HAL
open science

Infinite linear programming and online searching with turn cost

Spyros Angelopoulos, Diogo Arsénio, Christoph Dürr

To cite this version:

Spyros Angelopoulos, Diogo Arsénio, Christoph Dürr. Infinite linear programming and online searching with turn cost. Theoretical Computer Science, 2017, 10.1016/j.tcs.2017.01.013 . hal-01452876

HAL Id: hal-01452876

https://hal.sorbonne-universite.fr/hal-01452876v1

Submitted on 2 Feb 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ACCEPTED MANUSCRIPT

Infinite linear programming and online searching with turn cost

Spyros Angelopoulos[a,*], Diogo Arsénio[b], Christoph Dürr[a,*]

[a] *Sorbonne Universités, UPMC Univ Paris 06, CNRS, LIP6, Paris, France*
[b] *CNRS and Institut de Mathématiques, Université Paris Diderot, France*

Abstract

We consider the problem of searching for a hidden target in an environment that consists of a set of concurrent rays. Every time the searcher turns direction, it incurs a fixed cost. The objective is to derive a search strategy for locating the target as efficiently as possible, and the performance of the strategy is evaluated by means of the well-established competitive ratio. In this paper we revisit an approach due to Demaine *et al.* [TCS 2006] based on infinite linear-programming formulations of this problem. We first demonstrate that their definition of duality in infinite LPs can lead to erroneous results. We then provide a non-trivial correction which establishes the optimality of a certain round-robin search strategy.

Keywords: Search and exploration problems, infinite linear programming, competitive analysis of online algorithms.

*Research supported by project ANR-11-BS02-0015 "New Techniques in Online Computation" (NeTOC)

Email addresses: `spyros.angelopoulos@lip6.fr` (Spyros Angelopoulos), `arsenio@math.univ-paris-diderot.fr` (Diogo Arsénio), `christoph.durr@lip6.fr` (Christoph Dürr)

ACCEPTED MANUSCRIPT

1. Introduction

Searching for a hidden object is a task that is often encountered in everyday situations. It is thus not surprising that computational aspects of search problems have attracted significant attention. The broad setting can be described by three components: a *domain* (i.e., environment) which may be known or unknown; a hidden, immobile target that lies in some unknown position of the environment; and a mobile searcher (e.g. a robot), initially placed at some predefined position of the domain. The objective is to develop a search strategy for locating the target as efficiently as possible. As in [8] we are interested in the case of unbounded domains.

One of the earliest examples of search problems is the *linear search problem*, proposed in [6] and independently in [4]. Here, the target is hidden at some unknown position of the infinite line, and at distance h from a given point designated as the origin, whereas the searcher is initially located at the origin. The objective is to design a search strategy (namely, an algorithm that describes the movement of the searcher on the infinite line) that minimizes the *competitive ratio* of the strategy: the latter is defined as the worst-case ratio of the overall travel cost of the searcher divided by the distance h. A natural generalization of the linear-search problem is the *star search* or *ray search* problem. In this setting, we are given a set of m infinite rays with a common origin O, and a searcher which is initially placed at the origin. The target is located at distance h from O, however the searcher has no knowledge of the ray on which the target lies. A search strategy is an algorithm that specifies how the searcher traverses the rays, and the competitive ratio is defined as the worst-case ratio of the first time a searcher

ACCEPTED MANUSCRIPT

locates the target, over the optimal distance h.

In this paper we study the setting in which the searcher incurs a fixed *turn cost* upon changing direction, and the overall travel cost for a searcher is the sum of the individual search and turn costs. This formulation models the often-encountered setting in which changing a searcher's direction is a time-consuming operation which cannot be ignored; for instance, a robot cannot turn instantaneously. Following [8], we assume that there are costs d_1 and d_2 for turning at a ray and at the origin, respectively; hence the turn cost incurred by a searcher on a single ray exploration is $d = d_1 + d_2$.

1.1. Related work

It has long been known that *geometric* strategies are optimal for linear search [5], a result that was extended initially in [9] as well as in [3] and [11] to the m-ray setting. In this class of strategies, the searcher performs a round-robin exploration of rays with distances forming a geometric sequence (i.e., of the form $a^0, a^1, a^2, \ldots$ for some $a > 1$). In particular, Gal [9] showed an optimal geometric strategy of competitive ratio

$$1 + 2M, \text{ with } M = \frac{a^m}{a-1} \text{ and } a = \frac{m}{m-1}. \tag{1}$$

Other related work includes the study of randomization [20], [14], multi-searcher strategies [18], the variant in which an upper bound is known on the distance of the target from the origin [17] [7], the variant in which some probabilistic information on target placement is known [11], [12], the related problem of designing *hybrid algorithms* [13], and more recently, the study of new performance measures [16], [19]. For an overview of results on

ACCEPTED MANUSCRIPT

ray-searching we refer the reader to Chapter 9 of the textbook by Gal and Alpern [1].

The above results assume no turn cost. For given turn cost d, [8] studies ray searching using an approach based on infinite linear-program (LP) formulations. More specifically, in order to lower-bound the cost of any search strategy, they define an infinite series of linear programs, with each linear program describing a (progressively better) set of adversarial target placements. At the limit, the optimal value of the infinite LP gives the strongest lower bound. The approach of [8] consists of solving experimentally this series of finite LPs, then guessing a solution to the infinite LP, and finally providing a proof of optimality based on appropriate duality properties of infinite LP formulations. More precisely, they claim that for every search strategy there is a placement of the target at a certain distance h from the origin such that the strategy incurs a (tight) cost of $(1+2M)h+(M-m)d$. Furthermore, they show that this bound is tight, by providing a matching round-robin (and near-cyclical) strategy.

1.2. Contribution of this paper

We begin by revisiting the technique of infinite-LP formulations by Demaine *et al.* in the context of m-ray searching with turn cost, and by pointing out some subtle pitfalls. Specifically, we give a dual solution that is feasible for the infinite LP of [8] and whose objective value is larger than the upper bound on the search cost that is shown in [8]. This contradiction clearly demonstrates that we cannot rely on dual solutions to the infinite LP; instead one must insist on solutions that are feasible for any finite formulation, and evaluate the objective value that is attained at the limit.

ACCEPTED MANUSCRIPT

In more technical terms, in order to establish the feasibility of a crucial dual constraint, one needs to study the infinite sequence that is generated by a specific linear recurrence relation. In particular, we seek appropriate initial data for the recurrence relation (at a trade-off relation with the objective value of the LP) such that the generated sequence observes certain limit properties. We prove that the choice of initial data must be such that the sequence in question eventually becomes negative, in stark contrast to [8] which stipulates that the sequence must be strictly positive. This gives rise to a problem related to linear recurrences, which we address using tools from linear algebra and complex analysis.

The remainder of the paper is structured as follows. Section 2 demonstrates the caveats of infinite LPs and shows that duality in infinite LP formulations of the problem is not upheld. Section 3 shows how to remedy this problem; more precisely, we show how to obtain a feasible dual solution for every finite LP formulation, which suffices for obtaining the desired result. Section 4 addresses the technical details behind the construction of the dual solution, and provides a self-contained study of the underlying recurrence relation.

