-axes $\bar{a}, \bar{b}, \bar{c} = 0$, is unfortunately much more involved. It will require distinguishing between the generic case where $a > b > c$ and three special cases where $a > b = c$, $a = b > c$ or $a = b = c$, corresponding to voids of arbitrary ellipsoidal, prolate spheroidal, oblate spheroidal and spherical shapes respectively.

### *5.2 Generic case*

In the generic case where $a > b > c$, the asymptotic study of $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ and $\langle \mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ for surfaces $\mathcal{E}_\lambda$ close to the completely flat confocal ellipsoid is made easier by considering the ellipsoidal coordinates $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ associated to the semi-axes $\bar{a}, \bar{b}, \bar{c} = 0$ of this ellipsoid. These new coordinates are related to the old ones $(\lambda, \mu, \nu)$ through the formulae

$$\bar{\lambda} \equiv \lambda + c^2 \quad ; \quad \bar{\mu} \equiv \mu + c^2 \quad ; \quad \bar{\nu} \equiv \nu + c^2, \tag{36}$$

and the Cartesian coordinates $(x, y, z)$ are related to them through formulae analogous to (3), with $\bar{a}, \bar{b}, \bar{c} = 0, \bar{\lambda}, \bar{\mu}, \bar{\nu}$ instead of $a, b, c, \lambda, \mu, \nu$. The surface $\mathcal{E}_\lambda$ is close to the completely flat confocal ellipsoid when $\bar{\lambda} \to 0$; therefore, what we are interested in is the value of the limits

$$\begin{cases} L_x \equiv \lim\limits_{\bar{\lambda}\to 0} \langle d^0_{xx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \\ L_y \equiv \lim\limits_{\bar{\lambda}\to 0} \langle d^0_{yy}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \\ L_z \equiv \lim\limits_{\bar{\lambda}\to 0} \langle d^0_{zz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \end{cases} \quad ; \quad L^2 \equiv \lim_{\bar{\lambda}\to 0} \langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}. \tag{37}$$

To evaluate these limits, the first task is to derive the asymptotic expressions of the Cartesian coordinates $x, y, z$, the function $v(\lambda)$ and the quantity $T$ for $\bar{\lambda} \to 0$. The expressions $(3)_{1,2,3}$ of $x, y, z$ and (4) of $v(\lambda)$ become in this limit

$$\begin{cases} x \sim \pm \left( \dfrac{(\bar{a}^2 + \bar{\mu})(\bar{a}^2 + \bar{\nu})}{\bar{a}^2 - \bar{b}^2} \right)^{1/2} \\ y \sim \pm \left( \dfrac{(\bar{b}^2 + \bar{\mu})(\bar{b}^2 + \bar{\nu})}{\bar{b}^2 - \bar{a}^2} \right)^{1/2} \\ z \sim \pm \dfrac{\sqrt{\bar{\lambda}\bar{\mu}\bar{\nu}}}{\bar{a}\bar{b}} \end{cases} \quad ; \quad v(\lambda) \sim \bar{a}\bar{b}\sqrt{\bar{\lambda}}\ . \tag{38}$$

These equations imply that $x$ and $y$ are of order $O(\bar{\lambda}^0 = 1)$ while $z$ is of order $O(\bar{\lambda}^{1/2})$, so that the expression $(6)_4$ of the quantity $T$ becomes

$$T \sim \frac{x^2}{\bar{a}^4} + \frac{y^2}{\bar{b}^4} + \frac{z^2}{\bar{\lambda}^2} \sim \frac{z^2}{\bar{\lambda}^2} \quad \left(= O\left(\frac{1}{\bar{\lambda}}\right)\right). \tag{39}$$

The next task is to derive the asymptotic expressions of the diagonal components of the tensor $\mathbf{D}^0(\lambda)$. The coordinates $\bar{\lambda}, \bar{\mu}, \bar{\nu}$ being used instead of $\lambda, \mu, \nu$, the general expressions $(9)_{1,2,3}$ of $D^0_{xx}(\lambda), D^0_{yy}(\lambda), D^0_{zz}(\lambda)$ become in the limit $\bar{\lambda} \to 0$:

$$\begin{cases} D^0_{xx}(\lambda) \sim \displaystyle\int_0^{+\infty} \frac{d\bar{\rho}}{2\sqrt{(\bar{a}^2+\bar{\rho})^3\left(\bar{b}^2+\bar{\rho}\right)\bar{\rho}}} \\ D^0_{yy}(\lambda) \sim \displaystyle\int_0^{+\infty} \frac{d\bar{\rho}}{2\sqrt{(\bar{a}^2+\bar{\rho})\left(\bar{b}^2+\bar{\rho}\right)^3\bar{\rho}}} \\ D^0_{zz}(\lambda) \sim \displaystyle\int_0^{+\infty} \frac{d\bar{\rho}}{2\sqrt{(\bar{a}^2+\bar{\rho})\left(\bar{b}^2+\bar{\rho}\right)\bar{\rho}^3}} \end{cases}$$

where equation (4) has been used. The first two integrals here are provided by formulae (3.133.18) and (3.133.12) of Gradshteyn and Ryzhik (1980), and the third is obviously divergent at the origin. One thus gets

$$\lim_{\bar{\lambda}\to 0} D^0_{xx}(\lambda) = \frac{K'-E'}{k'^2\bar{a}^3} \quad ; \quad \lim_{\bar{\lambda}\to 0} D^0_{yy}(\lambda) = \frac{E'/k^2-K'}{k'^2\bar{a}^3} \quad ; \quad \lim_{\bar{\lambda}\to 0} D^0_{zz}(\lambda) = +\infty \tag{40}$$

where $E'$ and $K'$ denote classical complete elliptic integrals:

$$E' \equiv E(k') \equiv \int_0^{\pi/2} \sqrt{1-k'^2\sin^2\phi}\; d\phi \quad ; \quad K' \equiv K(k') \equiv \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k'^2\sin^2\phi}}. \tag{41}$$

It now becomes possible to evaluate the limits $L_x$, $L_y$ and $L_z$ defined by equations $(37)_{1,2,3}$. By equations $(38)_{1,2,4}$ and (39), the quantities $x^2/((a^2+\lambda)^2 Tv(\lambda)) \sim x^2/(\bar{a}^4 Tv(\lambda))$ and $y^2/((b^2+\lambda)^2 Tv(\lambda)) \sim y^2/(\bar{b}^4 Tv(\lambda))$ are both of order $O(\bar{\lambda}^{1/2})$ for $\bar{\lambda} \to 0$, so that they can be neglected with respect to $D^0_{xx}(\lambda)$ and $D^0_{yy}(\lambda)$ in the general expressions $(30)_{1,2}$ of $d^0_{xx}(\lambda)$ and $d^0_{yy}(\lambda)$ which become

$$d^0_{xx}(\mathbf{r}) \sim D^0_{xx}(\lambda) \quad ; \quad d^0_{yy}(\mathbf{r}) \sim D^0_{yy}(\lambda) \qquad \text{for } \bar{\lambda} \to 0. \tag{42}$$

The components $d^0_{xx}(\mathbf{r})$ and $d^0_{yy}(\mathbf{r})$ thus become asymptotically constant over the ellipsoid $\mathcal{E}_\lambda$ in the limit $\bar{\lambda} \to 0$. It then follows from equations $(37)_{1,2}$, $(40)_{1,2}$ and (42) that

$$L_x = \lim_{\bar{\lambda}\to 0} D^0_{xx}(\lambda) = \frac{K'-E'}{k'^2\bar{a}^3} \quad ; \quad L_y = \lim_{\bar{\lambda}\to 0} D^0_{yy}(\lambda) = \frac{E'/k^2-K'}{k'^2\bar{a}^3}. \tag{43}$$

The limit $L_z$ defined by equation $(37)_3$ cannot be evaluated in a similar way because the two terms in the right-hand side of equation $(30)_3$ can be checked to be of the same order. The simplest solution here consists in using the incompressibility of the velocity field $\mathbf{v}^0(\mathbf{r})$, which implies that $d^0_{zz}(\mathbf{r}) = -d^0_{xx}(\mathbf{r}) - d^0_{yy}(\mathbf{r})$ and therefore, by equations (43), that

$$L_z = -L_x - L_y = -\frac{E'}{k^2\bar{a}^3}. \tag{44}$$

(Note that $L_z \neq \lim_{\bar{\lambda}\to 0} D^0_{zz}(\lambda) = +\infty$).

