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**Abstract**

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**Résumé**

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## Version française abrégée

## 1 Introduction

Drop impact is crucial in many multiphase flows ranging from raindrops to combustion chambers or ink-jet printing and it has become an emblematic problem of surface flows [1,2]. Depending on the impact parameters (drop diameter, velocity), fluid properties (viscosity, density, surface tension) and impacted surface (liquid deep or thin film, solid substrate), it can lead to many different outcomes: spreading, rebound, prompt splash, crown splash, cavity and jet formation to cite the most famous ones [3]. Often, the influence of the surrounding gas is neglected in the analysis because of the high density and viscosity ratios between the gas and drop liquid. Indeed, beside the entrapment of a gas bubble at impact due to lubrication effect underneath the drop [4,5] and some aerodynamic corrections to the corolla dynamics, no significant effects of the gas was noticed so far. However, the situation has suddenly changed recently in a striking experiments on drop impacts on a smooth solid substrate [6]: there, by changing only the operating pressure, they observe that splashes were suppressed as the pressure was lower below a critical level, emphasizing thus the crucial role played by the gas in the splashing dynamics. Although different *scenari* has been proposed to explain such effect, involving in particular gas compressibility [6], singular bubble entrapment dynamics [7,8], thin film skating [9] and film wetting dynamics [10,11], the surrounding gas influence remains yet a vibrant question of scientific debates.

Of particular interest is the coupled dynamics between the drop and the gas underneath it just before the impact. In this case, it can be shown that the thin film air dynamics can be considered within the lubrication approximation while the liquid viscosity can be neglected as far as thin liquid jets are not formed [7,9,12]. Then neglecting the surface tension one can see that a finite time singularity arises as a gas bubble is entrapped by the dynamics. This singularity behavior has to be regularized physically at least by the surface tension and the liquid viscosity but is has been argued that the resulting violent dynamics might be relevant in the splashing dynamics. Within the lubrication approximation for the gas film, a liquid jet is then formed that skates on the very thin (but non zero) gas layer. Eventually it has been shown experimentally that the liquid wets the solid substrate [10,11], something that cannot be explained within such lubrication approximation when surface tension is present. In fact, different effects can be proposed to explain the liquid contact with the substrate when the gas layer is very thin: notably surface (Van der Waals for instance) interaction between the liquid interface and the substrate, interface fluctuations and/or surface roughness, and finite size (or rarefied gas) effect in the gas layer. In this paper, we focus on this latter case, that is when the gas layer thickness becomes of the order of the mean free path of the gas, leading firstly to a corrected lubrication equation for the thin film. In the next section, we recall the general scaling argument obtained for drop impact on a solid substrate using the classical lubrication equation for incompressible fluids. Then in section 3 we introduce the correction when the gas thickness is of the order of the mean-free path. Finally we discuss in section 4 the properties of the singularity in this case, in the absence of surface tension and we draw some perspectives for this work.

## 2 Scaling analysis

We consider the impact of a liquid drop of diameter $D$ with a vertical velocity $U$ on a smooth solid substrate. The liquid and surrounding gas densities are noted $\rho_l$ and $\rho_g$, their dynamical viscosity $\mu_l$ and $\mu_g$ respectively, the surface tension $\gamma$. Considering the high Reynolds and Froude numbers of the drop

$$\mathrm{Re} = \frac{\rho_l U D}{\mu_l} \quad \mathrm{Bo} = \frac{U}{\sqrt{gD}}$$

we assume that we can neglect the liquid viscosity and the gravity in the dynamics of the cushioning air film beneath the drop at impact.
Therefore, considering incompressible fluids we obtain the following set of equations describing the evolution of the drop surface $h(r,t)$ in axisymmetric geometry when it is approaching the substrate:

$$(\partial\Omega)\ \partial_t\varphi + \frac{1}{2}\nabla\varphi^2 + \frac{p}{\rho_l} + \frac{\gamma}{\rho_l}\kappa = C(t), \tag{1}$$

$$(\partial\Omega)\qquad \partial_t h = \frac{1}{12r\mu_g}\partial_r(rh^3\partial_r p), \tag{2}$$