2. LP formulations and the caveats of infinite LPs

2.1. Preliminaries and definitions

We first provide some preliminary facts and definitions concerning m-ray searching. We say that a search algorithm is (α, β)-*competitive* if for any placement of the target at distance h, the search cost is at most $\alpha h + \beta$. In particular, under the assumptions that the target is never placed

ACCEPTED MANUSCRIPT

within distances smaller than a specified fixed constant and for zero turn cost, previous work has established optimal $(1 + 2M, 0)$-competitive algorithms. The question we address is then the following: What is the smallest B such that an algorithm is $(1 + 2M, B)$ competitive, with no assumptions on the target placement? Note that this question becomes non-trivial only in the presence of turn costs, since otherwise B is zero, as argued in [8]. Note also that M is upper-bounded by a linear function in m, since $M = \frac{m^m}{(m-1)^{m-1}}$, therefore $m < M \leq \mathrm{e} \cdot m$, where e is the Euler constant.

We say that a strategy S is *no worse* than strategy S' if S is $(1+2M, B)$-competitive and S' is $(1 + 2M, B')$-competitive, with $B' \geq B$. The *optimal* strategy is the strategy that achieves the minimum possible B. It is also easy to show that the worst-case positions for placing the target are right after the turn point of each ray exploration (since every other possible placement cannot affect the worst-case competitiveness).

2.2. LP formulations of star search with turn cost

We begin with a review of the approach in [8]. First, it is easy to argue, by an exchange argument, that the optimal strategy must be round-robin (cyclic). Let $x_1, x_2, \ldots$ denote the (infinite) sequence of distances in which the algorithm cycles through the m rays. Suppose that the adversary places the target just beyond the turn point that corresponds to distance x_{i+1}, with $i \geq 0$. In this case, the cost of locating the target is $2\sum_{j=1}^{m+i} x_j + x_{i+1} + (m+i)d$. Therefore, in order for the algorithm to be $(1+2M, B)$-competitive we require that $2\sum_{j=1}^{m+i} x_j + x_{i+1} + (m+i)d \leq (1+2M)x_{i+1} + B$, or equivalently, that $2\sum_{j=1}^{m+i} x_j + (m+i)d \leq 2Mx_{i+1} + B$. In addition, the adversary may choose to place the target arbitrarily close to the origin, on the last ray to be explored

ACCEPTED MANUSCRIPT

in the first round, which implies the condition $2(\sum_{j=1}^{m-1} x_j) + (m-1)d \leq B$. Combining the above requirements, we obtain a family of linear programs. The k-th LP of this family is

$$\begin{aligned} \min \quad & B && (\mathcal{P}) \\ \text{s.t.} \quad & 2\sum_{j=1}^{m-1} x_j - B \leqslant -d(m-1) \\ & 2\sum_{j=1}^{m+i} x_j - 2Mx_{i+1} - B \leqslant -d(m+i) && \forall i = 0 \dots k \\ & B, x_1, \dots, x_{m+k} \geqslant 0, \end{aligned}$$

with dual LP

$$\begin{aligned} \max \quad & \Big((m-1)z + \sum_{i=0}^{k} y_i(m+i)\Big) d && (\mathcal{D}) \\ \text{s.t.} \quad & z + \sum_{i=0}^{k} y_i \leqslant 1 && (B\text{-constraint}) \\ & \left\{ \begin{array}{ll} z\ , & j \leq m-1 \\ 0\ , & \text{otherwise} \end{array} \right\} + \sum_{i=\max(0,j-m)}^{k} y_i \\ & \qquad - \left\{ \begin{array}{ll} My_{j-1}\ , & j \leq k+1 \\ 0\ , & \text{otherwise} \end{array} \right\} \geqslant 0 \quad \forall j = 1 \dots m+k \\ & && (x_j\text{-constraint}) \\ & z, y_0, \dots y_k \geqslant 0 \end{aligned}$$

We call the dual constraint that corresponds to the variable B of the primal the *B-constraint* and the dual constraint that corresponds to variable x_j the *x_j-constraint*, or simply the *j-th constraint*. We also call k the *index* of the LP.

We note that we slightly deviate from the notation of [8], in that we denote by z the dual variable that corresponds to the B-constraint of the

ACCEPTED MANUSCRIPT

primal; in contrast, in [8] this variable is denoted by y_{m-1}. We do so in order to emphasize that this variable corresponds to the B-constraint, and not to any of the x_j constraints. We also observe that the dual variables y_i are indexed starting from $i = 1$; in contrast, [8] uses the notation y_{n+m}, with $n \geq 0$.

For every fixed k, the objective value of any feasible solution to the LP $(\mathcal{D})$ is a lower bound on B. Thus, the best lower bound on B is obtained by the optimal solution to the dual LP, when $k \to \infty$. In [8], a different approach is proposed: it is argued that a lower bound on B can be obtained by finding a feasible solution to the following LP, which we call the *infinite dual LP*. Essentially, $(\mathcal{D}^\infty)$ is obtained as the dual of a (primal) infinite LP $(\mathcal{P}^\infty)$ which in turn is derived from $(\mathcal{P})$ by setting $k = \infty$.

$$
\begin{array}{lll}
\max & \left((m-1)z + \sum_{i=0}^{\infty} y_i(m+i)\right) d & (\mathcal{D}^\infty) \\
\text{s.t.} & z + \sum_{i=0}^{\infty} y_i \leqslant 1 & \\
\left\{\begin{array}{ll} z\ , & j \leq m-1 \\ 0\ , & \text{otherwise} \end{array}\right\} + \sum_{i=\max(0,j-m)}^{\infty} y_i - M y_{j-1} \geqslant 0 & \forall j = 1, 2, \ldots & \\
 & z, y_0, y_1, \ldots \geqslant 0 &
\end{array}
$$

In particular, [8] proposes the following recursively defined solution

$$
\begin{aligned}
z \quad &= \quad \frac{m}{M}, \quad y_0 = y_1 = y_{m-2} = \ldots = \frac{1}{M}, \quad y_{m-1} = \frac{1}{M}(1-z), \\
&\text{and} \quad y_i = y_{i-1} - \frac{1}{M} y_{i-m}, \text{ for all } i \geq m,
\end{aligned} \tag{2}
$$

which is then shown to be feasible for the infinite dual LP, and has objective value equal to $(M - Mz)d = (M - m)d$.

ACCEPTED MANUSCRIPT

We now argue that relying on a solution that is only feasible for the infinite dual LP ($\mathcal{D}^\infty$) can lead to erroneous results. In particular, Theorem 1 proves that one can give a solution that is feasible for the infinite dual LP, and whose objective value equals Md, which exceeds the upper bound of $(M-m)d$ shown in [8]. This demonstrates that one cannot necessarily rely on solutions to the infinite dual LP, and instead must first provide feasible solutions for finite LP formulations, then evaluate their objective value when $k \to \infty$.

