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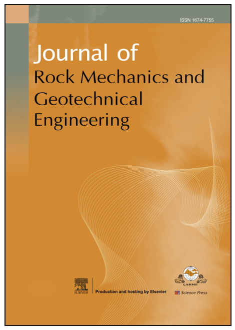

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Strength criterion of porous media: Application of homogenization techniques

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Abstract: The present paper aims at giving some general ideas concerning the micromechanical approach of the strength of a porous material. It is shown that its determination theoretically amounts to solving a nonlinear boundary value problem defined on a representative elementary volume (REV). The principle of nonlinear homogenization is illustrated based on the case of a solid phase having a Green's strength criterion. An original refinement of the so-called secant method (based on two reference strains) is also provided. The paper also describes the main feature of the Gurson's model which implements the principle of limit analysis on a conceptual model of hollow sphere. The last part of the paper gives some ideas concerning poromechanical couplings.

Keywords: strength criterion; porous media; homogenization techniques; nonlinear behavior; limit analysis; Gurson's model

1. Introduction

In civil engineering, characterization of material's strength is traditionally of paramount importance. The yield design of structures which takes place chronologically at the first step is based on precise input data concerning the strength. This general statement is true in particular for porous media with two specific features.

The first specificity of porous media which deserves being mentioned is that the pore space may be saturated by one or several fluids. The question is then how the fluid pressure(s) affect(s) the strength. This traditionally raises the question of the existence of a so-called effective stress, that is, a function of the stress and the pressure(s) which possibly captures the poromechanical coupling.

The second specificity is related to the so-called contraction or dilatant behavior of the porous material in association with the changes of porosity. The influence of porosity on strength is well-known and has been early incorporated in phenomenological models such as the Cam-Clay model and micromechanical models such as the Gurson's model. In some cases, it may be relevant to interpreting the strength criterion for a given porosity as a yield criterion in terms of plasticity and to regarding the porosity as a hardening (or softening) parameter.

The present paper aims at giving some general ideas concerning the micromechanical approach of the strength of a porous material. First, the mathematical definition of the macroscopic strength is presented. It is shown that its determination theoretically amounts to solving a nonlinear boundary value problem defined on a representative elementary volume (REV). On the methodological side, the available mathematical techniques of resolution are briefly introduced. On one hand, the principle of nonlinear homogenization is illustrated based on the case of a solid having a Green's strength criterion. An original refinement of the so-called secant method (based on two reference strains) is also provided. On the other hand, the paper describes the main feature of the Gurson's model which implements the principle of limit analysis on a conceptual model of hollow sphere.

The last part of the paper gives some ideas concerning the poromechanical coupling. Various assumptions are made concerning the strength of the solid phase and, in each case, the macroscopic counterpart in terms of effective stress is identified.

2. Macroscopic strength of an empty porous material

Concerning the upscaling of strength behavior, a good starting point is the empty or non-pressurized pore space. In this case, the determination of the overall strength of a porous material requires a description of the strength of the solid phase only, together with morphological information concerning the geometry of the microstructure. To this end, we present some classical results of convex analysis. But before doing this, some basic results of linear homogenization that will turn out useful in the forthcoming developments are briefly reviewed.

2.1. Some results of linear homogenization

Considering a REV (Ω) of a porous medium, $\Omega = \Omega^{\mathrm{s}} \cup \Omega^{\mathrm{p}}$, where the solid phase Ω^{s} is homogeneous, and Ω^{p} is the empty pore domain. The position vector at the microscopic scale in Ω is denoted by $\underline{z}$, and $\boldsymbol{\sigma}(\underline{z})$ (resp. $\boldsymbol{\varepsilon}(\underline{z})$ or $\underline{\xi}(\underline{z})$) is the microscopic stress (resp. strain or displacement) field in Ω. The average on Ω (resp. Ω^{s}) of a field $a(\underline{z})$ is denoted by $\overline{a}$ (resp. $\overline{a}^{\mathrm{s}}$):

$$\overline{a} = \frac{1}{|\Omega|}\int_{\Omega} a(\underline{z})\mathrm{d}V,\ \overline{a}^{\mathrm{s}} = \frac{1}{|\Omega^{\mathrm{s}}|}\int_{\Omega^{\mathrm{s}}} a(\underline{z})\mathrm{d}V \tag{1}$$

The local state equation is linear and reads

$$\boldsymbol{\sigma}(\underline{z}) = \mathbb{C}(\underline{z}) : \boldsymbol{\varepsilon}(\underline{z}) \tag{2a}$$

$$\mathbb{C}(\underline{z}) = \begin{cases} \mathbb{C}^{\mathrm{s}} & (\underline{z} \in \Omega^{\mathrm{s}}) \\ 0 & (\underline{z} \in \Omega^{\mathrm{p}}) \end{cases} \tag{2b}$$

Considering the so-called uniform strain boundary conditions, the boundary problem at the scale of the REV is defined by the following set of equations:

$$\operatorname{div}\boldsymbol{\sigma} = 0 \quad (\Omega) \tag{3a}$$

$$\boldsymbol{\sigma}(\underline{z}) = \mathbb{C}(\underline{z}) : \boldsymbol{\varepsilon}(\underline{z}) \quad (\Omega) \tag{3b}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\operatorname{grad}\underline{\xi} + {}^{\mathrm{t}}\operatorname{grad}\underline{\xi}) \quad (\Omega) \tag{3c}$$

$$\underline{\xi}(\underline{z}) = \boldsymbol{E}\underline{z} \quad (\partial\Omega) \tag{3d}$$

where $\boldsymbol{E}$ represents the macroscopic strain applied to the REV. It is related to the microscopic strain field by the average rule $\boldsymbol{E} = \overline{\varepsilon}$. In this paper, $\overline{a}$ is the average over the whole REV. Due to linearity, the local strain $\boldsymbol{\varepsilon}(\underline{z})$ is directly related to $\boldsymbol{E}$ by means of the fourth-order strain concentration tensor $\mathbb{A}(\underline{z})$ by $\boldsymbol{\varepsilon}(\underline{z}) = \mathbb{A}(\underline{z}) : \boldsymbol{E}$. In turn, the macroscopic stress $\boldsymbol{\Sigma}$ is determined from the stress average rule $\boldsymbol{\Sigma} = \overline{\boldsymbol{\sigma}}$. It is therefore related to $\boldsymbol{E}$ by the homogenized state equation:

$$\boldsymbol{\Sigma} = \mathbb{C}^{\mathrm{hom}} : \boldsymbol{E},\ \mathbb{C}^{\mathrm{hom}} = \overline{\mathbb{C} : \mathbb{A}} \tag{4}$$

In the case of a porous medium (with empty pore space), the stress average rule reads $\boldsymbol{\Sigma} = (1-\varphi)\overline{\boldsymbol{\sigma}}^{\mathrm{s}}$. This yields

$$\boldsymbol{\Sigma} = (1-\varphi)\mathbb{C}^{\mathrm{s}} : \overline{\mathbb{A}}^{\mathrm{s}} : \boldsymbol{E} \Rightarrow \mathbb{C}^{\mathrm{hom}} = (1-\varphi)\mathbb{C}^{\mathrm{s}} : \overline{\mathbb{A}}^{\mathrm{s}} \tag{5}$$

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where φ is the pore volume fraction or porosity. The notation $\overline{a}^{\alpha}$ refers to the average of the quantity a over the phase α. As a consequence of the strain average rule, it follows that $\bar{A} = I$ (fourth- order identity tensor). Accordingly, the homogenized stiffness tensor $\square^{\mathrm{hom}}$ can be put in the form:

$$C^{\mathrm{hom}} = C^{\mathrm{s}} : (I - \varphi \bar{A}^{\mathrm{p}}) \tag{6}$$

Whatever the chosen definition in Eq. (5) or (6) for the definition of the homogenized stiffness, the determination, or at least the estimate, of the average of the local strain over the solid phase (or the pore phase) is required. A direct estimate may be derived from Eq. (5) as

$$\bar{\boldsymbol{\varepsilon}}^{\mathrm{s}} = \frac{1}{1-\varphi} (C^{\mathrm{s}})^{-1} : C^{\mathrm{hom}} : \boldsymbol{E} \tag{7}$$

For further discussions, it may be proved to be more convenient to interpret the local strain in terms of its invariants. Let I_1' (resp. J_2') be the first (resp. second) invariant of the local strain (resp. deviatoric strain) tensor. It is readily seen that

$$\left.\begin{array}{l} \boldsymbol{\varepsilon} = \frac{1}{3} I_1' \mathbf{1} + \grave{\boldsymbol{o}} \\ I_1' = \mathrm{tr}\,\boldsymbol{\varepsilon} \\ J_2' = \frac{1}{2} \grave{\boldsymbol{o}} : \grave{\boldsymbol{o}} \end{array}\right\} \tag{8}$$

Averages of I_1' and J_2' over the solid phase will prove useful. From a direct application of Eq. (7), the average of I_1' over the solid phase reads

$$\overline{I_1'}^{\mathrm{s}} = \frac{1}{1-\varphi} \mathbf{1} : (C^{\mathrm{s}})^{-1} : C^{\mathrm{hom}} : \boldsymbol{E} \tag{9}$$

In the particular case of a porous medium with a homogeneous solid phase, the average of the second invariant J_2' in the solid phase reads (Kreher, 1990; Dormieux et al., 2001):

$$\overline{J_2'}^{\mathrm{s}} = \frac{1}{2(1-\varphi)} \left[\frac{1}{2} (\mathrm{tr}\boldsymbol{E})^2 \frac{\partial k^{\mathrm{hom}}}{\partial \mu^{\mathrm{s}}} + \boldsymbol{\varDelta} : \boldsymbol{\varDelta} \frac{\partial \mu^{\mathrm{hom}}}{\partial \mu^{\mathrm{s}}} \right] \tag{10}$$

where $\boldsymbol{E} = [(\mathrm{tr}\boldsymbol{E})\mathbf{1}]/3 + \boldsymbol{\varDelta}$. In Eq. (10), an isotropic macroscopic behavior is assumed: k^{hom} and μ^{hom} denote the macroscopic bulk and shear moduli, respectively. Implementation of this equation requires determining the derivatives of these quantities with respect to μ^{s}. This can be done by means of linear homogenization schemes such as the Mori-Tanaka scheme or the self-consistent scheme, depending on the type of microstructure at stake. With the same reasoning, the quadratic average of the volume strain in the solid phase is derived as

$$\overline{(I_1')^2}^{\mathrm{s}} = \frac{1}{1-\varphi} \left[(\mathrm{tr}\boldsymbol{E})^2 \frac{\partial k^{\mathrm{hom}}}{\partial k^{\mathrm{s}}} + 2\boldsymbol{\varDelta} : \boldsymbol{\varDelta} \frac{\partial \mu^{\mathrm{hom}}}{\partial k^{\mathrm{s}}} \right] \tag{11}$$

Either Eq. (9) or (11) can be used to estimate the volume strain in the solid.

2.2. Microscopic strength of the solid phase

There are two equivalent ways to define the strength of the solid phase, which can be referred to as the direct definition and the dual one.

The direct approach consists in defining the convex set G^{s} of strength-compatible (microscopic) stress states. From a mathematical point of view, this is achieved by means of a (convex) strength criterion $f^{\mathrm{s}}(\boldsymbol{\sigma})$:

$$G^{\mathrm{s}} = \{\boldsymbol{\sigma}, f(\boldsymbol{\sigma}) \le 0\} \tag{12}$$

The boundary ∂G^{s} is characterized by the condition $f^{\mathrm{s}}(\boldsymbol{\sigma}) = 0$, and the zero stress state $\boldsymbol{\sigma} = 0$ is assumed to be strength-compatible, i.e. $f^{\mathrm{s}}(\boldsymbol{\sigma}) \le 0$.