To finally evaluate the limit $L^2$ defined by equation $(37)_4$, one must calculate the limits of the average values $\langle d^{0\,2}_{xx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, $\langle d^{0\,2}_{yy}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, $\langle d^{0\,2}_{zz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, $\langle d^{0\,2}_{xy}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, $\langle d^{0\,2}_{yz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, $\langle d^{0\,2}_{zx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$

for $\bar{\lambda} \to 0$. From the fact that in this limit $d^0_{xx}(\mathbf{r})$, $d^0_{yy}(\mathbf{r})$ and $d^0_{zz}(\mathbf{r}) = -d^0_{xx}(\mathbf{r}) - d^0_{yy}(\mathbf{r})$ become asymptotically constant and equal to $L_x$, $L_y$ and $L_z = -L_x - L_y$ over the surface $\mathcal{E}_\lambda$, it follows that

$$\lim_{\bar{\lambda}\to 0} \langle d^{0\ 2}_{xx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = L_x^2 \quad ; \quad \lim_{\bar{\lambda}\to 0} \langle d^{0\ 2}_{yy}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = L_y^2 \quad ; \quad \lim_{\bar{\lambda}\to 0} \langle d^{0\ 2}_{zz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = L_z^2. \tag{45}$$

The limits of the three remaining average values may be computed by using equation (26) to write the integrals explicitly with the ellipsoidal coordinates $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$, and using various changes of variables and formulae of Gradshteyn and Ryzhik (1980). When one calculates the limits of $\langle d^{0\ 2}_{yz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ and $\langle d^{0\ 2}_{zx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, all complete elliptic integrals $E \equiv E(k)$, $K \equiv K(k)$, $E' \equiv E(k')$, $K' \equiv K(k')$ momentarily appear but finally cancel out upon use of Gradshteyn and Ryzhik (1980)'s formula (8.122) connecting them all. The very simple final results are as follows:

$$\lim_{\bar{\lambda}\to 0} \langle d^{0\ 2}_{xy}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = 0 \quad ; \quad \lim_{\bar{\lambda}\to 0} \langle d^{0\ 2}_{yz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = \frac{1}{k^4 \bar{a}^6} \quad ; \quad \lim_{\bar{\lambda}\to 0} \langle d^{0\ 2}_{zx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = \frac{1}{k^2 \bar{a}^6}. \tag{46}$$

Gathering equations $(37)_4$ and (43 - 46), one gets upon a straightforward calculation:

$$L^2 = \frac{4}{3\bar{a}^6}\left\{\frac{1}{k'^4}\left[\left(1 - \frac{1}{k^2} + \frac{1}{k^4}\right) E'^2 + K'^2 - \left(1 + \frac{1}{k^2}\right) E'K'\right] + \frac{1}{k^2} + \frac{1}{k^4}\right\}. \tag{47}$$

The conclusion is that *in the generic case where $a > b > c$, all limits $L_x, L_y, L_z, L^2$ defined by equations (37) are finite.*

### *5.3 Prolate spheroidal case*

The prolate spheroidal case where $a > b = c$ is special for both mathematical and physical reasons. From the mathematical viewpoint, the calculations presented above, based on the use of ellipsoidal coordinates, become invalid since the representation of 3D space by such coordinates breaks down, as is clear from the presence of the factor $b^2 - c^2 = 0$ in the denominators of the expressions $(3)_{2,3}$ of $y$ and $z$. From the physical viewpoint, the values of the limits looked for are intimately tied to the geometric properties of the completely flat confocal ellipsoid; now this ellipsoid, which is a flat elliptic disk in the generic case, becomes a straight segment in the prolate spheroidal case, which is obviously a very different geometric object.

It is important to note that although the coordinates $\mu, \nu$ can no longer be used, the coordinate $\lambda$ (or $\bar{\lambda}$ defined by equation $(36)_1$) still makes sense, as a definer of the semi-axes $\sqrt{a^2+\lambda}$, $\sqrt{b^2+\lambda} = \sqrt{c^2+\lambda}$ of the spheroid $\mathcal{E}_\lambda$. The definitions (37) of the limits $L_x, L_y, L_z, L^2$ also still make sense.

In the limit $\bar{\lambda} \to 0$, the expression (4) of $v(\lambda) \equiv v$ becomes

$$v \sim \bar{a}\bar{\lambda}. \tag{48}$$

Also, equation $(3)_4$ becomes $x^2/\bar{a}^2 + (y^2+z^2)/\bar{\lambda} = 1$, which implies that $x^2$ is of order $O(\bar{\lambda}^0 = 1)$, while $y^2+z^2$ is of order $O(\bar{\lambda}^1 = \bar{\lambda})$. Hence the expression $(6)_4$ of $T$ becomes

$$T \sim \frac{x^2}{\bar{a}^4} + \frac{y^2+z^2}{\bar{\lambda}^2} \sim \frac{y^2+z^2}{\bar{\lambda}^2} \quad \left(= O\left(\frac{1}{\bar{\lambda}}\right)\right). \tag{49}$$

With regard to the asymptotic expressions of the components of the tensor $\mathbf{D}^0(\lambda)$, the expression $(9)_1$ of $D^0_{xx}(\lambda)$ reads in the case considered

$$D^0_{xx}(\lambda) = \int_{\bar{\lambda}}^{+\infty} \frac{d\bar{\rho}}{2\left(\bar{a}^2+\bar{\rho}\right)^{3/2}\bar{\rho}},$$

which obviously implies that

$$D^0_{xx}(\lambda) \sim -\frac{\ln\bar{\lambda}}{2\bar{a}^3} \sim -\frac{\ln v}{2\bar{a}^3} \qquad \text{for } v \to 0 \tag{50}$$

where equation (48) has been used. Also, equations (10) and (50), combined with the rotational symmetry about the direction $x$, imply that

$$D^0_{yy}(\lambda) = D^0_{zz}(\lambda) = \frac{1}{2}\left[\frac{1}{v} - D^0_{xx}(\lambda)\right] \sim \frac{1}{2v} \qquad \text{for } v \to 0. \tag{51}$$

Hence $D^0_{xx}(\lambda)$ is negligible with respect to $D^0_{yy}(\lambda) = D^0_{zz}(\lambda)$ in the limit $v \to 0$.

Now the expression $(30)_1$ of $d^0_{xx}(\mathbf{r})$ becomes in the limit $\bar{\lambda} \to 0$

$$d^0_{xx}(\mathbf{r}) \sim D^0_{xx}(\lambda) - \frac{x^2}{\bar{a}^4 T v}$$

and it then follows from the fact that $x^2 = O(1)$ plus equations (48), (49) and (50) that

$$d^0_{xx}(\mathbf{r}) \sim D^0_{xx}(\lambda) \sim -\frac{\ln v}{2\bar{a}^3} \qquad \text{for } v \to 0. \tag{52}$$

Thus $d^0_{xx}(\mathbf{r})$ again becomes asymptotically constant and equal to $D^0_{xx}(\lambda)$ over the surface $\mathcal{E}_\lambda$ in the limit $v \to 0$. This implies that

$$\langle d^0_{xx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \sim -\frac{\ln v}{2\bar{a}^3} \qquad \text{for } v \to 0. \tag{53}$$

The asymptotic expressions of the other diagonal components of the tensor $\langle \mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ are then trivially deduced from the fact that it is traceless, combined with the rotational symmetry:

$$\langle d^0_{yy}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = \langle d^0_{zz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = -\frac{1}{2}\langle d^0_{xx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \sim \frac{\ln v}{4\bar{a}^3} \qquad \text{for } v \to 0. \tag{54}$$

It only remains to determine the asymptotic expressions of the average values $\langle {d^0_{xx}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, $\langle {d^0_{yy}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = \langle {d^0_{zz}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, $\langle {d^0_{xy}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = \langle {d^0_{xz}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ and $\langle {d^0_{yz}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ to get that of $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$.

