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Asymptotic Modeling of the Axisymmetric Flow of a Thermal Binary Gas Mixture in a Circular Micro-channel

Croizet Cédric^{1,a)} and Renée Gatignol¹

¹*Sorbonne Universités, UPMC Univ Paris 06, CNRS, UMR 7190, Institut Jean le Rond d'Alembert, F-75005, Paris, France*

^{a)}cedric.croizet@upmc.fr

Abstract. Sub-millimeter-sized channels are present in many medical and industrial tools such as micro-filters. In order to describe the gas flows in these micro-channels, the DSMC methods are frequently used but a large computation time is usually required to obtain the solutions [1, 2]. Consequently, it is of main interest to develop alternative methods to describe these flows. In this contribution, we are interested in flows at low Mach numbers and with low to moderate Knudsen numbers so that the flow is in the slip regime. We propose an asymptotic model for the axisymmetric flow of a mixture of two compressible gases in circular microchannels with a temperature gradient at the wall. The model is obtained from the Navier-Stokes-Fourier equations. The results of the model are compared to DSMC simulations and the influence of the temperature gradient which is present along the walls is investigated.

INTRODUCTION

Micro-metric devices are involved in technologies with broad potential application: they are present in various industrial manufacturing processes, medical tools and in measuring instruments. Consequently, a large number of works are devoted to the analysis of gas flows in micro-channels involved in these systems. Various approaches exist: theoretical, numerical and experimental [1, 2, 3, 4]. In a numerical point of view, the DSMC methods are very well adapted but they are expensive in computation time [1, 2]. Consequently, the development of asymptotic models describing these flows is an interesting alternative. This approach has been used in the case of a single gas in [4, 5] for instance and for a gas mixture in a plane channel [6] or in a cylindrical pipe [7]. In a previous work [8] the mixing length in a micro-channel of two parallel gas streams has been analyzed from DSMC simulations.

In this contribution, our purpose is to model the flow of a mixture of two compressible gases in a circular micro-channel. The geometry is showed in Fig. 1. The temperature of the channel wall depends on the longitudinal space variable z . The aim is to model these micro-flows with a macroscopic approach and to obtain an asymptotic model for describing the main physical phenomena. The two components are compressible. We are interested in flows at low Mach numbers and at low to moderate Knudsen numbers that corresponds to the slip-flow regime. Consequently, the movement equations are the usual macroscopic balance laws of mass, momentum and energy (Navier-Stokes-Fourier equations) with additional coupling terms in the momentum and energy equations [9, 10, 7]. In this weakly rarefied regime, the usual no-slip wall boundary conditions are replaced with slip conditions. In this paper, first order jump boundary conditions for the velocity and the temperature are written [4] at the walls.

The flow is assumed to be axisymmetric and steady. At the circular channel inlet, each gas has its own pressure and velocity. In the next section we present the model equations. For each gas, the balance equations are written with dimensional and dimensionless quantities. The characteristic velocities in each direction are identical for the two gases. A small parameter ϵ related to the geometry of the micro-pipe is built from the longitudinal and radial characteristic length scales. We introduce, for each gas, the Mach number and the Reynolds number built with the longitudinal length scale. The Knudsen number can be expressed in terms of these two dimensionless numbers. In the following sections, the solution at the first order in ϵ is investigated and compared to DSMC simulations in the same configuration. The last section is devoted to a short conclusion.

FIGURE 1. Channel geometry

MODEL EQUATIONS

The micro-channel has the length $L = 10 \mu\text{m}$ and the radius $R = 1 \mu\text{m}$ (Fig. 1). The channel wall at $\bar{r} = R$ is at rest and has a temperature depending on the longitudinal coordinate $\bar{T}_w(\bar{z})$. The movement equations are the Navier-Stokes-Fourier equations for a mixture of two ideal gases denoted with "a" and "b", with constant heat capacities c_{pa} and c_{pb} at constant pressure and c_{va} and c_{vb} at constant volume. The shear viscosities μ_a and μ_b and thermal conductivities κ_a and κ_b depend on the temperature only and are such as the Prandtl numbers are constant. The model equations are the usual balance equations with additional coupling terms. The "bars" are used for the quantities with dimension except when they are constant. For the species a , we have the three following balance equations for the mass, the momentum and the energy:

$$\frac{\partial \bar{\rho}_a}{\partial \bar{t}} + \bar{\nabla} \cdot (\bar{\rho}_a \bar{\mathbf{u}}_a) = 0, \quad (1)$$

$$\frac{\partial \bar{\rho}_a \bar{\mathbf{u}}_a}{\partial \bar{t}} + \bar{\nabla} \cdot (\bar{\rho}_a \bar{\mathbf{u}}_a \bar{\mathbf{u}}_a - \bar{\sigma}_a) = \bar{v}_{ab} \bar{\rho}_a (\bar{\mathbf{u}}_a - \bar{\mathbf{u}}_a^*), \quad (2)$$

$$\frac{\partial}{\partial \bar{t}} (\bar{\rho}_a \bar{\mathbf{u}}_a^2 + \bar{\rho}_a \bar{e}_a) + \bar{\nabla} \cdot \left[\bar{\rho}_a \left(\frac{1}{2} \bar{\mathbf{u}}_a \cdot \bar{\mathbf{u}}_a + \bar{e}_a \right) \bar{\mathbf{u}}_a - \bar{\sigma}_a \cdot \bar{\mathbf{u}}_a + \bar{\mathbf{q}}_a \right] = \bar{S}_a^E. \quad (3)$$

The equations for the species b are obtained by replacing a with b and vice versa. We introduce the notation "θ" ($\theta = a, b$) in order to denote one of the species. Because of the axisymmetry, the velocity field of the gas "θ" ($\theta = a, b$) is: $\bar{\mathbf{u}}_\theta = \bar{u}_\theta(\bar{r}, \bar{z}) \mathbf{e}_r + \bar{v}_\theta(\bar{r}, \bar{z}) \mathbf{e}_z$. As well, ρ_θ is the volumetric mass, $\bar{\sigma}_\theta$ is the Navier-Stokes stress tensor of the species θ , $\bar{\mathbf{q}}_\theta$ is the heat flux vector and $\bar{e}_\theta$ is the specific internal energy. In addition to the usual equations, some coupling terms appear in the previous equations. These terms are derived from a BGK-type model for binary gas mixture (see for instance [10, 11]). From this modeling we have:

$$\bar{\mathbf{u}}_a^* = \bar{\mathbf{u}}_b^* = \frac{m_a \bar{\mathbf{u}}_a + m_b \bar{\mathbf{u}}_b}{m_a + m_b}, \quad (4)$$

$$\bar{v}_{ab} = \bar{v}_a \left(\frac{d_a + d_b}{2 d_a} \right)^2 \left(\frac{m_a + m_b}{2 m_b} \right)^{\frac{1}{2}} \frac{\bar{n}_b}{\bar{n}_a} \psi_{ab}, \quad (5)$$

with:

$$\psi_{ab} = \left(\left(\frac{d_a}{d_b} \right)^2 \sqrt{\frac{m_b}{m_a} \frac{\Omega_b \bar{\mu}_a \bar{T}_b}{\Omega_a \bar{\mu}_b \bar{T}_a}} \right)^{1/2} \quad (6)$$

where m_θ is the molecular mass of the species θ and where d_θ , $\bar{v}_\theta$ and $\bar{n}_\theta$ are respectively the molecular diameter, the collision frequency and the numerical density of the species θ . With these definitions, $\bar{v}_{ba}$ is obtained from $\bar{v}_{ab}$ by replacing "a" with "b" and the total number of collisions between species a and b is balanced:

$$\bar{n}_a \bar{v}_{ab} = \bar{n}_b \bar{v}_{ba}. \quad (7)$$

As a consequence, the conservation of the global momentum is satisfied. Additionally:

$$\bar{v}_\theta = \frac{\bar{P}_\theta}{\bar{\mu}_\theta} \Omega_\theta \quad \text{with} \quad \bar{\mu}_\theta = \mu_\theta^{ref} \left(\frac{\bar{T}_\theta}{T_\theta^{ref}} \right)^{\omega_\theta}, \quad (8)$$