In contrast to the direct approach, a dual definition of the strength criterion consists in introducing the support function $\pi^{\mathrm{s}}(\boldsymbol{d})$ of G^{s}, which is defined on the set of symmetric second-order tensors $\boldsymbol{d}$ and is convex with respect to $\boldsymbol{d}$:

$$\pi^{\mathrm{s}}(\boldsymbol{d}) = \sup(\boldsymbol{\sigma} : \boldsymbol{d}, \boldsymbol{\sigma} \in G^{\mathrm{s}}) \tag{13}$$

where $\pi^{\mathrm{s}}(\boldsymbol{d})$ represents the maximum "plastic" dissipation capacity that the material can afford. The fact that the zero stress is strength- compatible, i.e. $0 \in G^{\mathrm{s}}$, implies the non-negativity of $\pi^{\mathrm{s}}(\boldsymbol{d}) \ge 0$. Furthermore, it is readily seen that

$$\pi^{\mathrm{s}}(t\boldsymbol{d}) = t\pi^{\mathrm{s}}(\boldsymbol{d}) \quad (\forall t \in R^{+}) \tag{14}$$

The dual definition of the solid strength thus takes the form:

$$\boldsymbol{\sigma} \in G^{\mathrm{s}} \Leftrightarrow \boldsymbol{\sigma} : \boldsymbol{d} \le \pi^{\mathrm{s}}(\boldsymbol{d}) \quad (\forall \boldsymbol{d}) \tag{15}$$

For a given value of $\boldsymbol{d}$, we recognize that the condition $\boldsymbol{\sigma} : \boldsymbol{d} = \pi^{\mathrm{s}}(\boldsymbol{d})$ defines a hyperplane H($\boldsymbol{d}$) in the stress space. This hyperplane is tangent to the boundary ∂G^{s} at the point $\boldsymbol{\sigma}$ at which the normal to ∂G^{s} is parallel to $\boldsymbol{d}$ (see Fig. 1).

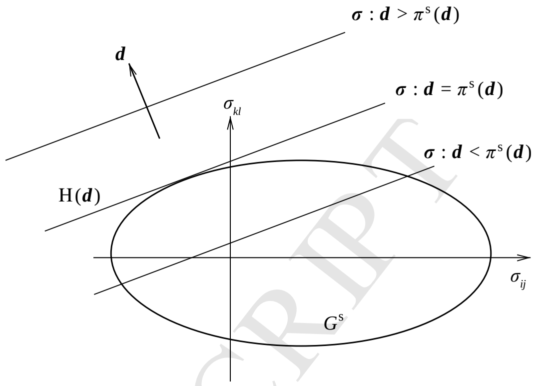

Fig. 1. Geometrical interpretation of the support function.

Moreover, differentiating Eq. (14) with respect to $t > 0$ yields

$$\frac{\partial \pi^{\mathrm{s}}}{\partial \boldsymbol{d}}(\boldsymbol{d}) : \boldsymbol{d} = \pi^{\mathrm{s}}(\boldsymbol{d}) \tag{16}$$

It follows that the stress state $\boldsymbol{\sigma} = \partial \pi^{\mathrm{s}} / \partial \boldsymbol{d}(\boldsymbol{d})$ is located on H($\boldsymbol{d}$). Furthermore, the convexity of the support function reads

$$\pi^{\mathrm{s}}(\boldsymbol{d}') - \pi^{\mathrm{s}}(\boldsymbol{d}) \ge \frac{\partial \pi^{\mathrm{s}}}{\partial \boldsymbol{d}}(\boldsymbol{d}) : (\boldsymbol{d}' - \boldsymbol{d}) \tag{17}$$

Combining Eqs. (16) and (17) yields

$$\pi^{\mathrm{s}}(\boldsymbol{d}') \ge \frac{\partial \pi^{\mathrm{s}}}{\partial \boldsymbol{d}}(\boldsymbol{d}) : \boldsymbol{d}' \quad (\forall \boldsymbol{d}') \tag{18}$$

According to the dual definition presented in Eq. (15) of G^{s}, Eq. (18) ensures that $\boldsymbol{\sigma} = \partial \pi^{\mathrm{s}} / \partial \boldsymbol{d}(\boldsymbol{d})$ is located at the intersection of H($\boldsymbol{d}$) with G^{s}.

2.3. Strength-compatible macroscopic stress states

A microscopic stress field $\boldsymbol{\sigma}(\underline{z})$ defined on the REV is statically compatible with a given macroscopic stress state $\boldsymbol{\Sigma}$ provided it satisfies:

(1) The momentum balance condition div$\boldsymbol{\sigma} = 0$;

(2) The average rule $\boldsymbol{\Sigma} = \bar{\boldsymbol{\sigma}}$; and

(3) A zero stress in the pore space, i.e. $\boldsymbol{\sigma} = 0 \ (\forall \underline{z} \in \Omega^{\mathrm{p}})$.

In turn, a macroscopic stress state $\boldsymbol{\Sigma}$ is compatible with the material strength if a microscopic stress field $\boldsymbol{\sigma}(\underline{z})$ exists that is statically admissible with $\boldsymbol{\Sigma}$ and compatible with the strength of the solid. Let G^{hom} denote the set of such strength-compatible macroscopic stress states:

$$G^{\mathrm{hom}} = \{\boldsymbol{\Sigma}, \exists \boldsymbol{\sigma} \text{ stat. adm. with } \boldsymbol{\Sigma}, \boldsymbol{\sigma}(\underline{z}) \in G^{\mathrm{s}} \ (\forall \underline{z} \in \Omega^{\mathrm{s}})\} \tag{19}$$

For a given macroscopic strain rate tensor $\boldsymbol{D}$, let us define the set V($\boldsymbol{D}$) of kinematically admissible microscopic velocity fields $\underline{v}(\underline{z})$:

$$\mathrm{V}(\boldsymbol{D}) = \{\underline{v}, \ \underline{v}(\underline{z}) = \boldsymbol{D}\underline{z} \ (\forall \underline{z} \in \partial U)\} \tag{20}$$

For $\boldsymbol{\Sigma} \in G^{\mathrm{hom}}$, let $\boldsymbol{\sigma}$ comply with the conditions of Eq. (19). Furthermore, let us consider an arbitrary element $\underline{v} \in \mathrm{V}(\boldsymbol{D})$. The Hill lemma (see e.g. Dormieux et al., 2006) states that

$$\boldsymbol{\Sigma} : \boldsymbol{D} = \overline{\boldsymbol{\sigma} : \boldsymbol{d}} = (1-\varphi)\overline{\boldsymbol{\sigma} : \boldsymbol{d}}^{\mathrm{s}} \tag{21}$$

where $\boldsymbol{d}$ denotes the microscopic strain rate associated with the velocity field $\underline{v}$. Recalling that $\boldsymbol{\sigma}$ is compatible with the strength of the solid, it follows from Eqs. (15) and (21) that

$$\left.\begin{array}{l} \boldsymbol{\Sigma} : \boldsymbol{D} \le \Pi^{\mathrm{hom}}(\boldsymbol{D}) \\ \Pi^{\mathrm{hom}}(\boldsymbol{D}) = (1-\varphi) \inf\limits_{\underline{v} \in \mathrm{V}(\boldsymbol{D})} \overline{\pi^{\mathrm{s}}(\boldsymbol{d})}^{\mathrm{s}} \end{array}\right\} \tag{22}$$

Eq. (22) shows that G^{hom} is located in a half-space bounded by the hyperplane $\boldsymbol{\Sigma} : \boldsymbol{D} = \Pi^{\mathrm{hom}}(\boldsymbol{D})$. In particular, if $\boldsymbol{\Sigma}$ belongs to both of this hyperplane and G^{hom}, it is located on the boundary $\partial G^{\mathrm{hom}}$ at a point at which the normal to $\partial G^{\mathrm{hom}}$ is parallel to $\boldsymbol{D}$:

$$\left.\begin{array}{l}\boldsymbol{\Sigma}:\boldsymbol{D}\le \Pi^{\mathrm{hom}}(\boldsymbol{D})\\ \boldsymbol{\Sigma}\in G^{\mathrm{hom}}\end{array}\right\}\Rightarrow \boldsymbol{\Sigma}\in \partial G^{\mathrm{hom}} \tag{23}$$

2.4. Determination of $\partial G^{\mathrm{hom}}$

We here present a strategy for the determination of $\partial G^{\mathrm{hom}}$. This strategy is based on a systematic method (Leblond et al., 1994) for deriving microscopic stress fields $\boldsymbol{\sigma}$ associated in the sense of Eq. (19) with the macroscopic stresses located on $\partial G^{\mathrm{hom}}$.

More precisely, in the dual definition of the solid strength (Eq. (15)), we have seen that the microscopic stress field $\boldsymbol{\sigma} = \partial\pi^{\mathrm{s}}/\partial\boldsymbol{d}(\boldsymbol{d})$ is located on the boundary ∂G^{s}. It is intriguing to explore this property in the sense of a nonlinear viscous behavior of the solid phase in the REV, by defining this property $\boldsymbol{\sigma} = \partial\pi^{\mathrm{s}}/\partial\boldsymbol{d}(\boldsymbol{d})$ as a viscous state equation, which is non-zero only in the solid phase and $\boldsymbol{\sigma} = 0$ in the pore space. For a given macroscopic strain rate $\boldsymbol{D}$, consider the microscopic stress field $\boldsymbol{\sigma}(\underline{z})$ and velocity field $\underline{v}(\underline{z})$ which are solutions to the mechanical problem defined on the REV by the Hashin boundary conditions $\underline{v}(\underline{z}) = \boldsymbol{D}\underline{z}$ on $\partial\Omega$:

$$\mathrm{div}\,\boldsymbol{\sigma}=0 \quad (\Omega) \tag{24a}$$

$$\boldsymbol{\sigma}=\frac{\partial\pi^{\mathrm{s}}}{\partial\boldsymbol{d}}(\boldsymbol{d}) \quad (\Omega^{\mathrm{s}}) \tag{24b}$$

$$\boldsymbol{\sigma}=0 \quad (\Omega^{\mathrm{p}}) \tag{24c}$$

$$\boldsymbol{d}=(\mathrm{grad}\,\underline{v}+{}^{\mathrm{t}}\,\mathrm{grad}\,\underline{v})/2 \quad (\Omega) \tag{24d}$$

$$\underline{v}(\underline{z})=\boldsymbol{D}\underline{z} \quad (\partial\Omega) \tag{24e}$$

According to the conclusion of Section 2.2, the stress field solution to Eq. (24) is compatible with the strength of the solid phase. Eq. (19) implies that $\boldsymbol{\Sigma}=\overline{\boldsymbol{\sigma}}\in G^{\mathrm{hom}}$. In particular, Eq. (22) holds.