Theorem 1. *Consider the solution that is recursively defined as in* (2)*, with the exception that* $z = 0$*. Then this solution is feasible for the infinite dual LP, and its objective value is equal to* Md*.*

Proof. We first need to establish that the sequence obtained by the linear recurrence

$$y_0 = y_1 = \ldots y_{m-1} = \frac{1}{M}, \text{ and } y_i = y_{i-1} - \frac{1}{M} y_{i-m}, \text{ for all } i \geq m,$$

is positive. In particular, we prove the following lemma[1].

Lemma 2. *Consider the linear recurrence* $y_i = y_{i-1} - \frac{1}{M} y_{i-m}$*, for* $i \geq m$*, with* $y_0 = y_1 = \ldots = y_{m-1} > 0$*. Then* $y_i > 0$ *for all* i*. Moreover,* $\lim_{i \to \infty} y_i = 0$*.*

Proof. We will show, by induction on i, that $\frac{y_i}{y_{i-1}} \geq \frac{1}{a}$, where $a = \frac{m}{m-1}$, which suffices to prove the lemma. (Recall that from (1) we have that $M = \frac{a^m}{a-1}$.)

[1]Lemma 2 also follows from Lemma 5 that will be proven later in the context of the study of the recurrence relation.

ACCEPTED MANUSCRIPT

Note that the claim holds for all $i \leq m$; in particular $\frac{y_m}{y_{m-1}} = 1 - \frac{1}{M} = 1 - \frac{a-1}{a^m} \geq 1 - \frac{a-1}{a} = \frac{1}{a}$. Suppose that the claim holds for all $i < j$, we will show it then holds for $i = j$. Since $y_i > 0$ for all $i < j$ (from the induction hypothesis), we obtain that

$$\frac{y_j}{y_{j-1}} = \frac{y_{j-1} - \frac{1}{M}y_{j-m}}{y_{j-1}} = 1 - \frac{1}{M}\frac{y_{j-m}}{y_{j-1}} = 1 - \frac{1}{M} \cdot \frac{y_{j-m}}{y_{j-m+1}}\frac{y_{j-m+1}}{y_{j-m+2}} \cdots \frac{y_{j-m+m-2}}{y_{j-m+m-1}}$$
$$\geq 1 - \frac{1}{M}a^{m-1} = 1 - \frac{a-1}{a^m}\frac{a^m}{a} = \frac{1}{a}.$$

Since we showed that the sequence y_i is positive, it follows that it is also decreasing. It is easy now to show that $\lim_{i\to\infty} y_i = 0$. Note that

$$\frac{y_i}{y_{i-m}} = \frac{y_i - \frac{1}{M}y_{i-m}}{y_{i-m}} < 1 - \frac{1}{M},$$

hence the sequence decreases geometrically, and its limit is 0. □

We now prove that y satisfies all remaining constraints of the infinite dual LP. Summing the parts of the equations $y_{m+i} = y_{m+i-1} - \frac{1}{M}y_i$, for all $i = 0, 1, 2 \ldots$, and using the fact that $\lim_{i\to\infty} y_i = 0$, we obtain that $\sum_{i=0}^{\infty} y_i = My_{m-1} = M\frac{1}{M} = 1$. Hence $z + \sum_{i=0}^{\infty} y_i = 1$, and the B-constraint is tight. It remains to show that the j-th constraint is satisfied, for every $j \geq 1$.

To this end we will consider three cases. When $j < m$, the LHS of the constraint is equal to $z + \sum_{i=0}^{\infty} y_i - M\frac{1}{M} = 1 - M\frac{1}{M} = 0$. When $j = m$, the LHS of the constraint becomes $\sum_{i=0}^{\infty} y_i - M\frac{1-z}{M} = \frac{1-z}{M} - \frac{1-z}{M} = 0$ (in both cases we used the fact that the B-constraint is tight). It remains to consider the j-th constraint, for all $j > m$. We can use induction to prove that the LHS of each such constraint is equal to zero. More specifically, subtracting the LHS of the $(j-1)$-th constraint from the LHS of the j-th constraint we obtain a difference equal to $-My_{j-1} + My_{j-2} - y_{j-m-1} = -M\left(y_{j-1} - y_{j-2} + \frac{1}{M}y_{j-m-1}\right)$,

ACCEPTED MANUSCRIPT

which in turn is equal to zero, from the recursive definition of y_{j-1}. Therefore, every j-constraint is tight.

Last, we need to evaluate the objective value of the above solution. It can be shown (using exactly the same approach as in [8]) that the dual objective of this solution, namely the value $((m-1)z + \sum_{i=0}^{\infty} y_i(m+i))\, d = \sum_{i=0}^{\infty} y_i(m+i)$ is equal to $(M - Mz)d = Md$, which exceeds the upper bound of $(M-m)d$ for this problem, also shown in [8]. $\square$

We conclude this section with two observations. First, the reason we arrive at this contradictory result is that this dual solution, while feasible for $\mathcal{D}^{\infty}$, does not give rise to a feasible solution for any finite dual LP (i.e., for any given k, the solution $\{z = 0, y_0, \ldots, y_k\}$ is infeasible for the dual LP of index k). To see this, note that by summing the parts of equations $y_{m+i} = y_{m+i-1} - \frac{1}{M} y_i$, for all $i = 0, 1, 2 \ldots k$, we obtain that $\sum_{i=0}^{k} y_i = M(y_{m-1} - y_{m+k}) = M\frac{1-z}{M} - My_{m+k}$; hence $z + \sum_{i=0}^{k} y_i = 1 - My_{m+k}$. Since $y_{m+k} > 0$ (as shown in Lemma 2), it follows that $z + \sum_{i=0}^{k} y_i$ is strictly less than 1. However, in this case, the solution clearly violates the very first constraint (i.e., the x_1-constraint, since its LHS is equal to $z + \sum_{i=0}^{k} y_i - M\frac{1}{M}$, which in turn is less than 0). The same holds for the solution given in [8], namely the one described by (2).

Second, it is worth pointing out that strong duality, and often weak duality, are not always satisfied in infinite LPs. We refer the reader to the work of Karney [15] and Romeijn *et al.* [10] for a discussion of conditions under which (weak or strong) duality can be established for certain families of infinite LPs.

ACCEPTED MANUSCRIPT

3. Finding a feasible solution to the dual LP

In this section we argue how to find a feasible dual solution to the LP $(\mathcal{D})$ for every index $k \geq m$. The best dual solution, namely the one that yields the best lower bound will be derived when $k \to \infty$. What complicates things is that, given indices k_1, k_2, with $k_1 < k_2$, the dual solution to the LP of index k_1 cannot be obtained, in a straightforward manner, from the dual solution to the LP of index k_2 (informally, the latter is not "contained" in the former). We thus fix k and propose a feasible dual solution that is parameterized on k. To this end, we need first to establish certain properties of the recurrence relation

$$y_i = y_{i-1} - \frac{1}{M} y_{i-m}, \qquad \text{with } i \in \mathbb{N}, \tag{3}$$

where $M = \frac{m^m}{(m-1)^{m-1}}$ and the initial data are

$$y_0 = y_1 = \ldots = y_{m-2} = \frac{1}{M}, \qquad y_{m-1} = \frac{1-z}{M}, \qquad \text{with } 0 \leq z \leq 1. \tag{4}$$

We define y as the sequence $\{y_i\}_{i=0}^{\infty}$. Note that since m and M are fixed, the only parameter that influences the initial data, and, implicitly, the asymptotic behavior of the sequence y, is z, which we call the *seed* of y.