$$\Omega_\theta = \frac{5(\alpha_\theta + 1)(\alpha_\theta + 2)}{\alpha_\theta(7 - 2\omega_\theta)(5 - 2\omega_\theta)}, \quad (9)$$

where $\bar{p}_\theta$ is the pressure of the species θ , α_θ is a constant depending on the collision potential model and where ω_θ is the power exponent of the viscosity law (see [1, 10]). In the energy balance equation the coupling term may be expressed as:

$$\bar{S}_a^E = \frac{1}{2} \bar{\rho}_a \bar{v}_{ab} \left[\bar{\mathbf{u}}_a^{*2} - \bar{\mathbf{u}}_a^2 + \frac{2 r_a}{\gamma_a - 1} (\bar{T}_a^* - \bar{T}_a) \right], \quad (10)$$

with $r_a = k/m_a$ where k is the Boltzmann constant and where $\bar{T}_a$ is the temperature of the species a . In this expression, the temperature $\bar{T}_a^*$ is given by a Morse formula [10] extended to polyatomic ideal gases:

$$\bar{T}_a^* = \bar{T}_a + \frac{3}{2} (\gamma_a - 1) \frac{2 m_a m_b}{(m_a m_b)^2} \left(\bar{T}_b - \bar{T}_a + \frac{m_b}{6k} (\bar{\mathbf{u}}_b - \bar{\mathbf{u}}_a)^2 \right). \quad (11)$$

According to this definition and equation (7): $\bar{S}_a^E + \bar{S}_b^E = 0$ that coincides with energy conservation. In addition, we have the ideal gas equations:

$$\bar{p}_\theta = r_\theta \bar{\rho}_\theta \bar{T}_\theta \quad (12)$$

and the usual Fourier and Navier-Stokes constitutive laws. For the viscosity coefficients, we adopt the Stokes hypothesis. The first order wall boundary conditions are [3, 4]:

$$\bar{v}_\theta|_{\bar{r}=R} = \left(-\frac{\bar{\mu}_\theta \sigma_{\theta p}}{\bar{p}_\theta} \sqrt{2 r_\theta \bar{T}_\theta} \frac{\partial \bar{v}_\theta}{\partial \bar{r}} + \frac{\sigma_{\theta T} r_\theta \bar{\mu}_\theta}{\bar{p}_\theta} \frac{\partial \bar{T}_\theta}{\partial \bar{z}} \right)_{\bar{r}=R}, \quad (13)$$

$$\bar{u}_\theta|_{\bar{r}=R} = 0, \quad (14)$$

$$\bar{T}_\theta|_{\bar{r}=R} = \bar{T}_w(\bar{z}) - \left(\xi_{\theta T} \sqrt{2 r_\theta \bar{T}_\theta} \frac{\bar{\mu}_\theta}{\bar{p}_\theta} \frac{\partial \bar{T}_\theta}{\partial \bar{r}} \right)_{\bar{r}=R} \quad (15)$$

where the constant coefficients $\sigma_{\theta p}$, $\sigma_{\theta T}$ and $\xi_{\theta T}$ characterize the viscous and thermal slip of the species θ on the walls [12]. The upstream and downstream boundary conditions for the pressures will be specified further.

In order to discuss the magnitude orders of the terms in the model equations, we write the corresponding dimensionless equations. We choose L as the longitudinal length scale and R as the radial length scale and we set $\epsilon = R/L \ll 1$. Additionally, the characteristic values of the radial and longitudinal velocities, the temperature, the pressure and the volumetric mass are respectively: U and V , T_c (for the two species), $p_{\theta c}$ and $\rho_{\theta c}$ (with $p_{\theta c} = r_\theta \rho_{\theta c} T_c$, $\theta = a, b$). With these characteristic scales, we built the following dimensionless numbers:

$$\gamma_a = \frac{c_{pa}}{c_{va}}, \quad Pr_a = \frac{\mu_a c_{pa}}{\kappa_a}, \quad Re_a = \frac{\rho_a c_{pa} V L}{\mu_a}, \quad Ma_a = \frac{V}{\sqrt{\gamma_a r_a T_c}}, \quad (16)$$