Now combining Eqs. (16) and (24b), one obtains

$$\pi^{\mathrm{s}}(\boldsymbol{d})=\frac{\partial\pi^{\mathrm{s}}}{\partial\boldsymbol{d}}(\boldsymbol{d}):\boldsymbol{d}=\boldsymbol{\sigma}:\boldsymbol{d} \tag{25}$$

Taking the average of Eq. (25) over the solid phase yields

$$(1-\varphi)\overline{\pi^{\mathrm{s}}(\boldsymbol{d})}^{\mathrm{s}}=(1-\varphi)\overline{\boldsymbol{\sigma}:\boldsymbol{d}}^{\mathrm{s}}=\boldsymbol{\Sigma}:\boldsymbol{D}\Rightarrow \Pi^{\mathrm{hom}}(\boldsymbol{D})\le\boldsymbol{\Sigma}:\boldsymbol{D} \tag{26}$$

The combination of Eqs. (22) and (26) proves that $\Pi^{\mathrm{hom}}(\boldsymbol{D})=\boldsymbol{\Sigma}:\boldsymbol{D}$, which means (see Eq. (23)) that $\boldsymbol{\Sigma}$ is located on the boundary $\partial G^{\mathrm{hom}}$. The determination of $\partial G^{\mathrm{hom}}$ therefore reduces to finding the effective behavior of a porous medium made up of a nonlinear viscous solid phase (see Eq. (24b) and (24c)).

2.5. Solid strength depending on the first two stress invariants

From now on, we assume that the strength of the solid phase is controlled by the mean stress and the equivalent deviatoric stress:

$$\left.\begin{array}{l}f^{\mathrm{s}}(\boldsymbol{\sigma})=\mathrm{F}\,(I_1,J_2)\\ I_1=\mathrm{tr}\,\boldsymbol{\sigma},\; J_2=\dfrac{1}{2}\boldsymbol{s}:\boldsymbol{s}\end{array}\right\} \tag{27}$$

where $\boldsymbol{s}=\boldsymbol{\sigma}-I_1\mathbf{1}/3$ is the deviatoric stress tensor. Similarly to Eq. (8), let us introduce the volume strain rate I_1' and the equivalent deviatoric strain rate $\sqrt{J_2'}$ associated with the strain rate tensor $\boldsymbol{d}$:

$$\left.\begin{array}{l}\boldsymbol{d}=\dfrac{1}{3}I_1'\mathbf{1}+\boldsymbol{\delta}\\ I_1'=\mathrm{tr}\boldsymbol{d},\, J_2'=\dfrac{1}{2}\boldsymbol{\delta}:\boldsymbol{\delta}\end{array}\right\} \tag{28}$$

According to definition in Eq. (13), the support function now reads

$$\pi^{\mathrm{s}}(\boldsymbol{d})=\sup\left(\frac{1}{3}I_1I_1'+\boldsymbol{s}:\boldsymbol{\delta},\;\mathrm{F}\,(I_1,J_2)\le 0\right) \tag{29}$$

For a given value of J_2, the choice of $\boldsymbol{s}$ which maximizes $\boldsymbol{s}:\boldsymbol{\delta}$ is parallel to $\boldsymbol{\delta}$, namely, $\boldsymbol{s}=\boldsymbol{\delta}\sqrt{J_2/J_2'}$.

Eq. (29) thus takes the following form:

$$\pi^{\mathrm{s}}(\boldsymbol{d})=\sup\left(\frac{1}{3}I_1I_1'+2\sqrt{J_2J_2'},\;\mathrm{F}\,(I_1,J_2)\le 0\right) \tag{30}$$

It then turns out that the support function only depends on the invariants I_1' and J_2' of $\boldsymbol{d}$:

$$\pi^{\mathrm{s}}(\boldsymbol{d})=\pi^{\mathrm{s}}(I_1',J_2') \tag{31}$$

The state equation (Eq. (24b)) therefore reads

$$\boldsymbol{\sigma}=\frac{\partial\pi^{\mathrm{s}}}{\partial I_1'}(I_1',J_2')\mathbf{1}+\frac{\partial\pi^{\mathrm{s}}}{\partial J_2'}(I_1',J_2')\boldsymbol{\delta}=C^{\mathrm{s}}(\boldsymbol{d}):\boldsymbol{d} \tag{32}$$

The fictitious viscous behavior of the solid phase is found to be defined by an isotropic secant “stiffness” tensor □$^{\mathrm{s}}(\boldsymbol{d})$, that is, by secant bulk and shear moduli $k^{\mathrm{s}}(I_1',J_2')$ and $\mu^{\mathrm{s}}(I_1',J_2')$:

$$\left.\begin{array}{l}C^{\mathrm{s}}(\boldsymbol{d})=3k^{\mathrm{s}}(I_1',J_2')J+2\mu^{\mathrm{s}}(I_1',J_2')K\\ k^{\mathrm{s}}(I_1',J_2')=\dfrac{1}{I_1'}\dfrac{\partial\pi^{\mathrm{s}}}{\partial I_1'}(I_1',J_2')\\ 2\mu^{\mathrm{s}}(I_1',J_2')=\dfrac{\partial\pi^{\mathrm{s}}}{\partial J_2'}(I_1',J_2')\end{array}\right\} \tag{33}$$

2.6. Principle of nonlinear homogenization

Taking Eq. (32) into account, we note that Eq. (24b) and (24c) can be summarized as follows:

$$\boldsymbol{\sigma}(\underline{z})=C\,(\underline{z}):\boldsymbol{d}(\underline{z}) \tag{34}$$

$$C\,(\underline{z})=\begin{cases}C^{\mathrm{s}}(\boldsymbol{d}(\underline{z})) & (\underline{z}\in\Omega^{\mathrm{s}})\\ 0 & (\underline{z}\in\Omega^{\mathrm{p}})\end{cases}$$

Accordingly, the boundary value problem (Eq. (24)) now reads

$$\mathrm{div}\,\boldsymbol{\sigma}=0 \quad (\Omega) \tag{35a}$$

$$\boldsymbol{\sigma}(\underline{z})=C\,(\underline{z}):\boldsymbol{d}(\underline{z}) \quad (\Omega) \tag{35b}$$

$$\boldsymbol{d}=\frac{1}{2}(\mathrm{grad}\,\underline{v}+{}^{\mathrm{t}}\,\mathrm{grad}\,\underline{v}) \quad (\Omega) \tag{35c}$$

$$\underline{v}(\underline{z})=\boldsymbol{D}\underline{z} \quad (\partial\Omega) \tag{35d}$$

In this form, Eq. (35a)-(35d) is formally identical to the problem shown in Eq. (3) introduced in Section 2.1, provided that the strain $\boldsymbol{\varepsilon}$ (resp. the displacement $\underline{\xi}$) is replaced by the strain rate $\boldsymbol{d}$ (resp. the velocity $\underline{v}$).

Still, two essential differences exist between Eqs. (3) and (35). In Eq. (3), the elastic stiffness is homogeneous in the solid phase and is independent of the loading. By contrast, like the strain rate $\boldsymbol{d}(\underline{z})$, the tensor $C^{\mathrm{s}}(\boldsymbol{d}(\underline{z}))$ which appears in Eqs. (34), (35a) and (35b) is heterogeneous and depends on the load level. The so-called secant methods in nonlinear homogenization aim at capturing the dependence of $C\,(\underline{z})=C^{\mathrm{s}}(\boldsymbol{d}(\underline{z}))$ on the loading level in an average way. The idea consists in introducing a so-called reference strain rate field $\boldsymbol{d}^{\mathrm{r}}$ in Ω^{s} and in approximating the “real” heterogeneous stiffness by a uniform value in the whole solid phase:

$$C\,(\underline{z})=C^{\mathrm{s}}(\boldsymbol{d}(\underline{z}))\approx C^{\mathrm{s}}(\boldsymbol{d}^{\mathrm{r}}) \quad (\forall\underline{z}\in\Omega^{\mathrm{s}}) \tag{36}$$

Accordingly, $\boldsymbol{d}^{\mathrm{r}}$ is looked for in the form of an average of the strain rate field $\boldsymbol{d}(\underline{z})$ over Ω^{s}, that of course should depend on the load level. Indeed, there are various ways to implement Eq. (36) that differ in the choice of the reference strain rate. For a more complete presentation of nonlinear homogenization, one can refer to Suquet (1997). The simplest choice consists in defining $\boldsymbol{d}^{\mathrm{r}}$ as the intrinsic average of the strain rate over the solid phase:

$$\boldsymbol{d}^{\mathrm{r}}=\overline{\boldsymbol{d}(\underline{z})}^{\mathrm{s}} \tag{37}$$

In particular, a reference volume strain rate can be defined as

$$d_{\mathrm{v}}^{\mathrm{r}}=\overline{I_1'}^{\mathrm{s}}=\mathrm{tr}\overline{\boldsymbol{d}}^{\mathrm{s}} \tag{38}$$

Alternatively, a second-order moment of the type introduced in Eq. (11) can be used:

$$(d_{\mathrm{v}}^{\mathrm{r}})^2=\overline{(I_1')^2}^{\mathrm{s}}=\overline{(\mathrm{tr}\boldsymbol{d})^2}^{\mathrm{s}} \tag{39}$$

In turn, the reference deviatoric strain rate will be defined from the second-order moment of the type introduced in Eq. (10):

$$d_{\mathrm{d}}^{\mathrm{r}}=\sqrt{\overline{J_2'}^{\mathrm{s}}}=\sqrt{\frac{1}{2}\overline{\boldsymbol{\delta}:\boldsymbol{\delta}}^{\mathrm{s}}} \tag{40}$$

The definitions in Eqs. (38) and (40) are first adopted. For comparison purposes, the definitions in Eqs. (39) and (40) will also be considered. With these elements in hand, let us summarize the successive steps of the secant approach of nonlinear homogenization:

(1) With the approximation of Eq. (36), Eq. (35) reduces to a standard problem of heterogeneous linear elasticity of the type in Eq. (3), which reads

$$\left.\begin{array}{ll} \boldsymbol{v}(\boldsymbol{z}) = \boldsymbol{D}\boldsymbol{z} & (\partial\Omega) \\ \operatorname{div}\boldsymbol{\sigma} = 0 & (\Omega) \\ \boldsymbol{\sigma} = C^{\mathrm{s}}(d_{\mathrm{v}}^{\mathrm{r}}, d_{\mathrm{d}}^{\mathrm{r}}) : \boldsymbol{d} & (\Omega^{\mathrm{s}}) \\ \boldsymbol{\sigma} = 0 & (\Omega^{\mathrm{p}}) \end{array}\right\} \qquad (41)$$

It is therefore possible to determine the macroscopic stress $\boldsymbol{\Sigma} = \bar{\boldsymbol{\sigma}}$ as that in Eqs. (4) and (5):

$$\left.\begin{array}{l} \boldsymbol{\Sigma} = C^{\mathrm{hom}} : \boldsymbol{D} \\ C^{\mathrm{hom}} = C^{\mathrm{s}}(d_{\mathrm{v}}^{\mathrm{r}}, d_{\mathrm{d}}^{\mathrm{r}}) : (I - \varphi \bar{A}^{\mathrm{p}}(d_{\mathrm{v}}^{\mathrm{r}}, d_{\mathrm{d}}^{\mathrm{r}})) \end{array}\right\} \qquad (42)$$

Eq. (42) represents the first step of the nonlinear homogenization problem.

(2) The second step consists in determining the reference strain as a function of the loading level according to the adopted definition. This step can be performed using the results of Section 2.1 concerning the first- and second-order moments of the strain field, applied here to the strain rate field. Formally, they yield $d_{\mathrm{v}}^{\mathrm{r}}$ and $d_{\mathrm{d}}^{\mathrm{r}}$ as a function of $\boldsymbol{D}$:

$$d_{\mathrm{v}}^{\mathrm{r}} = d_{\mathrm{v}}^{\mathrm{r}}(\boldsymbol{D}),\ d_{\mathrm{d}}^{\mathrm{r}} = d_{\mathrm{d}}^{\mathrm{r}}(\boldsymbol{D}) \qquad (43)$$

It is worth underlining that these developments have been obtained in a linear framework.