The following is the main technical theorem concerning the sequence y. In particular, the theorem shows that we can choose appropriate seed values z such that either y is a positive sequence, or y eventually becomes negative, respectively. This will allow us to argue that $y_{k+m} = 0$ for some seed value, which in turn yields the tightness of the crucial B-constraint (see Lemma 4).

Theorem 3. *Let $0 < \zeta < 1$ be given by*

$$\zeta = \left(\frac{m-1}{m}\right)^{m-1} = \frac{m}{M}, \tag{5}$$

ACCEPTED MANUSCRIPT

and consider the infinite sequence $y = \{y_i\}_{i=0}^{\infty}$ *with seed* $0 \leq z \leq 1$.

Then, $y_i > 0$ *for every* $i \in \mathbb{N}$ *if and only if* $0 \leq z \leq \zeta$. *Furthermore, for every* $i_0 \geq m$, *there exists* z *with* $\zeta < z < 1$, *such that* $y_{i_0} = 0$, $y_i > 0$ *for all* $i < i_0$, *and* $y_i < 0$ *for all* $i > i_0$.

Proof. For convenience, the technical details of the proof are deferred to Section 4.

Lemma 7 therein, which is obtained by studying the geometric behavior generated by the sequence, asserts the existence of some $0 < \zeta < 1$ such that $y_i > 0$ for every $i \in \mathbb{N}$ if and only if $0 \leq z \leq \zeta$. Furthermore, if $\zeta < z \leq 1$, then there exists $j \in \mathbb{N}$ such that $y_i \geq 0$, for all $i \leq j$, and $y_i < 0$, for all $i > j$.

Next, consider any $i_0 \geq m$. Accordingly, setting $z = 1$ yields $y_{i_0} < 0$, whereas $y_{i_0} > 0$ when $z = \zeta$. Therefore, since the value of y_{i_0} depends continuously on z, there exists z with $\zeta < z < 1$, such that $y_{i_0} = 0$, $y_i > 0$ for all $i < i_0$, and $y_i < 0$ for all $i > i_0$.

There only remains to establish the explicit formula (5) for ζ, which requires the use of linear algebra and complex analysis. More precisely, in Lemma 10, the study of subspaces which are invariant under the recurrence relation (3) (provided by the analysis of Jordan decompositions of matrices) shows that

$$\zeta = t_*^{m-1}, \tag{6}$$

where $t_* \in \mathbb{C}$ is the largest root (in modulus) of the characteristic polynomial $t^m - t^{m-1} + \frac{1}{M}$ associated to the recurrence.

Finally, applying essential tools from complex analysis, in particular Rouché's theorem, we identify in Lemma 9 that the principal root t_* is real and that

ACCEPTED MANUSCRIPT

it satisfies

$$t_* = \frac{m-1}{m}. \tag{7}$$

The combination of (6) and (7) yields the precise value (5) of ζ, which concludes the proof of the theorem. □

We now show how to apply Theorem 3 so as to obtain a feasible solution to the dual LP of index k.

Theorem 4. *For every fixed index $k \geq m$, there exists $\zeta < z^* \leq 1$ such that the solution $\{z^*, y_0, \ldots, y_k\}$ is feasible for the dual LP of index k.*

Proof. First, from Theorem 3, we know that there is some $z^* \in (\zeta, 1)$ for which $y_{m+k} = 0$ and $y_0, \ldots, y_k > 0$, whence the solution $\{z^*, y_0, \ldots, y_k\}$ satisfies the non-negativity constraints.

Next, we show that the B-constraint is tight. To this end, by summing up the equations of the form $y_i = y_{i-1} - \frac{1}{M} y_{i-m}$ for $i = m, m+1, \ldots, m+k$ we obtain that $\sum_{i=0}^{k} y_i = M(y_{m-1} - y_{k+m})$. Moreover, we have that $y_{k+m} = 0$ for seed $z = z^*$. Hence, $\sum_{i=0}^{k} y_i = M y_{m-1} = M\left(\frac{1}{M} - \frac{1}{M} z^*\right) = 1 - z^*$. Thus we obtain

$$z^* + \sum\nolimits_{i=0}^{k} y_i = z^* + 1 - z^* = 1, \tag{8}$$

thus the B-constraint is tight.

It remains to show that the j-th constraint is satisfied for all $j \in [1, m+k]$. Let us denote by LHS(j) the LHS of the j-th constraint; we also define

$$\mathrm{L}(j) := \left\{ \begin{array}{ll} z\ , & j \leq m-1 \\ 0\ , & \text{otherwise} \end{array} \right\} + \sum\nolimits_{i=\max(0,j-m)}^{k} y_i - M y_{j-1}.$$

Clearly LHS(j) $\geq$ L(j), therefore in order to show that that the j-th constraint is satisfied, it suffices to show that L(j) ≥ 0. When $j \leq m-1$, then

ACCEPTED MANUSCRIPT

$\mathrm{L}(j) = z^* + \sum_{i=0}^{k} y_i - M y_{j-1} = z^* + \sum_{i=0}^{k} y_i - M\frac{1}{M} = 1-1 = 0$, from (8). Similarly, when $j = m$, we have $\mathrm{L}(m) = \sum_{i=0}^{k} y_i - M y_{m-1} = \sum_{i=0}^{k} y_i - M\frac{1-z^*}{M} = 0$, again from (8).

Last, we will use induction to show that $\mathrm{L}(j) = 0$ when $j > m$. For any $j > m$ we have

$$\begin{aligned} \mathrm{L}(j+1) - \mathrm{L}(j) &= -My_j + My_{j-1} - y_{j-m} = \\ &- M(y_{j-1} - \frac{1}{M}y_{j-m}) + My_{j-1} - y_{j-m} = 0, \end{aligned}$$

where the second-to-last equation follows from the definition of the recurrence relation. □

We are now ready to formally prove the main claim of [8], namely that for every $(1+2M, B)$-competitive strategy, $B \geq (M-m)d$. Consider the dual LP of a given index $k \geq m$, and the feasible dual solution $\{z^*, y_0, \ldots, y_k\}$ obtained in the proof of Theorem 4. It is easy to see that the sequence y is strictly decreasing in the seed z (as we will prove in Lemma 6), which also implies that z^* is also decreasing in k. Thus, when $k \to \infty$, z^* converges to ζ from above, for ζ as defined in the statement of Theorem 3. In other words, for $k \to \infty$, $z^* \to \frac{m}{M}$. Informally, we have obtained a dual solution which is identical to the dual solution given for the infinite-index LP in [8]. A formal evaluation of the dual objective can be done as in Theorem 7 of [8], from which we obtain that $B \geq (M-m)d$.

We conclude this section by observing that, as shown in [8], there exists a cyclic $(1+2M, (M-m)d)$ strategy, therefore the lower bound we established is tight.