Obviously, similar dimensionless numbers are built for the second species. Since Pr_θ is constant, μ_θ and κ_θ are proportional to $T_\theta^{\omega_\theta}$. Now we are interested in the steady case. From the Principe of Least Degeneracy [13], we must keep the two terms in the dimensionless mass laws. Consequently $U = \epsilon V$. In order to investigate the significative degeneracies, we assume that the Reynolds number and that the Mach number are small or of order unity; we set $Re_a = R_a \epsilon^\alpha$ and $Ma_a = M_a \epsilon^\beta$ where both R_a and M_a are of order of the unity. The analysis of the magnitudes of the terms in the dimensionless balance laws derived from (1), (2) and (3) leads to a Mach number of order ϵ and to a Reynolds number of order unity that is to $\alpha = 0$ and $\beta = 1$ [11]. At the first approximation order, we have the following dimensionless equations:

$$\frac{1}{r} \frac{\partial}{\partial r} (r \rho_a u_a) + \frac{\partial}{\partial z} (\rho_a v_a) = 0, \quad (17)$$

for the mass. For the longitudinal and radial balance of momentum, we have:

$$0 = -\frac{1}{\gamma_a M_a^2} \frac{\partial p_a}{\partial z} + \frac{1}{R_a} \frac{1}{r} \frac{\partial}{\partial r} \left(r \mu_a \frac{\partial v_a}{\partial r} \right) + F_{ab} \frac{\rho_{bc}}{\rho_{ac}} \left(\frac{\mu_{ac}}{\mu_{bc}} \right)^{\frac{1}{2}} \frac{R_a}{\gamma_a M_a^2} p_a p_b \frac{(v_b - v_a)}{T(z)^{1+\frac{\omega_\theta+\omega_2}{2}}}, \quad (18)$$

$$0 = -\frac{1}{\gamma_a M_a^2} \frac{\partial p_a}{\partial r}, \quad (19)$$

with:

$$F_{ab} = \left(\frac{m_b}{m_a} \right)^{\frac{1}{4}} \frac{m_a}{\sqrt{2 m_b (m_a + m_b)}} \left(\frac{d_a + d_b}{2 d_a} \right)^2 \frac{d_a}{d_b} \sqrt{\Omega_a \Omega_b}. \quad (20)$$

For the species a , the first order dimensionless energy balance equation is:

$$0 = \frac{\gamma_a}{R_a P r_a} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \kappa_a \frac{\partial T_a}{\partial r} \right) \right] + G_{ab} \frac{R_a}{\gamma_a M_a^2} \frac{\rho_b p_a}{\sqrt{\mu_a \mu_b}} \sqrt{\frac{T_b}{T_a}} (T_b - T_a), \quad (21)$$

with:

$$G_{ab} = \frac{3 m_a (\gamma_a - 1)}{m_a + m_b} \frac{\rho_{bc}}{\rho_{ac}} \frac{\mu_{ac}}{\mu_{bc}} F_{ab} \quad (22)$$

For the sake of simplicity, we do not write here the dimensionless equations for the other species but they are easily obtained by changing a in b and vice versa in the previous equations. First we consider the energy balance equation (21) and we assume that the two gases are in thermal equilibrium. In that case, the equation (21) with the boundary condition (15) admits a unique solution:

$$T_a(r, z) = T_w(z) = T_b(r, z) = T(z) \quad (23)$$

Finally, the dimensionless wall boundary conditions for the velocities are:

$$v_\theta(r=1, z) = -K_\theta \frac{[T_\theta(z)]^{\omega_a + \frac{1}{2}}}{p_\theta(z)} \frac{\partial v_\theta}{\partial r}(r=1, z) + \frac{\sigma_{\theta T}}{R_\theta} [T_\theta(z)]^{\omega_a} \frac{dT_\theta}{dz}(z) \quad u_\theta(r=1, z) = 0 \quad (24)$$

with: $K_\theta = \sigma_{\theta p} \sqrt{2\gamma_\theta} \frac{M_\theta}{R_\theta}$. At the first order, we obtain a set of six equations: the mass and the momentum equations for the two species. In the next section we will investigate the solution of this set of equations.