$$(\partial\Omega)\qquad \partial_t h = \partial_z\varphi - \partial_r\varphi\partial_r h, \tag{3}$$

$$(\Omega)\qquad \Delta\varphi = 0, \tag{4}$$

where $\varphi$ is the liquid velocity potential (the velocity in the liquid is $\mathbf{u} = \nabla\varphi$), and $\Omega$ is the drop volume, $\partial\Omega$ its interface. The first equation (1) is the Bernoulli equation valid in the liquid and written at the interface, while the last one (4) is the incompressible condition for such potential flow. The interface motion is described by the advection equation (3) while the lubrication equation (2) allows the determination of the pressure $p(r,t)$ in the gas film. The $1/12$ numerical prefactor in this equation was obtained by considering no slip condition for the gas velocity both on the solid substrate and on the fluid interface. This system of equation can be solved numerically using boundary integral method and we resort with numerical simulation on the interface only [13,9]. Notice that lubrication is only valid where the slope of the interface is small enough and additional terms should be considered otherwise.
As the drop is approaching the solid substrate, cushioning of the gas leads to high pressure gradient beneath the drop. The high pressure created at the bottom of the drop deforms it shape so that a gas pocket is entrapped around the impact center. Simple scaling arguments can help to estimate the typical sizes of this entrapped gas bubble. Indeed, considering the time $t = 0$ as the time of impact in the absence of air, one can estimate the typical vertical $H$ and horizontal $R$ scales of the drop deformation as $H(t) \sim Ut$ and $R(t) \sim \sqrt{DH}$ thanks to a geometrical argument based on the intersection between the falling drop and the substrate. Introducing these scaling in the lubrication equation

yields the following scaling for the lubrication pressure in the gas layer:

$$P_l \sim \frac{\mu_g R^2 U}{H^3}.$$

On the other hand, considering the impact pressure in the liquid needed to deviate horizontally a volume of liquid of typical size $R$ (this argument was first given in [14]), one obtain:

$$P_i \sim \frac{1}{R^2}\frac{d}{dt}\left(\rho_l R^3 U\right) \sim \rho_l U^2 \frac{D}{R}.$$

Thus, the lubrication pressure $P_l$ is strong enough to deviate the liquid until a critical height $H^*$ where the liquid somehow has to touches the solid substrate, that is when $P_i = P_l$, yielding:

$$H^* \sim D\left(\frac{\mu_g}{\rho_l U D}\right)^{2/3} = D\mathrm{St}^{2/3}$$

introducing the Stokes number as the inverse of a Reynolds number balancing the liquid inertia with the gas viscous effects:

$$\mathrm{St} = \frac{\mu_g}{\rho_l U D}.$$

Therefore, one expects the liquid to contact the substrate for $t^* \sim \mathrm{St}^{2/3} D/U$ entrapping a gas bubble of height $H^*$ and radius $R^* \sim D\mathrm{St}^{1/3}$. Notice that in practical (experimental) situations, the Stokes number is very small: for instance for a 2 mm diameter droplet of water impacting at a velocity of 1 meter per second, we obtain $\mathrm{St} \sim 10^{-8}$, so that the entrapped bubble radius is only few thousands of the drop diameter.

However, it has been observed in numerical simulations of the set of equations (1,2,3,4) that this contact would arise as a finite time singularity in the absence of surface tension (taking $\gamma = 0$ in the equation 1) [12,7,9]. Such singularity exhibits a divergence of the pressure following $P \sim h_{min}^{-1/2}$ and of the interface curvature $\kappa \sim h_{min}^{-2}$ which can be explained following [9]. Writing the set of governing equation in the frame moving radially with the geometrical intersection between the falling sphere and the substrate and using the dimensionless variables defined by

$$\tilde{x} = \frac{r - R(t)}{D}, \quad \tilde{z} = \frac{z}{D}, \quad \tilde{t} = \frac{Ut}{D}, \quad \tilde{h}(\tilde{x}, \tilde{t}) = \frac{h(r,t)}{D}, \quad \tilde{p} = \frac{p}{\rho_l U^2}, \quad \text{and} \quad \tilde{\varphi} = \frac{\varphi}{UD},$$