ACCEPTED MANUSCRIPT

4. Analysis of the recurrence relation

The main goal of this section is to establish Theorem 3. In particular, we will provide a self-contained and detailed analysis of the recurrence relation (3) with initial data (4). For the sake of clarity and for convenience of the reader, we have broken up the proof of this theorem into several lemmas established in the remainder of the present section. Thus, the first part of the theorem asserting the existence of ζ is contained in Lemmas 6 and 7 below, while the precise formula (5) for ζ clearly follows from Lemmas 9 and 10.

The following lemma characterizes the behavior of the recurrence (3) for some choice of initial data.

Lemma 5. *Suppose that $y_{i+1} \geq ty_i \geq 0$ (or $y_{i+1} \leq ty_i \leq 0$), for all $i \in \{0, \ldots, m-2\}$ and for some $t > 0$ such that $t^m - t^{m-1} + \frac{1}{M} \leq 0$.*

Then, it holds that $y_{i+1} \geq ty_i \geq 0$ (or $y_{i+1} \leq ty_i \leq 0$, respectively), for all $i \in \mathbb{N}$. In particular, since $M = \frac{m^m}{(m-1)^{m-1}}$, it is always possible to choose $t = \frac{m-1}{m}$.

Proof. We argue by induction. Thus, let us suppose that

$$y_{i+1} \geq ty_i \geq 0, \qquad \text{for all } i \leq N,$$

for some integer $N \geq m-2$. We only have to show that the above inequalities remain true for $i = N+1$.

To this end, note first that

$$y_i \geq t^{i-j} y_j \geq 0, \qquad \text{for all } j \leq i \leq N+1.$$

ACCEPTED MANUSCRIPT

Therefore, employing the recurrence relation (3), we find that

$$\begin{aligned} y_{N+2} &= y_{N+1} - \frac{1}{M} y_{N+1-(m-1)} \geq y_{N+1} - \frac{1}{M t^{m-1}} y_{N+1} \\ &= t y_{N+1} - \frac{t^m - t^{m-1} + \frac{1}{M}}{t^{m-1}} y_{N+1} \geq t y_{N+1}, \end{aligned}$$

which concludes the proof of the lemma. □

Considering the specific choice of initial data (4) and applying Lemma 5, we have the following lemmas.

Lemma 6. *Let the sequence* $\{y_i\}_{i=0}^{\infty} \subset \mathbb{R}$ *be generated by the recurrence relation* (3) *with initial data* (4).

Then, for any given $i \geq m-1$*, the value of* y_i *is strictly decreasing with respect to the variable* $z \in [0,1]$.

Proof. For each $i \in \mathbb{N}$, we denote by y_i' the derivative (or any finite difference quotient) of y_i with respect to z. Then, clearly, by linearity, the sequence $\{y_i'\}_{i=0}^{\infty}$ verifies the same recurrence relation

$$y_i' = y_{i-1}' - \frac{1}{M} y_{i-m}', \qquad \text{for all } i \in \mathbb{N}.$$

Furthermore, it is clear that $y_i' = -\frac{1}{M}$, for every $i = m-1, m, \ldots, 2m-2$. Therefore, in view of Lemma 5, we conclude that $y_i' < 0$, for every $i \geq m-1$. □

Lemma 7. *Let the sequence* $\{y_i\}_{i=0}^{\infty} \subset \mathbb{R}$ *be generated by the recurrence relation* (3) *with initial data* (4).

Then, there exists $0 < \frac{1}{m} \leq \zeta \leq \frac{M-1}{M} < 1$ *such that* $y_i > 0$*, for all* $i \in \mathbb{N}$*, if and only if* $0 \leq z \leq \zeta$*. Furthermore, if* $\zeta < z \leq 1$*, then there exists* $j \in \mathbb{N}$ *such that* $y_i \geq 0$*, for all* $i \leq j$*, and* $y_i < 0$*, for all* $i > j$.

ACCEPTED MANUSCRIPT

Proof. Let us denote by $U^+ \subset [0,1]$ the set of values of $z \in [0,1]$ such that the corresponding sequence $\{y_i\}_{i=0}^{\infty}$ remains always non-negative, and by $U^- \subset [0,1]$ its complement, that is the set of values of $z \in [0,1]$ such that the corresponding sequence $\{y_i\}_{i=0}^{\infty}$ eventually becomes negative.

Then, in view of Lemma 5, we find that if

$$0 \leq z \leq \frac{1}{m},$$

then the whole sequence $\{y_i\}_{i=0}^{\infty}$ remains non-negative. Hence $\left[0, \frac{1}{m}\right] \subset U^+$. Furthermore, for any given $i \in \mathbb{N}$, the value of y_i clearly depends continuously on z. Therefore, it follows that the set U^+ is closed. Finally, since each $y_i \geq 0$ is monotonic with respect to z according to Lemma 6, one shows easily that the set U^+ is convex.

Thus, the set U^+ is closed, convex and contains $\left[0, \frac{1}{m}\right]$. It follows that U^+ is a closed interval and so, there exists $\zeta \in \left[\frac{1}{m}, 1\right]$ such that $U^+ = [0, \zeta]$. Moreover, $y_m < 0$ whenever $z > \frac{M-1}{M}$, whence $\zeta \in \left[\frac{1}{m}, \frac{M-1}{M}\right]$.

Finally, let us consider any $z \in U^-$. Then, there exists $j \in \mathbb{N}$ such that $y_i \geq 0$, for all $i \leq j$, and $y_{j+1} < 0$. It follows that

$$0 > y_{j+1} \geq y_{j+2} \geq \ldots \geq y_{j+m-1} \geq y_{j+m}.$$

Therefore, applying Lemma 5, we conclude that $y_i < 0$, for all $i > j$.

The proof of the lemma is complete. □

The previous lemma provides important information on the behavior of the sequence $\{y_i\}_{i=0}^{\infty} \subset \mathbb{R}$ generated by the recurrence relation (3) with initial data (4).

On the one hand, whenever $0 \leq z \leq \zeta$, the y_i's are strictly decreasing. On the other hand, whenever $\zeta < z \leq 1$, the y_i's are strictly increasing for

ACCEPTED MANUSCRIPT

large enough $i \in \mathbb{N}$. Therefore, as $i \to \infty$, the limit of y_i, which we denote by $y_\infty \in \mathbb{R}$, is well-defined. Then, letting $i \to \infty$ in (3), we deduce that $y_\infty = y_\infty - \frac{1}{M} y_\infty$, which implies that $y_\infty = 0$.

The remainder of this section aims at obtaining the explicit value of ζ. Employing linear algebra, we recall and apply now standard principles from the analysis of recurrence relations.

Accordingly, note that the recurrence relation (3) may be written, employing matrix notation, as

$$\begin{pmatrix} y_{i+1} \\ \vdots \\ y_{i+m} \end{pmatrix} = A \begin{pmatrix} y_i \\ \vdots \\ y_{i+m-1} \end{pmatrix}, \qquad \text{for all } i \in \mathbb{N},$$

where the square matrix $A \in \mathbb{R}^{m \times m}$ is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -\frac{1}{M} & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial $p_A(t) = \det(t\mathrm{Id} - A)$ of A, where $t \in \mathbb{C}$, is given by

$$p_A(t) = t^m - t^{m-1} + \frac{1}{M},$$

which is easily deduced upon noticing that A is the companion matrix of the monic polynomial $p_A(t)$.