FIRST ORDER SOLUTION

The radial momentum equations (19) for the two species lead to pressures depending only on the longitudinal space variable z . If we use, in addition, the dimensionless ideal gas law, we have: $p_a(z) = \rho_a(z) T(z)$ and $p_b(z) = \rho_b(z) T(z)$. Consequently, the problem equations are reduced to:

$$\frac{1}{r} \frac{\partial}{\partial r} (r p_a u_a) + \frac{\partial}{\partial z} (p_a v_a) = 0, \quad (25)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r p_b u_b) + \frac{\partial}{\partial z} (p_b v_b) = 0, \quad (26)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_a}{\partial r} \right) - \frac{B_a}{A_a} p_a p_b (v_a - v_b) = \frac{1}{A_a} \frac{dp_a}{dz}, \quad (27)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_b}{\partial r} \right) + \frac{B_b}{A_b} p_a p_b (v_a - v_b) = \frac{1}{A_b} \frac{dp_b}{dz}, \quad (28)$$

with:

$$A_\theta = \frac{\gamma_\theta M_\theta^2}{R_\theta} [T_\theta(z)]^{\omega_\theta} \quad \theta = a, b \quad (29)$$

and:

$$B_a = R_a \frac{\rho_{bc}}{\rho_{ac}} F_{ab} \frac{\mu_{ac}}{\mu_{bc}} [T_a(z)]^{-(1 + \frac{\omega_a + \omega_b}{2})}, \quad B_b = R_b \frac{\rho_{ac}}{\rho_{bc}} F_{ba} \frac{\mu_{bc}}{\mu_{ac}} [T_b(z)]^{-(1 + \frac{\omega_a + \omega_b}{2})} \quad (30)$$

In the expressions (29) and (30), A_a , A_b , B_a and B_b are functions depending on z . The longitudinal momentum equations (27) and (28) are solved to obtain the longitudinal velocities. They are the solutions of zero order modified Bessel equations. Since the longitudinal velocities of the two gases must be finite at the channel axis ($r = 0$), the Bessel function of the second kind vanishes in the solutions. With the wall boundary conditions (24), these velocities are explicitly given in terms of the pressures:

$$v_a(r, z) = \frac{1}{A_a B_b + A_b B_a} \left[A_b B_a C_1 I_o(\lambda_1 r) + \left(A_a \frac{dp_b}{dz} - A_b \frac{dp_a}{dz} \right) \frac{A_b B_a E_a}{E p_a p_b} \right]$$

$$\begin{aligned}
& + \left(B_a \frac{p_b}{dz} + B_b \frac{dp_a}{dz} \right) \left(\frac{r^2 - 1}{4} - \frac{K_b T^{\omega_b + \frac{1}{2}}}{2 p_b} \frac{A_b B_a E_a}{E} - \frac{K_a T^{\omega_a + \frac{1}{2}}}{2 p_a} \frac{A_a B_b E_b}{E} \right) \\
& + \left[\frac{A_a B_b E_b}{E} \frac{\sigma_{aT}}{R_a} T(z)^{\omega_a} + \frac{A_b B_a E_a}{E} \frac{\sigma_{bT}}{R_b} T(z)^{\omega_b} \right] \frac{dT}{dz}(z),
\end{aligned} \tag{31}$$

$$\begin{aligned}
v_b(r, z) &= \frac{1}{A_a B_b + A_b B_a} \left[-A_a B_b C_1 I_0(\lambda_1 r) + \left(A_b \frac{dp_a}{dz} - A_a \frac{dp_b}{dz} \right) \frac{A_a B_b E_b}{E p_a p_b} \right. \\
& + \left(B_a \frac{p_b}{dz} + B_b \frac{dp_a}{dz} \right) \left(\frac{r^2 - 1}{4} - \frac{K_b T^{\omega_b + \frac{1}{2}}}{2 p_b} \frac{A_b B_a E_a}{E} - \frac{K_a T^{\omega_a + \frac{1}{2}}}{2 p_a} \frac{A_a B_b E_b}{E} \right) \\
& + \left. \left[\frac{A_a B_b E_b}{E} \frac{\sigma_{aT}}{R_a} T(z)^{\omega_a} + \frac{A_b B_a E_a}{E} \frac{\sigma_{bT}}{R_b} T(z)^{\omega_b} \right] \frac{dT}{dz}(z), \right]
\end{aligned} \tag{32}$$

