Second order tensorial framework for 2D medium with open and closed cracks

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Abstract

The tensorial nature of crack density of an initially isotropic 2D medium with open and closed cracks is studied by means of polar decomposition rewriting of standard micro-mechanics results. The question of both indicial and constitutive symmetries of different crack density tensors is addressed: for instance the standard fourth order crack density tensor $\mathbb{D}^c$ is rari-constant (totally symmetric) and the fourth order closed cracks density tensor $\mathbb{\Delta}^c$ by which closed cracks are acting is found to have the square symmetry. The effect of cracks closure and sliding is accordingly shown to be represented by a second order tensor ($\boldsymbol{\delta}^c$) so that only two second order crack density tensors, $\mathbf{d}^o$ and $\boldsymbol{\delta}^c$, are needed for 2D medium with open and closed sliding cracks. Similarly to the open cracks case, any arbitrary closed crack system is shown to be represented by only two non orthogonal families of cracks. The question of macroscopic cracks closure conditions is finally studied. Present study leads to an approximate framework in which the only internal variable representative of physical cracks, open and closed, is second order cracks density tensor. Proposed second order tensorial framework is shown to be exact in the case of two orthogonal arrays of cracks, open and/or closed, it is approximate in the general case of many arrays of cracks, open and/or closed.

Keywords: micro-mechanics, crack density, closure effect, sliding, polar decomposition

Introduction

The micro-mechanical study of elastic materials in which cracks (or voids) are nucleated is not a new research subject [1]. Considering interacting or non-interacting crack arrays (i.e. regarding each crack as an isolated one), numerous micro-mechanical approaches to the definition of the change in compliance tensor or loss of stiffness tensor for initially isotropic [2, 3, 4, 5, 6, 7, 8, 9, 10] or anisotropic [11, 12, 13, 14, 15] materials were developed. The accuracy of the non interacting crack approximation was shown to be accurate even at high densities provided that mutual positions of cracks are random [12]. Those approaches may be based on micro-mechanics of two constituent materials in which inclusions are modelled as (elliptical) voids. They may also be treated by a direct approach [11, 2, 4, 6, 12, 13, 14] that uses closed form solutions of the displacements of the free faces of a crack embedded in a matrix of arbitrary anisotropy. The elastic solutions of [16, 17] are then used in order to define a crack opening displacement tensor. The micro-cracks spatial distribution defines an induced anisotropic behavior even for an initially isotropic material. Restriction to initially linear elastic isotropic materials with only open cracks has allowed for the definition of the concept of crack density tensor, this thanks to the fact that the crack compliance second order tensor is at first order approximation proportional to identity [18, 19].

On the other hand, depending on the stress state, micro-cracks may be open or closed, *i.e.* constrained against opening and allowed to slide or not [20, 21, 19, 22, 2]. The effect of induced anisotropy for closed cracks has been found of a different tensorial nature than for open cracks. When the closed cracks are considered to slide with no friction (lubricated approach) a fourth order density crack density tensor has been derived so far, for instance in the case of initially linear elastic isotropic materials [19, 22]. It is mentioned in the literature – see the review [4] and reference [23] – that in addition to classical second order crack density tensor there is a remaining so-called irreducible fourth order tensorial part, even in 2D non-interacting cracks approximation. Let us point out that in these works irreductibility is meant in the sense of tensors vector space [24, 25, 26, 27, 28, 29]: irreducible adjective is used as a synonymous for harmonic, *i.e.* for traceless and totally symmetric fourth order tensors. As no practical decomposition of irreducible/harmonic part of crack density fourth order tensor did exist in the literature, a second order closed crack density tensor still had to be defined.

A main objective of micro-mechanics studies of media with open and closed cracks is at the end to build an unilateral damage model at macroscopic scale

– which satisfies to the micro-mechanical requirements,

– which avoids modeling difficulties such as discontinuities of stress-strain response or non-uniqueness of the thermodynamical potential [30, 31, 32, 33],

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– which avoids the need to write the cracks closure conditions at the cracks microscale.

The question that arises is then how to define the open/closed crack status when the cracking pattern is not described by the individual sets of arrays of cracks. Only the case of non interacting frictionless sliding cracks is considered next. A complete model should of course account for the interaction between cracks and for dissipative sliding based behavior. Nevertheless, as simple closed form solutions exist with the assumptions of non interacting crack arrays and of closed cracks sliding without friction, those assumptions are an interesting first-step approach.

We investigate the induced anisotropy and the associated framework with second order crack density tensors only defined for an initially 2D isotropic elastic medium in the case of open and closed cracks (sliding without friction). In order to do so we use the polar decomposition of 2D fourth order tensors [34, 35], decomposition already applied to layered composite materials (but not to cracked media). It is worth pointing out that the polar method has recently been shown to include in 2D a decomposition of harmonic fourth order tensors (so-called irreductible fourth order tensors) [36].

In section 1, we present standard 2D micro-mechanics results. In section 2 is briefly recalled the essentials of the polar formalism. In section 3, the type of induced anisotropy of the change in compliance tensor is investigated for open and closed cracks cases. In a consistent manner with recent Tensorial Polar Decomposition [36] the rewriting of standard micro-mechanics results done in section 4 includes the decomposition of harmonic part of fourth order crack density tensor for closed cracks. The corresponding calculations allow us to define two second order crack density tensors: a first one, standard $\mathbf{d}^o$, for open cracks, and a second one, novel $\boldsymbol{\delta}^c$, for closed cracks. One proves in section 5 that any arbitrary closed cracks system may be represented by only two non orthogonal families of cracks. One shows finally in section 6 and 7, how their use by a proper projection on the stress tensor – instead of the use of standard fourth order crack density tensor as made in [10] for instance – ends up to a thermodynamics second order tensorial framework written at macroscale only.

1. Standard 2D micro-mechanics with open and closed cracks

We present in this section standard results for elastic effective properties of solids [4, 6] with non-interacting cracks expressed in terms of compliance tensor. The matrix material is assumed to be isotropic linear elastic with compliance tensor $\mathbb{S}_0$ and elastic energy density $\rho\psi_0^\star$ with ρ the density. The change in compliance and elastic energy densities due to open and closed cracks are respectively $\hat{\mathbb{S}}_{\text{open}}$, $\hat{\mathbb{S}}_{\text{closed}}$, $\rho\hat{\psi}^\star_{\text{open}}$ and $\rho\hat{\psi}^\star_{\text{closed}}$. The elastic energy density of cracked solid reads:

$$\rho\psi^\star = \rho\psi_0^\star + \rho\hat{\psi}^\star_{\text{open}} + \rho\hat{\psi}^\star_{\text{closed}} = \frac{1}{2}\boldsymbol{\sigma} : \mathbb{S}_0 : \boldsymbol{\sigma} + \frac{1}{2}\,\boldsymbol{\sigma} : \hat{\mathbb{S}}_{\text{open}} : \boldsymbol{\sigma} + \frac{1}{2}\,\boldsymbol{\sigma} : \hat{\mathbb{S}}_{\text{closed}} : \boldsymbol{\sigma} \tag{1}$$

from which the elasticity law is obtained as:

$$\boldsymbol{\epsilon} = \rho\frac{\partial\psi^\star}{\partial\boldsymbol{\sigma}} = \left(\mathbb{S}_0 + \hat{\mathbb{S}}_{\text{open}} + \hat{\mathbb{S}}_{\text{closed}}\right) : \boldsymbol{\sigma} \tag{2}$$

We consider rectilinear sets of cracks of length $2l^{(p)}$. The normal $\mathrm{n}^{(p)}$ of one array of cracks is turned by an angle $\varphi^{(p)}$ in the reference frame. The representative area element is noted A and the crack density of one array of cracks is defined as $d^{(p)} = \pi l^{(p)}/A$.

1.1. Elastic energy density for open cracks

For open arrays of cracks, the change in elastic energy density is

$$\rho\hat{\psi}^\star_{\text{open}} = \frac{1}{2}\,\frac{\pi}{E_0}\,\boldsymbol{\sigma} : \left[2\sum_{\text{open } p}\left(\frac{l^2}{A}\mathrm{n}\otimes\mathbf{1}\otimes\mathrm{n}\right)^{(p)}\right] : \boldsymbol{\sigma} \tag{3}$$

which allows to define the change in compliance due to open cracks:

$$\hat{\mathbb{S}}_{\text{open}} = 2\frac{\pi}{E_0}\sum_{\text{open } p}\left(\frac{l^2}{A}\,\frac{\mathbf{1}\,\underline{\overline{\otimes}}\,\mathrm{n}\otimes\mathrm{n} + \mathrm{n}\otimes\mathrm{n}\,\underline{\overline{\otimes}}\,\mathbf{1}}{2}\right)^{(p)} \tag{4}$$

Equation (3) can be recast as

$$\rho\hat{\psi}^\star_{\text{open}} = \frac{1}{E_0}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) : \mathbf{d}^o = \frac{1}{E_0}\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\mathbf{d}^o\cdot\boldsymbol{\sigma}) = \frac{1}{E_0}\boldsymbol{\sigma} : \frac{1}{2}\left(\mathbf{1}\,\underline{\overline{\otimes}}\,\mathbf{d}^o + \mathbf{d}^o\,\underline{\overline{\otimes}}\,\mathbf{1}\right) : \boldsymbol{\sigma} \tag{5}$$

introducing the standard second order crack density tensor $\mathbf{d}^o$ (with the sum over open cracks) [18]:

$$\mathbf{d}^o = \sum_{\text{open } p} d^{(p)}\mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)} \tag{6}$$

1.2. Elastic energy density for closed cracks

For arrays of closed cracks, the change in elastic energy density is

$$\rho\hat{\psi}^\star_{\text{closed}} = \frac{\pi}{E_0}\,\boldsymbol{\sigma} : \left[\sum_{\text{closed } p}\left(\frac{l^2}{A}(\mathrm{n}\otimes\mathbf{1}\otimes\mathrm{n} - \mathrm{n}\otimes\mathrm{n}\otimes\mathrm{n}\otimes\mathrm{n})\right)^{(p)}\right] : \boldsymbol{\sigma} \tag{7}$$

which allows to define the change in compliance due to closed cracks:

$$\hat{\mathbb{S}}_{\text{closed}} = 2\frac{\pi}{E_0}\sum_{\text{closed } p}\left[\frac{l^2}{A}\left(\frac{\mathbf{1}\,\overline{\underline{\otimes}}\,\mathrm{n}\otimes\mathrm{n} + \mathrm{n}\otimes\mathrm{n}\,\overline{\underline{\otimes}}\,\mathbf{1}}{2} - \mathrm{n}\otimes\mathrm{n}\otimes\mathrm{n}\otimes\mathrm{n}\right)\right]^{(p)} \tag{8}$$

Equation (7) can be recast as

$$\rho\hat{\psi}^\star_{\text{closed}} = \frac{1}{E_0}\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\mathbf{d}^c\cdot\boldsymbol{\sigma}) - \frac{1}{E_0}\boldsymbol{\sigma} : \mathbb{D}^c : \boldsymbol{\sigma} \tag{9}$$

equivalent to

$$\rho\hat{\psi}^\star_{\text{closed}} = \frac{1}{E_0}\boldsymbol{\sigma} : \mathbb{\Delta}^c : \boldsymbol{\sigma} \tag{10}$$

introducing three crack density tensors, standard second order tensor $\mathbf{d}^c$ and fourth order tensor $\mathbb{D}^c$ (here with the sum over closed cracks):

$$\mathbf{d}^c = \sum_{\text{closed } p} d^{(p)}\mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)} \qquad \mathbb{D}^c = \sum_{\text{closed } p} d^{(p)}\mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)} \tag{11}$$

but also symmetric fourth order crack density tensor $\mathbb{\Delta}^c$, which contains the whole cracks closure/sliding actions,

$$\mathbb{\Delta}^c = \frac{E_0}{2}\hat{\mathbb{S}}_{\text{closed}} = \sum_{\text{closed } p} d^{(p)}\left(\frac{\mathbf{1}\,\overline{\underline{\otimes}}\,\mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)} + \mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)}\,\overline{\underline{\otimes}}\,\mathbf{1}}{2} - \mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)}\right) \tag{12}$$

As $\mathbf{d}^c$ and $\mathbb{D}^c$ result from the same (closed) micro-cracking pattern the second order density tensor $\mathbf{d}^c$ is obtained from the knowledge of fourth order density tensor $\mathbb{D}^c$ as

$$\mathbf{d}^c = \mathbb{D}^c : \mathbf{1} = \mathbf{1} : \mathbb{D}^c \tag{13}$$

and

$$\mathrm{tr}\,\mathbf{d}^c = \mathbf{1} : \mathbf{d}^c = \sum_i d^c_{ii} = \mathbf{1} : \mathbb{D}^c : \mathbf{1} = \sum_i D^c_{iiii} \tag{14}$$

One has the orthogonality property

$$\mathbb{\Delta}^c : \mathbf{1} = \mathbf{1} : \mathbb{\Delta}^c = 0 \tag{15}$$

From equations (9), (10) and (13), a classical conclusion [4] is that the change in elastic energy density for closed cracks is given by the knowledge of one fourth order tensor ($\mathbb{D}^c$ or $\mathbb{\Delta}^c$).

1.3. Elastic energy density for open and closed cracks

Introducing the second order crack density tensor

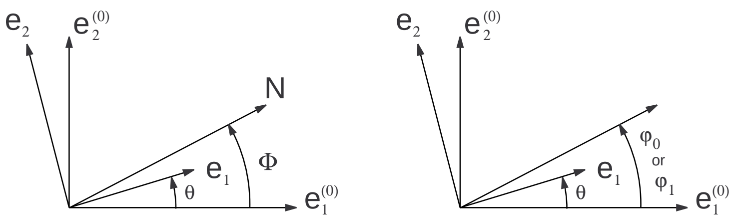
$$\mathbf{d} = \sum_{\text{all } p} d^{(p)}\mathrm{n}^{(p)}\otimes\mathrm{n}^{(p)} = \mathbf{d}^o + \mathbf{d}^c \tag{16}$$

defined over the entire set of open and closed cracks and considering equations (5) and (9) and (10), the change in elastic energy density $\rho\hat{\psi}^\star = \rho\hat{\psi}^\star_{\text{closed}} + \rho\hat{\psi}^\star_{\text{open}}$ for open and closed cracks arrays classically reads

$$\rho\hat{\psi}^\star = \frac{1}{E_0}\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\mathbf{d}^o\cdot\boldsymbol{\sigma}) + \frac{1}{E_0}\boldsymbol{\sigma} : \mathbb{\Delta}^c : \boldsymbol{\sigma} = \frac{1}{E_0}\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\mathbf{d}\cdot\boldsymbol{\sigma}) - \frac{1}{E_0}\boldsymbol{\sigma} : \mathbb{D}^c : \boldsymbol{\sigma} \tag{17}$$

2. The Polar method

We present in this section standard results from the polar method initially introduced by Verchery [34] and summarized in [35]. This method will next be used in order to study the tensorial nature of crack density and compliance tensors introduced in previous section 1.



(a) 2nd order tensor case (b) 4th order tensor case

Figure 1: Angles and reference frames definitions (frame vectors are of unit norm).