the following system of equation is obtained:

$$\partial_{\tilde{t}}\tilde{\varphi} - \dot{\tilde{R}}\partial_{\tilde{x}}\tilde{\varphi} + \frac{1}{2}\tilde{\nabla}\tilde{\varphi}^2 + \tilde{p} = C(t), \tag{5}$$

$$\partial_{\tilde{t}}\tilde{h} - \dot{\tilde{R}}\partial_{\tilde{x}}\tilde{h} = \frac{1}{12\mathrm{St}(\tilde{x}+\mathrm{St}^{1/3})}\partial_{\tilde{x}}\left((\tilde{x}+\mathrm{St}^{1/3})\tilde{h}^3\partial_{\tilde{x}}\tilde{p}\right), \tag{6}$$

$$\partial_{\tilde{t}}\tilde{h} - \dot{\tilde{R}}\partial_{\tilde{x}}\tilde{h} = \partial_z\varphi - \partial_r\varphi\partial_r h, \tag{7}$$

$$\tilde{\Delta}\tilde{\varphi} = 0. \tag{8}$$

The dimensionless radius $\tilde{R} = \sqrt{H/D}$ gives for the singularity dimensionles radial velocity $\dot{\tilde{R}} \sim \sqrt{D/H}$.
Thus developing the former set of equation near the singularity where $\dot{R} \sim \mathrm{St}^{-1/3}$ (we drop the ˜ thereafter for the sake of simplicity) and assuming that the time derivatives are subdominant compared with the $\dot{R}\partial_x$ terms, in good agreement with the numerics [9]. Then seeking for a self-similar structure for the interface near the singularity in the form:

$$h(x,t) = h_{min} f(\frac{x}{l(t)}),\;\; p(x,t) = P_0(t) g(\frac{x}{l(t)}) \;\text{ and }\; \varphi(x,t) = \varphi_0(t)\Phi(\frac{x}{l(t)},\frac{z}{l(t)})$$

we obtain the following system of equation at the interface at the dominant order in the self similar variable $\xi = x/l(t)$ and $\chi = z/l(t)$:

$$-\mathrm{St}^{-1/3}\frac{\varphi_0(t)}{l(t)}\partial_\xi\Phi + \frac{\varphi_0(t)^2}{2l(t)^2}\left(\nabla\Phi\right)^2 + P_0(t)g = C(t) \tag{9}$$

$$-\mathrm{St}^{-1/3}\frac{h_{min}(t)}{l(t)}f' = \frac{1}{12\mathrm{St}}\frac{h_{min}(t)^3 P_0(t)}{l(t)^2}(f^3 g')' \tag{10}$$

$$-\mathrm{St}^{-1/3}\frac{h_{min}(t)}{l(t)}f' = \frac{\varphi_0(t)}{l(t)}\left(\partial_\chi\Phi - \frac{h_{min}(t)}{l(t)}\partial_\xi\Phi f'\right) \tag{11}$$

where the prime stands for the derivative of the function to the variable $\xi$. The lubrication equation (10) gives the followinf relation:

$$P_0(t) \sim \mathrm{St}^{2/3}\frac{l(t)}{h_{min}(t)^2},$$

and from the interface dynamics eq. (11) one can see that two situations have to be considered:

— $h_{min}(t) \ll l(t)$ leading to the observed numerical scalings since $\varphi_0(t) \sim \mathrm{St}^{-1/3}h_{min}(t)$ that gives $P_0(t) \sim \mathrm{St}^{-2/3}h_{min}(t)/l(t)$ and thus:

$$l(t) \sim \mathrm{St}^{-2/3}h_{min}(t)^{3/2},\;\; P_0(t) \sim h_{min}(t)^{-1/2} \;\text{ and }\; \kappa \sim \mathrm{St}^{4/3}h_{min}(t)^{-2}.$$

and this regime is found to be valid for *thick* films, $h_{min} \gg \mathrm{St}^{4/3}$.