(3) The last step consists in solving the nonlinearity of the problem shown in Eqs. (42) and (43), which comes from the dependence of $\square^{\mathrm{hom}}$ on $d_{\mathrm{v}}^{\mathrm{r}}$ and $d_{\mathrm{d}}^{\mathrm{r}}$, with the latter being functions of $\boldsymbol{D}$. Combining these equations, the macroscopic state equation takes the form:

$$\boldsymbol{\Sigma} = C^{\mathrm{hom}}(\boldsymbol{D}) : \boldsymbol{D} \qquad (44)$$

It is important to note that the result of this nonlinear homogenization technique depends on the linear homogenization scheme which is chosen for relating $\square^{\mathrm{hom}}$ to $C^{\mathrm{s}}(d_{\mathrm{v}}^{\mathrm{r}}, d_{\mathrm{d}}^{\mathrm{r}})$ (first step). In particular, this choice incorporates morphological assumptions concerning the geometry of the microstructure (matrix-inclusion concept or polycrystal-like microstructure). These assumptions yield very different estimates of the effective stiffness. Similar results are therefore expected as regards the effective strength.

3. Green's strength criterion for the solid phase

3.1. The equivalent viscous behavior

We want to apply the method of Sections 2.4 and 2.6 to the case of a solid of the Green type, defined by the strength criterion:

$$f^{\mathrm{s}}(\boldsymbol{\sigma}) = \left(\frac{I_1}{L}\right)^2 + (\sqrt{J_2})^2 - k^2 \le 0 \qquad (45)$$

In the $(I_1, \sqrt{J_2})$ space, the set G^{s} of strength-compatible stress states associated with Eq. (45) is an ellipse centered at the origin. Note that the von Mises solid criterion is obtained asymptotically as $L \to \infty$:

$$f_{\mathrm{VM}}^{\mathrm{s}}(\boldsymbol{\sigma}) = J_2 - k^2 \le 0 \qquad (46)$$

We seek the set G^{hom} of macroscopic stress states compatible with the strength of the solid defined by Eq. (45). The methodology of Section 2.6 then yields an estimate for the domain G^{hom}.

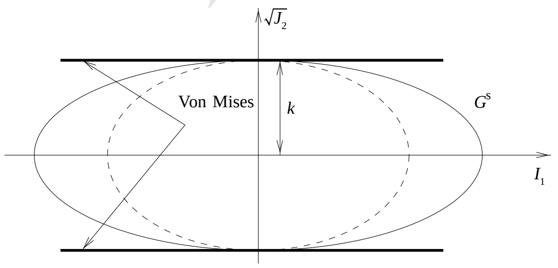

Fig. 2. Green's elliptic criterion. The von Mises criterion as an asymptotic case.

3.2. Homogenization of the fictitious viscous behavior

We first have to derive the support function $\pi^{\mathrm{s}}(\boldsymbol{d})$ of G^{s}. For a given value of the strain rate $\boldsymbol{d}$, we recall (see Section 2.2 and Fig. 1) that the maximum value of $\boldsymbol{\sigma} : \boldsymbol{d}$ is reached at the point $\boldsymbol{\sigma}^*$ where $\boldsymbol{d}$ is normal to ∂G^{s}. $\boldsymbol{d}$ is therefore parallel to $\partial f^{\mathrm{s}} / \partial\boldsymbol{\sigma}(\boldsymbol{\sigma}^*)$:

$$\boldsymbol{d} = \dot{\lambda}\frac{\partial f^{\mathrm{s}}}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}^*) \qquad (47)$$

where $\dot{\lambda}$ is a positive scalar.

Using Eq. (45), we successively obtain:

$$\boldsymbol{d} = \dot{\lambda}\left(2\frac{I_1^*}{L^2}\boldsymbol{1} + \boldsymbol{s}^*\right) \qquad (48)$$

$$\pi^{\mathrm{s}}(\boldsymbol{d}) = \dot{\lambda}\boldsymbol{\sigma}^* : \frac{\partial f^{\mathrm{s}}}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}^*) = 2\dot{\lambda}k^2 \qquad (49)$$

With the same notations as those in Eq. (28), we now observe from Eq. (48) that

$$I_1' = \dot{\lambda}\frac{6I_1^*}{L^2},\ J_2' = \dot{\lambda}^2 J_2^* \qquad (50)$$

Finally, a combination of the previous equations with Eq. (45) allows us to eliminate $\dot{\lambda}$, and the support function becomes

$$\pi^{\mathrm{s}}(\boldsymbol{d}) = 2k\sqrt{\frac{L^2}{36}(I_1')^2 + J_2'} \qquad (51)$$

The secant stiffness $C^{\mathrm{s}}(\boldsymbol{d})$ is then given by Eq. (33), with the following secant bulk and shear moduli:

$$k^{\mathrm{s}}(I_1', J_2') = k\frac{L^2/18}{\sqrt{J_2' + L^2(I_1')^2/36}} \qquad (52a)$$

$$2\mu^{\mathrm{s}}(I_1', J_2') = k\frac{1}{\sqrt{J_2' + L^2(I_1')^2/36}} \qquad (52b)$$

Interestingly, the ratio $\rho = k^{\mathrm{s}}/\mu^{\mathrm{s}} = L^2/9$ does not depend on either I_1' or J_2'. Let us now apply the method proposed in Section 2.6 to the porous material composed of the fictitious solid with stiffness $C^{\mathrm{s}}(\boldsymbol{d})$ (problem presented in Eqs. (34) and (35)). We start by writing the macroscopic behavior in the form of Eq. (42):

$$\frac{1}{3}\mathrm{tr}\boldsymbol{\Sigma} = k^{\mathrm{hom}}\mathrm{tr}\boldsymbol{D},\ \boldsymbol{\Sigma}_{\mathrm{d}} = 2\mu^{\mathrm{hom}}\boldsymbol{\Delta} \qquad (53)$$

where $\boldsymbol{\Sigma}_{\mathrm{d}}$ (resp. $\boldsymbol{\Delta}$) is the macroscopic deviatoric stress (resp. strain rate):

$$\boldsymbol{\Sigma} = \frac{1}{3}(\mathrm{tr}\boldsymbol{\Sigma})\boldsymbol{1} + \boldsymbol{\Sigma}_{\mathrm{d}},\ \boldsymbol{D} = \frac{1}{3}(\mathrm{tr}\boldsymbol{\Sigma})\boldsymbol{1} + \boldsymbol{\Delta} \qquad (54)$$

In the first step of the nonlinear homogenization process, we now need to relate k^{hom} and μ^{hom} to k^{s}, μ^{s} and the porosity φ. This requires selecting a linear homogenization scheme. The dimensional analysis shows that

$$k^{\mathrm{hom}} = K(\varphi,\rho)\mu^{\mathrm{s}},\ \mu^{\mathrm{hom}} = M(\varphi,\rho)\mu^{\mathrm{s}} \qquad (55)$$

where K and M are the dimensionless functions. For forthcoming use, we note that

$$\left.\begin{array}{l} \dfrac{\partial k^{\mathrm{hom}}}{\partial\mu^{\mathrm{s}}} = K(\varphi,\rho) - \rho\dfrac{\partial K}{\partial\rho}(\varphi,\rho) \\ \dfrac{\partial\mu^{\mathrm{hom}}}{\partial\mu^{\mathrm{s}}} = M(\varphi,\rho) - \rho\dfrac{\partial M}{\partial\rho}(\varphi,\rho) \end{array}\right\} \qquad (56)$$

$$\frac{\partial k^{\mathrm{hom}}}{\partial k^{\mathrm{s}}} = \frac{\partial K}{\partial\rho}(\varphi,\rho),\ \frac{\partial\mu^{\mathrm{hom}}}{\partial k^{\mathrm{s}}} = \frac{\partial M}{\partial\rho}(\varphi,\rho) \qquad (57)$$

As already stated, the Mori-Tanaka estimate of the effective behavior is implicitly associated with a matrix-inclusion morphology, in which the pores play the role of the inclusion phase. In contrast, the perfectly disordered microstructure can be addressed within the framework of a self-consistent approach. The Mori-Tanaka estimates of M and K read

$$\left.\begin{array}{l} K_{\mathrm{mt}}(\varphi,\rho) = \dfrac{4(1-\varphi)\rho}{3\rho\varphi + 4} \\ M_{\mathrm{mt}}(\varphi,\rho) = \dfrac{(1-\varphi)(9\rho+8)}{9\rho(1+2\varphi/3) + 8(1+3\varphi/2)} \end{array}\right\} \qquad (58)$$

For simplicity, the self-consistent estimates of M and K are given only in the asymptotic case where $L \to \infty$:

$$K_{\rm sc}(\varphi)=4\frac{(1-2\varphi)(1-\varphi)}{\varphi(3-\varphi)},\ M_{\rm sc}(\varphi)=3\frac{1-2\varphi}{3-\varphi} \tag{59}$$

The second step of the nonlinear homogenization procedure deals with the determination of the reference strain as a function of the macroscopic loading. It is recalled that this step is performed in the framework of linear elasticity. We need estimates of $d_{\rm v}^{\rm r}$ and $d_{\rm d}^{\rm r}$. As for $d_{\rm d}^{\rm r}$ defined by Eq. (40), the average of J_2' over the solid given by Eq. (10) is considered:

$$(d_{\rm d}^{\rm r})^2=\frac{1}{2(1-\varphi)}\left[\frac{1}{2}({\rm tr}\boldsymbol{D})^2\frac{\partial k^{\rm hom}}{\partial\mu^{\rm s}}+\boldsymbol{\Delta}:\boldsymbol{\Delta}\frac{\partial\mu^{\rm hom}}{\partial\mu^{\rm s}}\right] \tag{60}$$

where the derivatives of $k^{\rm hom}$ and $\mu^{\rm hom}$ are obtained from Eq. (56). In turn, an expression of $d_{\rm d}^{\rm r}$ as a function of the macroscopic stress is obtained:

$$(d_{\rm d}^{\rm r})^2=\frac{1}{4(1-\varphi)(\mu^{\rm s})^2}\left[\left(\frac{\Sigma_{\rm m}}{K}\right)^2\frac{\partial k^{\rm hom}}{\partial\mu^{\rm s}}+\left(\frac{\Sigma_{\rm d}}{M}\right)^2\frac{\partial\mu^{\rm hom}}{\partial\mu^{\rm s}}\right] \tag{61}$$

where the following notations have been used:

$$\Sigma_{\rm m}=\frac{1}{3}{\rm tr}\boldsymbol{\Sigma},\ \Sigma_{\rm d}=\sqrt{\frac{1}{2}\boldsymbol{\Sigma}_{\rm d}:\boldsymbol{\Sigma}_{\rm d}} \tag{62}$$

Similarly, as for $d_{\rm v}^{\rm r}$ as defined by Eq. (38), we replace the macroscopic strain $\boldsymbol{E}$ by the macroscopic strain rate $\boldsymbol{D}$, and select the average of I_1' over the solid provided by Eq. (9):

$$(1-\varphi)I_1'^{\rm r}=\frac{\Sigma_{\rm m}}{k^{\rm s}} \tag{63}$$

Owing to Eq. (52), this yields

$$(1-\varphi)I_1'^{\rm r}=\frac{9}{L^2}\frac{\Sigma_{\rm m}}{\mu^{\rm s}} \tag{64}$$

Alternatively, if the definition in Eq. (39) is used, we obtain from Eq. (11) that

$$(d_{\rm v}^{\rm r})^2=\frac{1}{1-\varphi}\left[({\rm tr}\boldsymbol{D})^2\frac{\partial k^{\rm hom}}{\partial k^{\rm s}}+2\boldsymbol{\Delta}:\boldsymbol{\Delta}\frac{\partial\mu^{\rm hom}}{\partial k^{\rm s}}\right] \tag{65}$$