2.1. *Polar formalism for second order tensors*

The decomposition of a symmetric second order tensor $\mathbf{s}$ introduces two invariants s_m and s_{eq} and one unit vector N ($||\mathrm{N}|| = \sqrt{\mathrm{N}\cdot\mathrm{N}} = 1$):

$$\mathbf{s} = s_m \mathbf{1} + 2 s_{eq} (\mathrm{N}\otimes\mathrm{N})' \tag{18}$$

First and second invariants are defined as 2D mean value $s_m = \frac{1}{2}\operatorname{tr}\mathbf{s}$ and 2D von Mises norm $s_{eq} = \sqrt{\frac{1}{2}\mathbf{s}' : \mathbf{s}'}$.

Let us introduce the angle Φ to define the vector components in reference frame $(\mathrm{e}_1^{(0)}, \mathrm{e}_2^{(0)})$: $\mathrm{N} = \cos\Phi\, \mathrm{e}_1^{(0)} + \sin\Phi\, \mathrm{e}_2^{(0)}$ (see figure 1). In a frame $(\mathrm{e}_1, \mathrm{e}_2)$ rotated by an angle θ with respect to the frame $(\mathrm{e}_1^{(0)}, \mathrm{e}_2^{(0)})$, $\mathrm{N} = \cos(\Phi-\theta)\mathrm{e}_1 + \sin(\Phi-\theta)\mathrm{e}_2$, and equation (18) reads (all matrix expressions or components are next given in working frame $(\mathrm{e}_1, \mathrm{e}_2)$)

$$\mathbf{s} = s_m\mathbf{1} + s_{eq}\begin{bmatrix} \cos 2(\Phi-\theta) & \sin 2(\Phi-\theta) \\ \sin 2(\Phi-\theta) & -\cos 2(\Phi-\theta) \end{bmatrix} \quad \text{or} \quad \begin{cases} s_{11}(\theta) = s_m + s_{eq}\cos 2(\Phi-\theta) \\ s_{22}(\theta) = s_m - s_{eq}\cos 2(\Phi-\theta) \\ s_{12}(\theta) = s_{eq}\cos 2(\Phi-\theta) \end{cases} \tag{19}$$

2.2. *Polar decomposition of fourth order tensors*

Let consider a fourth order tensor $\mathbb{T}$ having minor and major symmetries:

$$T_{ijkl} = T_{jikl} = T_{ijlk} = T_{klij} \tag{20}$$

In the polar formalism, five invariants are defined: four of them are polar moduli (t_0, t_1, r_0, r_1) and the last one is the angular difference $\varphi_0 - \varphi_1$. The basic result of the polar formalism is the expression of the Cartesian components of $\mathbb{T}$ in terms of polar parameters, in working frame $(\mathrm{e}_1, \mathrm{e}_2)$ rotated by an angle θ with respect to the reference frame $(\mathrm{e}_1^{(0)}, \mathrm{e}_2^{(0)})$ (see figure 1):

$$\begin{aligned}
T_{1111}(\theta) &= t_0 + 2t_1 + r_0\cos 4(\varphi_0-\theta) + 4r_1\cos 2(\varphi_1-\theta), \\
T_{1112}(\theta) &= r_0\sin 4(\varphi_0-\theta) + 2r_1\sin 2(\varphi_1-\theta), \\
T_{1122}(\theta) &= -t_0 + 2t_1 - r_0\cos 4(\varphi_0-\theta), \\
T_{1212}(\theta) &= t_0 - r_0\cos 4(\varphi_0-\theta), \\
T_{1222}(\theta) &= -r_0\sin 4(\varphi_0-\theta) + 2r_1\sin 2(\varphi_1-\theta), \\
T_{2222}(\theta) &= t_0 + 2t_1 + r_0\cos 4(\varphi_0-\theta) - 4r_1\cos 2(\varphi_1-\theta).
\end{aligned} \tag{21}$$

By inverting equation (21), it is possible to find the polar parameters as a function of the cartesian components:

$$\begin{aligned}
8t_0 &= T_{1111}(\theta) - 2T_{1122}(\theta) + 4T_{1212}(\theta) + T_{2222}(\theta) \\
8t_1 &= T_{1111}(\theta) + 2T_{1122}(\theta) + T_{2222}(\theta) \\
8r_0 e^{4i(\varphi_0-\theta)} &= T_{1111}(\theta) - 2T_{1122}(\theta) - 4T_{1212}(\theta) + T_{2222}(\theta) + 4i[T_{1112}(\theta) - T_{1222}(\theta)] \\
8r_1 e^{2i(\varphi_1-\theta)} &= T_{1111}(\theta) - T_{2222}(\theta) + 2i[T_{1112}(\theta) + T_{1222}(\theta)]
\end{aligned} \tag{22}$$

with i pure imaginary number ($i^2 = -1$). We emphasize here that t_0, t_1, r_0, r_1 and $\varphi_0 - \varphi_1$ are invariants and are thus not functions of frame angle θ.

2.3. Constitutive symmetries

Constitutive symmetries (elastic symmetries in case of elasticity tensors) are defined in 2D by special values of one or two invariants:

- *ordinary orthotropy*:
$$\varphi_0 - \varphi_1 = k\,\frac{\pi}{4},\ k \in \{0, 1\} \tag{23}$$
- r_0*–orthotropy*:
$$r_0 = 0 \tag{24}$$
- *square symmetry*:
$$r_1 = 0 \tag{25}$$
- *isotropy*:
$$r_0 = r_1 = 0 \tag{26}$$

The lack of any constitutive symmetry is $r_0 \neq 0$ and $r_1 \neq 0$ and $\varphi_0 - \varphi_1 \neq k\,\frac{\pi}{4}$.

2.4. Rari-constant tensors

A rari-constant tensor is a fourth order tensor having minor and major tensorial symmetries (20) and the Cauchy tensorial symmetry [37, 38, 39, 29]
$$T_{ijkl} = T_{ikjl} \tag{27}$$
Such a tensor may be fully anisotropic but has then 5 independant elasticity parameters instead of 6 in the present 2D case.

For fourth order tensor with minor and major symmetries, we have the equivalence
$$T_{ijkl} = T_{ikjl} \quad \Leftrightarrow \quad t_0 = t_1 \tag{28}$$
This equivalence is proven considering $T_{1122} = T_{1212}$ in equation (22) and $t_0 = t_1$ in equation (21) [40].

*2.5. **Positivity (semi-)definitness***

The tensor $\mathbb{T}$ is positive definite if and only if [41]
$$t_0 > r_0 \qquad t_1(t_0^2 - r_0^2) > 2r_1^2\,[t_0 - r_0 \cos 4(\varphi_0 - \varphi_1)] \qquad r_0 \geqslant 0 \qquad r_1 \geqslant 0 \tag{29}$$
this condition (29) implying necessarily that $t_1 \geqslant 0$.

The positive semi-definitness of the tensor $\mathbb{T}$ reads

- for $t_0 = 0$:
$$t_1 \geqslant 0 \qquad r_0 = 0 \qquad r_1 = 0 \tag{30}$$
- for $t_1 = 0$:
$$t_0 \geqslant r_0 \qquad r_0 \geqslant 0 \qquad r_1 = 0 \tag{31}$$
- for $t_0 > 0$ and $t_1 > 0$:
$$t_0 \geqslant r_0 \qquad t_1(t_0^2 - r_0^2) \geqslant 2r_1^2\,[t_0 - r_0 \cos 4(\varphi_0 - \varphi_1)] \qquad r_0 \geqslant 0 \qquad r_1 \geqslant 0 \tag{32}$$
which in the particular case $\varphi_0 = \varphi_1$ and $t_0 = r_0$ simplifies into:
$$t_0 t_1 \geqslant r_1^2 \qquad r_1 \geq 0 \tag{33}$$

This conditions for positive semi-definiteness of tensor $\mathbb{T}$ is simplified:

$$\text{for } r_0\text{–orthotropic materials}(r_0 = 0) : \qquad t_0 \geqslant 0 \qquad t_1 \geqslant 0 \qquad t_1 t_0 \geqslant 2r_1^2 \qquad r_1 \geqslant 0 \tag{34}$$

$$\text{for square symmetric materials } (r_1 = 0) : \qquad t_0 \geqslant r_0 \qquad t_1 \geqslant 0 \qquad r_0 \geqslant 0 \tag{35}$$

2.6. Quadratic form

Using the polar decompositions (19) and (21) of a second order tensor **s** and of a fourth order tensor $\mathbb{T}$, the expression of associated quadratic form is
$$\frac{1}{2}\mathbf{s} : \mathbb{T} : \mathbf{s} = 2t_0 s_{eq}^2 + 4t_1 s_m^2 + 2r_0 s_{eq}^2 \cos 4\,(\varphi_0 - \Phi) + 8r_1 s_m s_{eq} \cos 2\,(\varphi_1 - \Phi) \tag{36}$$

2.7. *Sum of fourth order tensors*

Let consider fourth order tensors $\mathbb{T}^{(p)}$ each of polar parameters $t_0^{(p)}$, $t_1^{(p)}$, $r_0^{(p)}$, $r_1^{(p)}$, $\varphi_1^{(p)}$ and $\varphi_1^{(p)}$. The polar parameters t_0, t_1, r_0, r_1, φ_0 and φ_1 of the tensor $\mathbb{T} = \sum_p \mathbb{T}^{(p)}$ satisfies to the conditions (expressed here with the choice $\theta = 0$):

$$t_0 = \sum_k t_0^{(p)} \qquad t_1 = \sum_p t_1^{(p)} \qquad r_0 \mathrm{e}^{4i\varphi_0} = \sum_p r_0^{(p)} \mathrm{e}^{4i\varphi_0^{(p)}} \qquad r_1 \mathrm{e}^{2i\varphi_1} = \sum_p r_1^{(p)} \mathrm{e}^{2i\varphi_1^{(p)}} \tag{37}$$

so that

$$\begin{aligned} t_0 &= \sum_p t_0^{(p)} \qquad r_0 = \left| \sum_p r_0^{(p)} \mathrm{e}^{4i\varphi_0^{(p)}} \right| \qquad \varphi_0 = \frac{1}{4} \arg \left[\sum_p r_0^{(p)} \mathrm{e}^{4i\varphi_0^{(p)}} \right] \\ t_1 &= \sum_p t_1^{(p)} \qquad r_1 = \left| \sum_p r_1^{(p)} e^{2i\varphi_1^{(p)}} \right| \qquad \varphi_1 = \frac{1}{2} \arg \left[\sum_p r_1^{(p)} e^{2i\varphi_1^{(p)}} \right] \end{aligned} \tag{38}$$

where $|z|$ and $\arg z$ stand for modulus and argument of complex number z.

From the last equation, we get the following properties:

- the sum of r_0-orthotropic tensors is r_0-orthotropic,
- the sum of square symmetric tensors is square symmetric,
- the sum of rari-constant tensors is rari-constant.

3. Polar decomposition and symmetry analysis of open and closed cracks contributions

The uncracked material being isotropic, the induced anisotropy in the 2D cracked material will result from anisotropy of the change of compliance tensors with open cracks, closed cracks or with both open and closed cracks. We propose in this section to investigate the type of induced anisotropy using the polar method.

3.1. *r_0-orthotropic material with open cracks*

Considering one array of open cracks (density d and angle φ), the change in compliance $\hat{\mathbb{S}}_{\text{open}} = \frac{d}{E_0} \left(\mathbf{1} \underline{\overline{\otimes}}\, \mathrm{n} \otimes \mathrm{n} + \mathrm{n} \otimes \mathrm{n} \,\underline{\overline{\otimes}}\, \mathbf{1} \right)$ (see eq. (4)) has the following polar parameters (obtained using equation (22)):

$$\begin{aligned} \hat{t}_0^o &= \frac{d}{2E_0} \qquad \hat{r}_0^o = 0 \\ \hat{t}_1^o &= \frac{d}{4E_0} \qquad \hat{r}_1^o = \frac{d}{4E_0} \qquad \hat{\varphi}_1^o = \varphi \end{aligned} \tag{39}$$

Using equation (38), the change in compliance tensor $\hat{\mathbb{S}}_{\text{open}}$ in the case of multiple arrays of open cracks has the following polar parameters:

$$\begin{aligned} \hat{t}_0^o &= \frac{1}{2E_0} \sum_{\text{open } p} d^{(p)} \qquad \hat{r}_0^o = 0 \\ \hat{t}_1^o &= \frac{1}{4E_0} \sum_{\text{open } p} d^{(p)} \qquad \hat{r}_1^o = \frac{1}{4E_0} \left| \sum_{\text{open } p} d^{(p)} e^{2i\varphi^{(p)}} \right| \qquad \hat{\varphi}_1^o = \frac{1}{2} \arg \left(\sum_{\text{open } p} d^{(p)} e^{2i\varphi^{(p)}} \right) \end{aligned} \tag{40}$$

Equation (40) shows that the change in compliance $\hat{\mathbb{S}}_{\text{open}}$ is r_0-orthotropic and positive semi-definite (see equation (34)). A complete presentation of the r_0-orthotropic symmetry can be found in [42, 43]. This symmetry presents all the properties already discussed in [4] for cracked media.

3.2. Square symmetric material with closed cracks

Considering one array of open cracks (density d and angle φ), the change in compliance (see eq. (8))

$$\hat{\mathbb{S}}_{\text{closed}} = \frac{2d}{E_0}\left(\frac{\mathbf{1}\,\overline{\underline{\otimes}}\,\mathrm{n}\otimes\mathrm{n} + \mathrm{n}\otimes\mathrm{n}\,\overline{\underline{\otimes}}\,\mathbf{1}}{2} - \mathrm{n}\otimes\mathrm{n}\otimes\mathrm{n}\otimes\mathrm{n}\right)$$

has the following polar parameters (obtained using equation (22)):

$$\begin{aligned} &\hat{t}_0^c = \frac{d}{4E_0} \qquad \hat{r}_0^c = \frac{d}{4E_0} \qquad \hat{\varphi}_0^c = \varphi + \frac{\pi}{4} \\ &\hat{t}_1^c = 0 \qquad \hat{r}_1^c = 0 \end{aligned} \tag{41}$$

Using equation (38), the change in compliance tensor $\hat{\mathbb{S}}_{\text{closed}}$ in the case of multiple arrays of closed cracks has the following polar parameters:

$$\begin{aligned} &\hat{t}_0^c = \frac{1}{4E_0}\sum_{\text{closed } p} d^{(p)} \qquad \hat{r}_0^c = \frac{1}{4E_0}\left|\sum_{\text{closed } p} d^{(p)} e^{4i\varphi^{(p)}}\right| \qquad \hat{\varphi}_0^c = \frac{1}{4}\arg\left(\sum_{\text{closed } p} d^{(p)} e^{4i\varphi^{(p)}}\right) + \frac{\pi}{4} \\ &\hat{t}_1^c = 0 \qquad \hat{r}_1^c = 0 \end{aligned} \tag{42}$$

Equation (42) shows that the change in compliance $\hat{\mathbb{S}}_{\text{closed}}$ in the case of multiple arrays of closed cracks is square symmetric and positive semi-definite (see equation (31) or (35)). As a direct consequence, the tensor $\mathbb{A}^c = \frac{E_0}{2}\hat{\mathbb{S}}_{\text{closed}}$ defined in equation (12) is also square symmetric and positive semi-definite.