where I_0 is the first kind modified Bessel function of order zero. Moreover, the following functions depending on z appear in (31) and (32):

$$\lambda_1(z) = \sqrt{\left(\frac{B_a}{A_a} + \frac{B_b}{A_b} \right) p_a(z) p_b(z)}, \tag{33}$$

$$E_\theta(z) = I_0(\lambda_1(z)) + \frac{K_\theta T(z)^{\omega_\theta + \frac{1}{2}}}{p_\theta(z)} \lambda_1(z) I_1(\lambda_1(z)) \quad \theta = a, b \tag{34}$$

$$E(z) = A_a B_b E_b(z) + A_b B_a E_a(z), \tag{35}$$

$$\begin{aligned}
C_1(z) &= \frac{B_b}{E(z)} \left[\frac{A_b}{B_b p_a(z) p_b(z)} + \frac{K_b T^{\omega_b + \frac{1}{2}}}{2 p_b(z)} - \frac{K_a T^{\omega_a + \frac{1}{2}}}{2 p_a(z)} \right] \frac{dp_a}{dz}(z) \\
& + \frac{B_a}{E(z)} \left[-\frac{A_a}{B_a p_a(z) p_b(z)} + \frac{K_b T^{\omega_b + \frac{1}{2}}}{2 p_b(z)} - \frac{K_a T^{\omega_a + \frac{1}{2}}}{2 p_a(z)} \right] \frac{dp_b}{dz}(z) \\
& + \frac{A_a B_b + A_b B_a}{E} \left(\frac{\sigma_{aT}}{R_a} T(z)^{\omega_a} - \frac{\sigma_{bT}}{R_b} T(z)^{\omega_b} \right) \frac{dT}{dz}(z),
\end{aligned} \tag{36}$$

where I_1 is the first kind modified Bessel function of order one and where K_θ , A_θ and B_θ ($\theta = a, b$) are given in (24), (29) and (30).

As a consequence of the mass balance equations, the pressure of each gas is given by an ordinary differential equation. In order to obtain these equations, we integrate the mass balance equations of each species (25) and (26) from $r = 0$ to $r = 1$. We use the wall boundary conditions for the transversal velocities and we obtain:

$$\int_0^1 \rho_a(z) v_a(r, z) r dr = \frac{Q_{ma}}{2}, \tag{37}$$

$$\int_0^1 \rho_b(z) v_b(r, z) r dr = \frac{Q_{mb}}{2}, \tag{38}$$

where Q_{ma} and Q_{mb} are the constant dimensionless mass flow rate of each species ($Q_{m\theta} = Q_{m\theta}^* / (\rho_{\theta c} \pi R^2 V)$, $\theta = a, b$). The equations (37) and (38) are integrated in r and then we obtain two ordinary differential equations for the two pressures $p_a(z)$ and $p_b(z)$. After a few calculations, these equations are written as:

$$\begin{aligned}
X_a \frac{dp_a}{dz} + Y_a \frac{dp_b}{dz} &= (A_a B_b + A_b B_a) \left[\frac{T Q_{ma}}{p_a} - \left(\frac{A_a B_b E_b \sigma_{aT}}{E R_a} T^{\omega_a} + \frac{A_b B_a E_a \sigma_{bT}}{E R_b} T^{\omega_b} \right) \frac{dT}{dz} \right. \\
& - \left. \frac{2 A_b B_a}{E \lambda_1} I_1(\lambda_1) \left(\frac{\sigma_{aT}}{R_a} T^{\omega_a} - \frac{\sigma_{bT}}{R_b} T^{\omega_b} \right) \frac{dT}{dz} \right],
\end{aligned} \tag{39}$$

$$\begin{aligned}
Y_b \frac{dp_a}{dz} + X_b \frac{dp_b}{dz} &= (A_a B_b + A_b B_a) \left[\frac{T Q_{mb}}{p_b} - \left(\frac{A_a B_b E_b \sigma_{aT}}{E R_a} T^{\omega_a} + \frac{A_b B_a E_a \sigma_{bT}}{E R_b} T^{\omega_b} \right) \frac{dT}{dz} \right. \\
& + \left. \frac{2 A_a B_b}{E \lambda_1} I_1(\lambda_1) \left(\frac{\sigma_{aT}}{R_a} T^{\omega_a} - \frac{\sigma_{bT}}{R_b} T^{\omega_b} \right) \frac{dT}{dz} \right],
\end{aligned} \tag{40}$$