It first follows from equation (52) that

$$\langle d^{0\,2}_{xx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \sim \frac{(\ln v)^2}{4\bar{a}^6} \qquad \text{for } v \to 0. \tag{55}$$

Also, by equations $(30)_{2,3,5}$, (48), (49) and (51),

$$\begin{aligned} d^{0\,2}_{yy}(\mathbf{r}) + d^{0\,2}_{zz}(\mathbf{r}) + 2d^{0\,2}_{yz}(\mathbf{r}) &= \left(D^0_{yy}(\lambda) - \frac{y^2}{\bar{\lambda}^2 T v}\right)^2 + \left(D^0_{zz}(\lambda) - \frac{z^2}{\bar{\lambda}^2 T v}\right)^2 + 2\left(\frac{yz}{\bar{\lambda}^2 T v}\right)^2 \\ &\sim \frac{1}{v^2}\left[\left(\frac{1}{2} - \frac{y^2}{y^2+z^2}\right)^2 + \left(\frac{1}{2} - \frac{z^2}{y^2+z^2}\right)^2 + 2\left(\frac{yz}{y^2+z^2}\right)^2\right] \\ &= \frac{1}{2v^2} \end{aligned}$$

so that

$$\langle d^{0\,2}_{yy}(\mathbf{r}) + d^{0\,2}_{zz}(\mathbf{r}) + 2d^{0\,2}_{yz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \sim \frac{1}{2v^2} \qquad \text{for } v \to 0. \tag{56}$$

Finally, by equations $(30)_{4,6}$, (48) and (49),

$$d^{0\,2}_{xy}(\mathbf{r}) + d^{0\,2}_{xz}(\mathbf{r}) = \left(\frac{xy}{\bar{a}^2\bar{\lambda}Tv}\right)^2 + \left(\frac{xz}{\bar{a}^2\bar{\lambda}Tv}\right)^2 \sim \frac{x^2}{\bar{a}^6(y^2+z^2)} = O\left(\frac{1}{v}\right)$$

since $x^2$ and $y^2+z^2$ are of order $O(1)$ and $O(v)$ respectively, and it follows that

$$\langle d^{0\,2}_{xy}(\mathbf{r}) + d^{0\,2}_{xz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = O\left(\frac{1}{v}\right) \qquad \text{for } v \to 0. \tag{57}$$

Combination of equations (55), (56) and (57) implies that

$$\langle d^{0\,2}_{eq}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \sim \frac{1}{3v^2} \qquad \text{for } v \to 0. \tag{58}$$

This result coincides with that derived by Gologanu *et al.* (1993) for the velocity field $\mathbf{v}^0(\mathbf{r})$ in question, account being taken of the different notations used.

In conclusion, equations (53), (54) and (58) show that *in the prolate spheroidal case where $a > b = c$, all limits $L_x$, $L_y$, $L_z$, $L^2$ defined by equations (37) are infinite.*

### *5.4 Oblate spheroidal case*

From a mathematical viewpoint, the oblate spheroidal case where $a = b > c$ seems just as special as the prolate spheroidal case, the representation of 3D space by ellipsoidal coordinates becoming again invalid because of the presence of the term $a^2 - b^2 = 0$ in the denominators of the expressions $(3)_{1,2}$ of $x$ and $y$. From a physical viewpoint, however, one may expect that nothing particular will in fact occur in this case. Indeed the completely flat confocal ellipsoid, which is an elliptic disk in the generic case, simply becomes a circular disk in the oblate spheroidal case. Now one may fix one semi-axis of this disk, say $\bar{a}$, and vary the other one, $\bar{b}$, from values smaller than $\bar{a}$ to values larger than it, with clearly nothing special occurring when $\bar{a} = \bar{b}$.

This means that one may expect the limits $L_x$, $L_y$, $L_z$, $L^2$ provided by formulae (43), (44) and (47) in the generic case to simply go to finite values in the oblate spheroidal case, in spite of the presence of the diverging terms $1/k'^2$ and $1/k'^4$ in these formulae. (Recall that $k' = 0$ in the oblate spheroidal case). Indeed, this is fully confirmed by a calculation of the limits of $L_x$, $L_y$, $L_z$, $L^2$ for $k' \to 0$, based on Gradshteyn and Ryzhik (1980)'s asymptotic expressions (8.113.1) and (8.114.1) of the elliptic integrals $E'$ and $K'$. The values of $L_x$, $L_y$, $L_z$, $L^2$ for $k' = 0$ are thus found to be

$$L_x = L_y = \frac{\pi}{4\bar{a}^3} \quad ; \quad L_z = -\frac{\pi}{2\bar{a}^3} \quad ; \quad L^2 = \frac{3\pi^2 + 32}{12\bar{a}^6}. \tag{59}$$

Again, these results coincide with those found by Gologanu *et al.* (1994) for the velocity field $\mathbf{v}^0(\mathbf{r})$ in question, with the necessary changes of notation.

One may thus conclude that *in the oblate spheroidal case where $a = b > c$, the limits $L_x$, $L_y$, $L_z$, $L^2$ are finite just like in the generic case.*

### 5.5 Spherical case

The spherical case where $a = b = c$ is special for both mathematical and physical reasons: the system of ellipsoidal coordinates once again breaks down, and the completely flat confocal ellipsoid becomes a mere point. The results looked for are easily found by a direct reasoning analogous to that pertaining to ellipsoids $\mathcal{E}_\lambda$ of very large dimensions, presented at the end of Subsection 5.1; the only difference being that here formulae are exact instead of being just asymptotic. One thus gets

$$\langle d^0_{xx}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = \langle d^0_{yy}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = \langle d^0_{zz}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = 0 \quad ; \quad \langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = \frac{4}{9v^2} \tag{60}$$

for all values of $r$ or $v = r^3$.

Thus, *in the spherical case where $a = b = c$, the limits $L_x$, $L_y$, $L_z$ are zero whereas $L^2$ is infinite.*

### 5.6 Comments

Some qualitative comments are in order with regard to the asymptotic behavior of the average value $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ when the surface $\mathcal{E}_\lambda$ gets close to the completely flat confocal ellipsoid.

With approximation $\mathcal{A}_1$, the quantity $\sqrt{\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}}$ represents the average value over the ellipsoidal surface $\mathcal{E}_\lambda$ of the plastic dissipation associated to the trial velocity field $\mathbf{v}^0(\mathbf{r})$ used to represent the expansion of a confocal ellipsoidal void contained in $\mathcal{E}_\lambda$. Thus its asymptotic behavior for surfaces $\mathcal{E}_\lambda$ close to the completely flat confocal ellipsoid is connected to the value of the plastic dissipation necessary to open a void coinciding with the interior of this ellipsoid.

Now this void is an elliptic crack in the generic case, a needle in the prolate spheroidal case, a circular crack in the oblate spheroidal case, and a mere point in the spherical case. Thus equations (47) and $(59)_3$ imply that opening an elliptic or circular crack only requires a locally finite plastic dissipation, whereas equations (58) and $(60)_2$ imply that opening a needle- or point-shaped void requires a locally infinite dissipation. In other words, these equations mean, quite reasonably, that *it is much easier to open an elliptic or circular crack than a needle- or point-shaped void.*

## 6 Approximate overall plastic dissipation and yield function

We shall now use the results of the preceding Section to propose a reasonable approximate expression of the average value $\langle d_{eq}^{0\;2}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, allowing for an analytic calculation of the integral in equation $(27)_2$ defining $\Pi^{++}(\mathbf{D})$ and the associated approximate yield function.