This symmetry result may be surprising because the change in compliance $\hat{\mathbb{S}}_{\text{closed}}$ is the difference of two terms, the first one being driven by a r_0−orthotropic compliance matrix (as it is the same for open cracks). It is worth studying the polar decomposition of the fourth order crack density tensor $\mathbb{D}^c$ involved in the definition of the second term of the change in elastic energy (see equations (9) and (11)). Considering one array of closed cracks (density d and angle φ), tensor $\mathbb{D}^c = d\ \mathrm{n}\otimes\mathrm{n}\otimes\mathrm{n}\otimes\mathrm{n}$ has the following polar parameters:

$$\begin{aligned} &t_0^{\mathbb{D}^c} = \frac{d}{8} \qquad r_0^{\mathbb{D}^c} = \frac{d}{8} \qquad \varphi_0^{\mathbb{D}^c} = \varphi \\ &t_1^{\mathbb{D}^c} = \frac{d}{8} \qquad r_1^{\mathbb{D}^c} = \frac{d}{8} \qquad \varphi_1^{\mathbb{D}^c} = \varphi \end{aligned} \tag{43}$$

Using equation (38), the tensor $\mathbb{D}^c$ in the case of multiple arrays of closed cracks has the following polar parameters:

$$\begin{aligned} &t_0^{\mathbb{D}^c} = \frac{1}{8}\sum_{\text{closed } p} d^{(p)} \qquad r_0^{\mathbb{D}^c} = \frac{1}{8}\left|\sum_{\text{closed } p} d^{(p)} e^{4i\varphi^{(p)}}\right| \qquad \varphi_0^{\mathbb{D}^c} = \frac{1}{4}\arg\left(\sum_{\text{closed } p} d^{(p)} e^{4i\varphi^{(p)}}\right) \\ &t_1^{\mathbb{D}^c} = \frac{1}{8}\sum_{\text{closed } p} d^{(p)} \qquad r_1^{\mathbb{D}^c} = \frac{1}{8}\left|\sum_{\text{closed } p} d^{(p)} e^{2i\varphi^{(p)}}\right| \qquad \varphi_1^{\mathbb{D}^c} = \frac{1}{2}\arg\left(\sum_{\text{closed } p} d^{(p)} e^{2i\varphi^{(p)}}\right) \end{aligned} \tag{44}$$

Noticing that $t_0^{\mathbb{D}c} = t_1^{\mathbb{D}c}$ and considering the property (28), the polar decomposition (44) shows that $\mathbb{D}^c$ is a rari-constant fourth order tensor usually anisotropic.

3.3. Fully anisotropic materials with open and closed cracks

Using equations (38), (40) and (42), the change in compliance tensor $\hat{\mathbb{S}}_{\text{open}} + \hat{\mathbb{S}}_{\text{closed}}$ in the case of multiple arrays of open and closed cracks has the following polar parameters:

$$\begin{aligned} &\hat{t}_0 = \frac{1}{4E_0}\left(2\sum_{\text{open } p} d^{(p)} + \sum_{\text{closed } p} d^{(p)}\right) \qquad \hat{r}_0 = \frac{1}{4E_0}\left|\sum_{\text{closed } p} d^{(p)} e^{4i\varphi^{(p)}}\right| \qquad \hat{\varphi}_0 = \frac{1}{4}\arg\left[\sum_{\text{closed } p} d^{(p)} e^{4i\varphi^{(p)}}\right] + \frac{\pi}{4} \\ &\hat{t}_1 = \frac{1}{4E_0}\left(\sum_{\text{open } p} d^{(p)}\right) \qquad \hat{r}_1 = \frac{1}{4E_0}\left|\sum_{\text{open } p} d^{(p)} e^{2i\varphi^{(p)}}\right| \qquad \hat{\varphi}_1 = \frac{1}{2}\arg\left[\sum_{\text{open } p} d^{(p)} e^{2i\varphi^{(p)}}\right] \end{aligned} \tag{45}$$

This tensor is fully anisotropic in the general case, the $\hat{r}_0$ and $\hat{r}_1$ terms being driven respectively only by closed or open cracks. It is not rari-constant as $\hat{t}_0 \neq \hat{t}_1$.

4. Rewriting of micro-mechanics results with 2 second order open and closed crack tensors

4.1. Energy density expression from second order tensors $\mathbf{d}^o$ and $\boldsymbol{\delta}^c$

Let us set

$$\hat{t}_0 = \frac{1}{E_0}d^o_m + \frac{1}{2E_0}\delta^c_m \qquad \hat{r}_0 = \frac{1}{2E_0}\delta^c_{eq} \qquad \hat{\varphi}_0 = \varphi^c_\delta + \frac{\pi}{4}$$
$$\hat{t}_1 = \frac{1}{2E_0}d^o_m \qquad \hat{r}_1 = \frac{1}{2E_0}d^o_{eq} \qquad \hat{\varphi}_1 = \varphi^o_{\mathrm{d}} \tag{46}$$

with

$$d^o_m = \frac{1}{2}\sum_{\text{open } p} d^{(p)} \qquad d^o_{eq} = \frac{1}{2}\left|\sum_{\text{open } p} d^{(p)}e^{2i\varphi^{(p)}}\right| \qquad \varphi^o_{\mathrm{d}} = \frac{1}{2}\arg\left[\sum_{\text{open } p} d^{(p)}e^{2i\varphi^{(p)}}\right] \tag{47}$$

$$\delta^c_m = \frac{1}{2}\sum_{\text{closed } p} d^{(p)} \qquad \delta^c_{eq} = \frac{1}{2}\left|\sum_{\text{closed } p} d^{(p)}e^{4i\varphi^{(p)}}\right| \qquad \varphi^c_\delta = \frac{1}{4}\arg\left[\sum_{\text{closed } p} d^{(p)}e^{4i\varphi^{(p)}}\right] \tag{48}$$

This means that one introduces two second order tensors to represent micro-mechanics results, first standard second order tensor $\mathbf{d}^o$ (with $\mathrm{N}^{\mathrm{o}}_{\mathrm{d}} = \cos(\varphi^{\mathrm{o}}_{\mathrm{d}} - \theta)\,\mathrm{e}_1 + \sin(\varphi^{\mathrm{o}}_{\mathrm{d}} - \theta)\,\mathrm{e}_2$, frame angle denoted θ, all matrix expressions being given in working frame $(\mathrm{e}_1, \mathrm{e}_2)$, Fig. 1)

$$\mathbf{d}^o = d^o_m\mathbf{1} + 2d^o_{eq}(\mathrm{N}^{\mathrm{o}}_{\mathrm{d}} \otimes \mathrm{N}^{\mathrm{o}}_{\mathrm{d}})' \quad \text{or} \quad \mathbf{d}^o = d^o_m\mathbf{1} + d^o_{eq}\begin{bmatrix} \cos 2(\varphi^o_d - \theta) & \sin 2(\varphi^o_d - \theta) \\ \sin 2(\varphi^o_d - \theta) & -\cos 2(\varphi^o_d - \theta) \end{bmatrix} \tag{49}$$

second, novel second order tensor $\boldsymbol{\delta}^c$, representative of the effect of closed cracks (with $\mathrm{N}^{\mathrm{c}}_\delta = \cos(\varphi^{\mathrm{c}}_\delta - \theta)\,\mathrm{e}_1 + \sin(\varphi^{\mathrm{c}}_\delta - \theta)\,\mathrm{e}_2$)

$$\boldsymbol{\delta}^c = \delta^c_m\mathbf{1} + 2\delta^c_{eq}(\mathrm{N}^{\mathrm{c}}_\delta \otimes \mathrm{N}^{\mathrm{c}}_\delta)' \quad \text{or} \quad \boldsymbol{\delta}^c = \delta^c_m\mathbf{1} + \delta^c_{eq}\begin{bmatrix} \cos 2(\varphi^c_\delta - \theta) & \sin 2(\varphi^c_\delta - \theta) \\ \sin 2(\varphi^c_\delta - \theta) & -\cos 2(\varphi^c_\delta - \theta) \end{bmatrix} \tag{50}$$

Introducing the decomposition (19) of the stress tensor (with $\mathrm{N} = \cos(\Phi - \theta)\,\mathrm{e}_1 + \sin(\Phi - \theta)\,\mathrm{e}_2$)

$$\boldsymbol{\sigma} = \sigma_m\mathbf{1} + 2\tau_{eq}(\mathrm{N} \otimes \mathrm{N})' \quad \text{or} \quad \boldsymbol{\sigma} = \sigma_m\mathbf{1} + \tau_{eq}\begin{bmatrix} \cos 2(\Phi - \theta) & \sin 2(\Phi - \theta) \\ \sin 2(\Phi - \theta) & -\cos 2(\Phi - \theta) \end{bmatrix} \tag{51}$$

the change in elastic energy density (17) is written using equation (36) as

$$\rho\hat{\psi}^\star = 2\hat{t}_0\tau^2_{eq} + 4\hat{t}_1\sigma^2_m + 2\hat{r}_0\tau^2_{eq}\cos 4\,(\hat{\varphi}_0 - \Phi) + 8\hat{r}_1\sigma_m\tau_{eq}\cos 2\,(\hat{\varphi}_1 - \Phi) \tag{52}$$

It can be rewritten in a tensorial form, according to results of appendix A and of Tensorial Polar Decomposition [36],

$$\rho\hat{\psi}^\star = \frac{1}{E_0}\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\mathbf{d}^o\cdot\boldsymbol{\sigma}) + \frac{1}{E_0}\left(\tau^2_{eq}(\delta^c_m + \delta^c_{eq}) - \frac{1}{2\delta^c_{eq}}(\boldsymbol{\sigma}' : \boldsymbol{\delta}^c)^2\right) \tag{53}$$

where closed/sliding cracks act equivalently by means of fourth order tensor $\mathbb{\Delta}^c$ or by means of second order tensor $\boldsymbol{\delta}^c$.

The case of open cracks only is $\boldsymbol{\delta}^c = 0$ and the case of closed cracks only is $\mathbf{d}^o = 0$. Note that the definitions of open and closed crack densities d^o_m and δ^c_m (equations (47)-(48)) are strictly identical. They are not confusing because d^o_m only represents open cracks when δ^c_m only represents closed cracks.

4.2. Properties and expressions of tensor $\mathbb{\Delta}^c$

From equations (53) and (17), tensor $\mathbb{\Delta}^c$ is

$$\mathbb{\Delta}^c = \frac{1}{2}\left(\delta^c_m + \delta^c_{eq}\right)\mathbb{J} - \frac{1}{2\delta^c_{eq}}\boldsymbol{\delta}^{c\prime}\otimes\boldsymbol{\delta}^{c\prime} \tag{54}$$

where $\mathbb{J} = \mathbb{I} - \frac{1}{2}\,\mathbf{1}\otimes\mathbf{1}$ takes deviatoric part of any tensor in 2D, $\mathbb{J} : \mathbf{s} = \mathbf{s}'$. The tensor $\mathbb{\Delta}^c$ has the classical property

$$\boldsymbol{\sigma} : \mathbb{\Delta}^c : \boldsymbol{\sigma} = \boldsymbol{\sigma}' : \mathbb{\Delta}^c : \boldsymbol{\sigma}' \tag{55}$$

which states that only the deviatoric part of stress tensor is acting in closed cracks contribution.

One can make $\boldsymbol{\delta}^c$ minimum principal value $\delta^c_{min} = \delta^c_m - \delta^c_{eq}$ appear in the expression for energy density

$$\rho\hat{\psi}^\star_{\text{closed}} = \frac{\tau^2_{eq}}{E_0}\delta^c_{\min} + \frac{1}{E_0}\left[2\tau^2_{eq}\delta^c_{eq} - \frac{1}{2\delta^c_{eq}}(\boldsymbol{\sigma}' : \boldsymbol{\delta}^c)^2\right] \tag{56}$$

and rewrite $\mathbb{\Delta}^c$ as

$$\mathbb{\Delta}^c = \frac{1}{2}\delta^c_{\min}\mathbb{J} + \delta^c_{eq}\,(\mathbb{J} - \mathbf{N}_{\boldsymbol{\delta}^{c\prime}}\otimes\mathbf{N}_{\boldsymbol{\delta}^{c\prime}}) = \frac{1}{2}\delta^c_{\min}\mathbb{J} + \delta^c_{eq}\mathbb{P}^{\perp\boldsymbol{\delta}^{c\prime}} \tag{57}$$

by introducing unit normal $\mathbf{N}_{\boldsymbol{\delta}^{c\prime}} = \boldsymbol{\delta}^{c\prime}/\|\boldsymbol{\delta}^{c\prime}\| = \boldsymbol{\delta}^{c\prime}/\sqrt{2}\delta^c_{eq}$ and fourth order tensor $\mathbb{P}^{\perp\boldsymbol{\delta}^{c\prime}}$ which is the projector on plane perpendicular to $\boldsymbol{\delta}^{c\prime}$ in deviatoric space.

4.3. Tensor $\mathbb{D}^c$

The tensor $\mathbf{d}^c$ defined as the (geometric) complementary to $\mathbf{d}^o$ (see equation (16)) is introduced and expressed by use of polar formalism as (matrix expressed in working frame $(\mathrm{e}_1, \mathrm{e}_2)$)

$$\mathbf{d}^c = \sum_{\text{closed } p} d^{(p)} \mathrm{n}^{(p)} \otimes \mathrm{n}^{(p)} = d_m^c \mathbf{1} + 2 d_{eq}^c (\mathrm{N}_\mathrm{d}^\mathrm{c} \otimes \mathrm{N}_\mathrm{d}^\mathrm{c})' \quad \text{or} \quad \mathbf{d}^\mathrm{c} = d_m^c \mathbf{1} + d_{eq}^c \begin{bmatrix} \cos 2(\varphi_\mathrm{d}^c - \theta)) & \sin 2(\varphi_\mathrm{d}^c - \theta) \\ \sin 2(\varphi_\mathrm{d}^c - \theta) & -\cos 2(\varphi_\mathrm{d}^c - \theta) \end{bmatrix} \tag{58}$$

with $\mathrm{N}_\mathrm{d}^\mathrm{c} = \cos(\varphi_\mathrm{d}^\mathrm{c} - \theta)\, \mathrm{e}_1 + \sin(\varphi_\mathrm{d}^\mathrm{c} - \theta)\, \mathrm{e}_2$ and where one has set, in a consistent manner with set of equations (47),

$$d_m^c = \frac{1}{2} \sum_{\text{closed } p} d^{(p)}, \qquad d_{eq}^c = \frac{1}{2} \left| \sum_{\text{closed } p} d^{(p)} e^{2i\varphi^{(p)}} \right|, \qquad \varphi_\mathrm{d}^c = \frac{1}{2} \arg \left[\sum_{\text{closed } p} d^{(p)} e^{2i\varphi^{(p)}} \right]. \tag{59}$$

Last equation gives the mean densities d_m^c and δ_m^c as equal, and equal to the sum over closed cracks $\frac{1}{2} \sum_p d_{\text{closed}}^{(p)}$.