FIGURE 2. Comparison to DSMC: Pressures along the micro-channel axis (a) - Velocity profiles at the channel outlet (b)

with:

$$X_a = \frac{A_b B_a B_b}{E \lambda_1} \left[\frac{2 A_b}{B_b p_a p_b} + \frac{K_b T^{\omega_b + \frac{1}{2}}}{p_b} - \frac{K_a T^{\omega_a + \frac{1}{2}}}{p_a} \right] I_1(\lambda_1) - \frac{A_b^2 B_a E_a}{E p_a p_b} - B_b \left[\frac{1}{8} + \frac{K_b T^{\omega_b + \frac{1}{2}}}{2 p_b} \frac{A_b B_a E_a}{E} + \frac{K_a T^{\omega_a + \frac{1}{2}}}{2 p_a} \frac{A_a B_b E_b}{E} \right], \quad (41)$$

$$Y_a = + \frac{A_b B_a^2}{E \lambda_1} \left[- \frac{2 A_a}{B_a p_a p_b} + \frac{K_b T^{\omega_b + \frac{1}{2}}}{p_b} - \frac{K_a T^{\omega_a + \frac{1}{2}}}{p_a} \right] I_1(\lambda_1) + \frac{A_a A_b B_a E_a}{E p_a p_b} - B_a \left[\frac{1}{8} + \frac{K_b T^{\omega_b + \frac{1}{2}}}{2 p_b} \frac{A_b B_a E_a}{E} + \frac{K_a T^{\omega_a + \frac{1}{2}}}{2 p_a} \frac{A_a B_b E_b}{E} \right]. \quad (42)$$

The expressions of X_b and Y_b , respectively, are obtained from X_a and Y_a by changing a in b and vice versa. The equations (39) and (40) are coupled and linear in term of the longitudinal pressure gradients but non linear in z .

NUMERICAL SOLUTIONS

In order to solve the two equations (39) and (40), we need the wall temperature and four boundary conditions: for instance, the pressures p_a and p_b at the channel inlet and the dimensionless mass flow rates $\bar{Q}_{ma}$ and $\bar{Q}_{mb}$. These values may be set from DSMC simulation in the same geometry. We have: $\bar{p}_a(-L/2) = 2.378 \cdot 10^4$ Pa, $\bar{p}_b(-L/2) = 2.387 \cdot 10^4$ Pa, $\bar{Q}_{ma} = 7.861 \cdot 10^{-13}$ kg.m⁻¹.s⁻¹ and $\bar{Q}_{mb} = 7.881 \cdot 10^{-13}$ kg.m⁻¹.s⁻¹. The wall temperature decreases linearly from 300 K to 200 K. From DSMC simulations, we set equally the characteristic values of the longitudinal velocity and of the pressures: $V = 1$ m.s⁻¹ and $p_{ac} = p_{bc} = 10^5$ Pa. In addition, we have to set the values of the various constants related to the nature of the gases. We are interested in gases with the same molecular mass so we assume that the species a is nitrogen and that the species b is Carbon monoxide. Consequently, from [1], we have: $m_a = m_b = 46.5 \cdot 10^{-27}$ kg, $d_a = 4.11 \cdot 10^{-10}$ m, $\gamma_a = \gamma_b = 1.4$, $[\mu_a]_{VSS} = 1.725 \cdot 10^{-5}$ Pa.s, $\alpha_a = 1.36$, $\omega_a = 0.74$, $r_a = r_b = 296.77$ J.kg⁻¹.K⁻¹, $d_b = 4.12 \cdot 10^{-10}$ m, $[\mu_b]_{VSS} = 1.702 \cdot 10^{-5}$ Pa.s, $\alpha_b = 1.49$ and $\omega_b = 0.73$. These values are those used in the DSMC simulations with Bird's DS2V program [14]. The two equations (39) and (40) are solved with MATLAB using a variable step Runge-Kutta method of fourth order. With these solutions for the pressures, the longitudinal velocities are computed from (31) and (32). The numerical solutions of the analytical models are compared to the results of the DSMC simulations in Fig. 2. Obviously, the computation time is strongly decreased for the asymptotic model ; while DSMC need a few days to produce a solution, only a few seconds are required for our model. We have a good agreement between the DSMC simulation and the asymptotic solution for the pressure of the two species. Since the pressure difference between the inlet and the outlet of the channel, for the two species, is weak, the pressures, in both calculations, seem to be quasi linear in z . Nevertheless the pressures are underestimated at the channel outlet by our model: the difference is about 4%. This difference between the asymptotical approach and DSMC simulations is a consequence of the strong non linearity of the equations (39) and (40): They are very sensitive to the boundary conditions. As a consequence, a small variation in the inlet pressure and in the mass flow rate leads