### *6.1 Simplification of the "crossed term" in the expression of $\langle d_{eq}^{0\;2}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$*

It has already been noted that in the "crossed term" $\frac{4}{3}\mathcal{A}\,\langle\mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda} : \mathbf{\Delta}$ of the expression (33) of $\langle d_{eq}^{0\;2}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, the off-diagonal components of the tensor $\langle\mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ make no contribution. It follows that

$$\langle\mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda} : \mathbf{\Delta} = \langle\mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}^{\mathrm{dg}} \,.\, \mathbf{\Delta}^{\mathrm{dg}} \tag{61}$$

where, for any symmetric second-rank tensor $\mathbf{T}$, $\mathbf{T}^{\mathrm{dg}} \in \mathbb{R}^3$ denotes the vector made from its diagonal components:

$$\mathbf{T}^{\mathrm{dg}} \equiv \begin{pmatrix} T_{xx} \\ T_{yy} \\ T_{zz} \end{pmatrix}. \tag{62}$$

Now let $\mathcal{P} \subset \mathbb{R}^3$ denote the plane consisting of vectors whose components have a zero sum. Both vectors $\langle\mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}^{\mathrm{dg}}$ and $\mathbf{\Delta}^{\mathrm{dg}}$ are in $\mathcal{P}$ since both tensors $\mathbf{d}^0(\mathbf{r})$ and $\mathbf{\Delta}$ are traceless. Also, let $(\mathbf{U},\mathbf{V})$ denote the orthonormal basis of $\mathcal{P}$ consisting of the vectors [3]

$$\mathbf{U} \equiv \frac{1}{\sqrt{L_x^2+L_y^2+L_z^2}}\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} \quad ; \quad \mathbf{V} \equiv \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \mathbf{U} = \frac{1}{\sqrt{3(L_x^2+L_y^2+L_z^2)}}\begin{pmatrix} L_z - L_y \\ L_x - L_z \\ L_y - L_x \end{pmatrix}. \tag{63}$$

[3] It can be checked that $\mathbf{U}$ and $\mathbf{V}$ are well-defined even in the prolate spheroidal case where $L_x$, $L_y$, $L_z$ are infinite; this is tied to the fact that $\mathbf{U}$ gives the direction of the vector $\langle\mathbf{d}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}^{\mathrm{dg}}$ in $\mathbb{R}^3$ in the limit $v \to 0$, and that in this special case this direction is fixed and independent of $v$ for symmetry reasons ($\langle d_{yy}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}^{\mathrm{dg}} = \langle d_{zz}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}^{\mathrm{dg}} = -\frac{1}{2}\langle d_{xx}^0(\mathbf{r})\rangle_{\mathcal{E}_\lambda}^{\mathrm{dg}}$). In the spherical case $\mathbf{U}$ and $\mathbf{V}$ are ill-defined but may harmlessly be considered as zero.

This basis may be used to express the vector $\mathbf{\Delta}^{\rm dg}$ :

$$\mathbf{\Delta}^{\rm dg} \equiv \mathcal{B}\mathbf{U} + \mathcal{C}\mathbf{V} \quad , \quad \mathcal{B} \equiv \mathbf{\Delta}^{\rm dg}.\mathbf{U} \quad , \quad \mathcal{C} \equiv \mathbf{\Delta}^{\rm dg}.\mathbf{V}; \tag{64}$$

then

$$\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\,\mathbf{\Delta}^{\rm dg} = \mathcal{B}\,\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{U} + \mathcal{C}\,\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{V}$$

so that, by equations (33) and (61),

$$\langle d^2_{eq}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} = \mathcal{A}^2\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} + \Delta^2_{eq} + \frac{4}{3}\mathcal{A}\mathcal{B}\,\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{U} + \frac{4}{3}\mathcal{A}\mathcal{C}\,\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{V}. \tag{65}$$

Equation (65) shows that there are in fact *two* crossed terms in the expression of $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$, proportional to the components $\mathcal{B}$ and $\mathcal{C}$ of the vector $\mathbf{\Delta}^{\rm dg}$ in the plane $\mathcal{P}$. But the basis $(\mathbf{U},\mathbf{V})$ of the plane $\mathcal{P}$ has precisely been defined so as to allow for the following approximation:

$\mathcal{A}_2$: *The term* $\frac{4}{3}\mathcal{A}\mathcal{C}\,\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{V}$ *may be discarded in the expression (65) of* $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$.

Some explanatory comments are in order here. An indispensable preliminary remark is that in an arbitrary quadratic form $A_{11}X_1^2 + A_{22}X_2^2 + 2A_{12}X_1X_2$ with positive coefficients $A_{11}$ and $A_{22}$, the "crossed term" $2A_{12}X_1X_2$ is negligible for all values of the arguments $X_1, X_2$ if and only if $A_{12}^2 \ll A_{11}A_{22}$. (The easy proof is left to the reader). Applied to the terms proportional to $\mathcal{A}^2$, $\mathcal{C}^2$ and $\mathcal{A}\mathcal{C}$ in the quadratic form (65) defining $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$,[4] this property implies that the term $\frac{4}{3}\mathcal{A}\mathcal{C}\,\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{V}$ is negligible if and only if

$$\left[\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{V}\right]^2 \ll \langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}. \tag{66}$$

One may then note that:

- For surfaces $\mathcal{E}_\lambda$ of large dimensions, the left-hand side of condition (66) is $o(1/v^2)$ by equations (35), whereas the right-hand side is $O(1/v^2)$ by equation (34); hence condition (66) is satisfied.
- For surfaces $\mathcal{E}_\lambda$ close to the completely flat confocal ellipsoid, the vector $\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}$ goes to a limit collinear to the vector $\mathbf{U}$, by the very definition $(63)_1$ of the latter vector; since $\mathbf{U}$ and $\mathbf{V}$ are orthogonal, the left-hand side of condition (66) goes to zero. In contrast, the right-hand side goes to some nonzero finite or infinite limit, depending on the case considered (see equations $(37)_4$, (47), (58), $(59)_3$ and $(60)_2$). Hence condition (66) is again fulfilled.
- Condition (66) being thus satisfied when the surface $\mathcal{E}_\lambda$ is either distant from or close to the completely flat confocal ellipsoid, should be approximately satisfied for all positions of this surface.
- In addition, in the spheroidal, prolate and oblate cases, the left-hand side of (66) is exactly zero. Indeed the direction of the vector $\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}$ in $\mathbb{R}^3$ is independent of $v$ in both cases for symmetry reasons ($\langle d^0_{yy}(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda} = \langle d^0_{zz}(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda} = -\frac{1}{2}\langle d^0_{xx}(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}$ in the prolate

[4] The term in $\mathcal{C}^2$ in this quadratic form arises from the term $\Delta^2_{eq}$.

case, $\langle d^0_{xx}(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda} = \langle d^0_{yy}(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda} = -\frac{1}{2}\langle d^0_{zz}(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}$ in the oblate case); and it follows that this vector is orthogonal to $\mathbf{V}$ not only in the limit $v \to 0$, but for all values of $v$.

Hence condition (66) should quite generally be met, implying that the error resulting from approximation $\mathcal{A}_2$ should be small.