This yields

$$(d_{\rm v}^{\rm r})^2=\frac{1}{(1-\varphi)(\mu^{\rm s})^2}\left[\left(\frac{\Sigma_{\rm m}}{K}\right)^2\frac{\partial k^{\rm hom}}{\partial k^{\rm s}}+\left(\frac{\Sigma_{\rm d}}{M}\right)^2\frac{\partial\mu^{\rm hom}}{\partial k^{\rm s}}\right] \tag{66}$$

The third and last steps consist in dealing with the nonlinearity which comes from the fact that $\mu^{\rm s}$ depends on $J_2'^{\rm r}$ and $I_1'^{\rm r}$ as stated by Eq. (52). Introducing Eqs. (61) and (64) to Eq. (52) yields

$$(1-\varphi)k^2=\Sigma_{\rm m}^2\left[\frac{9}{L^2(1-\varphi)}+\frac{1}{K^2}\frac{\partial k^{\rm hom}}{\partial\mu^{\rm s}}\right]+\Sigma_{\rm d}^2\frac{1}{M^2}\frac{\partial\mu^{\rm hom}}{\partial\mu^{\rm s}} \tag{67}$$

Alternatively, if the definition in Eq. (39) is retained, substituting Eqs. (61) and (66) into Eq. (52) yields

$$(1-\varphi)k^2=\left(\frac{\Sigma_{\rm m}}{K}\right)^2\left(\frac{\partial k^{\rm hom}}{\partial\mu^{\rm s}}+\frac{L^2}{9}\frac{\partial k^{\rm hom}}{\partial k^{\rm s}}\right)+\left(\frac{\Sigma_{\rm d}}{M}\right)^2\left(\frac{\partial\mu^{\rm hom}}{\partial\mu^{\rm s}}+\frac{L^2}{9}\frac{\partial\mu^{\rm hom}}{\partial k^{\rm s}}\right) \tag{68}$$

Owing to the fact that $\rho=L^2/9$, together with Eqs. (56) and (57), Eq. (68) reduces to

$$(1-\varphi)k^2=\frac{\Sigma_{\rm m}^2}{K}+\frac{\Sigma_{\rm d}^2}{M} \tag{69}$$

Within the framework of the secant approximation, Eqs. (67) (reference strains in Eqs. (38)-(40)) and (69) (reference strains in Eqs. (39) and (40)) represent the asymptotic locations in the stress space of the macroscopic stress state solutions of Eq. (35), for arbitrary orientations of $\boldsymbol{D}$. In other words, it defines the boundary of $G^{\rm hom}$ which is found to be a closed elliptic domain centered at the origin of the ($\Sigma_{\rm m}$, $\Sigma_{\rm d}$) plane. It is recalled that Eqs. (56) and (57) are to be used together with $\rho=L^2/9$.

3.3. The case of a von Mises solid ($L\to\infty$)

For simplicity, the limit case $L\to\infty$ is now considered. In this case, it is first emphasized that Eqs. (67) and (69) yield identical results. Let us discuss the influence of the morphology of the microstructure and of the corresponding homogenization scheme. In particular, for the matrix-inclusion morphology, use of the Mori-Tanaka scheme yields (see Eq. (58) (for a discussion on this type of criterion as compared to the one derived by Gurson (1977), one can see Gologanu et al. (1997)):

$$\frac{3\varphi}{4}\Sigma_{\rm m}^2+\left(1+\frac{2\varphi}{3}\right)\Sigma_{\rm d}^2=k^2(1-\varphi)^2 \tag{70}$$

First, we note that Eq. (46) is retrieved for $\varphi=0$. The other limit case corresponds to $\varphi\to 1$ for which we observe that the effective strength vanishes. Conversely, some strength is available even for high values of the porosity, provided that $\varphi<1$. This should be attributed to the matrix-inclusion morphology which has been considered here through the use of the Mori-Tanaka estimate.

Consider next the self-consistent (or polycrystal) scheme which captures morphology of a perfectly disordered solid phase intermixed with porosity. Introducing Eq. (59) into Eq. (69) yields the following self-consistent estimate of the homogenized strength criterion:

$$\frac{3\varphi}{4}\Sigma_{\rm m}^2+(1-\varphi)\Sigma_{\rm d}^2=k^2\frac{(1-\varphi)^2(1-2\varphi)}{1-\varphi/3} \tag{71}$$

As seen in the previous case, Eq. (46) is retrieved for $\varphi=0$. However, the homogenized strength now vanishes for $\varphi\geq 1/2$. As for the stiffness (see Eq. (59)), the macroscopic strength exhibits a percolation threshold of the pore space at $\varphi=1/2$.

The domains of admissible macroscopic stress states corresponding to Eqs. (70) and (71), respectively, are shown in Fig. 3.

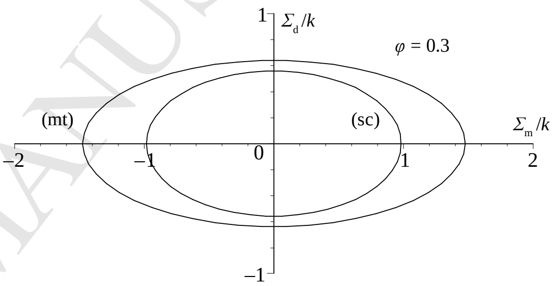

Fig. 3. Mori-Tanaka (mt) and self-consistent (sc) estimates of the domain of admissible macroscopic stress states (von Mises solid).

3.4. Validation

It is instructive to compare the results obtained with the Mori-Tanaka scheme in Section 3.2 with the ones of the hollow sphere model. In fact, the geometry of the hollow sphere in which the cavity is surrounded by the solid is a very particular form of the matrix-inclusion morphology captured by the Mori-Tanaka scheme. Furthermore, despite its limitation, the hollow sphere model provides a reasonable estimate of the strength under hydrostatic compression or traction of both microscopic and macroscopic isotropic materials. The strength domain of the hollow sphere under isotropic loading reads

$$|\Sigma_{\rm m}|\leq\frac{2k}{\sqrt{3}}\ln\varphi \tag{72}$$

In turn, the Mori-Tanaka estimate (Eq. (70)) yields the following hydrostatic strength limits of the empty porous material (contraction or compression):

$$|\Sigma_{\rm m}|\leq\frac{2k}{\sqrt{3}}\frac{1-\varphi}{\sqrt{\varphi}} \tag{73}$$

Fig. 4 displays an excellent agreement between the estimates from Eqs. (72) and (73), except for infinitesimal values of the porosity. For such small values, high strain rates are expected to concentrate around the pores, which cannot be captured by the reference strain rate concept (Eq. (36)). In fact, an average value over the whole solid phase fails to provide an accurate estimate of the local strain rate level. This is why we observe a divergence of the estimate (Eq. (73)) from the more accurate estimate (Eq. (72)).

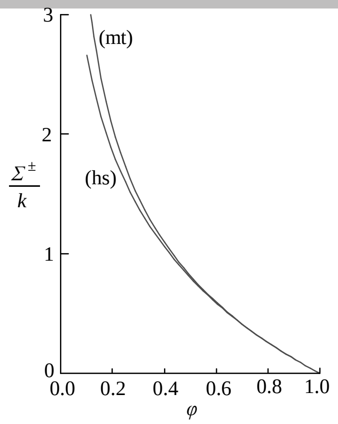

Fig. 4. Von Mises solid: Hydrostatic strength predicted by the hollow sphere (hs) model and the Mori-Tanaka (mt) scheme.

3.5. Theoretical background

Following Ponte Castaneda (1997), a theoretical support to the secant method presented in Section 3.2 can be obtained if the reference strains are defined by Eqs. (38)-(40). In this case, the homogenized strength predicted by the secant method is a rigorous upper bound of G^{hom}. In order to prove this result, the definition of its support function is considered:

$$\Pi^{\text{hom}}(\boldsymbol{D}) = f_{\text{s}}\left(\inf_{\boldsymbol{v}k.a.\boldsymbol{D}} \frac{1}{|\Omega^{\text{s}}|}\int_{\Omega^{\text{s}}} \pi(\boldsymbol{d})\,\mathrm{d}V\right) \tag{74}$$

where $f_{\text{s}} = 1-\varphi$ is the volume fraction of the solid in the REV. Incorporating the expression of $\pi(\boldsymbol{d})$ (Eq. (51)), it is observed that an upper bound of $\Pi^{\text{hom}}(\boldsymbol{D})$ can be obtained in the form:

$$\Pi^{\text{hom}}(\boldsymbol{D}) = f_{\text{s}} \inf_{\boldsymbol{v}k.a.\boldsymbol{D}} k\sqrt{\frac{1}{|\Omega^{\text{s}}|}\int_{\Omega^{\text{s}}}\left(\frac{L^2}{9}d_{\text{v}}^2 + 2\boldsymbol{\delta}:\boldsymbol{\delta}\right)\mathrm{d}V} \tag{75}$$

which also reads

$$\Pi^{\text{hom}}(\boldsymbol{D}) \le \Pi_1(\boldsymbol{D}) = f_{\text{s}} \inf_{\boldsymbol{v}k.a.\boldsymbol{D}} k\left(\sqrt{\frac{L^2}{9}(d_{\text{v}}^{\text{r}})^2 + 4(d_{\text{d}}^{\text{r}})^2}\right) = f_{\text{s}} \inf_{\boldsymbol{v}k.a.\boldsymbol{D}} \pi(d_{\text{v}}^{\text{r}}, d_{\text{d}}^{\text{r}}) \tag{76}$$

where the subscript “1” recalls that the method resorts to averages defined on a unique zone, namely, the whole solid phase Ω^{s}. In the next section, Ω^{s} will be divided into two subdomains, with a specific average defined on each.

In Eq. (76), the definitions in Eqs. (39) and (40) of the reference strains are adopted (quadratic averages). The important result lies in the fact that Eq. (76) then defines a variational problem whose solution is also the one to the problem of elasticity defined by Eq. (41). The latter is none but the problem imposed by the modified secant method. Since $\Pi_1(\boldsymbol{D})$ is an upper bound of $\Pi^{\text{hom}}(\boldsymbol{D})$, Eq. (69) is in turn an upper bound of the exact domain G^{hom}. For this reason, Eq. (76) provides a theoretical justification to the modified secant method. We therefore have to focus on the solution to Eq. (41), and more precisely on how to determine the averages of d_{v}^{r} and d_{d}^{r} in Eqs. (39) and (40), respectively.

For practical implementation, estimates for $\rho = L^2/9$ of the quantities $K(\varphi, \rho)$ and $M(\varphi, \rho)$ introduced in Eq. (55) are due. In general, Eq. (69) can lose its interpretation as an upper bound when the exact functions $K(\varphi, \rho)$ and $M(\varphi, \rho)$ are replaced by estimates. Clearly enough, if the estimates at stake are upper bounds, the corresponding estimated boundary Eq. (69) will remain an upper bound. This is in particular the case if the Hashin-Shtrikman upper bound is used.