Recalling that in case of closed cracks equation (53) determines

$$\rho \hat{\psi}^\star_{\text{closed}} = \frac{1}{E_0} \left[\tau_{eq}^2 (\delta_m^c + \delta_{eq}^c) - \frac{1}{2} \frac{(\boldsymbol{\sigma}' : \boldsymbol{\delta}^c)^2}{\delta_{eq}^c} \right] \tag{60}$$

the change in elastic energy density defined in equation (17) gives, using $\tau_{eq}^2 = \frac{1}{2} \boldsymbol{\sigma}' : \boldsymbol{\sigma}'$,

$$\boldsymbol{\sigma} : \mathbb{D}^c : \boldsymbol{\sigma} = \operatorname{tr}(\boldsymbol{\sigma} \cdot \mathbf{d}^c \cdot \boldsymbol{\sigma}) - \frac{1}{2} (\delta_m^c + \delta_{eq}^c) \boldsymbol{\sigma}' : \boldsymbol{\sigma}' + \frac{1}{2} \frac{(\boldsymbol{\sigma}' : \boldsymbol{\delta}^c)(\boldsymbol{\delta}^c : \boldsymbol{\sigma}')}{\delta_{eq}} \tag{61}$$

or

$$\mathbb{D}^c = \frac{1}{2} \left(\mathbf{1} \underline{\overline{\otimes}}\, \mathbf{d}^c + \mathbf{d}^c \underline{\overline{\otimes}}\, \mathbf{1} \right) - \frac{1}{2} (\delta_m^c + \delta_{eq}^c) \mathbb{J} + \frac{1}{2} \frac{\boldsymbol{\delta}^{c\prime} \otimes \boldsymbol{\delta}^{c\prime}}{\delta_{eq}^c} \tag{62}$$

It can be checked using this tensorial expression that tensor $\mathbb{D}^c$ is rari-constant (see Appendix B).

4.4. On the extension to other homogenization schemes

In 2D, the Tensorial Polar Decomposition [36] does apply to any fourth order tensor having both minor and major indicial symmetries. The decomposition leads to the definition of two saclars and two deviatoric second order tensors, which allows to define the whole fourth order tensor considered. Applied to change in compliance tensors $\hat{\mathbb{S}} = \mathbb{S} - \mathbb{S}_0$ given by any other homogenization scheme for cracked media than the one considered in present work, obtained for example

- from schemes that consider continuous distributions of microcracks [44, 45, 46, 47] (for which the linear dependency of effective elastic properties with the microcracks density parameters is kept),
- also from nonlinear homogenization schemes (as derived in [7]),

proposed approach (by Eq. (B.3)) allows for the representation of the crack distribution (open and/or closed) by only two second order tensors. In the cases of homogenization schemes that give a tensor $E_0 \hat{\mathbb{S}}$ independent of ν_0 (see for instance [48]), the two second tensors obtained by such a decomposition are interpreted as crack density tensors.

5. Representation of any arbitrary closed cracks system by two non orthogonal families of microcracks

Let us consider multiple arrays of closed cracks. The non zero polar parameters of the change in compliance tensor $\hat{\mathbb{S}}_{\text{closed}}$ are $\hat{t}_0^c$, $\hat{r}_0^c$ and $\hat{\varphi}_0^c$, see equation (42). Let us then consider two unknown arrays of closed cracks with densities $d^{(1)}$ and $d^{(2)}$ and angles $\varphi^{(1)}$ and $\varphi^{(2)}$. In order to retrieve the same change in compliance tensor $\hat{\mathbb{S}}_{\text{closed}}$ by means of only those two arrays of closed cracks, one has to solve, using equations (41)-(42),

$$\frac{1}{4E_0} \left(d^{(1)} + d^{(2)} \right) = \hat{t}_0^c \qquad \frac{1}{4E_0} \left(d^{(1)} e^{4i(\varphi^{(1)} + \frac{\pi}{4})} + d^{(2)} e^{4i(\varphi^{(2)} + \frac{\pi}{4})} \right) = \hat{r}_0^c \mathrm{e}^{4i\hat{\varphi}_0^c} \tag{63}$$

with unknowns $d^{(1)}$, $d^{(2)}$, $\varphi^{(1)}$ and $\varphi^{(2)}$ at given values $\hat{t}_0^c$, $\hat{r}_0^c$ and $\hat{\varphi}_0^c$. Setting $\varphi^{(1)} = \hat{\varphi}_0^c$ and $\varphi^{(2)} = \hat{\varphi}_0^c + \frac{\pi}{4}$, one gets

$$d^{(1)} + d^{(2)} = 4E_0 \hat{t}_0^c \qquad d^{(2)} - d^{(1)} = 4E_0 \hat{r}_0^c \tag{64}$$

simply inverted into

$$d^{(1)} = 2E_0 (\hat{t}_0^c - \hat{r}_0^c) \qquad d^{(2)} = 2E_0 (\hat{t}_0^c + \hat{r}_0^c) \tag{65}$$

This proves that two arrays of closed cracks with densities and angles

$$d^{(1)} = 2E_0(\hat{t}_0^c - \hat{r}_0^c) \qquad \varphi^{(1)} = \hat{\varphi}_0^c \qquad d^{(2)} = 2E_0(\hat{t}_0^c + \hat{r}_0^c) \qquad \varphi^{(2)} = \hat{\varphi}_0^c + \frac{\pi}{4} \tag{66}$$

allow to represent the effects of any arbitrary closed microcracks systems. This is made possible by the fact that such complex effects are fully taken into account by the values of $\hat{t}_0^c$, $\hat{r}_0^c$ and $\hat{\varphi}_0^c$ of the change in compliance tensor $\hat{\mathbb{S}}_{\text{closed}}$, as shown in section 3 (the values of $d^{(1)}$ and $d^{(2)}$ are positive thanks to the positive semi-definiteness condition of square symmetric tensor $\hat{\mathbb{S}}_{\text{closed}}$, equation (35)).

The angle between the to equivalent families of closed cracks is not $\frac{\pi}{2}$, it is

$$\varphi^{(2)} - \varphi^{(1)} = \frac{\pi}{4} \tag{67}$$

Standard case with open cracks. In case of open cracks, the fact that only two arrays of open cracks are needed to represent for the effects of any arbitrary open microcracks systems is a standard result [6]. Using the polar formalism in the same manner than for previous case with closed cracks, this standard result is stated as following: two arrays of open cracks with densities and angles

$$d^{(1)} = 2E_0(\hat{t}_1^o + \hat{r}_1^o) \qquad \varphi^{(1)} = \hat{\varphi}_1^o \qquad d^{(2)} = 2E_0(\hat{t}_1^o - \hat{r}_1^o) \qquad \varphi^{(2)} = \hat{\varphi}_1^o + \frac{\pi}{2} \tag{68}$$

allows to represent for the effects of any arbitrary opened microcracks systems, fully taken into account by the values of $\hat{t}_0^o = 2\hat{t}_1^o$, $\hat{r}_1^o$ and $\hat{\varphi}_1^o$ of the change in compliance tensor $\hat{\mathbb{S}}_{\text{open}}$ (the values of $d^{(1)}$ and $d^{(2)}$ are positive thanks to the positive semi-definiteness condition of r_0-orthotropic tensor $\hat{\mathbb{S}}_{\text{open}}$, equation (34)).

To sum up, in each cases, open cracks only or closed cracks only, two families of cracks allow to represent the effects of any arbitrary (non interacting) microcracks systems, either open (standard result) or closed (new result). The angle made by the two families of cracks are different: angle of $\frac{\pi}{2}$ for open cracks and angle of $\frac{\pi}{4}$ for closed cracks sliding with no friction.

6. On tensorial nature of microcracks state representation when crack density evolution laws are considered

One has so far studied the state coupling. Let us now consider the case of crack growth expressed in a tensorial framework. Crack density evolution laws are rate form expressions ensuring definite positiveness of either fourth order tensor $\dot{\mathbb{D}}$ or of second order tensor $\dot{\mathbf{d}}$,

$$\dot{\mathbb{D}} = \cdots \geq 0 \quad \text{or} \quad \dot{\mathbf{d}} = \cdots \geq 0 \tag{69}$$

They allows to model micro-cracks growth and (loading oriented) cracks nucleation.

6.1. Geometric definition of a fourth order crack density tensor

The pure geometric definition of a fourth order crack density tensor $\mathbb{D}$ may be defined for all cracks, open and closed, as

$$\mathbb{D} = \sum_{\text{all } p} d^{(p)}\, \mathrm{n}^{(p)} \otimes \mathrm{n}^{(p)} \otimes \mathrm{n}^{(p)} \otimes \mathrm{n}^{(p)} = \frac{1}{2}\left(\mathbf{1} \underline{\overline{\otimes}}\, \mathbf{d} + \mathbf{d}\, \underline{\overline{\otimes}}\, \mathbf{1}\right) - \frac{1}{2}(\delta_m + \delta_{eq})\mathbb{J} + \frac{1}{2}\frac{\boldsymbol{\delta}' \otimes \boldsymbol{\delta}'}{\delta_{eq}} \tag{70}$$

if second order tensor $\boldsymbol{\delta}$ is defined as well in a pure geometric manner, over all cracks, open and closed:

$$\delta_m(= d_m) = \frac{1}{2}\sum_{\text{all } p} d^{(p)} \qquad \delta_{eq} = \frac{1}{2}\left|\sum_{\text{all } p} d^{(p)} e^{4i\varphi^{(p)}}\right| \qquad \varphi_\delta = \frac{1}{4}\arg\left[\sum_{\text{all } p} d^{(p)} e^{4i\varphi^{(p)}}\right] \tag{71}$$

The expression of the polar parameters of $\mathbb{D}$ as a function of the densities and orientations of all (open and closed) cracks (obtained in an exact similar way than for $\mathbb{D}^c$, equation (44)) reads:

$$\begin{aligned} t_0^{\mathbb{D}} &= \frac{1}{8}\sum_{\text{all } p} d^{(p)} \qquad & r_0^{\mathbb{D}} &= \frac{1}{8}\left|\sum_{\text{all } p} d^{(p)} e^{4i\varphi^{(p)}}\right| \qquad & \varphi_0^{\mathbb{D}} &= \frac{1}{4}\arg\left(\sum_{\text{all } p} d^{(p)} e^{4i\varphi^{(p)}}\right) \\ t_1^{\mathbb{D}} &= \frac{1}{8}\sum_{\text{all } p} d^{(p)} \qquad & r_1^{\mathbb{D}} &= \frac{1}{8}\left|\sum_{\text{all } p} d^{(p)} e^{2i\varphi^{(p)}}\right| \qquad & \varphi_1^{\mathbb{D}} &= \frac{1}{2}\arg\left(\sum_{\text{all } p} d^{(p)} e^{2i\varphi^{(p)}}\right) \end{aligned} \tag{72}$$

The knowledge of fourth order tensor $\mathbb{D}$ is thus equivalent to the knowledge of 2 second order crack density tensors $\mathbf{d}$ and $\boldsymbol{\delta}$. The definition (70) of $\mathbb{D}$ implies that second order tensor $\mathbf{d}$ and total crack density d_m are derived from $\mathbb{D}$ as

$$\mathbf{d} = \mathbb{D} : \mathbf{1} = \mathbf{1} : \mathbb{D} \qquad \operatorname{tr}\mathbf{d} = 2d_m = \mathbf{1} : \mathbb{D} : \mathbf{1} \tag{73}$$

Both tensors $\mathbf{d}$ and $\boldsymbol{\delta}$ have the same mean (first invariant) value $d_m = \delta_m$ (see eq. (71)). Using Polar Method terminology, $\mathbf{d}$ corresponds to isotropic (constant) terms in t_0, t_1 and to linear r_1-term of polar decomposition of fourth order tensor $\mathbb{D}$.

Considering equations (71) and (72), the knowledge of $\boldsymbol{\delta}'$ is found equivalent to the knowledge of r_0-term of polar decomposition of $\mathbb{D}$ as, with k integer,

$$\delta_{eq} = 4r_0^{\mathbb{D}} \qquad \varphi_\delta = \varphi_0^{\mathbb{D}} \tag{74}$$

This means, using the Tensorial Polar Decomposition [36] of $\mathbb{D}$ (see Appendix B), that deviatoric second order tensor $\boldsymbol{\delta}'$ represents the harmonic – sometimes called irreducible – part $\mathbb{H}$ of fourth order tensor $\mathbb{D}$,

$$\mathbb{H} = \frac{\delta_{eq}}{2}\left(\frac{\boldsymbol{\delta}'}{\delta_{eq}} \otimes \frac{\boldsymbol{\delta}'}{\delta_{eq}} - \mathbb{J}\right) \tag{75}$$

and can be obtained from Kelvin decomposition of $\mathbb{H}$, as shown i [36].

Remark 1. Harmonic part $\mathbb{H}$ of $\mathbb{D}$ is orthogonal to and independent from its complementary term $\frac{1}{2}\left(\mathbf{1} \overline{\underline{\otimes}}\, \mathbf{d} + \mathbf{d} \,\overline{\underline{\otimes}}\, \mathbf{1}\right) - \frac{1}{2} d_m \mathbb{J}$ built from the knowledge of second order tensor $\mathbf{d}$ only (see Appendix B, with $\delta_m = d_m$). Both tensors $\mathbf{d}'$ and $\boldsymbol{\delta}'$ are thus independent.

Remark 2. Altogether with property (28), the equality $t_0^{\mathbb{D}} = t_1^{\mathbb{D}}$ exhibited thanks to equation (72) shows that tensor $\mathbb{D}$ is rari-constant (property also retrieved from Tensorial Polar Decomposition in Appendix B).

Remark 3. All the derivation of present section also apply to $\mathbb{D}^c$ and to related second order tensors $\mathbf{d}^c$ and $\boldsymbol{\delta}^{c\prime}$.

Remark 4. While the relation $\mathbf{d} = \mathbf{d}^o + \mathbf{d}^c$ is true, it is worth to note that $\boldsymbol{\delta} \neq \boldsymbol{\delta}^o + \boldsymbol{\delta}^c$.

6.2. On fourth order tensorial evolution law

There is *a priori* more information in a fourth order tensorial framework. One has nevertheless to be careful if one wants to satisfy the micro-mechanics requirement of polar invariants equality $t_0^{\mathbb{D}} = t_1^{\mathbb{D}}$ making $\mathbb{D} = \sum_p d^{(p)} \mathrm{n}^{(p)} \otimes \mathrm{n}^{(p)} \otimes \mathrm{n}^{(p)} \otimes \mathrm{n}^{(p)}$ rari-constant (i.e. with $D_{ikjl} = D_{ijkl}$). The sum of fourth order tensors has for polar modulus t_0 (respectively t_1) the sum of polar modulus t_0 (respectively t_1) of each contribution (see equation (38)). One has the property that if $t_0^{\dot{\mathbb{D}}}$ and $t_1^{\dot{\mathbb{D}}}$ are first polar invariants of $\dot{\mathbb{D}}$ then the first polar invariants of $\mathbb{D}$ are:

$$t_0^{\mathbb{D}} = \int t_0^{\dot{\mathbb{D}}}\, \mathrm{d}\tau \qquad t_1^{\mathbb{D}} = \int t_1^{\dot{\mathbb{D}}}\, \mathrm{d}\tau \tag{76}$$

Because the sum – and by extension the integral over time – of rari constant tensors is rari constant (see section 2.7), ensuring rari-constant equality $t_0^{\dot{\mathbb{D}}} = t_1^{\dot{\mathbb{D}}}$ for $\dot{\mathbb{D}}$ at any time τ implies rari-constant equality $t_0^{\mathbb{D}} = t_1^{\mathbb{D}}$ for $\mathbb{D}$.