With this approximation, equation (65) becomes

$$\langle d^2_{eq}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \simeq \mathcal{A}^2\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} + \Delta^2_{eq} + \frac{4}{3}\mathcal{A}\mathcal{B}\,\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{U}. \tag{67}$$

### 6.2 *Simplification of the spatial dependence of* $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$

With respect to Gologanu *et al.* (1993, 1994)'s treatment of the prolate and oblate spheroidal cases, approximation $\mathcal{A}_2$ is new. (There was no need in the spheroidal cases to disregard the term $\frac{4}{3}\mathcal{A}\mathcal{C}\,\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{V}$ which was zero anyway). In contrast, we shall essentially follow from now on Gologanu *et al.* (1994)'s treatment of the oblate spheroidal case; the transition to the general ellipsoidal case will not introduce any further major novelty.

Denoting by $L$ the square root of the limit defined by equation $(37)_4$, we define the variable

$$w \equiv \frac{1}{3v/2 + \chi/L} \tag{68}$$

where $\chi$ is a dimensionless constant of order unity. The advantage of introducing such a change of variable, with such an adjustable constant, will appear below. Note that when $v$ varies from 0 to $+\infty$, $w$ varies from $L/\chi$ to 0.

We then write the quantities $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ and $\frac{4}{3}\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{U}$ in the following form, which defines the functions $F(w)$ and $G(w)$:

$$\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \equiv F^2(w)w^2 \quad ; \quad \frac{4}{3}\langle \mathbf{d}^0(\mathbf{r})\rangle^{\rm dg}_{\mathcal{E}_\lambda}.\mathbf{U} \equiv 2F(w)G(w)w^2. \tag{69}$$

Also, we note that by equation$(64)_1$, since the basis $(\mathbf{U},\mathbf{V})$ is orthonormal,

$$\Delta^2_{eq} = \frac{2}{3}\,\boldsymbol{\Delta}^{\rm dg}\,.\,\boldsymbol{\Delta}^{\rm dg} + \frac{4}{3}\left(\Delta^2_{xy} + \Delta^2_{yz} + \Delta^2_{zx}\right) = \frac{2}{3}\left(\mathcal{B}^2 + \mathcal{C}^2\right) + \frac{4}{3}\left(\Delta^2_{xy} + \Delta^2_{yz} + \Delta^2_{zx}\right).$$

It follows that with the notations just introduced, equation (67) becomes

$$\langle d^2_{eq}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \simeq \mathcal{A}^2F^2(w)w^2 + \frac{2\mathcal{B}^2}{3} + 2\mathcal{A}\mathcal{B}F(w)G(w)w^2 + \frac{2\mathcal{C}^2}{3} + \frac{4}{3}\left(\Delta^2_{xy} + \Delta^2_{yz} + \Delta^2_{zx}\right),$$

which can be rewritten in the form

$$\langle d^2_{eq}(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \simeq \left[\mathcal{A}F(w) + \mathcal{B}G(w)\right]^2 w^2 + \frac{2}{3}\mathcal{B}^2H^2(w) + \frac{2\mathcal{C}^2}{3} + \frac{4}{3}\left(\Delta^2_{xy} + \Delta^2_{yz} + \Delta^2_{zx}\right) \tag{70}$$

where

$$H^2(w) \equiv 1 - \frac{3}{2}G^2(w)w^2. \tag{71}$$

(The quantity $H^2(w)$ thus defined is positive since the quadratic form of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\Delta_{xy}$, $\Delta_{yz}$, $\Delta_{zx}$ defined by equation (70) is obviously positive-definite).

We then introduce the following final approximation:

$\mathcal{A}_3$: *The functions $F(w)$, $G(w)$, $H(w)$ in equation (70) may be replaced by suitable average values $\bar{F}$, $\bar{G}$, $\bar{H}$.*

Explanatory comments are again in order here.

► Consider first the function $F(w)$, defined by the expression $(69)_1$ of $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$. Near infinity ($v \to +\infty$), $w \sim \frac{2}{3v} \to 0$ by equation (68), so that equation $(69)_1$ becomes $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \sim \frac{4F^2(0)}{9v^2}$. Comparison with equation (34) then reveals that

$$F(0) = 1. \tag{72}$$

Consideration of the other limiting case ($v \to 0$) now requires to distinguish between the different geometrical cases.

- In the generic and oblate spheroidal cases, close to the completely flat confocal ellipsoid ($v \to 0$), $w$ goes to the limit $L/\chi$ so that $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ goes to the limit $F^2(L/\chi)\frac{L^2}{\chi^2}$, which must be equal to $L^2$ by equation $(37)_4$; it follows that

$$F(L/\chi) = \chi. \tag{73}$$

  Equations (72) and (73) show that when $w$ varies within its interval of definition $(0, L/\chi)$, $F(w)$ varies from 1 to $\chi$. Provided that $\chi$ is chosen of order unity, this variation is modest, which justifies the replacement of the function $F(w)$ by a constant.
- In the prolate spheroidal case, $L = +\infty$ so that equation (68) becomes $w = \frac{2}{3v}$ for all values of $v$. Therefore, close to the completely flat confocal ellipsoid ($v \to 0$), $w$ goes to $+\infty$ and $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \sim \frac{4F^2(+\infty)}{9v^2}$. Comparison with equation (58) then shows that

$$F(+\infty) = \frac{\sqrt{3}}{2}. \tag{74}$$

  Hence the function $F(w)$ again varies modestly over the interval of definition $(0, +\infty)$ of $w$, from 1 to $\sqrt{3}/2 \simeq 0.87$, so that it may again be replaced by a constant.
- In the spherical case, again, $w = \frac{2}{3v}$ since $L = +\infty$; equations $(60)_2$ and $(69)_1$ then imply that $F(w)$ is rigorously constant, equal to 1 for all values of $w$.

Note that these nice properties are intimately connected to the definition (68) of the new variable $w$, which justifies this definition except, momentarily, for the introduction of the adjustable constant $\chi$.

► Consider now the function $G(w)$, defined by the expression $(69)_2$ of $\frac{4}{3}\langle \mathbf{d}^0(\mathbf{r})\rangle^{\mathrm{dg}}_{\mathcal{E}_\lambda}.\mathbf{U}$. First, near infinity ($v \to +\infty$), note that by equation (72), $\frac{4}{3}\langle \mathbf{d}^0(\mathbf{r})\rangle^{\mathrm{dg}}_{\mathcal{E}_\lambda}.\mathbf{U} \sim 2G(w)w^2$.

Comparison with equations (35) then shows that

$$G(w)w^2 = o\left(\frac{1}{v}\right) = o(w) \qquad \text{for } w \to 0. \tag{75}$$

Therefore in this limit, $\left[\frac{4}{3}\langle \mathbf{d}^0(\mathbf{r})\rangle^{\mathrm{dg}}_{\mathcal{E}_\lambda}.\mathbf{U}\right]^2 \sim [2G(w)w^2]^2 = o(w^2)$ is much smaller than $\langle {d^0_{eq}}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \sim \frac{4}{9v^2} \sim w^2$. By the remark on quadratic forms made in Subsection 6.1, this implies that the crossed term proportional to $\frac{4}{3}\langle \mathbf{d}^0(\mathbf{r})\rangle^{\mathrm{dg}}_{\mathcal{E}_\lambda}.\mathbf{U}$ in the expression (67) of $\langle d^2_{eq}(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ is negligible near infinity. This property has been established for the true function $G(w)$, but it remains true if it is replaced by some constant $\bar{G}$, since again $\left(2\bar{G}w^2\right)^2 \ll w^2$ in the limit $w \to 0$. Hence, near infinity, replacing $G(w)$ by $\bar{G}$ just means replacing a negligible term by an equally negligible approximation, which is harmless whatever the value chosen for $\bar{G}$.

Close to the completely flat confocal ellipsoid, it is again necessary to distinguish between the different geometrical cases.