4. Implementation of the secant method with two zones

The originality of the implementation of the secant method presented hereafter lies in the fact that the solid phase (s) is arbitrarily split in two subdomains, respectively denoted by Ω_1^{s} and Ω_2^{s}. The purpose is to take into account the heterogeneity of the strains in the solid phase that is induced by the nonlinear behavior, while the classical secant method is based on a single reference strain d_{d} for the whole solid phase. Hence, the idea is to introduce two distinct estimates $d_{\text{d}}^{(1)}$ and $d_{\text{d}}^{(2)}$, respectively, for Ω_1^{s} and Ω_2^{s}. Ω_1^{s} is qualitatively defined as a set of solid domains surrounding the pores while the complementary region Ω_2^{s} can be viewed as a matrix in which composite inclusions (pores and the surrounding solid) are embedded. The problem to be solved now reads

$$\left.\begin{array}{l} \boldsymbol{v}(\boldsymbol{z}) = \boldsymbol{D}\boldsymbol{z} \quad (\partial\Omega) \\ \mathrm{div}\,\boldsymbol{\sigma} = 0 \quad (\Omega) \\ \boldsymbol{\sigma} = \mathbb{C}\,(d_{\text{d}}^{(1)}) : \boldsymbol{d} \quad (\Omega_1^{\text{s}}) \\ \boldsymbol{\sigma} = \mathbb{C}\,(d_{\text{d}}^{(2)}) : \boldsymbol{d} \quad (\Omega_2^{\text{s}}) \\ \boldsymbol{\sigma} = 0 \quad (\Omega^{\text{p}}) \end{array}\right\} \tag{77}$$

In order to solve Eq. (77), we have to introduce two distinct shear moduli μ_1 and μ_2 respectively for Ω_1^{s} and Ω_2^{s}. The average rule in Eq. (60) is applied twice:

$$2 f_1 \overline{(d_{\text{d}}^{(1)})^2}^{\text{s}} = \frac{1}{2}\frac{\partial k^{\text{hom}}}{\partial \mu_1}(\mathrm{tr}\boldsymbol{D})^2 + \frac{\partial \mu^{\text{hom}}}{\partial \mu_1}\boldsymbol{\Delta}:\boldsymbol{\Delta} \tag{78}$$

$$2 f_2 \overline{(d_{\text{d}}^{(2)})^2}^{\text{s}} = \frac{1}{2}\frac{\partial k^{\text{hom}}}{\partial \mu_2}(\mathrm{tr}\boldsymbol{D})^2 + \frac{\partial \mu^{\text{hom}}}{\partial \mu_2}\boldsymbol{\Delta}:\boldsymbol{\Delta} \tag{79}$$

Owing to the definition of Ω_1^{s}, it is convenient to represent the discrete components of the latter by a so-called morphological representative pattern (MRP), namely a solid sphere with a spherical cavity. Let f denote the volume fraction of the cavity in the MRP. In turn, Ω_2^{s} plays the role of a matrix in which a set of such MRPs are embedded. Let ϕ denote the volume fraction of the MRPs in Ω. The volume fractions of Ω_1^{s} and Ω_2^{s} are respectively $f_1 = \phi(1-f)$ and $f_2 = 1-\phi$ ($f_1 + f_2 = 1-\varphi$). Note that the porosity φ and the variables ϕ and f are related by $\varphi = f\phi$. This morphological model thus includes a degree of freedom to be optimized (either f or ϕ).

Prior to implement the method, let us show that this new approach based on a partition of the solid provides a better upper bound than the one derived by the classical modified secant method.

Starting from Eqs. (74) and (75), we may write

$$\Pi^{\text{hom}}(\boldsymbol{D}) \le k \inf_{\boldsymbol{v}k.a.\boldsymbol{D}}\left[f_1\sqrt{\frac{1}{|\Omega_1^{\text{s}}|}\int_{\Omega_1^{\text{s}}}\left(\frac{L^2}{9}d_{\text{v}}^2 + 2\boldsymbol{\delta}:\boldsymbol{\delta}\right)\mathrm{d}V} + f_2\sqrt{\frac{1}{|\Omega_2^{\text{s}}|}\int_{\Omega_2^{\text{s}}}\left(\frac{L^2}{9}d_{\text{v}}^2 + 2\boldsymbol{\delta}:\boldsymbol{\delta}\right)\mathrm{d}V}\right] \tag{80}$$

or equivalently,

$$\Pi^{\text{hom}}(\boldsymbol{D}) \le \Pi_2(\boldsymbol{D}) \tag{81}$$

$$\Pi_2(\boldsymbol{D}) = \inf_{\boldsymbol{v}k.a.\boldsymbol{D}}\left[f_1\pi(d_{\text{v}}^{(1)}, d_{\text{d}}^{(1)}) + f_2\pi(d_{\text{v}}^{(2)}, d_{\text{d}}^{(2)})\right] \tag{82}$$

where the index “2” stands for the “2 zones” method.

It is also readily seen that

$$f_1\pi(d_{\text{v}}^{(1)}, d_{\text{d}}^{(1)}) + f_2\pi(d_{\text{v}}^{(2)}, d_{\text{d}}^{(2)}) \le f_{\text{s}}\pi(d_{\text{v}}^{\text{r}}, d_{\text{d}}^{\text{r}}) \tag{83}$$

$$\Pi_2(\boldsymbol{D}) \le \Pi_1(\boldsymbol{D}) \tag{84}$$

imply that $\Pi_2(\boldsymbol{D})$ is a better upper bound of $\Pi^{\text{hom}}(\boldsymbol{D})$ than $\boldsymbol{\Pi_1}(\boldsymbol{D})$. The theoretical proof of this result lies in the fact that the solution to the variational problem defined by Eq. (82) is the velocity solution to Eq. (77).

At that stage, estimates for $\Pi_2(\boldsymbol{D})$ still goes through the choice made for the MRPs volume fraction ϕ. Indeed, the best upper bound for $\Pi^{\text{hom}}(\boldsymbol{D})$ is obtained as the minimum value of $\Pi_2(\boldsymbol{D})$ with respect to ϕ.

In the following sections, hydrostatic and deviatoric loadings are respectively considered in the limit case $L \to \infty$. Thus, concerning Eq. (84), it may be sufficient to only consider the d_{d} term in the definition of the support function π, leading to $\pi(\boldsymbol{d}) \approx 2kd_{\text{d}}$ and

$$\Pi_2(\boldsymbol{D}) = 2k \inf_{\boldsymbol{v}k.a.\boldsymbol{D}}\left(f_1 d_{\text{d}}^{(1)} + f_2 d_{\text{d}}^{(2)}\right) \tag{85}$$

4.1. Hydrostatic loading

Let us consider the macroscopic strain rate tensor $\boldsymbol{D} = D\mathbf{1}$ with $D > 0$. Owing to the definition of $\Pi^{\mathrm{hom}}(\boldsymbol{D})$, the isotropic tensile strength $\boldsymbol{\Sigma} = \Sigma_{\mathrm{m}}\mathbf{1}$ is characterized by

$$\Sigma_{\mathrm{m}}^{+} = \frac{1}{3D}\Sigma^{\mathrm{hom}}(\boldsymbol{D}) \tag{86}$$

An upper bound of Σ_{m}^{+} is determined by using $\Pi_2(\boldsymbol{D})$ as an upper bound of $\Pi^{\mathrm{hom}}(\boldsymbol{D})$. Combining Eqs. (78) and (79) yields

$$d_{\mathrm{d}}^{(1)} = \frac{3D}{2}\sqrt{\frac{1}{f_1}\frac{\partial k^{\mathrm{hom}}}{\partial \mu_1}},\; d_{\mathrm{d}}^{(2)} = \frac{3D}{2}\sqrt{\frac{1}{f_2}\frac{\partial k^{\mathrm{hom}}}{\partial \mu_2}} \tag{87}$$

which have to be used in Eq. (85) in order to get

$$\Pi_2(\boldsymbol{D}) = 3Dk\left(\sqrt{f_1\frac{\partial k^{\mathrm{hom}}}{\partial \mu_1}} + \sqrt{f_2\frac{\partial k^{\mathrm{hom}}}{\partial \mu_2}}\right) \tag{88}$$

The homogenized bulk modulus takes the definition derived in Eq. (A8) in Appendix, so that Eq. (88) yields

$$\Pi_2(\boldsymbol{D}) = 2\sqrt{3}Dk\left(\frac{1-f}{\sqrt{f}} + \frac{1-\phi}{\sqrt{\phi}}\right) \tag{89}$$

Recalling that $f = \varphi/\phi$, the optimal value $\phi = \sqrt{\varphi}$ is then derived. The latter eventually gives the "2 zones" estimate of Σ_{m}^{+}:

$$\Sigma_{\mathrm{m}}^{+(2)} = \frac{4k}{\sqrt{3}}\frac{1-\sqrt{\varphi}}{\varphi^{1/4}} \tag{90}$$

which may be compared to the "1 zone" estimate:

$$\Sigma_{\mathrm{m}}^{+(1)} = \frac{2k}{\sqrt{3}}\frac{1-\varphi}{\sqrt{\varphi}} \tag{91}$$

together with the exact solution:

$$\Sigma_{\mathrm{m}}^{+(1)} = -\frac{2k}{\sqrt{3}}\log_{10}\varphi \tag{92}$$

4.2. Deviatoric loading

Using the same notations, the deviatoric strength reads

$$\Sigma_{\mathrm{d}}^{+} = \frac{\Pi^{\mathrm{hom}}(\boldsymbol{D})}{\sqrt{2\boldsymbol{D}:\boldsymbol{D}}} \tag{93}$$

where $\boldsymbol{D}$ is a purely deviatoric strain rate.

An upper bound of Σ_{d}^{+} is determined by using $\Pi_2(\boldsymbol{D})$ as an upper bound of $\Pi^{\mathrm{hom}}(\boldsymbol{D})$. Using Eq. (78) together with Eq. (79) yields

$$\left.\begin{aligned} d_{\mathrm{d}}^{(1)} &= \frac{1}{2}\sqrt{2\boldsymbol{D}:\boldsymbol{D}}\sqrt{\frac{1}{f_1}\frac{\partial \mu^{\mathrm{hom}}}{\partial \mu_1}} \\ d_{\mathrm{d}}^{(2)} &= \frac{1}{2}\sqrt{2\boldsymbol{D}:\boldsymbol{D}}\sqrt{\frac{1}{f_2}\frac{\partial \mu^{\mathrm{hom}}}{\partial \mu_2}} \end{aligned}\right\} \tag{94}$$

Introducing Eq. (94) in Eq. (85) allows to write

$$\Pi_2(\boldsymbol{D}) = k\sqrt{2\boldsymbol{D}:\boldsymbol{D}}\left(\sqrt{f_1\frac{\partial \mu^{\mathrm{hom}}}{\partial \mu_1}} + \sqrt{f_2\frac{\partial \mu^{\mathrm{hom}}}{\partial \mu_2}}\right) \tag{95}$$

which is nothing but the "deviatoric" formulation of Eq. (89). However, contrary to Eq. (89), Eq. (95) depends upon the shear moduli μ_1 and μ_2 through the ratio $\rho = \mu_1/\mu_2$. This difficulty was not accounted for in the hydrostatic loading case since the shear moduli μ_i (i = 1, 2) were not part of the derivatives of k^{hom}. The complementary equation is derived from Eq. (52b):

$$\rho = d_{\mathrm{d}}^{(2)} / d_{\mathrm{d}}^{(1)} \tag{96}$$

which, accounting for Eq. (94), also reads

$$\rho = \sqrt{\frac{f_1}{f_2}\frac{\partial \mu^{\mathrm{hom}}/\partial \mu_2}{\partial \mu^{\mathrm{hom}}/\partial \mu_1}} \tag{97}$$

where the homogenized shear modulus is derived from Eq. (A7).

The analytical resolution to this fourth-order polynomial equation in ρ is a complicated task in the general case. Recalling the optimal value $\phi = \sqrt{\varphi}$ derived in the hydrostatic loading case, we may consider the asymptotic development of ρ for low porosities:

$$\rho = 1 - \frac{5}{9}\sqrt{\varphi} + \frac{485}{162}\varphi - \frac{40}{9}\varphi^{4/3} - \frac{4345}{10206}\varphi^{3/2} \tag{98}$$

which has to be considered in Eq. (95).