FIGURE 3. Velocity profiles at different channel sections

FIGURE 4. Pressure (Pa) along the channel axis with $T_{in} = 300$ K and different values for T_{out}

to a great variation of the solution. In addition, these boundary conditions are obtained from DSMC simulations. Consequently, because of the numerical noise in the DSMC results, it is not easy to set the boundary conditions of the analytical set of equations at a convenient level of accuracy. Moreover, the statistical error for the velocities in DSMC computations is inversely proportional to the Mach number [15] which is of order $2 \cdot 10^{-2}$ in the present case. The result is an important DSMC error on the velocities (of order 2%) and, consequently, on the mass flow rate. In order to limit the effects of this error, we set the mass flow rate to the mean of three values calculated in different sections of the channel.

The evolution of the longitudinal velocities is plotted on Fig. 3. We denote a small difference for the two species. This is explained by the differences in the two gases properties and by the difference of inlet conditions. As an effect of the thermal creep, we observe that the longitudinal velocities decrease with the pressures along the micro-channel.

For the same gases, we focus on the effect of the constant gradient of temperature. The inlet temperature is still 300K and we set different values of the outlet temperature. The boundary conditions are: $\bar{p}_{ain} = 2.48 \cdot 10^4$ Pa, $\bar{p}_{bin} = 2.47 \cdot 10^4$ Pa, $\bar{Q}_{ma} = 1.866 \cdot 10^{-13}$ kg.m⁻¹.s⁻¹ and $\bar{Q}_{mb} = 1.845 \cdot 10^{-13}$ kg.m⁻¹.s⁻¹. Figure 4 shows the evolution of the pressure of each species for three different outlet temperatures. The pressures decrease with z. For a given value of z, the pressures increase with the outlet temperature. This result is in agreement with the observations in the plane case [6]. The velocity profiles at the channel inlet and outlet are plotted in Fig. 5. In the isothermal case, as expected [7], the longitudinal velocity increases along the channel as the pressure decreases. In the non isothermal cases, the effect of the thermal creep is observed. It induces a movement of the gases from the cold end of the channel to the hot one: when the inlet temperature is greater than the outlet temperature (case $T_{out} = 200$ K on Fig. 5), the effects of the pressure gradient and of the temperature gradient are opposite and, as a consequence, we observe that the velocity decreases along the channel. When the inlet temperature is lower than the outlet temperature, both the pressure and temperature gradients accelerate the gases along the channel.

FIGURE 5. Longitudinal velocities at the channel inlet (a) and outlet (b) for $T_{in} = 300$ K and different values for T_{out}

CONCLUSION

In this contribution, we present an original asymptotic study to analyze the flow of a gaseous mixture in a micro-channel. The model is obtained from Navier-Stokes-Fourier equations with additional coupling terms. The longitudinal velocities are given explicitly in term of the pressures that are computed from a first order set of differential equations. The effect of the wall temperature gradient has been investigated. The advantage of this method, compared to DSMC simulations, is a very short computation time.

The results have been compared to DSMC simulations and the two methods are in good agreement. The main difficulty in comparing the two methods is to set the correct boundary conditions for the analytical model which is strongly non linear. This non linearity comes from the coupling terms. In the future, less penalizing coupling terms could be investigated, in the case of gases with very different molecular masses for instance.

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