The following lemma counts the multiplicity of the roots of $p_A(t)$.

ACCEPTED MANUSCRIPT

Lemma 8. *The polynomial $p_A(t)$ has exactly $m-1$ distinct roots. Only one of these roots, namely $\frac{m-1}{m}$, is positive. Moreover, it has multiplicity two.*

Proof. Let us focus first on the real roots of $p_A(t)$, that is to say on the set $\{t \in \mathbb{R} \subset \mathbb{C} \ : \ p_A(t) = 0\}$.

Clearly, $p_A'(t) = mt^{m-2}\left(t - \frac{m-1}{m}\right)$. Therefore, when m is even, the function $p_A(t)$ is decreasing on $(-\infty, 0)$ and $\left(0, \frac{m-1}{m}\right)$, increasing on $\left(\frac{m-1}{m}, \infty\right)$, and reaches its global minimum at $\frac{m-1}{m}$, where it takes on the value $p_A\left(\frac{m-1}{m}\right) = \frac{1}{M} - \frac{(m-1)^{m-1}}{m^m} = 0$. We conclude that $p_A(t)$ has exactly one real root at $\frac{m-1}{m}$. More precisely, since $p_A\left(\frac{m-1}{m}\right) = p_A'\left(\frac{m-1}{m}\right) = 0$, while $p_A''\left(\frac{m-1}{m}\right) = m\left(\frac{m-1}{m}\right)^{m-2} \neq 0$, it holds that $\frac{m-1}{m}$ is a root of $p_A(t)$ of multiplicity two.

In fact, a straightforward computation shows the explicit decomposition

$$p_A(t) = \left(t - \frac{m-1}{m}\right)^2 q_A(t),$$

where

$$q_A(t) = \frac{(m-1)^{m-3}}{m^{m-2}} \sum_{k=1}^{m-1} k \left(\frac{m}{m-1} t\right)^{k-1},$$

and $q_A\left(\frac{m-1}{m}\right) \neq 0$.

When m is odd, the function $p_A(t)$ is decreasing on $\left(0, \frac{m-1}{m}\right)$, increasing on $(-\infty, 0)$ and $\left(\frac{m-1}{m}, \infty\right)$, and reaches a local minimum at $\frac{m-1}{m}$, where it takes on the value $p_A\left(\frac{m-1}{m}\right) = \frac{1}{M} - \frac{(m-1)^{m-1}}{m^m} = 0$. Therefore, the situation is similar and $p_A(t)$ has one positive root of multiplicity two at $\frac{m-1}{m}$. However, the situation differs now from the preceding case by the fact that $p_A(t)$ has a third real root which is negative.

Let us consider now the non-real roots of $p_A(t)$, that is to say the set $\{t \in \mathbb{C} \setminus \mathbb{R} \ : \ p_A(t) = 0\}$. Since we have already identified two real roots (with multiplicity) when m is even, and three real roots (with multiplicity)

ACCEPTED MANUSCRIPT

when m is odd, there can only be at most $2\left(\left[\frac{m}{2}\right]-1\right)$ remaining complex roots. In fact, we are about to show that there are exactly $2\left(\left[\frac{m}{2}\right]-1\right)$ distinct complex roots with non-trivial imaginary part, which will thereby conclude the justification of the lemma.

To this end, we may write $t=\rho e^{i\varphi}$, with $\rho>0$ and $\varphi\in[0,2\pi)$. Moreover, since the coefficients of $p_A(t)$ are real, its complex roots are pairwise conjugate and so, it suffices to consider the restriction $\varphi\in(0,\pi)$.

Thus, we find that $t=\rho e^{i\varphi}$ is a root of $p_A(t)$ if and only if

$$\begin{cases} \rho^m\cos(m\varphi)-\rho^{m-1}\cos((m-1)\varphi)+\dfrac{1}{M}=0, \\ \rho\sin(m\varphi)-\sin((m-1)\varphi)=0. \end{cases} \tag{9}$$

Note that, if $\sin(m\varphi)=0$, then, according to the second equation of the above system, necessarily $\sin((m-1)\varphi)=0$, whence $\sin\varphi=0$. However, this case is ruled out by the fact that we have restricted the range of φ to $(0,\pi)$. Therefore, $\sin(m\varphi)\neq 0$, $\sin((m-1)\varphi)\neq 0$ and $\sin\varphi\neq 0$, so that, employing basic trigonometry, one can show that (9) is equivalent to

$$\begin{cases} (\sin((m-1)\varphi))^{m-1}\sin\varphi=\dfrac{1}{M}(\sin(m\varphi))^m, \\ \rho\sin(m\varphi)=\sin((m-1)\varphi). \end{cases} \tag{10}$$

Next, considering the continuous function

$$f(\varphi)=\sin((m-1)\varphi)^{m-1}\sin\varphi-\frac{1}{M}\sin(m\varphi)^m,$$

one easily checks, for each integer $1\leq k\leq\left[\frac{m}{2}\right]-1$, since

$$\frac{2k}{m}<\frac{2k}{m-1}<\frac{2k+1}{m}<\frac{2k+1}{m-1},$$

ACCEPTED MANUSCRIPT

that

$$f\left(\frac{2k}{m-1}\pi\right) < 0 < f\left(\frac{2k+1}{m}\pi\right),$$

whence there exists at least one $\varphi \in \left(\frac{2k}{m-1}\pi, \frac{2k+1}{m}\pi\right)$ such that $f(\varphi) = 0$ and, thus, solves the first equation of (10). As for the second equation of (10), it is readily satisfied for some $\rho > 0$ upon noticing that $\sin(m\varphi) > 0$ and $\sin((m-1)\varphi) > 0$, for any $\varphi \in \left(\frac{2k}{m-1}\pi, \frac{2k+1}{m}\pi\right)$.

Thus, we have shown that, for each integer $1 \leq k \leq \left[\frac{m}{2}\right] - 1$, there exists a distinct root $t = \rho e^{i\varphi}$ of $p_A(t)$ with positive imaginary part satisfying $\varphi \in \left(\frac{2k}{m-1}\pi, \frac{2k+1}{m}\pi\right)$. As mentioned previously, the complex roots are pairwise conjugate and so, there are exactly $2\left(\left[\frac{m}{2}\right] - 1\right)$ distinct non-real roots, which concludes the proof of the lemma. □

The following lemma determines the principal root $t_* \in \mathbb{C}$ of $p_A(t)$.

Lemma 9. *The real root $t_* = \frac{m-1}{m}$ of $p_A(t)$ is strictly larger (in modulus) than any other root (real or complex) of $p_A(t)$.*

Proof. Since $p_A\left(\frac{m-1}{m}\right) = 0$, $p_A(1) = \frac{1}{M}$ and $p_A(t)$ is strictly increasing in $\left(\frac{m-1}{m}, 1\right)$, we have that, for any $0 < \epsilon < \frac{1}{M}$, there is exactly one root t_{**} of $p_A(t) - \epsilon$ in $\left(\frac{m-1}{m}, 1\right)$. Moreover, $\lim_{\epsilon \to 0} t_{**} = t_*$.