The intrinsic rate forms $\dot{\mathbb{D}} = \ldots$ should then satisfy $t_0^{\dot{\mathbb{D}}} = t_1^{\dot{\mathbb{D}}}$. Nevertheless, one may encounter $t_0^{\dot{\mathbb{D}}} \neq t_1^{\dot{\mathbb{D}}}$ for the fourth order crack density laws for initially isotropic materials of classical form $\dot{\mathbb{D}} = \dot{\lambda}\, \mathbf{P} \otimes \mathbf{P}$, with $\dot{\lambda}$ a positive multiplier, as tensor $\dot{\lambda}\, \mathbf{P} \otimes \mathbf{P}$ may not satisfy Cauchy relationships $(\mathbf{P} \otimes \mathbf{P})_{ikjl} = (\mathbf{P} \otimes \mathbf{P})_{ijkl}$.

6.3. On the use of second order tensorial crack growth and nucleation

Let us show here how the knowledge at time τ of second order crack density tensor $\mathbf{d}(\tau)$, of deviatoric part $\boldsymbol{\delta}'(\tau)$ and of the rate $\dot{\mathbf{d}}$ by means of an evolution law (for example any of the form $\dot{\mathbf{d}} = \dot{\lambda}\, \mathbf{P}$, $\dot{\lambda} \geq 0$, $\mathbf{P}$ positive semi-definite) allows to build full fourth order crack density tensor $\mathbb{D}$ (at time $\tau + \mathrm{d}\tau$) with the rari-constant micro-mechanics requirement $t_0^{\mathbb{D}} = t_1^{\mathbb{D}}$ at any time τ.

In the following, we consider that crack growth and nucleation between times τ and $\tau + \mathrm{d}\tau$ are represented by second order tensor $\boldsymbol{\Delta}\mathbf{d}$:

$$\mathbf{d}(\tau + \mathrm{d}\tau) = \mathbf{d}(\tau) + \boldsymbol{\Delta}\mathbf{d} \quad \text{with} \quad \boldsymbol{\Delta}\mathbf{d} = d\tau\, \dot{\mathbf{d}} \tag{77}$$

In order to derive the polar parameters of $\mathbb{D}(\tau + \mathrm{d}\tau)$, we first introduce the information on the type of physical arrays of cracks (namely "new cracks") that are added to the existing pattern of arrays of cracks at time τ (namely "old cracks"): we consider that the physical arrays of new cracks are characterized by densities $d^{(q)}$ and normal orientations $\varphi^{(q)}$.

We thus have by micro-mechanical definition (equations (16), (47), (59), (71)):

$$d_m(\tau) = \frac{1}{2} \sum_{\text{all old } p} d^{(p)} \qquad d_{eq}(\tau) = \frac{1}{2} \left| \sum_{\text{all old } p} d^{(p)} e^{2i\varphi^{(p)}} \right| \qquad \varphi_d(\tau) = \frac{1}{2} \arg\left(\sum_{\text{all old } p} d^{(p)} e^{2i\varphi^{(p)}} \right) \tag{78}$$

$$\delta_{eq}(\tau) = \frac{1}{2} \left| \sum_{\text{all old } p} d^{(p)} e^{4i\varphi^{(p)}} \right| \qquad \varphi_\delta(\tau) = \frac{1}{4} \arg\left(\sum_{\text{all old } p} d^{(p)} e^{4i\varphi^{(p)}} \right) \tag{79}$$

$$\Delta d_m = \frac{1}{2} \sum_{\text{all new } q} d^{(q)} \qquad (\boldsymbol{\Delta}\mathbf{d})_{eq} = \frac{1}{2} \left| \sum_{\text{all new } q} d^{(q)} e^{2i\varphi^{(q)}} \right| \qquad \varphi_{\Delta d} = \frac{1}{2} \arg\left(\sum_{\text{all new } q} d^{(q)} e^{2i\varphi^{(q)}} \right) \tag{80}$$

From equation (72):

$$t_0^{\mathbb{D}}(\tau + \mathrm{d}\tau) = \left(t_1^{\mathbb{D}}(\tau + \mathrm{d}\tau) =\right) \frac{1}{8} \sum_{\text{all old } p} d^{(p)} + \frac{1}{8} \sum_{\text{all new } q} d^{(q)} \tag{81}$$

thus

$$t_0^{\mathbb{D}}(\tau + \mathrm{d}\tau) = \left(t_1^{\mathbb{D}}(\tau + \mathrm{d}\tau) =\right) \frac{1}{4} \left(d_m(\tau) + \Delta d_m\right) \tag{82}$$

Equations (82) shows that the knowledge of $\mathbf{d}(\tau)$ and $\dot{\mathbf{d}}$ allows to determine the polar invariants $t_0^{\mathbb{D}} = t_1^{\mathbb{D}}$ of tensor $\mathbb{D}$ at time $\tau + \mathrm{d}\tau$.

Recalling that equation (37) and (38) are equivalent, equation (72) gives

$$r_1^{\mathbb{D}}(\tau + \mathrm{d}\tau) e^{2i\varphi_1^{\mathbb{D}}(\tau+\mathrm{d}\tau)} = \frac{1}{8} \sum_{\text{all old } p} d^{(p)} e^{2i\varphi^{(p)}} + \frac{1}{8} \sum_{\text{all new } q} d^{(q)} e^{2i\varphi^{(q)}} \tag{83}$$

thus

$$r_1^{\mathbb{D}}(\tau + \mathrm{d}\tau) e^{2i\varphi_1^{\mathbb{D}}(\tau+\mathrm{d}\tau)} = \frac{1}{4} \left(d_{eq}(\tau) e^{2i\varphi_d(\tau)} + (\mathbf{\Delta d})_{eq} e^{2i\varphi_{\Delta d}}\right) \tag{84}$$

Equation (84) shows that the knowledge of $\mathbf{d}(\tau)$ and $\dot{\mathbf{d}}$ allows to determine the polar parameters $r_1^{\mathbb{D}}$ and $\varphi_1^{\mathbb{D}}$ of tensor $\mathbb{D}$ at time $\tau + \mathrm{d}\tau$.

And finally, equation (72) gives

$$r_0^{\mathbb{D}}(\tau + \mathrm{d}\tau) e^{4i\varphi_0^{\mathbb{D}}(\tau+\mathrm{d}\tau)} = \frac{1}{8} \sum_{\text{all old } p} d^{(p)} e^{4i\varphi^{(p)}} + \frac{1}{8} \sum_{\text{all new } q} d^{(q)} e^{4i\varphi^{(q)}} \tag{85}$$

thus

$$r_0^{\mathbb{D}}(\tau + \mathrm{d}\tau) e^{4i\varphi_0^{\mathbb{D}}(\tau+\mathrm{d}\tau)} = \frac{1}{4} \delta_{eq}(\tau) e^{4i\varphi_\delta(\tau)} + \frac{1}{8} \sum_{\text{all new } q} d^{(q)} e^{4i\varphi^{(q)}} \tag{86}$$

Equation (86) shows that in the most general case, it is *not* possible to determine the polar parameters $r_0^{\mathbb{D}}$ and $\varphi_0^{\mathbb{D}}$ of tensor $\mathbb{D}$ at time $\tau + \mathrm{d}\tau$ from the knowledge of $\mathbf{d}(\tau)$, $\boldsymbol{\delta}'(\tau)$ and $\dot{\mathbf{d}}$ only, as one needs to explicitely know the physical pattern of crack arrays in order to compute the second term of the right hand side of equation (86).

6.4. *Second order crack growth / nucleation modeling that satisfies rari-constancy micro-mechanics requirement*

Nevertheless, it is possible to propose a crack growth/nucleation modeling that satisfies rari-constancy micro-mechanics requirement in the second order tensorial evolution law framework. In order to do so, let now consider that the new crack arrays created from time τ to time $\tau + \mathrm{d}\tau$ are of the following types:

1. a set of arrays of cracks of uniform spatial orientation with a constant density d_H,
2. a pattern of at most two physical orthogonal arrays of cracks with densities d and $d_\perp$ and orientations φ and $\varphi + \frac{\pi}{2}$,

The corresponding increment of crack density $\mathbf{\Delta d}$ is

$$\mathbf{\Delta d} = \mathbf{\Delta d}^H + \mathbf{\Delta d}^\perp \tag{87}$$

with

$$\Delta d_m = \Delta d_m^H + \Delta d_m^\perp = \frac{1}{2} d_H + \frac{1}{2} \left(d + d_\perp\right) \tag{88}$$

$$(\mathbf{\Delta d})_{eq} = (\mathbf{\Delta d}^\perp)_{eq} = \frac{1}{2} \left(d - d_\perp\right) \tag{89}$$

$$\varphi_{\Delta d} = \varphi_{\Delta d^\perp} = \varphi \tag{90}$$

With this two types of new crack arrays,

$$\begin{aligned}
\frac{1}{8} \sum_{\text{all new } q} d^{(q)} e^{4i\varphi^{(q)}} &= \frac{1}{8} \left(d e^{4i\varphi} + d^\perp e^{4i(\varphi + \frac{\pi}{2})}\right) + \frac{1}{8\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d^H e^{4i\varphi^H} \mathrm{d}\varphi^{\mathrm{H}} \\
&= \frac{1}{8} \left(d + d^\perp\right) e^{4i\varphi} + 0 \\
&= \frac{1}{4} \Delta d_m^\perp \, e^{4i\varphi_{\Delta d}}
\end{aligned} \tag{91}$$

and equation (86) is changed into

$$r_0^{\mathbb{D}}(\tau + \mathrm{d}\tau)e^{4i\varphi_0^{\mathbb{D}}(\tau+\mathrm{d}\tau)} = \frac{1}{4}\delta_{eq}(\tau)e^{4i\varphi_\delta(\tau)} + \frac{1}{4}\Delta d_m^{\perp}\, e^{4i\varphi_{\Delta d}} \tag{92}$$

Equations (82), (84) and (92) show that it is possible to determine the full fourth order tensor $\mathbb{D}(\tau + \mathrm{d}\tau)$ at time $\tau + \mathrm{d}\tau$, i.e. standard crack density second order tensor $\mathbf{d}(\tau + \mathrm{d}\tau)$ and novel deviatoric second order tensor $\boldsymbol{\delta}'(\tau + \mathrm{d}\tau)$,

– with the rari-constant micro-mechanics requirement $t_0^{\mathbb{D}} = t_1^{\mathbb{D}}$,

– from the knowledge of $\mathbf{d}(\tau)$, $\boldsymbol{\delta}'(\tau)$ at previous time and of density rate $\dot{\mathbf{d}}$,

when new cracks arrays nucleated between time τ and time $\tau + \mathrm{d}\tau$ are either a set of arrays of cracks of uniform spatial orientation with a constant density or a pattern of at most two physical orthogonal arrays of cracks.

With this in mind we propose next to define cracks closure conditions from second order tensorial framework.

7. Cracks closure from second order tensorial framework

According to the results of section 4, the effect of open cracks is represented by second order density tensor $\mathbf{d}^o$, the effect of closed cracks (sliding with no friction) by second tensor tensor $\boldsymbol{\delta}^c$. An important question is wether one can use the proposed second order tensorial framework

$$\rho\hat{\psi}^\star = \frac{1}{E_0}\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\mathbf{d}^o\cdot\boldsymbol{\sigma}) + \frac{1}{E_0}\left[\tau_{eq}^2(\delta_m^c + \delta_{eq}^c) - \frac{1}{2\delta_{eq}^c}(\boldsymbol{\sigma}' : \boldsymbol{\delta}^c)^2\right] \tag{93}$$

to obtain the partition between open cracks and closed cracks contribution, with no need anymore of information at micro-scale: can we do this simply by proper projections on macroscopic stress directions ? In other words can we define the cracks densities $\mathbf{d}^o$ and $\boldsymbol{\delta}^c$ from the knowledge of stress tensor $\boldsymbol{\sigma}$ and of second order density tensors $\mathbf{d}$, $\boldsymbol{\delta}$ defined in a pure geometric manner on all cracks, open and closed ?

7.1. The special case of of one and two orthogonal arrays of cracks

Let consider the particular case of two arrays of orthogonal cracks (respectively of normal $\mathrm{n}^{(1)}$ and $\mathrm{n}^{(2)}$ and densities d_1 and d_2) given by standard 2D micro-mechanics of section 1. We assume with no loss of generality that $d_1 \geq d_2$. The case of one array of cracks is simply obtained by choosing $d_2 = 0$. It is shown in Appendix C that the crack opening conditions which define the open cracks density d_1^o and d_2^o from the knowledge of pure geometric cracks density d_1 and d_2 can be cast as

$$d_1^o = d_1\,\mathcal{H}(\sigma_{11}) \qquad\qquad d_2^o = d_2\,\mathcal{H}(\sigma_{22}) \tag{94}$$

with $\mathcal{H}(x)$ the Heaviside function and where direction 1 and 2 are the normals of the two cracks families.

It is also shown in the appendix that a continuous stress strain response is gained by a proper choice for second order deviatoric tensor $\boldsymbol{\delta}'$ with finally in present case of two known orthogonal arrays of cracks :

$$\rho\hat{\psi}^\star = \rho\hat{\psi}^\star(\boldsymbol{\sigma}, d_1, d_2) = \frac{1}{E_0}\left[d_1\left(\langle\sigma_{11}\rangle_+^2 + \sigma_{12}^2\right) + d_2\left(\langle\sigma_{22}\rangle_+^2 + \sigma_{12}^2\right)\right] \tag{95}$$

The density variable is of tensorial nature and cannot be replaced by the two scalars d_1 and d_2 in the general case of many arrays of cracks. Let us therefore consider the case of a symmetric second order tensorial crack density issued from the time integration of an evolution law $\dot{\mathbf{d}} = \cdots \geq 0$ written in a rate form, ensuring rate $\dot{\mathbf{d}}$ to be semi definite positive, ensuring also rari-constant equality of first polar invariants $t_0 = t_1$ of rebuild tensor $\mathbb{D}$, as shown in section 6.

An intrinsic expression for cracks closure is needed, which has to be exact for the cases of one array and of two orthogonal arrays of cracks.