The "2 zones" estimate of Σ_{d}^{+} is then defined as

$$\Sigma_{\mathrm{d}}^{+(2)} = k\left(1 - \frac{4}{3}\varphi - \frac{5}{54}\varphi^{3/2} + \frac{41}{27}\varphi^2\right) \tag{99}$$

which may be compared to the "1 zone" estimate obtained for the same asymptotic development:

$$\Sigma_{\mathrm{d}}^{+(1)} = k\left(1 - \frac{4}{3}\varphi + \frac{1}{2}\varphi^2\right) \tag{100}$$

5. Introduction to Gurson's model

In the context of the ductile failure of porous materials, the Gurson's model (Gurson, 1977) is well-known to provide an efficient approach of the strength reduction due to the porosity. The derivation of the Gurson's model presented below is based on the rigorous framework of limit analysis which can be found in Salencon (2001). Dormieux et al. (2006) also introduced the main concepts of this theory for the derivation of the macroscopic strength of ductile porous media.

The basic features of the classical Gurson approach are recalled. The latter deals with the case of a von Mises solid phase:

$$f^{\mathrm{s}}(\boldsymbol{\sigma}) = \frac{3}{2}\boldsymbol{s}:\boldsymbol{s} - \sigma_{\mathrm{o}}^2 \tag{101}$$

where $\boldsymbol{s}$ is the deviatoric part of $\boldsymbol{\sigma}$. The support function $\pi^{\mathrm{s}}(\boldsymbol{d})$ accordingly reads

$$\pi^{\mathrm{s}}(\boldsymbol{d}) = \sigma_{\mathrm{o}} d_{\mathrm{eq}},\; d_{\mathrm{eq}} = \sqrt{\frac{2}{3}\boldsymbol{d}:\boldsymbol{d}} \quad (\mathrm{tr}\boldsymbol{d} = 0) \tag{102a}$$

$$\pi^{\mathrm{s}}(\boldsymbol{d}) = +\infty \quad (\mathrm{tr}\boldsymbol{d} = 0) \tag{102b}$$

The Gurson's model introduces two simplifications. It first consists in representing the morphology of the porous material by a hollow sphere instead of the REV. Let R_{e} (resp. R_{i}) denote the external (resp. cavity) radius. The volume fraction of the cavity in the sphere is equal to the porosity $\varphi = (R_{\mathrm{i}}/R_{\mathrm{e}})^3$. Then, instead of seeking the infimum in Eq. (76), $\Pi^{\mathrm{hom}}(\boldsymbol{D})$ is estimated by a particular microscopic velocity field $\underline{v}(\underline{z})$. In the solid, the latter is defined as the sum of a linear part involving a uniform second-order tensor $\boldsymbol{A}$ and of the solution to an isotropic expansion in an incompressible medium. In spherical coordinates, it thus reads

$$\underline{v}^{\mathrm{G}}(\underline{z}) = \boldsymbol{A}\underline{z} + \alpha\frac{R_{\mathrm{i}}^3}{r^2}\underline{e}_r \tag{103}$$

In the pore domain, the strain rate is defined from the velocity at the cavity wall:

$$\boldsymbol{d}^{\mathrm{I}} = \boldsymbol{A} + \alpha\mathbf{1} \tag{104}$$

The local condition tr$\boldsymbol{d}$ = 0 has to be satisfied in the case of a von Mises material (see Eq. (102)). This implies that $\boldsymbol{A}$ is a deviatoric tensor: tr$\boldsymbol{A}$ = 0. Furthermore, the boundary condition Eq. (41) at $r = R_{\mathrm{e}}$ yields

$$\boldsymbol{D} = \boldsymbol{A} + \alpha\varphi\mathbf{1} \tag{105}$$

which reveals that $\boldsymbol{A}$ is the deviatoric part $\boldsymbol{\Delta}$ of $\boldsymbol{D}$, while α is related to its spherical part:

$$\boldsymbol{A} = \boldsymbol{\Delta},\; \alpha = \frac{1}{3\varphi}\mathrm{tr}\boldsymbol{D} \tag{106}$$

The combination of Eqs. (104) and (106) also yields

$$\boldsymbol{d}^{\mathrm{I}} = \boldsymbol{\Delta} + \frac{\mathrm{tr}\boldsymbol{D}}{3\varphi}\mathbf{1} \tag{107}$$

Recalling Eq. (22), the use of $\underline{v}^{\mathrm{G}}$ (giving strain rate $\boldsymbol{d}^{\mathrm{G}}$) provides an upper bound of Π^{hom}:

$$\Pi^{\mathrm{hom}}(\boldsymbol{D}) \le (1-\varphi)\overline{\pi^{\mathrm{s}}(\boldsymbol{d}^{\mathrm{G}})}^{\mathrm{s}} \tag{108}$$

Using Eq. (102), the derivation of the right-hand side in Eq. (108) requires determining the average of d_{eq} over Ω^{s}. In order to obtain an analytical expression, it is convenient to apply the following inequality to $\mathrm{G} = \boldsymbol{d}:\boldsymbol{d} = 3d_{\mathrm{eq}}^2/2$ (Gurson, 1977):

$$\int_{\Omega^{\mathrm{s}}}\sqrt{\mathrm{G}(r,\theta,\varphi)}\,\mathrm{d}V \le 4\pi\int_{R_{\mathrm{i}}}^{R_{\mathrm{e}}} r^2\left(<\mathrm{G}>_{\mathrm{S}(r)}\right)^{1/2}\mathrm{d}r \tag{109}$$

where $\mathrm{S}(r)$ is the sphere of radius r, and $<\mathrm{G}>_{\mathrm{S}(r)}$ is the average of

$\mathrm{G}(r,\theta,\varphi)$ over all the orientations:

$$<\mathrm{G}>_{\mathrm{S}(r)}=\frac{1}{4\pi}\int_{\mathrm{S}(r)}\mathrm{G}(r,\theta,\varphi)\,\mathrm{dS} \tag{110}$$

This eventually yields the following upper bound of $\Pi^{\mathrm{hom}}(\boldsymbol{D})$:

$$\Pi_{\mathrm{G}}^{\mathrm{hom}}(\boldsymbol{D})=\sigma_{\mathrm{o}}D_{\mathrm{eq}}\{\varphi\xi[\operatorname{arcsinh}\xi-\operatorname{arcsinh}(\varphi\xi)]+\sqrt{1+\varphi^2\xi^2}-\varphi\sqrt{1+\xi^2}\} \tag{111}$$

with $D_{\mathrm{eq}}=\sqrt{2\boldsymbol{\Delta}:\boldsymbol{\Delta}/3}$ and $\xi=2\alpha/D_{\mathrm{eq}}$. In the standard case (no interface effect), it is emphasized that the pore size R_{i} does not matter by itself since only the ratio $R_{\mathrm{i}}/R_{\mathrm{e}}=\varphi^{1/3}$ intervenes in Eq. (111).

The last step is the derivation of the limit states $\boldsymbol{\Sigma}=\partial\Pi_{\mathrm{G}}^{\mathrm{hom}}/\partial\boldsymbol{D}$. It is first observed that $\Pi_{\mathrm{G}}^{\mathrm{hom}}(\boldsymbol{D})$ is in fact a function of $\boldsymbol{D}$ through α and D_{eq}:

$$\boldsymbol{\Sigma}=\frac{\partial\Pi_{\mathrm{G}}^{\mathrm{hom}}}{\partial\alpha}\frac{\partial\alpha}{\partial\boldsymbol{D}}+\frac{\partial\Pi_{\mathrm{G}}^{\mathrm{hom}}}{\partial D_{\mathrm{eq}}}\frac{\partial D_{\mathrm{eq}}}{\partial\boldsymbol{D}} \tag{112}$$

where

$$\frac{\partial\alpha}{\partial\boldsymbol{D}}=\frac{1}{3\varphi}\mathbf{1},\ \frac{\partial D_{\mathrm{eq}}}{\partial\boldsymbol{D}}=\frac{2}{3D_{\mathrm{eq}}}\boldsymbol{\Delta} \tag{113}$$

The combination of Eqs. (112) and (113) also yields

$$\mathrm{tr}\boldsymbol{\Sigma}=\frac{1}{\varphi}\frac{\partial\Pi_{\mathrm{G}}^{\mathrm{hom}}}{\partial\alpha},\ \Sigma_{\mathrm{eq}}=\sqrt{3\boldsymbol{\Sigma}_{\mathrm{d}}:\boldsymbol{\Sigma}_{\mathrm{d}}/2}=\frac{\partial\Pi_{\mathrm{G}}^{\mathrm{hom}}}{\partial D_{\mathrm{eq}}} \tag{114}$$

In turn, Eq. (111) leads to

$$\left.\begin{aligned}&\mathrm{tr}\boldsymbol{\Sigma}=2\sigma_{\mathrm{o}}[\operatorname{arcsinh}\xi-\operatorname{arcsinh}(\varphi\xi)]\\&\Sigma_{\mathrm{eq}}=\sigma_{\mathrm{o}}(\sqrt{1+\varphi^2\xi^2}-\varphi\sqrt{1+\xi^2})\end{aligned}\right\} \tag{115}$$

Eliminating ξ between the spherical and deviatoric parts of $\boldsymbol{\Sigma}$ eventually leads to the well-known Gurson strength criterion:

$$\frac{\Sigma_{\mathrm{eq}}^2}{\sigma_{\mathrm{o}}^2}+2\varphi\cosh\left(\frac{\mathrm{tr}\boldsymbol{\Sigma}}{2\sigma_{\mathrm{o}}}\right)-1-\varphi^2=0 \tag{116}$$

This equation characterizes the boundary of the domain $G_{\mathrm{G}}^{\mathrm{hom}}$ whose support function is $\Pi_{\mathrm{G}}^{\mathrm{hom}}$. This domain is in fact an upper bound of the exact domain G^{hom} of macroscopic admissible stresses, that is, $G^{\mathrm{hom}}\subset G_{\mathrm{G}}^{\mathrm{hom}}$. It has to be emphasized that the derived macroscopic strength criterion for porous media (Eq. (116)) does not account by construction for the third invariant (or Lode angle) effect. A detailed analysis of this additional effect, especially in the context of Gurson's model, is available in Lemarchand et al. (2015).

6. Role of pore pressure on the macroscopic strength criterion

We now investigate the role of a fluid pressure P on the macroscopic strength criterion. The presence of such a fluid pressure does not affect the strength-compatible stress state definition in Eq. (19). On the other hand, the conditions for a microscopic stress field $\boldsymbol{\sigma}$ to be statically admissible with the macroscopic stress $\boldsymbol{\Sigma}$ now read

$$\left.\begin{aligned}&\mathrm{div}\boldsymbol{\sigma}=0\\&\boldsymbol{\Sigma}=\bar{\boldsymbol{\sigma}}\\&\boldsymbol{\sigma}=-P\mathbf{1}\quad(\forall\underline{z}\in\Omega^{\mathrm{p}})\end{aligned}\right\} \tag{117}$$

Let $G^{\mathrm{hom}}(P)$ denote the set of strength-compatible macroscopic stress states for the value P of the pore pressure. In particular, $G^{\mathrm{hom}}(0)$ is the domain obtained in the non-pressurized case, which has been studied in Section 2.