Next, for any $t \in \mathbb{C}$ such that $|t| = \frac{m-1}{m}$, we find that, since $p_A\left(\frac{m-1}{m}\right) = 0$,

$$\left|t^m + \frac{1}{M} - \epsilon\right| \leq |t|^m + \frac{1}{M} - \epsilon = p_A(|t|) - \epsilon + |t|^{m-1} < \left|t^{m-1}\right|.$$

Hence, by Rouché's theorem, the polynomial $p_A(t) - \epsilon$ has the same number of roots (with multiplicity), inside $\left\{|t| < \frac{m-1}{m}\right\}$, as t^{m-1}. That is to say, inside $\left\{|t| < \frac{m-1}{m}\right\}$, $p_A(t) - \epsilon$ has exactly $m-1$ roots and, therefore, t_{**} is the largest root of $p_A(t) - \epsilon$.

ACCEPTED MANUSCRIPT

Now, by the continuous dependence of the roots of a monic polynomial with respect to its coefficients, we find, letting $\epsilon \to 0$, that all m roots of $p_A(t)$ lie inside $\left\{|t| \leq \frac{m-1}{m}\right\}$. Thus, there only remains to show that the only possible root on $\left\{|t| = \frac{m-1}{m}\right\}$ is actually given by $t_* = \frac{m-1}{m}$. To this end, consider any $t = \rho e^{i\varphi} \in \mathbb{C}$ with $\rho = \frac{m-1}{m}$ and $\varphi \in [0, 2\pi)$. Supposing that t is a root of $p_A(t)$, we compute that

$$\frac{1}{M^2} = \left|t^m - t^{m-1}\right|^2 = \rho^{2m} - 2\rho^{2m-1}\cos\varphi + \rho^{2m-2},$$

which yields the identity

$$1 = (m-1)^2 - 2\left(\cos\varphi\right)(m-1)m + m^2 = 1 + 2\left(1 - \cos\varphi\right)(m-1)m,$$

whose unique solution is given by $\varphi = 0$. Thus, necessarily $t = \frac{m-1}{m}$, which is a root of multiplicity two according to Lemma 8, and all remaining $m-2$ roots lie inside $\left\{|t| < \frac{m-1}{m}\right\}$, which concludes the proof of the lemma. $\square$

Lemma 10. *Let $0 < \zeta < 1$ be the threshold value for $0 \leq z \leq 1$ determined in Lemma 7, and let $t_* = \frac{m-1}{m}$ be the principal root of $p_A(t)$ determined in Lemma 9. Then, $\zeta = t_*^{m-1}$.*

Proof. If $t \in \mathbb{C}$ is an eigenvalue of A, i.e. it is a root of $p_A(t)$, then it is easily verified that it has geometric multiplicity one and that the corresponding eigenspace is spanned by the sole vector

$$v(t) = \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{m-1} \end{pmatrix} \in \mathbb{C}^m.$$

ACCEPTED MANUSCRIPT

Furthermore, according to Lemma 8, the characteristic polynomial $p_A(t)$ has $m-1$ distinct roots. More precisely, $p_A(t)$ has exactly one root $t_1 = t_* = \frac{m-1}{m}$ of algebraic multiplicity two, whereas all remaining $m-2$ eigenvalues, $t_i \in \mathbb{C}$, $i = 2, \ldots, m-1$, are distinct. As previously mentioned, the geometric multiplicity of t_1 is necessarily one and so, the matrix A is not diagonalizable.

Instead, one verifies easily that the invariant Jordan subspace corresponding to the eigenvalue t_1 is spanned by the vectors

$$v(t_1) = \begin{pmatrix} 1 \\ t_1 \\ t_1^2 \\ \vdots \\ t_1^{m-1} \end{pmatrix}, \tilde{v}(t_1) = \begin{pmatrix} 0 \\ 1 \\ 2t_1 \\ \vdots \\ (m-1)t_1^{m-2} \end{pmatrix} \in \mathbb{C}^m.$$

More precisely, the action of A on this invariant subspace satisfies

$$Av(t_1) = t_1 v(t_1) \qquad \text{and} \qquad A\tilde{v}(t_1) = v(t_1) + t_1\tilde{v}(t_1).$$

It is therefore possible to show that any sequence $\{y_i\}_{i=0}^{\infty} \subset \mathbb{C}$ generated by the recurrence (3) can be expressed as

$$y_i = \tilde{\alpha}_1 i t_1^{i-1} + \alpha_1 t_1^i + \alpha_2 t_2^i + \ldots + \alpha_{m-1} t_{m-1}^i, \qquad \text{for every } i \in \mathbb{N},$$

where the coefficients $\tilde{\alpha}_1, \alpha_k \in \mathbb{C}$, $k = 1, \ldots, m-1$, are determined by the initial data. That is to say, they are the unique solution to the system

$$\widetilde{V} \begin{pmatrix} \tilde{\alpha}_1 \\ \alpha_1 \\ \vdots \\ \alpha_{m-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{pmatrix}, \tag{11}$$

ACCEPTED MANUSCRIPT

where the matrix $\widetilde{V} \in \mathbb{R}^{m\times m}$ is defined by

$$\widetilde{V} = (\tilde{v}(t_1), v(t_1), v(t_2), \ldots, v(t_{m-1})) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & t_1 & t_2 & \cdots & t_{m-1} \\ 2t_1 & t_1^2 & t_2^2 & \cdots & t_{m-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m-1)t_1^{m-2} & t_1^{m-1} & t_2^{m-1} & \cdots & t_{m-1}^{m-1} \end{pmatrix}.$$

Note that $\widetilde{V}$ is a variant of the Vandermonde matrix $V \in \mathbb{R}^{m\times m}$, which is defined by

$$V = (v(t_1), v(t_2), \ldots, v(t_m)) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \\ t_1^2 & t_2^2 & \cdots & t_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{m-1} & t_2^{m-1} & \cdots & t_m^{m-1} \end{pmatrix},$$

and recall that the well-known Vandermonde determinant satisfies

$$\det V = \prod_{1\le k<k'\le m} (t_{k'} - t_k)\,. \tag{12}$$

It follows that the determinant of $\widetilde{V}$ is given by

$$\det \widetilde{V} = -\prod_{k=2}^{m-1} (t_k - t_1)^2 \prod_{2\le k<k'\le m-1} (t_{k'} - t_k)\,,$$

which can be easily established by differentiating (12) in t_1 (recalling that the determinant is multilinear with respect to each column vector) and then setting $t_2 = t_1$.

Further note that $t_1 = t_* = \frac{m-1}{m}$ is the principal eigenvalue determined by Lemma 9. Therefore, writing

$$\frac{y_i}{it_*^{i-1}} = \tilde{\alpha}_1 + \alpha_1 \frac{t_*}{i} + \alpha_2 \frac{t_*}{i}\left(\frac{t_2}{t_*}\right)^i + \ldots + \alpha_{m-1}\frac{t_*}{i}\left(\frac{t_{m-1}}{t_*}\right)^i,$$

ACCEPTED MANUSCRIPT

we find that the sign of $\tilde{\alpha}_1$ (note that $\tilde{\alpha}_1 \in \mathbb{R}$ because $t_* \in \mathbb{R}$) determines the behavior of y_i according to Lemma 7. More precisely, if $\tilde{\alpha}_1 < 0$, then y_i will eventually become negative, as $i \to \infty$. However, if $\tilde{\alpha}_1 > 0$, then y_i will remain positive for all $i \in \mathbb{N}$.