7.2. Splitting of crack density tensor for open cracks $\mathbf{d}^o$

Let us define

$$\mathbf{M}^o = (\mathbf{M}^o)^2 = \frac{1}{2}\left(\mathbf{1} + \frac{\mathbf{d}^{o\prime}}{d_{eq}^o}\right) \qquad \mathbf{m}^o = (\mathbf{m}^o)^2 = \frac{1}{2}\left(\mathbf{1} - \frac{\mathbf{d}^{o\prime}}{d_{eq}^o}\right) \tag{96}$$

with unit quadratic norm $||\mathbf{M}^o|| = ||\mathbf{m}^o|| = 1$ and $M_{eq}^o = m_{eq}^o = \frac{1}{2}$, such as the second order crack density tensor $\mathbf{d}^o$ for open cracks may be split into two parts:

$$\mathbf{d}^o = d_{\mathrm{Max}}^o\,\mathbf{M}^o + d_{\mathrm{min}}^o\,\mathbf{m}^o \qquad \begin{cases} d_{\mathrm{Max}}^o = d_m^o + d_{eq}^o \\ d_{\mathrm{min}}^o = d_m^o - d_{eq}^o \end{cases} \tag{97}$$

Tensor $\mathbf{M}^o$ selects maximum eigen density $d^o_{\text{Max}} = d^o_1 \geq 0$ of tensor $\mathbf{d}^o$ in principal basis (in which $\mathbf{M}^o = \text{diag}[d^o_{\text{Max}}, 0]$), tensor $\mathbf{m}^o$ selects mimimum eigen density $d^o_{\text{min}} = d^o_2 \geq 0$ of $\mathbf{d}^o$ in principal basis (in which $\mathbf{m}^o = \text{diag}[0, d^o_{\text{min}}]$). One has then

$$\text{tr}\,(\boldsymbol{\sigma}\cdot\mathbf{d}^o\cdot\boldsymbol{\sigma}) = \text{tr}\left(\boldsymbol{\sigma}\cdot\left[\sqrt{d^o_{\text{Max}}}\,\mathbf{M}^o\right]^2\cdot\boldsymbol{\sigma}\right) + \text{tr}\left(\boldsymbol{\sigma}\cdot\left[\sqrt{d^o_{\text{min}}}\,\mathbf{m}^o\right]^2\cdot\boldsymbol{\sigma}\right) \tag{98}$$

One shows in Appendix D that for any second order tensor $\mathbf{h}$ in 2D (with von Mises norm $\tau_{eq} = \sqrt{\frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'}$)

$$\text{tr}\left(\boldsymbol{\sigma}\cdot\mathbf{h}^2\cdot\boldsymbol{\sigma}\right) = \text{tr}\,(\mathbf{h}\cdot\boldsymbol{\sigma}\cdot\mathbf{h}\cdot\boldsymbol{\sigma}) + 4\tau^2_{eq}h^2_{eq} - (\boldsymbol{\sigma}' : \mathbf{h}')^2 \tag{99}$$

Using equality (99) for either $\mathbf{h} = \sqrt{d^o_{\text{Max}}}\,\mathbf{M}^o$ or $\mathbf{h} = \sqrt{d^o_{\text{min}}}\,\mathbf{m}^o$, equation (98) reads

$$\begin{aligned}\text{tr}\,(\boldsymbol{\sigma}\cdot\mathbf{d}^o\cdot\boldsymbol{\sigma}) =& d^o_{\text{Max}}\,\text{tr}\,(\mathbf{M}^o\cdot\boldsymbol{\sigma}\cdot\mathbf{M}^o\cdot\boldsymbol{\sigma}) + d^o_{\text{Max}}\left(\tau^2_{eq} - (\boldsymbol{\sigma}' : \mathbf{M}^o)^2\right)\\ &+ d^o_{\text{min}}\,\text{tr}\,(\mathbf{m}^o\cdot\boldsymbol{\sigma}\cdot\mathbf{m}^o\cdot\boldsymbol{\sigma}) + d^o_{\text{min}}\left(\tau^2_{eq} - (\boldsymbol{\sigma}' : \mathbf{m}^o)^2\right)\end{aligned} \tag{100}$$

Using this last equation (100), the change in elastic energy density (93) becomes

$$\begin{aligned}\rho\hat{\psi}^\star =& \frac{1}{E_0}\left[d^o_{\text{Max}}\,\text{tr}\,(\mathbf{M}^o\cdot\boldsymbol{\sigma}\cdot\mathbf{M}^o\cdot\boldsymbol{\sigma}) + d^o_{\text{Max}}\left(\tau^2_{eq} - (\boldsymbol{\sigma}' : \mathbf{M}^o)^2\right)\right.\\ &\left.+\, d^o_{\text{min}}\,\text{tr}\,(\mathbf{m}^o\cdot\boldsymbol{\sigma}\cdot\mathbf{m}^o\cdot\boldsymbol{\sigma}) + d^o_{\text{min}}\left(\tau^2_{eq} - (\boldsymbol{\sigma}' : \mathbf{m}^o)^2\right)\right]\\ &+ \frac{1}{E_0}\left[\tau^2_{eq}(\delta^c_m + \delta^c_{eq}) - \frac{1}{2\delta^c_{eq}}(\boldsymbol{\sigma}' : \boldsymbol{\delta}^{c\prime})^2\right]\end{aligned} \tag{101}$$

At this stage, equation (101) is only a rewriting of equation (93).

7.3. *Special positive part*

Adapting mathematical tools developed in [49, 50], one defines special positive part of tensor $\mathbf{h}_+$ from positive eigenvalues λ_I and associated eigenvectors y^I of non symmetric matrix $\mathbf{h}\cdot\boldsymbol{\sigma}$,

$$\mathbf{h}_+ = \sum \langle\lambda_I\rangle_+\,\mathrm{y}^I\otimes\mathrm{y}^I \tag{102}$$

(the eigenvectors are normalized as $\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathrm{y}^J = \delta_{IJ}$). Using the special positive part (102), the term $\text{tr}\,(\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma})$ is continuously differentiable (see Appendix E):

$$\text{d}\,\text{tr}\,(\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma}) = 2\,(\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+) : \text{d}\boldsymbol{\sigma} + 2\,(\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma}) : \text{d}\mathbf{h} \tag{103}$$

Remark 1. For one array of cracks of normal n, of density d, one can simply set $\mathbf{h} = \mathbf{d}^{\frac{1}{2}} = \sqrt{d}\,\mathrm{n}\otimes\mathrm{n} = \mathbf{d}/\sqrt{d}$ so that special positive part (102) is nothing else than physical definition of open (squareroot) density tensor,

$$\mathbf{h}_+ = \mathbf{d}^{\frac{1}{2}}_+ = \mathcal{H}(\sigma_{nn})\begin{bmatrix}\sqrt{d} & 0\\ 0 & 0\end{bmatrix} \qquad \mathbf{d}^o = \mathbf{d}^{\frac{1}{2}}_+\cdot\mathbf{d}^{\frac{1}{2}}_+ \tag{104}$$

with $\mathcal{H}$ Heaviside function making $\mathbf{d}_+$ vanishing when cracks close.

Remark 2. The case of two orthogonal arrays of cracks can mathematically be handled by use of the same procedure, but the derivations do not give a satisfactory definition of open crack density tensor: they do not lead to expressions $d_{+\,11} = \mathcal{H}(\sigma_{11})\sqrt{d_1}$, $d_{+\,22} = \mathcal{H}(\sigma_{22})\sqrt{d_2}$, $d_{+\,12} = 0$.

7.4. *Derivation of open crack density tensor* $\mathbf{d}^o$ *from the geometric crack density tensor* $\mathbf{d}$

One proposes to use previous special positive parts (102) to define the open crack density tensor $\mathbf{d}^o$ from the second order crack density tensor $\mathbf{d}$ (geometrically defined for all cracks, open and closed). Using a similar decomposition than (97) for the crack density tensor $\mathbf{d}$

$$\mathbf{d} = d_{\text{Max}}\,\mathbf{M} + d_{\text{min}}\,\mathbf{m} \qquad \begin{cases} d_{\text{Max}} = d_m + d_{eq}\\ d_{\text{min}} = d_m - d_{eq}\end{cases} \tag{105}$$

Definition (102) gives the open crack density tensor $\mathbf{d}^o$ from the knowledge of pure geometric crack density $\mathbf{d}$ thanks to tensors $\mathbf{M}$ and $\mathbf{m}$, which have each only one non zero eigenvalue:

$$\begin{cases}\sqrt{d^o_{\text{Max}}}\,\mathbf{M}^o = \left[\sqrt{d_{\text{Max}}}\,\mathbf{M}\right]_+ = \sqrt{d_{\text{Max}}}\,\mathcal{H}(\boldsymbol{\sigma} : \mathbf{M})\,\mathbf{M}\\ \sqrt{d^o_{\text{min}}}\,\mathbf{m}^o = \left[\sqrt{d_{\text{min}}}\,\mathbf{m}\right]_+ = \sqrt{d_{\text{min}}}\,\mathcal{H}(\boldsymbol{\sigma} : \mathbf{m})\,\mathbf{m}\end{cases} \tag{106}$$

defining $\mathbf{M}^o = \mathbf{M}$, $\mathbf{m}^o = \mathbf{m}$, as well as

$$d^o_{\rm Max} = d_{\rm Max}\mathcal{H}(\boldsymbol{\sigma} : \mathbf{M}) \qquad d^o_{\rm min} = d_{\rm min}\mathcal{H}(\boldsymbol{\sigma} : \mathbf{m}) \tag{107}$$

and the density of open cracks $\mathbf{d}^o$ is finally gained from Eq. (97).

Equation (101) reads then:

$$\begin{aligned}\rho\hat{\psi}^\star =& \frac{1}{E_0} d_{\rm Max}\left[\langle\boldsymbol{\sigma} : \mathbf{M}\rangle_+^2 + \mathcal{H}(\boldsymbol{\sigma} : \mathbf{M})\left(\tau_{eq}^2 - (\boldsymbol{\sigma}' : \mathbf{M})^2\right)\right] \\ &+ \frac{1}{E_0} d_{\rm min}\left[\langle\boldsymbol{\sigma} : \mathbf{m}\rangle_+^2 + \mathcal{H}(\boldsymbol{\sigma} : \mathbf{m})\left(\tau_{eq}^2 - (\boldsymbol{\sigma}' : \mathbf{m})^2\right)\right] \\ &+ \frac{1}{E_0}\left[\tau_{eq}^2(\delta^c_m + \delta^c_{eq}) - 2\delta^c_{eq}\left(\boldsymbol{\sigma}' : \frac{\boldsymbol{\delta}^{c\prime}}{2\delta^c_{eq}}\right)^2\right]\end{aligned} \tag{108}$$

It can be checked that Eq. (108) extends the expression (C.7) for two orthogonal arrays of fixed cracks, to general case of many arrays of cracks due to possibly non proportional loading. The terms $\langle\boldsymbol{\sigma} : \mathbf{M}\rangle_+^2$ and $\langle\boldsymbol{\sigma} : \mathbf{m}\rangle_+^2$ are continuously differentiable but the Heaviside terms are not: they lead to strain discontinuity that have to be erased by slip term in $\boldsymbol{\delta}^c$.

Continuity of derivative of $d_{\rm Max}\mathcal{H}(\boldsymbol{\sigma} : \mathbf{M})\tau_{eq}^2 + d_{\rm min}\mathcal{H}(\boldsymbol{\sigma} : \mathbf{m})\tau_{eq}^2 + \tau_{eq}^2(\delta^c_m + \delta^c_{eq})$ with respect to τ_{eq} gives

$$\delta^c_m + \delta^c_{eq} = d_{\rm Max}\left(1 - \mathcal{H}(\boldsymbol{\sigma} : \mathbf{M})\right) + d_{\rm min}\left(1 - \mathcal{H}(\boldsymbol{\sigma} : \mathbf{m})\right) \tag{109}$$

Because in 2D $(\boldsymbol{\sigma}' : \mathbf{M})^2 = (\boldsymbol{\sigma}' : \mathbf{m})^2 = (\boldsymbol{\sigma}' : \mathbf{M}')^2 = (\boldsymbol{\sigma}' : \mathbf{m}')^2$ with $M_{eq} = m_{eq} = 1/2$, and $\left(\boldsymbol{\delta}^{c\prime}/2\delta^c_{eq}\right)_{eq} = 1/2$, the continuity of the derivative of $d_{\rm Max}(\boldsymbol{\sigma}' : \mathbf{M})^2 + d_{\rm min}(\boldsymbol{\sigma}' : \mathbf{m})^2 + 2\delta^c_{eq}\left(\boldsymbol{\sigma}' : \frac{\boldsymbol{\delta}^{c\prime}}{2\delta^c_{eq}}\right)^2$ with respect to $\boldsymbol{\sigma}'$ enforces then

$$\frac{\boldsymbol{\delta}^{c\prime}}{2\delta^c_{eq}} = \pm\mathbf{M}' = \mp\mathbf{m}' \quad \text{and} \quad 2\delta^c_{eq} = d_{\rm Max}\left(1 - \mathcal{H}(\boldsymbol{\sigma} : \mathbf{M})\right) + d_{\rm min}\left(1 - \mathcal{H}(\boldsymbol{\sigma} : \mathbf{m})\right) \tag{110}$$

which, combined with equation (109) leads to $\delta^c_m = \delta^c_{eq}$ and thus $\delta^c_{\rm min} = \delta^c_m - \delta^c_{eq} = 0$.

Finally, equation (108) with the strain continuity condition ends up to

$$\rho\hat{\psi}^\star = \frac{1}{E_0} d_{\rm Max}\left[\langle\boldsymbol{\sigma} : \mathbf{M}\rangle_+^2 + \frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}' - (\boldsymbol{\sigma}' : \mathbf{M})^2\right] + \frac{1}{E_0} d_{\rm min}\left[\langle\boldsymbol{\sigma} : \mathbf{m}\rangle_+^2 + \frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}' - (\boldsymbol{\sigma}' : \mathbf{m})^2\right] \tag{111}$$

which is convex and continuously differentiable and can be rewritten in terms of first and second invariants of $\mathbf{d}$

$$\rho\hat{\psi}^\star = \rho\hat{\psi}^\star(\boldsymbol{\sigma}, \mathbf{d}) = \frac{1}{E_0}\left[d_m\boldsymbol{\sigma}' : \boldsymbol{\sigma}' + \left(d_m + d_{eq}\right)\left(\langle\boldsymbol{\sigma} : \mathbf{M}\rangle_+^2 - (\boldsymbol{\sigma}' : \mathbf{M})^2\right) + \left(d_m - d_{eq}\right)\left(\langle\boldsymbol{\sigma} : \mathbf{m}\rangle_+^2 - (\boldsymbol{\sigma}' : \mathbf{m})^2\right)\right] \tag{112}$$

One obtains a tensorial second order density framework for a 2D medium with open and closed cracks. The cracks closure conditions are expressed in a tensorial expression at the macroscopic scale: there will be no need of a multiscale analysis when performing Finite Element computations. Non proportional loadings are naturally handled from state law $\boldsymbol{\epsilon} = \rho\frac{(\partial\psi^\star_0 + \hat{\psi}^\star)}{\partial\boldsymbol{\sigma}}$,

$$\boldsymbol{\epsilon} = \mathbb{S}_0 : \boldsymbol{\sigma} + \frac{2}{E_0}\left[d_m\boldsymbol{\sigma}' + \left(d_m + d_{eq}\right)\left(\langle\boldsymbol{\sigma} : \mathbf{M}\rangle_+\mathbf{M} - (\boldsymbol{\sigma}' : \mathbf{M})\,\mathbf{M}'\right) + \left(d_m - d_{eq}\right)\left(\langle\boldsymbol{\sigma} : \mathbf{m}\rangle_+\mathbf{m} - (\boldsymbol{\sigma}' : \mathbf{m})\,\mathbf{m}'\right)\right] \tag{113}$$

with mathematical equality $(\boldsymbol{\sigma}' : \mathbf{m})\,\mathbf{m}' = (\boldsymbol{\sigma}' : \mathbf{M})\,\mathbf{M}'$ in 2D.