- In the generic and oblate spheroidal cases, in the limit $v \to 0$, equation $(69)_2$ yields $\frac{4}{3}\langle \mathbf{d}^0(\mathbf{r})\rangle^{\mathrm{dg}}_{\mathcal{E}_\lambda}.\mathbf{U} \to 2F(L/\chi)G(L/\chi)\frac{L^2}{\chi^2} = 2G(L/\chi)\frac{L^2}{\chi}$ where equation (73) has been used. Now by the definitions $(37)_{1,2,3}$ of $L_x$, $L_y$, $L_z$ and $(63)_1$ of $\mathbf{U}$, the value of this limit is $\frac{4}{3}\sqrt{L_x^2 + L_y^2 + L_z^2}$, and it follows that

$$G(L/\chi) = \frac{2\chi}{3L^2}\sqrt{L_x^2 + L_y^2 + L_z^2}. \tag{76}$$

  Thus replacing the function $G(w)$ by some constant is reasonable near the completely flat confocal ellipsoid provided that the chosen value of this constant is close to that given by equation (76).
- In the prolate spheroidal case, equations $(69)_2$, (53) and (54) yield, in the limit $v \to 0$, $\frac{4}{3}\langle \mathbf{d}^0(\mathbf{r})\rangle^{\mathrm{dg}}_{\mathcal{E}_\lambda}.\mathbf{U} \sim \sqrt{3}\,G(w)w^2 = o(\ln v) = o(\ln w)$ where equation (74) has been used, so that $G(w) = o\left(\frac{\ln w}{w^2}\right)$ and consequently

$$G(+\infty) = 0. \tag{77}$$

  Hence the replacement of the function $G(w)$ by $\bar{G}$ is again reasonable close to the completely flat confocal ellipsoid provided that $\bar{G}$ is taken to be zero or very small. [5]
- In the spherical case, the function $G(w)$ is uniformly zero by equation $(60)_1$ so that $\bar{G} = 0$ is again a good value.

► Consider finally the function $H(w)$, defined by equation (71). Near infinity, it follows from equation (75) that

$$G^2(w)w^2 = \left[G(w)w^2\right]^2 w^{-2} = o(1) \to 0 \qquad \text{for } w \to 0$$

so that

$$H(0) = 1. \tag{78}$$

[5] Gologanu *et al.* (1993) made the first choice, and Gologanu (1997) and Gologanu *et al.* (1997) the second.

On the other hand, on the completely flat confocal ellipsoid, equation (76) implies that

$$H(L/\chi) = \left[1 - \frac{3}{2}G^2\left(L/\chi\right)\left(\frac{L}{\chi}\right)^2\right]^{1/2} = \left(1 - \frac{2}{3}\frac{L_x^2 + L_y^2 + L_z^2}{L^2}\right)^{1/2}. \tag{79}$$

Using equations (43), (44) and (47), one may check that when $k$ varies from 0 (prolate spheroidal case) to 1 (oblate spheroidal case), the value of $H(L/\chi)$ given by equation (79) varies from 1 to $\sqrt{\frac{32}{3\pi^2+32}} \simeq 0.72$. Hence $H(L/\chi)$ is always rather close to unity. By equation (78), it follows that the variation of the function $H(w)$ over the interval of definition of $w$ is modest. Hence it is again safe to replace it by a constant.

This concludes the justification of approximation $\mathcal{A}_3$. With this approximation, equation (70) becomes

$$\langle d_{eq}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda} \simeq \left(\mathcal{A}\bar{F} + \mathcal{B}\bar{G}\right)^2 w^2 + \frac{2}{3}\mathcal{B}^2\bar{H}^2 + \frac{2\mathcal{C}^2}{3} + \frac{4}{3}\left(\Delta_{xy}^2 + \Delta_{yz}^2 + \Delta_{zx}^2\right). \tag{80}$$

### *6.3 Approximate yield function*

Since $dv = -\frac{2}{3}\frac{dw}{w^2}$ by equation (68), the expression $(27)_2$ of $\Pi^{++}(\mathbf{D})$ may be rewritten in the form

$$\Pi^{++}(\mathbf{D}) \simeq \frac{2\sigma_0}{3\Omega}\int_{w_{\min}}^{w_{\max}} \sqrt{\langle d_{eq}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}}\,\frac{dw}{w^2}\,, \tag{81}$$

where

$$w_{\min} \equiv w(v(\lambda = \Lambda)) = \frac{1}{3\Omega/2 + \chi/L} \quad ; \quad w_{\max} \equiv w(v(\lambda = 0)) = \frac{1}{3\omega/2 + \chi/L} \tag{82}$$

and $\langle d_{eq}^2(\mathbf{r})\rangle_{\mathcal{E}_\lambda}$ is given by equation (80).

From there, the derivation of the approximate yield function associated to the estimate $\Pi^{++}(\mathbf{D})$ of the overall plastic dissipation is based on the following result, established in Appendix B:

*Gurson's lemma:*[6] *consider the integral*

$$I(\alpha,\beta) \equiv \int_{u_1}^{u_2} \sqrt{\alpha^2 + \beta^2 u^2}\,\frac{du}{u^2} \tag{83}$$

*where $u_1$ and $u_2$ denote positive parameters. Then the derivatives $\partial I/\partial\alpha$ and $\partial I/\partial\beta$ are connected through the following relation independent of $\alpha$ and $\beta$:*

$$\left(\frac{\partial I}{\partial\alpha}\right)^2 + \frac{2}{u_1 u_2}\cosh\left(\frac{\partial I}{\partial\beta}\right) - \frac{1}{u_1^2} - \frac{1}{u_2^2} = 0. \tag{84}$$

[6] This conventional denomination overlooks the fact that the use of this result was only implicit in Gurson (1977)'s work.

The procedure involves four steps:

(1) Using Gurson's lemma with $\alpha \equiv \left(\frac{2}{3}\mathcal{B}^2\bar{H}^2 + \frac{2\mathcal{C}^2}{3} + \frac{4}{3}(\Delta_{xy}^2 + \Delta_{yz}^2 + \Delta_{zx}^2)\right)^{1/2}$, $\beta \equiv \mathcal{A}\bar{F} + \mathcal{B}\bar{G}$, $I(\alpha,\beta) \equiv \frac{3\Omega}{2\sigma_0}\Pi^{++}(\mathbf{D}) \equiv \frac{3\Omega}{2\sigma_0}\Pi^{++}(\alpha,\beta)$, $u \equiv w$, $u_1 \equiv w_{\min}$ and $u_2 \equiv w_{\max}$, write equation (84) in terms of the derivatives $\partial\Pi^{++}/\partial\alpha$, $\partial\Pi^{++}/\partial\beta$.

(2) Relate the derivatives $\partial\Pi^{++}/\partial\mathcal{A}$, $\partial\Pi^{++}/\partial\mathcal{B}$, $\partial\Pi^{++}/\partial\mathcal{C}$, $\partial\Pi^{++}/\partial\Delta_{xy}$, $\partial\Pi^{++}/\partial\Delta_{yz}$, $\partial\Pi^{++}/\partial\Delta_{zx}$ to the derivatives $\partial\Pi^{++}/\partial\alpha$, $\partial\Pi^{++}/\partial\beta$ using the expressions of $\alpha$ and $\beta$.

(3) Use the formulae found to express $(\partial\Pi^{++}/\partial\alpha)^2$ and $\partial\Pi^{++}/\partial\beta$ as linear and quadratic forms, respectively, of the derivatives $\partial\Pi^{++}/\partial\mathcal{A}$, $\partial\Pi^{++}/\partial\mathcal{B}$, $\partial\Pi^{++}/\partial\mathcal{C}$, $\partial\Pi^{++}/\partial\Delta_{xy}$, $\partial\Pi^{++}/\partial\Delta_{yz}$, $\partial\Pi^{++}/\partial\Delta_{zx}$, with coefficients independent of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\Delta_{xy}$, $\Delta_{yz}$, $\Delta_{zx}$. Rewrite equation (84) in terms of these derivatives.