In order to evaluate the coefficient $\tilde{\alpha}_1$, we apply Cramer's rule to the system (11) with the initial data (4) to deduce that

$$\begin{aligned}
\tilde{\alpha}_1 M \det \widetilde{V} &= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & t_1 & \cdots & t_{m-1} \\ 1 & t_1^2 & \cdots & t_{m-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_1^{m-2} & \cdots & t_{m-1}^{m-2} \\ 1-z & t_1^{m-1} & \cdots & t_{m-1}^{m-1} \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & t_1 & \cdots & t_{m-1} \\ 1 & t_1^2 & \cdots & t_{m-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_1^{m-2} & \cdots & t_{m-1}^{m-2} \\ 1 & t_1^{m-1} & \cdots & t_{m-1}^{m-1} \end{pmatrix} - \det \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 0 & t_1 & \cdots & t_{m-1} \\ 0 & t_1^2 & \cdots & t_{m-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & t_1^{m-2} & \cdots & t_{m-1}^{m-2} \\ z & t_1^{m-1} & \cdots & t_{m-1}^{m-1} \end{pmatrix} \\
&= \left(\prod_{k=1}^{m-1} (t_k - 1) + (-1)^m z \right) \prod_{1 \le k < k' \le m-1} (t_{k'} - t_k) \,.
\end{aligned}$$

Then, since each t_k satisfies

$$t_k - 1 = \frac{-1}{t_k^{m-1} M} \qquad \text{and} \qquad t_1^2 \prod_{k=2}^{m-1} t_k = \det A = \frac{(-1)^m}{M},$$

we find that

$$\tilde{\alpha}_1 = \frac{t_*^{m-1} - z}{M \prod_{k=2}^{m-1} (t_* - t_k)}.$$

ACCEPTED MANUSCRIPT

Finally, employing the fact, from Lemma 9, that t_* has the largest real part of all eigenvalues, which are all pairwise conjugate, we obtain that $\prod_{k=2}^{m-1}(t_* - t_k) > 0$. Therefore, we conclude that $\tilde{\alpha}_1 > 0$ if and only if $z < t_*^{m-1}$ and that $\tilde{\alpha}_1 < 0$ if and only if $z > t_*^{m-1}$, whence

$$\zeta = t_*^{m-1} = \frac{(m-1)^{m-1}}{m^{m-1}} = \frac{m}{M}.$$

The proof of the lemma is now complete. □

5. Conclusion

In this paper we revisited the problem of online searching with turn cost in the domain of m unbounded concurrent rays. We demonstrated that duality in infinite LP formulations of this problem, as defined in [8], is not necessarily upheld, and we provided a correct proof using tools from the theory of linear recurrences. Since linear programming provides useful formulations in the context of search games, we expect that our findings may have further implications, especially in unbounded domains (as a concrete example, [2] studies the search problem with a mobile hider on a network via infinite-dimensional LPs). In a similar vein, infinite LP formulations can be applied to resource-allocation problems with an infinite decision-making horizon; for instance they can provide optimal randomized algorithms for the online ski rental problem.

An interesting topic for further research is extending the results to the multi-searcher problem with turn cost; namely, in the setting in which p identical searchers must locate a target in the m-ray domain. Note that here the optimal strategy is not necessarily round-robin (cyclic), in contrast to

ACCEPTED MANUSCRIPT

the single-searcher setting. This implies that the optimal strategy may have a complicated structure that is not conducive to obtaining a suitable LP formulation. In recent follow-up work we address these challenges.

[1] S. Alpern and S. Gal. *The theory of search games and rendezvous*. Kluwer Academic Publishers, 2003.

[2] E.J. Anderson and M.A. Aramendia. The search game on a network with immobile hider. *Networks*, 20(7):817–844, 1990.

[3] R. Baeza-Yates, J. Culberson, and G. Rawlins. Searching in the plane. *Information and Cmputation*, 106:234–244, 1993.

[4] A. Beck. On the linear search problem. *Naval Research Logistics*, 2:221–228, 1964.

[5] A. Beck and D.J. Newman. Yet more on the linear search problem. *Israel J. of Math.*, 8:419–429, 1970.

[6] R. Bellman. An optimal search problem. *SIAM Review*, 5:274, 1963.

[7] P. Bose, J. De Carufel, and S. Durocher. Searching on a line: A complete characterization of the optimal solution. *Theoretical Computer Science*, 569:24–42, 2015.

[8] E.D. Demaine, S.P. Fekete, and S. Gal. Online searching with turn cost. *Theoretical Computer Science*, 361:342–355, 2006.

[9] S. Gal. Minimax solutions for linear search problems. *SIAM J. on Applied Math.*, 27:17–30, 1974.

ACCEPTED MANUSCRIPT

[10] R.L. Smith H.E. Romeijn and J.C Bean. Duality in infinite dimensional linear programming. *Mathematical Programming*, 53:79–97, 1992.

[11] P. Jaillet and M. Stafford. Online searching. *Operations Research*, 49:234–244, 1993.

[12] M-Y. Kao and M.L. Littman. Algorithms for informed cows. In *Proceedings of the AAAI 1997 Workshop on Online Search*, 1997.

[13] M-Y. Kao, Y. Ma, M. Sipser, and Y.L. Yin. Optimal constructions of hybrid algorithms. *Journal of Algorithms*, 29(1):142–164, 1998.

[14] M-Y. Kao, J.H. Reif, and S.R. Tate. Searching in an unknown environment: an optimal randomized algorithm for the cow-path problem. *Inform. and Comp.*, 131(1):63–80, 1996.

[15] D.F. Karney. Duality gaps in semi-infinite linear programming–an approximation problem. *Mathematical Programming*, 20:129–143, 1981.

[16] D. G. Kirkpatrick. Hyperbolic dovetailing. In *Proceedings of the 17th Annual European Symposium on Algorithms (ESA)*, pages 616–627, 2009.

[17] A. López-Ortiz and S. Schuierer. The ultimate strategy to search on m rays? *Theoretical Computer Science*, 261(2):267–295, 2001.

[18] A. López-Ortiz and S. Schuierer. On-line parallel heuristics, processor scheduling and robot searching under the competitive framework. *Theor. Comp. Sci.*, 310(1–3):527–537, 2004.

ACCEPTED MANUSCRIPT

[19] A. McGregor, K. Onak, and R. Panigrahy. The oil searching problem. In *Proc. of the 17th European Symposium on Algorithms (ESA)*, pages 504–515, 2009.

[20] S. Schuierer. A lower bound for randomized searching on m rays. In *Computer Science in Perspective*, pages 264–277, 2003.

ACCEPTED MANUSCRIPT