7.5. Discussion

The change in elastic energy density (112) is a function of the stress and of the density tensor only (through invariants $d_m = \frac{1}{3}\,\mathrm{tr}\,\mathbf{d}$, $d_{eq} = (\frac{1}{2}\mathbf{d}' : \mathbf{d}')^{1/2}$ and through second order tensors $\mathbf{M} = \frac{1}{2}(\mathbf{1} + \frac{\mathbf{d}'}{d_{eq}})$ and $\mathbf{m} = \frac{1}{2}(\mathbf{1} - \frac{\mathbf{d}'}{d_{eq}})$). It is convex in $\boldsymbol{\sigma}$. It does not depend on tensors $\mathbf{d}^c$, $\boldsymbol{\delta}^c$ for closed cracks anymore. Theses tensors responsible for closed/sliding cracks are not thermodynamics (internal) variables in proposed framework: their values have been introduced within $\rho\hat{\psi}^\star$ by the strain continuity feature. The property that density tensor $\mathbf{d}$ remains as single thermodynamics internal variable is due to the assumption made for sliding: friction is neglected in present so-called "lubricated cracks" approach [20, 21, 19, 22, 2].

There is a drawback (the price to pay to get a modeling at macroscopic scale only): a vanishing minimum eigenvalue $\delta^c_{\rm min} = \delta^c_m - \delta^c_{eq} = 0$ is obtained when standard micro-mechanics allows for $\delta^c_{\rm min} > 0$, *i.e.* for a second positive eigenvalue (see equation (48) of section 4.1). The feature $\delta^c_{\rm min} = 0$ can nevertheless be qualified. It corresponds to a sliding tensor $\boldsymbol{\delta}^c$ responsible for the effect of closed cracks with only one non zero eigenvalue, *i.e.* it corresponds to only one equivalent family of closed cracks allowed to slide. Present approach may be requalified "rotating closed cracks approach": tensor $\boldsymbol{\delta}^c$ – and only it – instantaneously rotates in order to remain coaxial with full open+closed density tensor $\mathbf{d}$. The standard density tensor is itself loading history dependent, it is gained as $\int \dot{\mathbf{d}}\, d\tau$ for possibly non proportional loading: as the time integration, time increment by time increment, of crack density evolution law. This seems acceptable in many practical applications with either proportional loading (in which case proposed framework is exact) or with limited shear.

Note nevertheless that the error made compared to a given microcracking pattern and loading can be quantified: it is simply $\delta^c_{\min}$ calculated directly from exact micro-mechanics problem, as $\delta^c_{\min}\big|^{\text{model}} = 0$:

$$Error = \delta^c_{\min}\Big|^{\text{exact micro-mechanics}} - \delta^c_{\min}\Big|^{\text{model}} = \frac{1}{2}\sum_{\text{closed } p} d^{(p)} - \frac{1}{2}\left|\sum_{\text{closed } p} d^{(p)} e^{4i\varphi^{(p)}}\right| \tag{114}$$

where $|z|$ stands for the modulus of complex number z. The error vanishes in the case of two arrays of orthogonal cracks.

Conclusion

We have proposed a second order tensorial framework for 2D medium initially isotropic with open and closed sliding cracks (i.e. without friction).

In order to do so, we have studied, using the polar formalism, the particular indicial and constitutive symmetries of the change in compliance tensor in the case of open and closed cracks. The change in compliance tensor for open cracks is found to be r_0-orthotropic, and is classically expressed in terms of a second order crack density tensor $\mathbf{d}^o$. The change in compliance tensor for closed cracks is found to be square symmetric (a new result to the best of our knowledge), and is then expressed in terms of novel second order crack density tensor $\boldsymbol{\delta}^c$. One specific property of $\boldsymbol{\delta}^c$ is that it is not equal to the sum of second order cracks density tensors of each set of closed cracks arrays considered (as it is the case for $\mathbf{d}^o$ for open cracks). In the closed cracks case the fourth order crack density tensor $\mathbb{D}^c$ that appears as a supplementary term is found to be rari-constant. Those results lead to the fact that:

- with open cracks only , the microcracking state is represented by a single 2nd order crack density tensor $\mathbf{d}^o$ and the compliance tensor is (r_0-) orthotropic (standard results),
- with closed cracks only , the microcracking state is represented by a single 2nd order crack density tensor $\boldsymbol{\delta}^c$ and the compliance tensor is square symmetric,
- with open and closed cracks, the microcracking state is represented by the two second order tensors $\mathbf{d}^o$ and $\boldsymbol{\delta}^c$ and the compliance tensor can be fully anisotropic,
- two families of cracks allow to represent the effects of any arbitrary open or closed microcracks systems, the angle made by the two families of cracks being $\frac{\pi}{2}$ for open cracks (standard result) and $\frac{\pi}{4}$ for closed cracks (new result).

Using the microscopic representation of the effect of crack arrays on the fourth order crack density tensor $\mathbb{D}$ defined for all geometrical cracks (i.e. open and closed), we have shown that in common cases of cracks nucleation an evolution law ($\dot{\mathbf{d}}$ = ...) based on the second order crack density tensor $\mathbf{d}$ only (defined for all geometrical cracks) is sufficient in order to completely determine tensor $\mathbb{D}$, i.e. $\mathbf{d}$ and $\boldsymbol{\delta}$, at time step $\tau + d\tau$ with the rari-constancy property from the knowledge of $\mathbb{D}(\tau)$, i.e. $\mathbf{d}(\tau)$ and $\boldsymbol{\delta}(\tau)$, and of the crack density rate $\dot{\mathbf{d}}$.

We have finally defined a framework that build both second order second order crack density tensors for open cracks $\mathbf{d}^o$ and for closed cracks $\boldsymbol{\delta}^c$ from the knowledge of the stress tensor and of the crack density tensors $\mathbf{d}$ and $\boldsymbol{\delta}$ geometrically defined on all cracks. Proposed framework is exact for proportional loading cases. It is approximate for more complex loading cases as it leads to the alignment of the contribution $\boldsymbol{\delta}^c$ for cracks closure with the open cracks density tensor $\mathbf{d}$.

Acknowlegments

The authors would like to thanks D. Kondo from Université Paris 6 and P. Vannucci from Université Versailles Saint Quentin for their constructive comments.

Appendix A. Intrinsic expression of polar decomposition – Application to tensor $\boldsymbol{\Delta}^c$

Using the definitions (49) and (51) for $\mathbf{d}^o$ and $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma}' : \mathbf{d}^o = 2\tau_{eq} d^o_{eq} \cos 2(\varphi^o - \Phi) \tag{A.1}$$

therefore considering equation (46):

$$8\hat{r}_1 \sigma_m \tau_{eq} \cos 2(\hat{\varphi}_1 - \Phi) = \frac{2}{E_0} \boldsymbol{\sigma}' : \mathbf{d}^o \sigma_m \tag{A.2}$$

Starting from the same equality but expressed in $\boldsymbol{\delta}^c$ (defined in equation (50))

$$\begin{aligned}(\boldsymbol{\sigma}' : \boldsymbol{\delta}^c)^2 &= 4\tau_{eq}^2 (\delta^c_{eq})^2 \cos^2 2(\varphi^c - \Phi) \\ &= 2\tau_{eq}^2 (\delta^c_{eq})^2 \left(1 + \cos 4(\varphi^c - \Phi)\right)\end{aligned} \tag{A.3}$$

Therefore considering equation (46)

$$
\begin{aligned}
2\hat{r}_0\tau_{eq}^2\cos 4\left(\hat{\varphi}_0-\Phi\right) &= -2\hat{r}_0\tau_{eq}^2\cos 4\left(\varphi^c-\Phi\right)\\
&= -\frac{1}{2E_0}2\delta_{eq}^c\tau_{eq}^2\cos 4\left(\varphi^c-\Phi\right)\\
&= \frac{1}{E_0}\left[\tau_{eq}^2\delta_{eq}^c-\frac{1}{2}\frac{(\boldsymbol{\sigma}':\boldsymbol{\delta}^c)^2}{\delta_{eq}^c}\right]
\end{aligned}
\tag{A.4}
$$

Adding open cracks and closed cracks contribution allows to write equation (52) as

$$
\rho\hat{\psi}^\star = \frac{2}{E_0}\left(d_m^o(\tau_{eq}^2+\sigma_m^2)+\boldsymbol{\sigma}':\mathbf{d}^o\,\sigma_m\right)+\frac{1}{E_0}\left[\tau_{eq}^2(\delta_m^c+\delta_{eq}^c)-\frac{1}{2}\frac{(\boldsymbol{\sigma}':\boldsymbol{\delta}^c)^2}{\delta_{eq}}\right] \tag{A.5}
$$

In 2D the square of deviatoric stress is $\boldsymbol{\sigma}'^2=\tau_{eq}^2\mathbf{1}$ so that

$$
\operatorname{tr}(\boldsymbol{\sigma}\cdot\mathbf{d}^o\cdot\boldsymbol{\sigma}) = \operatorname{tr}\left(\boldsymbol{\sigma}^2\cdot\mathbf{d}^o\right) = 2\left(\tau_{eq}^2+\sigma_m^2\right)d_m^o+2\boldsymbol{\sigma}':\mathbf{d}^o\,\sigma_m \tag{A.6}
$$

Finally:

$$
\rho\hat{\psi}^\star = \frac{1}{E_0}\operatorname{tr}(\boldsymbol{\sigma}\cdot\mathbf{d}^o\cdot\boldsymbol{\sigma})+\frac{1}{E_0}\left[\tau_{eq}^2(\delta_m+\delta_{eq})-\frac{1}{2}\frac{(\boldsymbol{\sigma}':\boldsymbol{\delta}^c)^2}{\delta_{eq}}\right] \tag{A.7}
$$

where on recovers of course the case of open cracks only by setting $\boldsymbol{\delta}^c=0$ and the case of closed cracks from $\mathbf{d}^o=0$.

Appendix B. Link with Tensorial Polar Decomposition of $\mathbb{D}$

The independency of $\boldsymbol{\delta}'$ and $\mathbf{d}'$ can be retrieved in a tensorial manner by directly performing the Tensorial Polar Decomposition of fourth order tensor $\mathbb{D}$, starting from equation (70). Rewrite its first term as

$$
\frac{1}{2}\left(\mathbf{1}\,\underline{\overline{\otimes}}\,\mathbf{d}+\mathbf{d}\,\underline{\overline{\otimes}}\,\mathbf{1}\right) = d_m\mathbf{1}\,\underline{\overline{\otimes}}\,\mathbf{1}+\frac{d_{eq}}{2}\left(\mathbf{1}\,\underline{\overline{\otimes}}\,\frac{\mathbf{d}'}{d_{eq}}+\frac{\mathbf{d}'}{d_{eq}}\,\underline{\overline{\otimes}}\,\mathbf{1}\right) \tag{B.1}
$$

and because $d_m=\delta_m$ (eq. (71)), we get (with $\mathbf{1}\,\underline{\overline{\otimes}}\,\mathbf{1}=\mathbb{J}+\frac{1}{2}\mathbf{1}\otimes\mathbf{1}$ in 2D):

$$
\mathbb{D} = \frac{d_m}{2}\mathbb{J}+\frac{d_m}{2}\mathbf{1}\otimes\mathbf{1}+\frac{\delta_{eq}}{2}\left(\frac{\boldsymbol{\delta}'}{\delta_{eq}}\otimes\frac{\boldsymbol{\delta}'}{\delta_{eq}}-\mathbb{J}\right)+\frac{d_{eq}}{2}\left(\mathbf{1}\,\underline{\overline{\otimes}}\,\frac{\mathbf{d}'}{d_{eq}}+\frac{\mathbf{d}'}{d_{eq}}\,\underline{\overline{\otimes}}\,\mathbf{1}\right) \tag{B.2}
$$

which is the Tensorial Polar Decomposition of symmetric tensor $\mathbb{T}=\mathbb{D}$ written in reference [36] as

$$
\mathbb{T} = 2t_0\mathbb{J}+2t_1\mathbf{1}\otimes\mathbf{1}+2r_0\left[\mathbf{R}_0'\otimes\mathbf{R}_0'-\mathbb{J}\right]+2r_1\left(\mathbf{1}\,\underline{\overline{\otimes}}\,\mathbf{R}_1'+\mathbf{R}_1'\,\underline{\overline{\otimes}}\,\mathbf{1}\right) \tag{B.3}
$$

where the polar moduli t_0, t_1, r_0, r_1 are exactly those of Polar Method and where symmetric deviatoric second order tensors

$$
\mathbf{R}_0' = \frac{\boldsymbol{\delta}'}{\delta_{eq}} \qquad \mathbf{R}_1' = \frac{\mathbf{d}'}{d_{eq}} \tag{B.4}
$$

are of generic form $2(\mathrm{N}\otimes\mathrm{N})'$ (they are of unit von Mises norm $(\mathbf{R}_n')_{eq}=\sqrt{\frac{1}{2}\mathbf{R}_n':\mathbf{R}_n'}=1$).

The isotropic invariants $2t_0$ and $2t_1$ – the terms in factor of $\mathbb{J}$ and $\mathbf{1}\otimes\mathbf{1}$ in Tensorial Polar Decomposition – are found to be equal which proves (altogether with property (28)) the rari-constancy property of the tensor $\mathbb{D}$.
Note last that is shown in [36] that the term

$$
\mathbb{H} = 2r_0\left[\mathbf{R}_0'\otimes\mathbf{R}_0'-\mathbb{J}\right] = \frac{\delta_{eq}}{2}\left(\frac{\boldsymbol{\delta}'}{\delta_{eq}}\otimes\frac{\boldsymbol{\delta}'}{\delta_{eq}}-\mathbb{J}\right) \tag{B.5}
$$

is the harmonic (i.e. traceless and totally symmetric) part of tensor $\mathbb{D}$. This harmonic part is independent and orthogonal to the remaining term (isotropic terms at $t_0=t_1$ and linear term in $\mathbf{d}'$),

$$
\mathbb{H}::\left[2t_0\mathbb{J}+2t_1\mathbf{1}\otimes\mathbf{1}+2r_1\left(\mathbf{1}\,\underline{\overline{\otimes}}\,\mathbf{R}_1'+\mathbf{R}_1'\,\underline{\overline{\otimes}}\,\mathbf{1}\right)\right] = 0 \tag{B.6}
$$

with scalar product such as $\mathbb{H}::\mathbb{T}=H_{ijkl}T_{ijkl}$.