— $h_{min}(t) \gg l(t)$, which gives $\varphi_0(t) \sim \mathrm{St}^{-1/3}l(t)$ and yielding:

$$P_0(t) \sim \mathrm{St}^{-2/3},\;\; l(t) \sim \mathrm{St}^{-4/3}h_{min}(t)^2 \;\text{ and}\kappa \sim \mathrm{St}^{8/3}h_{min}(t)^{-3}.$$

Similarly, this regime is valid for *thin* films, $h_{min} \ll \mathrm{St}^{4/3}$. Finally, notice that this second regime has never been reached numerically because of the small values of the Stokes number considered.

## 3 Influence on the mean free path

It is interesting to observe that under typical experimental condition the Stokes number is very small so that the system of equations used has to be questionned. In particular, when the typical air layer becomes of the order of the mean free path $\lambda$ one has to considered rarefied gas correction to the gas dynamics. This can be quantified by the Knudsen number Kn defined as the ratio between the mean free path and the typical air layer thickness which gives here:

$$\mathrm{Kn} = \frac{\lambda}{D\mathrm{St}^{2/3}},$$

and one expect rarefied gas effects to enter into account for $\mathrm{Kn} > 10^{-3}$. From kinetic theory, we have that the mean free path is related to the gas pressure $P_g$ through:

$$\lambda = \frac{k_B T}{\sqrt{2}\pi d^2 P_g},$$

where $T$ is the temperature, $k_B$ the Boltzmann constant and $d$ the typical size of the atoms of the gas. For ambiant temperature $T = 300$ K and ambient pressure $P_{g0} = 10^5$ Pa on obtain the typical mean free path in the air $\lambda_0 = 70\,\mathrm{nm}$ so that one can write the simple relation:

$$\frac{\lambda}{\lambda_0} = \frac{P_{g0}}{P_g}.$$

Considering typical Stokes number of the order of $10^{-8}$, we obtain for millimetric droplet Knudsen numbers of the order one so that correction have to be introduced in the gas dynamics. Finally, let us remark that decreasing the external gas pressure enhance this effect since it increases $\lambda$. For such Knudsen numbers, a simple way to account for the correction due to the rarefied gas context is to introduce a slipping velocity for the gas so that the no-slip boundary condition at a solid interface transforms into the Navier-slip condition:

$$u_t = \lambda \frac{\partial u_t}{\partial n},$$

where $n$ is the normal direction at the interface and $u_t$ the tangiantial velocity. This condition comes from the fact that at the level of the mean free path the no-slip condition is meaningless and cannot be imposed. Solving the Stokes equation between the solid substrate and the drop interface located at $z = h(r,t)$ and assuming that the Navier-slip condition applies on both side we obtain the following relation for the radial velocity $u$ in the gas layer, under the thin film approximation where the interface slope is supposed to be small:

$$u(z,t) = -\frac{1}{2\mu_g}\partial_r p\left(hz - z^2 + \lambda h\right),$$

so that the lubrication equation (2) becomes:

$$\partial_t h = \frac{1}{12r\mu_g}\left(\partial_r(rh^3\partial_r p) + 6\lambda\partial_r(rh^2\partial_r p)\right).$$

Therefore, for film thickness $h \ll \lambda$ one expects that the second term in the lubrication dominates so that the singularity features should be changed.

## 4 Discussion

First of all, we will assume that the position of the singularity is not strongly affected by this correction, that is we can consider the gas bubble entrapped to have the same features than before. This assumption is reasonable since the Knudsen number is of order one but since the Knudsen number increases when the pressure is lowered, the other situation where the bubble size itself is selected by the rarefied gas lubrication term should also be considered in further work. Then writing the set of equation again in the moving frame and in dimensionless units and then seeking for similarity solution using the same change of variables, we obtain at the dominant order near the singularity:

$$-\mathrm{St}^{-1/3}\frac{\varphi_0(t)}{l(t)}\partial_\xi\Phi + \frac{\varphi_0(t)^2}{2l(t)^2}\left(\nabla\Phi\right)^2 + P_0(t)g = C(t) \quad (12)$$