Let us consider a given macroscopic stress state $\boldsymbol{\Sigma}\in G^{\mathrm{hom}}(P)$ and a microscopic stress field $\boldsymbol{\sigma}$ complying with Eq. (117) and with the strength criterion of the solid. We then introduce $\tilde{\boldsymbol{\sigma}}=\boldsymbol{\sigma}+P\mathbf{1}$. It follows that

$$\boldsymbol{\Sigma}\in G^{\mathrm{hom}}(P)\Leftrightarrow\exists\tilde{\boldsymbol{\sigma}}\begin{cases}\mathrm{div}\tilde{\boldsymbol{\sigma}}=0\\\boldsymbol{\Sigma}+P\mathbf{1}=\bar{\tilde{\boldsymbol{\sigma}}}\\\tilde{\boldsymbol{\sigma}}=0\quad(\forall\underline{z}\in\Omega^{\mathrm{p}})\\\tilde{\boldsymbol{\sigma}}\in G^{\mathrm{s}}+P\mathbf{1}\quad(\forall\underline{z}\in\Omega^{\mathrm{s}})\end{cases} \tag{118}$$

where $G^{\mathrm{s}}+P\mathbf{1}$ is the set obtained from G^{s} by application of the translation $\boldsymbol{a}\to\boldsymbol{a}+P\mathbf{1}$.

6.1. Von Mises or Tresca solid

In the case of a von Mises or Tresca solid, the strength is not influenced by the hydrostatic stress, that is, $G^{\mathrm{s}}+P\mathbf{1}=G^{\mathrm{s}}$. Accordingly, if $\boldsymbol{\Sigma}\in G^{\mathrm{hom}}(P)$, the properties of $\tilde{\boldsymbol{\sigma}}$ ensure that $\boldsymbol{\Sigma}+P\mathbf{1}\in G^{\mathrm{hom}}(0)$.

This reasoning can be summarized by

$$G^{\mathrm{hom}}(0)=G^{\mathrm{hom}}(P)+P\mathbf{1} \tag{119}$$

or, alternatively

$$\boldsymbol{\Sigma}\in G^{\mathrm{hom}}(P)\Leftrightarrow\boldsymbol{\Sigma}+P\mathbf{1}\in G^{\mathrm{hom}}(0) \tag{120}$$

Expression shows that the macroscopic strength criterion of the pressurized porous material can be formulated as a function of Terzaghi's effective stress. For a von Mises solid, the strength domain is obtained from Eq. (67) by replacing the mean stress of the empty porous material, Σ_{m}, by the mean effective stress of the pressurized medium, $\Sigma_{\mathrm{m}}+P$:

$$\frac{1}{K}(\Sigma_{\mathrm{m}}+P)^2+\frac{1}{M}\Sigma_{\mathrm{d}}^2=k^2(1-\varphi) \tag{121}$$

where $K(\varphi,\rho)$ and $M(\varphi,\rho)$ are still defined by Eq. (58) for a Mori-Tanaka morphology and by Eq. (59) for a (self-consistent) polycristal morphology. Eq. (119) allows for the following straightforward geometrical interpretation: $G^{\mathrm{hom}}(P)$ is obtained from $G^{\mathrm{hom}}(0)$ by a translation parallel to the Σ_{m}-axis in the $(\Sigma_{\mathrm{m}},\Sigma_{\mathrm{d}})$ plane (Fig. 5).

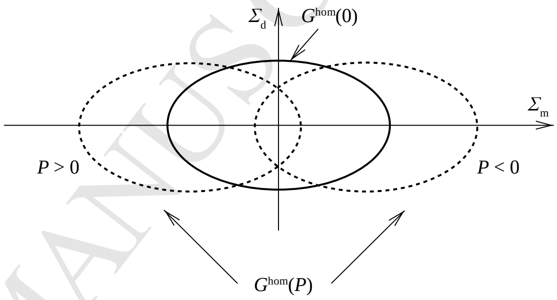

Fig. 5. $G^{\mathrm{hom}}(0)$ and $G^{\mathrm{hom}}(P)$ in the case of a von Mises solid.

6.2. Drucker-Prager solid

It is interesting to check whether the Terzaghi effective stress concept still holds in the case of a solid strength criterion that is sensitive to the mean stress. To this end, let us assume that the set of admissible stress states for the solid is a cone. Its apex lies on the line $\sigma_1=\sigma_2=\sigma_3$ in the space of principal stresses and represents an isotropic tensile stress state $h\mathbf{1}$. This set is denoted by G_h^{s}. The scalar $h>0$ can be referred to as the tensile strength. The corresponding set of macroscopic stress states in drained condition ($P=0$) is denoted by $G_h^{\mathrm{hom}}(0)$.

Such geometry of the domain of admissible stresses is characteristic of a Drucker-Prager material:

$$\sqrt{J_2}+\alpha(h-I_1/3)\leq 0 \tag{122}$$

Interestingly, we observe that the sets G_h^{s} and $G_{h'}^{\mathrm{s}}$ associated with two different values $h>0$ and $h'>0$ of the tensile strength can be deduced from one another by either homothety or translation of G_h^{s} (λG^{s} is the image of G^{s} by the homothety of which the center is located at the origin, with a ratio equal to $\lambda:\underline{z}\to\lambda\underline{z}$):

$$G_{h'}^{\mathrm{s}}=\frac{h'}{h}G_h^{\mathrm{s}} \tag{123a}$$

$$G_{h'}^{\mathrm{s}}=G_h^{\mathrm{s}}+(h'-h)\mathbf{1} \tag{123b}$$

According to the definition given in Eq. (19), Eq. (123a) implies that the set of admissible macroscopic stress states in drained condition linearly depends on the microscopic tensile strength h:

$$G_{h'}^{\mathrm{hom}}(0)=\frac{h'}{h}G_h^{\mathrm{hom}}(0) \tag{124}$$

It is readily seen from Eq. (123b) that

$$G_h^{\mathrm{s}}(0)+P\mathbf{1}=G_{h+P}^{\mathrm{s}} \tag{125}$$

Introducing this result into Eq. (118) shows that

$$\boldsymbol{\Sigma}\in G_h^{\mathrm{hom}}(P)\Leftrightarrow\boldsymbol{\Sigma}+P\mathbf{1}\in G_{h+P}^{\mathrm{hom}}(0) \tag{126}$$

Assuming that $P > -h$, a combination of Eqs. (124) and (126) then yields

$$G_h^{\text{hom}}(P) + P\mathbf{1} = (1 + P/h)G_h^{\text{hom}}(0) \quad (127)$$

That is

$$\boldsymbol{\Sigma} \in G_h^{\text{hom}}(P) \Leftrightarrow \frac{\boldsymbol{\Sigma} + P\mathbf{1}}{1 + P/h} \in G_h^{\text{hom}}(0) \quad (128)$$

The previous relations show that the macroscopic strength is controlled by the following effective stress:

$$\boldsymbol{\Sigma}^{\text{eff}} = \frac{\boldsymbol{\Sigma} + P\mathbf{1}}{1 + P/h} \quad (129)$$

Clearly enough, the definition of the effective stress depends on the solid behavior. For further example, one can see de Buhan and Dormieux (1996, 1999). In contrast to Terzaghi's effective stress relevant for a von Mises solid, the effective stress $\boldsymbol{\Sigma}^{\text{eff}}$ defined by Eq. (129) for a Drucker-Prager solid does not linearly depend on the pore pressure P. This result generalizes the one-dimensional result obtained for the hollow sphere model. In other words, it is sufficient to estimate the strength domain $G^{\text{hom}}(0)$ for the empty porous material, and determine $G^{\text{hom}}(P)$ from a straightforward application of Eq. (127).

Conflict of interest

The authors wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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Appendix. Effective properties of 3-phase heterogeneous material with matrix and composite inclusions

A generalized Eshelby problem is defined as follows. A composite spherical inclusion is embedded in an infinite linear elastic isotropic medium with bulk and shear moduli k_2 and μ_2 (stiffness tensor $\mathbb{C}_2$). The composite inclusion comprises a spherical cavity with volume fraction f. The solid of this composite inclusion is linearly elastic, isotropic, with bulk and shear moduli k_1 and μ_1, respectively. Some uniform strain (rate) boundary conditions are written at infinity in the form $\boldsymbol{v} = \boldsymbol{D}_o\boldsymbol{z}$. Analytical expressions of the average stress $\bar{\boldsymbol{\sigma}}^{\text{ci}}$ and average strain rate $\bar{\boldsymbol{d}}^{\text{ci}}$ over the composite inclusion (ci) are derived as functions of $\boldsymbol{D}_o$:

$$\bar{\boldsymbol{d}}^{\text{ci}} = \mathbb{A} : \boldsymbol{D}_o, \; \bar{\boldsymbol{\sigma}}^{\text{ci}} = \mathbb{B} : \boldsymbol{D}_o \quad \text{(A1)}$$

We now consider a REV of a 3-phase heterogeneous material: spherical composite inclusions of the previous type are embedded in a homogeneous matrix made up of a linear elastic isotropic incompressible medium with shear modulus μ_2. Let ϕ denote the volume fraction of the composite inclusions in the REV. In the extension of the Mori-Tanaka scheme, $\boldsymbol{D}_o$ is interpreted as the average strain rate in the matrix. The stress average rule yields the macroscopic stress tensor $\boldsymbol{\Sigma} = \bar{\boldsymbol{\sigma}}$:

$$\boldsymbol{\Sigma} = [\phi\mathbb{B} + (1-\phi)\mathbb{C}_2] : \boldsymbol{D}_o \quad \text{(A2)}$$

The strain average rule yields the macroscopic strain rate tensor $\boldsymbol{D} = \bar{\boldsymbol{d}}$:

$$\boldsymbol{D} = [\phi\mathbb{A} + (1-\phi)\mathbb{I}] : \boldsymbol{D}_o \quad \text{(A3)}$$

where $\mathbb{I}$ is the fourth-order unit tensor. Eliminating $\boldsymbol{D}_o$ between Eqs. (A2) and (A3) provides the expression of the effective stiffness tensor in the following form:

$$\mathbb{C}^{\text{hom}} = [\phi\mathbb{B} + (1-\phi)\mathbb{C}_2] : [\phi\mathbb{A} + (1-\phi)\mathbb{I}]^{-1} \quad \text{(A4)}$$

In the case of incompressibility (k_1 and $k_2 \to \infty$), the expressions of the effective bulk and shear moduli k^{hom} and μ^{hom} simplify. Let us introduce the following notations:

$$\left.\begin{aligned} A &= 19 - 75f + 112f^{5/3} - 75f^{7/3} + 19f^{10/3} \\ B &= 48 + 200f - 336f^{5/3} + 225f^{7/3} + 38f^{10/3} \\ C &= -9 + 250f - 672f^{5/3} + 450f^{7/3} - 19f^{10/3} \\ D &= 89 - 50f + 112f^{5/3} - 75f^{7/3} - 76f^{10/3} \end{aligned}\right\} \quad \text{(A5)}$$

and

$$\left.\begin{aligned} n_1 &= (2+3\phi)A \\ n_{12} &= C\phi + D \\ n_2 &= (1-\phi)B \\ d_1 &= 6(1-\phi)A \\ d_{12} &= -2C\phi + 3D \\ d_2 &= (2\phi + 3)B \end{aligned}\right\} \quad \text{(A6)}$$

One obtains

$$\mu^{\text{hom}} = 3\mu_2 \frac{n_1\mu_1^2 + n_2\mu_2^2 + n_{12}\mu_1\mu_2}{d_1\mu_1^2 + d_2\mu_2^2 + d_{12}\mu_1\mu_2} \quad \text{(A7)}$$

and

$$k^{\text{hom}} = \frac{4(1-f)}{3f\phi}\mu_1 + \frac{4f(1-\phi)}{3f\phi}\mu_2 \quad \text{(A8)}$$