(4) Use equations (20) and $(64)_1$ to express $\partial\Pi^{++}/\partial\mathcal{A}$, $\partial\Pi^{++}/\partial\mathcal{B}$, $\partial\Pi^{++}/\partial\mathcal{C}$, $\partial\Pi^{++}/\partial\Delta_{xy}$, $\partial\Pi^{++}/\partial\Delta_{yz}$, $\partial\Pi^{++}/\partial\Delta_{zx}$ in terms of the derivatives $\partial\Pi^{++}/\partial D_{ij}$, then in terms of the macroscopic stress components $\Sigma_{ij}$ using equation (28). Rewrite equation (84) in terms of these components.

The calculations involved are tedious but straightforward. The final output is the following Gurson-like approximate overall yield criterion of the representative cell considered:

$$\Phi(\mathbf{\Sigma}) \equiv \frac{\mathcal{Q}(\mathbf{\Sigma})}{\sigma_0^2} + 2(1+g)(f+g)\cosh\left[\frac{\mathcal{L}(\mathbf{\Sigma})}{\sigma_0}\right] - (1+g)^2 - (f+g)^2 = 0. \quad (85)$$

In this expression:

- The parameter $g$, which plays the role of a kind of "second porosity" (whence the notation), is given by

$$g \equiv \frac{2\chi}{3\Omega L}. \quad (86)$$

  Note that the constant $\chi$ appears in this formula; in Part II, advantage will be taken of the relative freedom of choice of this constant to select a value allowing for a simple, appealing geometric interpretation of the second porosity.
- $\mathcal{L}(\mathbf{\Sigma})$ is a linear form of the diagonal components of $\mathbf{\Sigma}$ given by

$$\mathcal{L}(\mathbf{\Sigma}) \equiv \kappa\Sigma_h\,, \quad \kappa \equiv \frac{3}{2\bar{F}}\,, \quad \Sigma_h \equiv H_x\Sigma_{xx} + H_y\Sigma_{yy} + H_z\Sigma_{zz}\,, \quad \begin{cases} H_x \equiv \Omega D^0_{xx}(\Lambda) \\ H_y \equiv \Omega D^0_{yy}(\Lambda) \\ H_z \equiv \Omega D^0_{zz}(\Lambda). \end{cases} \quad (87)$$

  Note that by equation (10), the coefficients $H_x$, $H_y$, $H_z$ here satisfy the equation

$$H_x + H_y + H_z = 1. \quad (88)$$

- $\mathcal{Q}(\mathbf{\Sigma})$ is a quadratic form of the components of $\mathbf{\Sigma}$ given by

$$\mathcal{Q}(\mathbf{\Sigma}) \equiv \frac{3}{2\bar{H}^2} \left\{ \left[ U_x - \frac{\bar{G}}{\bar{F}} D^0_{xx}(\Lambda) \right] \Sigma_{xx} + \left[ U_y - \frac{\bar{G}}{\bar{F}} D^0_{yy}(\Lambda) \right] \Sigma_{yy} + \left[ U_z - \frac{\bar{G}}{\bar{F}} D^0_{zz}(\Lambda) \right] \Sigma_{zz} \right\}^2 + \frac{3}{2} \left( V_x \Sigma_{xx} + V_y \Sigma_{yy} + V_z \Sigma_{zz} \right)^2 + 3 \left( \Sigma^2_{xy} + \Sigma^2_{yz} + \Sigma^2_{zx} \right). \tag{89}$$

Quite remarkably, the approximate yield criterion (85) for general ellipsoidal voids is of the same basic form as that in the GLD model for oblate spheroidal cavities (Gologanu *et al.*, 1994, 1997; Gologanu, 1997). Note however that expression (85) is not fully explicit yet since the constants $\chi$, $\bar{F}$, $\bar{G}$, $\bar{H}$ have not been ascribed precise values.

## 7 Conclusion

This work, which extends upon previous ones of Gurson (1977), Gologanu *et al.* (1993, 1994, 1997) and Gologanu (1997), represents a first step in the development of a Gurson-type model for plastic porous solids containing arbitrary ellipsoidal cavities. It was devoted to a limit-analysis of some representative elementary cell in such a medium, namely an ellipsoidal volume containing a confocal ellipsoidal void, and made of some rigid-ideal-plastic von Mises material. This cell was assumed to be loaded through conditions of homogeneous boundary strain rate (Mandel, 1964; Hill, 1967).

We first recalled the expression of the incompressible velocity fields recently discovered by Leblond and Gologanu (2008), satisfying conditions of homogeneous strain rate on an arbitrary family of confocal ellipsoids. We then explained the principle of a limit-analysis of the cell considered based on such a family of trial velocity fields.

The next step consisted in a thorough asymptotic study, both near infinity and near the origin, of the integrand appearing in the integral expression of the overall plastic dissipation, identical to some weighted average value of the local plastic dissipation over an arbitrary ellipsoid of the family considered. One notable finding was that this average value remains finite near the smallest, completely flat ellipsoid of the family when it is an elliptic or circular disk, that is in the generic and oblate spheroidal cases, but diverges to infinity when it becomes a needle or a point, that is in the prolate spheroidal and spherical cases. This is connected to the fact that it is obviously easier to open an elliptic or circular crack than a needle- or point-shaped void.

These results were finally used to define a few reasonable approximations leading to some simplified form of the integrand in the expression of the overall plastic dissipation. This allowed us to express this dissipation as an analytically calculable integral and derive from there an approximate, Gurson-like expression of the overall yield criterion of the cell considered. This yield criterion was found to be of the same basic form as that in the GLD model for oblate spheroidal cavities (Gologanu *et al.*, 1994, 1997; Gologanu, 1997).

It remains to ascribe precise expressions to all coefficients appearing in the yield criterion.

This will be the object of Part II.

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## A Appendix: derivation of Leblond and Gologanu (2008)'s family of velocity fields

This Appendix briefly recalls Leblond and Gologanu (2008)'s study of incompressible velocity fields satisfying conditions of homogeneous strain rate on confocal ellipsoids.

An arbitrary family of confocal ellipsoids is represented, in ellipsoidal coordinates, by the family of surfaces $\mathcal{E}_\lambda$ of constant $\lambda$. It follows that the velocity fields looked for must be of the form (7) for some family of symmetric second-rank tensors $\mathbf{D}(\lambda)$. Calculation of the divergence of such a field yields upon use of equations (6):

$$\begin{aligned}\operatorname{div}\mathbf{v}(\mathbf{r}) = \operatorname{tr}\mathbf{D}(\lambda) + \frac{2}{T}\Bigg[&\frac{dD_{xx}}{d\lambda}(\lambda)\frac{x^2}{a^2+\lambda} + \frac{dD_{yy}}{d\lambda}(\lambda)\frac{y^2}{b^2+\lambda} + \frac{dD_{zz}}{d\lambda}(\lambda)\frac{z^2}{c^2+\lambda} \\ &+\frac{dD_{xy}}{d\lambda}(\lambda)\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\right)xy + \frac{dD_{yz}}{d\lambda}(\lambda)\left(\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)yz \\ &+\frac{dD_{zx}}{d\lambda}(\lambda)\left(\frac{1}{c^2+\lambda}+\frac{1}{a^2+\lambda}\right)zx\Bigg].\end{aligned}$$