Appendix C. One and two orthogonal arrays of cracks: thermodynamics and continuity features

For two orthogonal arrays of open cracks of densities d_1^o, equations (47) gives

$$d_m^o = \frac{d_1^o + d_2^o}{2} \qquad d_{eq}^o = \frac{d_1^o - d_2^o}{2} \qquad \varphi_d^o = \varphi_1 \tag{C.1}$$

In crack density tensor $\mathbf{d}^o$ principal basis (see equation (49)), $\mathbf{d}^o = \mathrm{diag}[d_1^o, d_2^o]$ where $d_1^o = d_m^o + d_{eq}^o = d_{\mathrm{Max}}^o$ and $d_2^o = d_m^o - d_{eq}^o = d_{\mathrm{min}}^o$ are maximum and minimum principal densities. Thus, in the orthogonal arrays of cracks basis:

$$\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\mathbf{d}^o\cdot\boldsymbol{\sigma}) = d_1^o\left(\sigma_{11}^2+\sigma_{12}^2\right) + d_2^o\left(\sigma_{22}^2+\sigma_{12}^2\right) \tag{C.2}$$

For two arrays of closed cracks of densities d_1^c,d_2^c and orientations φ_1^c,φ_2^c, equations (48) gives

$$\delta_m^c = \delta_{eq}^c = \frac{d_1^c + d_2^c}{2} \qquad \varphi_\delta^c = \varphi_1 \tag{C.3}$$

This implies that $\delta_{\mathrm{Max}}^c = \delta_m^c - \delta_{eq}^c = 2\delta_{eq}^c = 2\delta_m^c$, $\delta_{\mathrm{min}}^c = \delta_m^c - \delta_{eq}^c = 0$ and that the crack density tensors $\mathbf{d}^o$ and $\boldsymbol{\delta}^c$ principal basis coincides. Thus, in the orthogonal arrays of cracks basis, $\delta_{12}^c = 0$, and

$$\tau_{eq}^2(\delta_m^c+\delta_{eq}^c) - \frac{1}{2\delta_{eq}^c}(\boldsymbol{\sigma}' : \boldsymbol{\delta}^c)^2 = \delta_{\mathrm{Max}}^c\,\sigma_{12}^2 \tag{C.4}$$

The crack opening conditions which define the open cracks density d_1^o and d_2^o from the knowledge of pure geometric cracks density d_1 and d_2 read

$$\begin{cases} \mathrm{n}^{(1)}\cdot\boldsymbol{\sigma}\cdot\mathrm{n}^{(1)} > 0 : \quad d_1^o = d_1 \\ \\ \mathrm{n}^{(2)}\cdot\boldsymbol{\sigma}\cdot\mathrm{n}^{(2)} > 0 : \quad d_2^o = d_2 \end{cases} \tag{C.5}$$

They can be cast as

$$d_1^o = d_1\,\mathcal{H}(\sigma_{11}) \qquad d_2^o = d_2\,\mathcal{H}(\sigma_{22}) \tag{C.6}$$

with $\mathcal{H}(x)$ the Heaviside function.

By rewriting (93) using equations (C.2), (C.4) and (94), the change in elastic energy density reads then, in the cracks principal basis (with positive part $\langle x\rangle_+ = x\,\mathcal{H}(x)$):

$$\rho\hat{\psi}^\star = \frac{1}{E_0}\left[d_1\left(\langle\sigma_{11}\rangle_+^2 + \mathcal{H}(\sigma_{11})\sigma_{12}^2\right) + d_2\left(\langle\sigma_{22}\rangle_+^2 + \mathcal{H}(\sigma_{22})\sigma_{12}^2\right) + \delta_{\mathrm{Max}}^c\,\sigma_{12}^2\right] \tag{C.7}$$

The state law $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_0 + \hat{\boldsymbol{\epsilon}}$ defines the total strain $\boldsymbol{\epsilon}$ as the sum of $\boldsymbol{\epsilon}_0 = \mathbb{S}_0 : \boldsymbol{\sigma}$ and of crack contribution $\hat{\boldsymbol{\epsilon}} = \rho\frac{\partial\hat{\psi}^\star}{\partial\boldsymbol{\sigma}}$ with as normal and shear components due to cracks contribution:

$$\hat{\epsilon}_{11} = \rho\frac{\partial\hat{\psi}^\star}{\partial\sigma_{11}} \qquad \hat{\epsilon}_{22} = \rho\frac{\partial\hat{\psi}^\star}{\partial\sigma_{22}} \qquad \hat{\gamma}_{nt} = \rho\frac{\partial\hat{\psi}^\star}{\partial\sigma_{12}} \tag{C.8}$$

They gives discontinuous strains if δ_{Max}^c is independent from both the cracks density d and the stress.

A continuous stress strain response is enforced by choosing $\delta_{\mathrm{Max}}^c = \delta_{\mathrm{Max}}^c(d,\sigma_{11},\sigma_{22}) = d_1\,(1-\mathcal{H}(\sigma_{11})) + d_2\,(1-\mathcal{H}(\sigma_{22}))$ giving in present case of two known orthogonal arrays of cracks :

$$\rho\hat{\psi}^\star = \rho\hat{\psi}^\star(\boldsymbol{\sigma}, d_1, d_2) = \frac{1}{E_0}\left[d_1\left(\langle\sigma_{11}\rangle_+^2 + \sigma_{12}^2\right) + d_2\left(\langle\sigma_{22}\rangle_+^2 + \sigma_{12}^2\right)\right] \tag{C.9}$$

$\rho\hat{\psi}^\star$ is a function of $\boldsymbol{\sigma}$ and of cracks densities d_1 and d_2 (and not of $\boldsymbol{\delta}^c$ anymore), it is convex in $\boldsymbol{\sigma}$ and is continuously differentiable.

Appendix D. Rewriting of tr($\sigma\cdot\mathbf{h}^2\cdot\sigma$)

Let us calculate

$$\mathrm{tr}\left(\boldsymbol{\sigma}\cdot\mathbf{h}^2\cdot\boldsymbol{\sigma}\right) = \mathrm{tr}\,(\boldsymbol{\sigma}\cdot\mathbf{h}\cdot\mathbf{h}\cdot\boldsymbol{\sigma}) = \mathrm{tr}\left[\left((\boldsymbol{\sigma}\cdot\mathbf{h})^S + (\boldsymbol{\sigma}\cdot\mathbf{h})^A\right)\cdot\mathbf{h}\cdot\boldsymbol{\sigma}\right] \tag{D.1}$$

where upper scripts ^{S} and ^{A} denote symmetric and antisymmetric parts of $\boldsymbol{\sigma}\cdot\mathbf{h}$,

$$(\boldsymbol{\sigma}\cdot\mathbf{h})^S = (\mathbf{h}\cdot\boldsymbol{\sigma})^S = \frac{1}{2}(\boldsymbol{\sigma}\cdot\mathbf{h} + \mathbf{h}\cdot\boldsymbol{\sigma}) \qquad (\boldsymbol{\sigma}\cdot\mathbf{h})^A = (\mathbf{h}\cdot\boldsymbol{\sigma})^{AT} = \frac{1}{2}(\boldsymbol{\sigma}\cdot\mathbf{h} - \mathbf{h}\cdot\boldsymbol{\sigma}) \tag{D.2}$$

and where T means the transpose. One has then

$$\mathrm{tr}\left(\boldsymbol{\sigma}\cdot\mathbf{h}^2\cdot\boldsymbol{\sigma}\right)=\frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\sigma}\cdot\mathbf{h}^2\cdot\boldsymbol{\sigma}\right)+\frac{1}{2}\,\mathrm{tr}\,(\mathbf{h}\cdot\boldsymbol{\sigma}\cdot\mathbf{h}\cdot\boldsymbol{\sigma})+\mathrm{tr}\left[(\boldsymbol{\sigma}\cdot\mathbf{h})^A\cdot(\mathbf{h}\cdot\boldsymbol{\sigma})^A\right] \tag{D.3}$$

so that

$$\mathrm{tr}\left(\boldsymbol{\sigma}\cdot\mathbf{h}^2\cdot\boldsymbol{\sigma}\right)=\mathrm{tr}\,(\mathbf{h}\cdot\boldsymbol{\sigma}\cdot\mathbf{h}\cdot\boldsymbol{\sigma})+A \tag{D.4}$$

Due to antisymmetry term A can be expressed with deviators only,

$$A=2\,\mathrm{tr}\left[(\boldsymbol{\sigma}\cdot\mathbf{h})^A\cdot(\mathbf{h}\cdot\boldsymbol{\sigma})^A\right]=2\,\mathrm{tr}\left[(\boldsymbol{\sigma}'\cdot\mathbf{h}')^A\cdot(\boldsymbol{\sigma}'\cdot\mathbf{h}')^A\right] \tag{D.5}$$

with

$$(\boldsymbol{\sigma}'\cdot\mathbf{h}')^A=(\mathbf{h}'\cdot\boldsymbol{\sigma}')^{AT}=\tau_{eq}h_{eq}\begin{bmatrix}0 & \sin 2(\varphi-\Phi)\\ -\sin 2(\varphi-\Phi) & 0\end{bmatrix} \tag{D.6}$$

and

$$\mathrm{tr}\left[(\boldsymbol{\sigma}\cdot\mathbf{h})^A\cdot(\mathbf{h}\cdot\boldsymbol{\sigma})^A\right]=2\tau_{eq}^2h_{eq}^2\sin^2 2(\varphi-\Phi) \tag{D.7}$$

One can calculate

$$\boldsymbol{\sigma}':\mathbf{h}'=2\tau_{eq}h_{eq}\cos 2(\varphi-\Phi) \tag{D.8}$$

so that antisymmetric term A is

$$A=2\,\mathrm{tr}\left[(\boldsymbol{\sigma}\cdot\mathbf{h})^A\cdot(\mathbf{h}\cdot\boldsymbol{\sigma})^A\right]=4\tau_{eq}^2h_{eq}^2-(\boldsymbol{\sigma}':\mathbf{h}')^2 \tag{D.9}$$

leading then to Eq. (99).

Appendix E – Special positive part of a second order tensor

Ladevèze special positive part $\boldsymbol{\sigma}_+$

Ladevèze [49, 50] uses the eigenvalue problem $\mathbf{h}\cdot\boldsymbol{\sigma}\cdot\mathrm{x}^I=\lambda_I\mathrm{x}^I$ to define special positive part $\boldsymbol{\sigma}_+$ of stress tensor,

$$\boldsymbol{\sigma}=\sum\langle\lambda_I\rangle_+\;\mathbf{h}^{-1}\cdot\mathrm{x}^I\otimes\mathrm{x}^I\cdot\mathbf{h}^{-1}\quad\text{with normalization}\quad\mathrm{x}^I\cdot\boldsymbol{\sigma}^{-1}\cdot\mathrm{x}^J=\delta_{IJ} \tag{D.10}$$

Proposed special positive part $\mathbf{h}_+$

To invert the role of second order tensor $\mathbf{h}$ and of stress tensor makes the invert of the stress tensor appear, which is not satisfactory. Let us consider instead the eigenvalue problem $\mathbf{h}\cdot\boldsymbol{\sigma}\cdot\mathrm{y}^I=\lambda_I\,\mathrm{y}^I$ (which is equivalent to Ladevèze one with equal eigenvalues when $\boldsymbol{\sigma}$ is invertible, as eigenvectors are changed into $\mathrm{y}^I=\boldsymbol{\sigma}^{-1}\cdot\mathrm{x}^I$). Special stress dependent positive part $\mathbf{h}_+$ does not need $\boldsymbol{\sigma}^{-1}$ anymore if one defines it directly as

$$\mathbf{h}_+=\sum\langle\lambda_I\rangle_+\,\mathrm{y}^I\otimes\mathrm{y}^I\quad\text{with normalization}\quad\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathrm{y}^J=\delta_{IJ} \tag{D.11}$$

The interesting property in definition (D.11) is that it makes $\mathrm{tr}\,(\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma})$ continuously differentiable, as shown below by adapting [49] proof to present case.

First, using normalization $\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathrm{y}^J=\delta_{IJ}$,

$$\mathrm{tr}\,(\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma})=\sum_I\sum_J\langle\lambda_I\rangle_+\langle\lambda_J\rangle_+\,\mathrm{tr}\left(\mathrm{y}^I\otimes\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathrm{y}^J\otimes\mathrm{y}^J\cdot\boldsymbol{\sigma}\right)=\sum_I\langle\lambda_I\rangle_+^2 \tag{D.12}$$

which is then continuously differentiable in $\mathrm{d\,tr}\,(\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma})=2\sum_I\langle\lambda_I\rangle_+\,\mathrm{d}\lambda_I$ (as the square of the positive part of a scalar $\langle x\rangle_+^2$ is continuously differentiable in $2\langle x\rangle_+\,\mathrm{d}x$).

Second, one has $\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathrm{y}^I=\langle\lambda_I\rangle_+\,\mathrm{y}^I$ from definition (D.11) and

$$\mathrm{d}\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathrm{y}^I+\mathbf{h}_+\cdot\mathrm{d}\boldsymbol{\sigma}\cdot\mathrm{y}^I+\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathrm{dy}^I=\mathrm{d}\langle\lambda_I\rangle_+\,\mathrm{y}^I+\langle\lambda_I\rangle_+\,\mathrm{dy}^I \tag{D.13}$$

Scalar product with $\langle\lambda_I\rangle_+\,\mathrm{y}^I\cdot\boldsymbol{\sigma}$ gives

$$\begin{aligned}&\langle\lambda_I\rangle_+\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathrm{d}\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathrm{y}^I+\langle\lambda_I\rangle_+\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\mathrm{d}\boldsymbol{\sigma}\cdot\mathrm{y}^I+\langle\lambda_I\rangle_+\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathrm{dy}^I\\&\qquad=\langle\lambda_I\rangle_+\mathrm{d}\langle\lambda_I\rangle_++\langle\lambda_I\rangle_+\langle\lambda_I\rangle_+\,\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathrm{dy}^I\end{aligned} \tag{D.14}$$

Let us take the sum over I and rearrange terms to recognize $\mathbf{h}_+$,

$$\begin{aligned}&\sum_I\left(\langle\lambda_I\rangle_+\mathrm{y}^I\otimes\mathrm{y}^I\right):(\boldsymbol{\sigma}\cdot\mathrm{d}\mathbf{h}_+\cdot\boldsymbol{\sigma})+\left(\langle\lambda_I\rangle_+\mathrm{y}^I\otimes\mathrm{y}^I\right):(\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\mathrm{d}\boldsymbol{\sigma})+\langle\lambda_I\rangle_+\,\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathrm{dy}^I\\&\qquad=\sum_I\langle\lambda_I\rangle_+\mathrm{d}\langle\lambda_I\rangle_++\langle\lambda_I\rangle_+^2\,\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathrm{dy}^I\end{aligned} \tag{D.15}$$

The transpose of initial eigenvalue problem reads $\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+=\langle\lambda_I\rangle_+\,y^{IT}$ so that equality $\langle\lambda_I\rangle_+\,\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathrm{dy}^I=\langle\lambda_I\rangle_+^2\,\mathrm{y}^I\cdot\boldsymbol{\sigma}\cdot\mathrm{dy}^I$ stands making $\mathrm{d}y^I$ terms cancel. One recognize finally the sought equality

$$\mathrm{d\,tr}\,(\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma})=2\sum_I\langle\lambda_I\rangle_+\,\mathrm{d}\langle\lambda_I\rangle_+=2\,(\mathbf{h}_+\cdot\boldsymbol{\sigma}\cdot\mathbf{h}_+):\mathrm{d}\boldsymbol{\sigma}+2\,(\boldsymbol{\sigma}\cdot\mathbf{h}_+\cdot\boldsymbol{\sigma}):\mathrm{d}\mathbf{h} \tag{D.16}$$

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