$$-\mathrm{St}^{-1/3}\frac{h_{min}(t)}{l(t)}f' = \frac{1}{12\mathrm{St}}\frac{h_{min}(t)^3 P_0(t)}{l(t)^2}\left(f^3g' + 6\frac{\lambda}{h_{min}(t)}f^2g'\right)' \quad (13)$$

$$-\mathrm{St}^{-1/3}\frac{h_{min}(t)}{l(t)}f' = \frac{\varphi_0(t)}{l(t)}\left(\partial_\chi\Phi - \frac{h_{min}(t)}{l(t)}\partial_\xi\Phi f'\right) \quad (14)$$

Notice that $\lambda$ is now the mean free path made dimensionless using the drop diameter. Now, two regimes can also be identified due to the two terms in the right hand side of the lubrication equation (13).

— $h_{min}(t) \gg \lambda$, the pressure relation due to the dominant term in the lubrication equation remains:

$$P_0(t) \sim \mathrm{St}^{2/3}\frac{l(t)}{h_{min}(t)^2}.$$

— $h_{min}(t) \ll \lambda$, the pressure relation is determined by the other term in the lubrication equation, yielding

$$P_0(t) \sim \mathrm{St}^{2/3} \frac{l(t)}{\lambda h_{min}(t)}.$$

Finally, we resort with two dynamical *scenari* for the singularity dynamics, depending on the ratio between the mean free path $\lambda$ and the critical thickness $\mathrm{St}^{4/3}$ separating the two regimes of the singularity without slip condition, namely:

— $\lambda \gg \mathrm{St}^{4/3}$, two dynamical regimes follow. First, when $h_{min} \gg \lambda$, then the "usual" scaling are valide:

$$l(t) \sim \mathrm{St}^{-2/3} h_{min}(t)^{3/2}, \;\; P_0(t) \sim h_{min}(t)^{-1/2} \;\text{ and }\; \kappa \sim \mathrm{St}^{4/3} h_{min}(t)^{-2}.$$

then it is followed by an other regime as $h_{min}$ decreases. When $h_{min} \ll \lambda$, the following scalings are obtained:

$$l(t) \sim \sqrt{\lambda} h_{min}(t) \mathrm{St}^{-2/3}, \;\; P_0(t) \sim \frac{1}{\sqrt{\lambda}} \;\text{ and }\; \kappa \sim \frac{\mathrm{St}^{4/3}}{\lambda h_{min}(t)}.$$

and we remains thereafter within the configuration where $h_{min}(t)/l(t) \sim \mathrm{St}^{2/3}/\sqrt{\lambda} \ll 1$. In this case, the regime where $h_{min}(t) \gg l(t)$ is not present.

— $\lambda \ll \mathrm{St}^{4/3}$, then for $h_{min} \gg \mathrm{St}^{4/3} \gg \lambda$ the "usual" scaling holds:

$$l(t) \sim \mathrm{St}^{-2/3} h_{min}(t)^{3/2}, \;\; P_0(t) \sim h_{min}(t)^{-1/2} \;\text{ and }\; \kappa \sim \mathrm{St}^{4/3} h_{min}(t)^{-2}.$$

It is followed by a regime where $\mathrm{St}^{4/3} \gg h_{min} \gg \lambda$ which gives the second scaling obtained in the beginning

$$P_0(t) \sim \mathrm{St}^{-2/3}, \;\; l(t) \sim \mathrm{St}^{-4/3} h_{min}(t)^2 \;\text{ and }\; \kappa \sim \mathrm{St}^{8/3} h_{min}(t)^{-3}.$$

Such regime is also folowed by a new regime when $\mathrm{St}^{4/3} \gg \lambda \gg h_{min}$ yielding:

$$l(t) \sim \lambda h_{min}(t) \mathrm{St}^{-4/3} \;\; P_0(t) \sim \mathrm{St}^{-2/3} \;\text{ and }\; \kappa \sim \frac{\mathrm{St}^{8/3}}{\lambda^2 h_{min}(t)}.$$

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