Writing the incompressibility condition in the form $T\operatorname{div}\mathbf{v}(\mathbf{r}) = 0$, accounting for the expression $(6)_4$ of $T$ and reordering terms, one gets

$$\begin{aligned}&\left[\frac{2}{a^2+\lambda}\frac{dD_{xx}}{d\lambda}(\lambda) + \frac{\operatorname{tr}\mathbf{D}(\lambda)}{(a^2+\lambda)^2}\right]x^2 + \left[\frac{2}{b^2+\lambda}\frac{dD_{yy}}{d\lambda}(\lambda) + \frac{\operatorname{tr}\mathbf{D}(\lambda)}{(b^2+\lambda)^2}\right]y^2 \\ &+\left[\frac{2}{c^2+\lambda}\frac{dD_{zz}}{d\lambda}(\lambda) + \frac{\operatorname{tr}\mathbf{D}(\lambda)}{(c^2+\lambda)^2}\right]z^2 + 2\frac{dD_{xy}}{d\lambda}(\lambda)\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\right)xy \\ &+2\frac{dD_{yz}}{d\lambda}(\lambda)\left(\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)yz + 2\frac{dD_{zx}}{d\lambda}(\lambda)\left(\frac{1}{c^2+\lambda}+\frac{1}{a^2+\lambda}\right)zx = 0.\end{aligned} \tag{A.1}$$

In this equation the triplet $(x, y, z)$ is tied to $\lambda$ through equation $(3)_4$. However, even if it is not, one can find a triplet of the form $(kx, ky, kz)$ satisfying equation $(3)_4$; this triplet must verify equation (A.1), and the homogeneity of this equation in $x$, $y$, $z$ implies that it must also be satisfied by the triplet $(x, y, z)$. Hence it is in fact verified for arbitrary triplets $(x, y, z)$, which implies that

$$\begin{cases} 2\dfrac{dD_{xx}}{d\lambda}(\lambda) + \dfrac{\operatorname{tr}\mathbf{D}(\lambda)}{a^2+\lambda} = 0 \\ 2\dfrac{dD_{yy}}{d\lambda}(\lambda) + \dfrac{\operatorname{tr}\mathbf{D}(\lambda)}{b^2+\lambda} = 0 \\ 2\dfrac{dD_{zz}}{d\lambda}(\lambda) + \dfrac{\operatorname{tr}\mathbf{D}(\lambda)}{c^2+\lambda} = 0 \end{cases} ; \quad \begin{cases} \dfrac{dD_{xy}}{d\lambda}(\lambda) = 0 \\ \dfrac{dD_{yz}}{d\lambda}(\lambda) = 0 \\ \dfrac{dD_{zx}}{d\lambda}(\lambda) = 0. \end{cases} \tag{A.2}$$

The solution of equations $(A.2)_{4,5,6}$ is obviously

$$\begin{cases} D_{xy}(\lambda) \equiv \Delta_{xy} \equiv Cst. \\ D_{yz}(\lambda) \equiv \Delta_{yz} \equiv Cst. \\ D_{zx}(\lambda) \equiv \Delta_{zx} \equiv Cst. \end{cases} \tag{A.3}$$

To solve equations $(A.2)_{1,2,3}$, the first step is to take their sum to get a differential equation on $\operatorname{tr}\mathbf{D}(\lambda)$ which is readily integrated into

$$\operatorname{tr}\mathbf{D}(\lambda) = \frac{\mathcal{A}}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} = \frac{\mathcal{A}}{v(\lambda)} \tag{A.4}$$

where $\mathcal{A}$ is an arbitrary constant. The second step is to reinsert this expression into equations $(A.2)_{1,2,3}$ and integrate to get the following expressions of $D_{xx}(\lambda)$, $D_{yy}(\lambda)$, $D_{zz}(\lambda)$:

$$\begin{cases} D_{xx}(\lambda) = \mathcal{A}\displaystyle\int_{\lambda}^{+\infty} \frac{d\rho}{2\,(a^2+\rho)\,v(\rho)} + \Delta_{xx} \\ D_{yy}(\lambda) = \mathcal{A}\displaystyle\int_{\lambda}^{+\infty} \frac{d\rho}{2\,(b^2+\rho)\,v(\rho)} + \Delta_{yy} \\ D_{zz}(\lambda) = \mathcal{A}\displaystyle\int_{\lambda}^{+\infty} \frac{d\rho}{2\,(c^2+\rho)\,v(\rho)} + \Delta_{zz} \end{cases} \tag{A.5}$$

where the constants $\Delta_{xx}$, $\Delta_{yy}$, $\Delta_{zz}$ are *a priori* arbitrary. But the expression of $\operatorname{tr}\mathbf{D}(\lambda)$ deduced from there,

$$\begin{aligned} \operatorname{tr}\mathbf{D}(\lambda) &= \mathcal{A}\int_{\lambda}^{+\infty} \left( \frac{1}{a^2+\rho} + \frac{1}{b^2+\rho} + \frac{1}{b^2+\rho} \right) \frac{d\rho}{2v(\rho)} + \Delta_{xx} + \Delta_{yy} + \Delta_{zz} \\ &= -\mathcal{A}\int_{\lambda}^{+\infty} \frac{d}{d\rho}\left(\frac{1}{v(\rho)}\right) d\rho + \Delta_{xx} + \Delta_{yy} + \Delta_{zz} = \frac{\mathcal{A}}{v(\lambda)} + \Delta_{xx} + \Delta_{yy} + \Delta_{zz}, \end{aligned}$$

must coincide with that given by equation (A.4), which implies that $\Delta_{xx} + \Delta_{yy} + \Delta_{zz}$ must be zero; hence the symmetric second-rank tensor $\mathbf{\Delta}$ of components $\Delta_{xx}$, $\Delta_{yy}$, $\Delta_{zz}$, $\Delta_{xy}$, $\Delta_{yz}$, $\Delta_{zx}$ must be traceless.

Gathering these elements, one gets the conclusions mentioned in Section 3 of the text.

An alternative method of construction of Leblond and Gologanu (2008)'s velocity fields, suggested to the authors by Kondo (2008), consists in adding to the solution for an ellipsoidal stress-free void in an infinite elastic matrix, that generated by a uniform "free strain" imposed in the void and adjusted so as to satisfy conditions of homogeneous strain on the outer boundary. (In essence, this idea was suggested in Chapter 6 of Monchiet (2006)'s thesis, in the special case of spheroidal voids).

## B Appendix: proof of Gurson's lemma

Differentiation of the expression (83) of the integral $I(\alpha, \beta)$ with respect to $\alpha$ and $\beta$ and calculation of the integrals yields

$$\begin{cases} \dfrac{\partial I}{\partial \alpha} = \displaystyle\int_{u_1}^{u_2} \frac{\alpha}{\sqrt{\alpha^2+\beta^2u^2}}\,\frac{du}{u^2} = \frac{1}{u_1}\sqrt{1+\frac{\beta^2u_1^2}{\alpha^2}} - \frac{1}{u_2}\sqrt{1+\frac{\beta^2u_2^2}{\alpha^2}} \\ \dfrac{\partial I}{\partial \beta} = \displaystyle\int_{u_1}^{u_2} \frac{\beta}{\sqrt{\alpha^2+\beta^2u^2}}\,du = \sinh^{-1}\left(\frac{\beta u_2}{\alpha}\right) - \sinh^{-1}\left(\frac{\beta u_1}{\alpha}\right). \end{cases}$$

It follows that

$$\begin{cases} \left(\dfrac{\partial I}{\partial \alpha}\right)^2 & = \dfrac{1}{u_1^2}\left(1+\dfrac{\beta^2 u_1^2}{\alpha^2}\right)+\dfrac{1}{u_2^2}\left(1+\dfrac{\beta^2 u_2^2}{\alpha^2}\right)-\dfrac{2}{u_1 u_2}\sqrt{1+\dfrac{\beta^2 u_1^2}{\alpha^2}}\sqrt{1+\dfrac{\beta^2 u_2^2}{\alpha^2}} \\ \cosh\left(\dfrac{\partial I}{\partial \beta}\right) & = \sqrt{1+\dfrac{\beta^2 u_1^2}{\alpha^2}}\sqrt{1+\dfrac{\beta^2 u_2^2}{\alpha^2}}-\dfrac{\beta^2 u_1 u_2}{\alpha^2}, \end{cases}$$

and the left-hand side of equation (84) is then trivially checked to be zero.