Tensorial Polar Decomposition of 2D fourth order tensors

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Décomposition Polaire Tensorielle des tenseurs 2D d'ordre 4

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Abstract

One studies the structure of 2D symmetric fourth order tensors, *i.e.* having both minor and major indicial symmetries. Verchery polar decomposition is rewritten in a tensorial form entitled Tensorial Polar Decomposition. Main result is that any 2D symmetric fourth order tensor can be written in terms of second order tensors only in a decomposition that makes explicitly appear invariants and symmetry classes. The link with harmonic decomposition is made thanks to Kelvin decomposition of its harmonic term.

On étudie la structure des tenseurs 2D symétriques d'ordre 4, *i.e.* ayant aussi bien la symétrie indicielle mineure que la symétrie majeure. La décomposition polaire de Verchery est réécrite sous forme tensorielle nommée Décomposition Polaire Tensorielle. Le résultat principal est que tout tenseur 2D symétrique d'ordre 4, peut s'écrire à l'aide de tenseurs d'ordre 2 uniquement dans une décomposition faisant apparaître explicitement les invariants et les classes de symétrie. Le lien avec la décomposition harmonique est fait en utilisant la décomposition de Kelvin de son terme harmonique.

Keywords: polar decomposition, invariants, harmonic decomposition, Kelvin decomposition
décomposition polaire, invariants, décomposition harmonique, décomposition de Kelvin

Introduction

The structure of 3D fourth order elasticity tensor has been intensively studied since XIXth century controversy on the number of independent elasticity constants. Major and minor symmetries reduce to 21 the number of material parameters (symmetric tensors $\mathbb{T}$ referred to as multi-constant tensors), when an elasticity tensor having all Cauchy indicial symmetries $\mathrm{T}_{ijkl} = \mathrm{T}_{ikjl}$ only has 15 material parameters (supersymmetric or rari-constant tensors).

A well known tool for the study of symmetry classes is isomorphic harmonic decomposition $2\mathbb{H}^0 \oplus 2\mathbb{H}^2 \oplus \mathbb{H}^4$ of symmetric tensors space[1, 2, 3, 4], defining scalar (real) space as $\mathbb{H}^0$, second order harmonic tensors $\mathbf{h} \in \mathbb{H}^2$ as traceless (deviatoric, $\sum_k h_{kk} = 0$) symmetric tensors and fourth order harmonic tensors $\mathbb{H} \in \mathbb{H}^4$ as traceless supersymmetric/rari-constant tensors ($\mathrm{H}_{ijkl} = \mathrm{H}_{ikjl}$, $\sum_k \mathrm{H}_{kkij} = \sum_k \mathrm{H}_{kikj} = 0$). In other words any symmetric tensor $\mathbb{T}$, such as triclinic elasticity tensor, can be represented by two Lamé isotropic constants $\in \mathbb{H}^0$, by two second order harmonic tensors $\in \mathbb{H}^2$ and by one fourth order harmonic tensor $\in \mathbb{H}^4$.

In 2D some simplifications arise as scalar expressions for the components $\mathrm{T}_{ijkl}(\theta)$ of symmetric tensor $\mathbb{T}$ (having both minor and major symmetries) may be derived by making explicitly appear the dependency upon frame angle θ [5] and upon invariants [6, 7, 8, 9]. Theses expressions do not have a complete tensorial counterpart in the literature [10].

In present note, we therefore propose a tensorial rewriting and an associated interpretation of Verchery polar decomposition for 2D fourth order tensors with both minor and major indicial symmetries. It is shown in section 2 that any 2D symmetric tensor $\mathbb{T}$ (resp. any 2D harmonic fourth order tensor $\mathbb{H} \in \mathbb{H}^{4(2D)}$) can be expressed by means of 2 scalar invariants and of 2 second order deviatoric tensors $\in \mathbb{H}^{2(2D)}$ (resp. of only 1 second order deviatoric tensor $\mathbf{h}_0 \in \mathbb{H}^{2(2D)}$). The link with harmonic decomposition is made in section 4. The general tensorial expression of harmonic elements $\in \mathbb{H}^{4(2D)}$ is retrieved in section 5 by use of Kelvin decomposition.

Tensorial products $\underline{\otimes}$, $\overline{\otimes}$, $\underline{\overline{\otimes}}$ will be used. They are defined as follows: $(\mathbf{X}\underline{\otimes}\mathbf{Y})_{ijkl} = X_{ik}Y_{jl}$, $(\mathbf{X}\overline{\otimes}\mathbf{Y})_{ijkl} = X_{il}Y_{jk}$, $\mathbf{X}\underline{\overline{\otimes}}\mathbf{Y} = \frac{1}{2}(\mathbf{X}\underline{\otimes}\mathbf{Y} + \mathbf{X}\overline{\otimes}\mathbf{Y})$.

1. 2D quadratic form using the polar formalism

Let us consider any 2D fourth order tensor $\mathbb{T}$ with minor and major symmetries. In the polar formalism [6, 7], 5 invariants are defined: 4 of them are elastic moduli (t_0, t_1, r_0, r_1) and the last one is the angular difference

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$\varphi_0 - \varphi_1$(each φ_n is not an invariant by itself, see discussion on joint invariant at end section 2). A basic result of the polar formalism is the expression of the Cartesian components of $\mathbb{T}$ in terms of polar parameters, in a frame rotated of an angle θ:

$$\begin{aligned}
&\mathrm{T}_{1111}(\theta)=t_0+2t_1+r_0\cos 4(\varphi_0-\theta)+4r_1\cos 2(\varphi_1-\theta)\\
&\mathrm{T}_{1112}(\theta)=r_0\sin 4(\varphi_0-\theta)+2r_1\sin 2(\varphi_1-\theta)\\
&\mathrm{T}_{1122}(\theta)=-t_0+2t_1-r_0\cos 4(\varphi_0-\theta)\\
&\mathrm{T}_{1212}(\theta)=t_0-r_0\cos 4(\varphi_0-\theta)\\
&\mathrm{T}_{1222}(\theta)=-r_0\sin 4(\varphi_0-\theta)+2r_1\sin 2(\varphi_1-\theta)\\
&\mathrm{T}_{2222}(\theta)=t_0+2t_1+r_0\cos 4(\varphi_0-\theta)-4r_1\cos 2(\varphi_1-\theta)
\end{aligned} \tag{1}$$

t_0 and t_1 terms are frame independent (they define isotropic part of $\mathbb{T}$ from a generalization of Lamé constants to anisotropy), r_1 terms rotates in $\cos 2(\varphi_1-\theta)$ and $\sin 2(\varphi_1-\theta)$ as second order tensors do (Eq. 2), r_0 term rotates twice more in $\cos 4(\varphi_0-\theta)$ and $\sin 4(\varphi_0-\theta)$. In given frame θ the knowledge of the 6 independent coefficients of any 2D symmetric tensor $\mathbb{T}$ is equivalent to the knowledge of the 5 invariants $(t_0, t_1, r_0, r_1, \varphi_0-\varphi_1)$ and of one angle, either $\varphi_0-\theta$ or $\varphi_1-\theta$.

Still in 2D, a general expression for any symmetric second order tensor $\mathbf{s}$, making appear explicitly frame angle θ, is

$$\mathbf{s} = s_m\mathbf{1} + \mathbf{s}' = s_m\mathbf{1} + s_{eq}\begin{bmatrix} \cos 2(\varphi-\theta)) & \sin 2(\varphi-\theta) \\ \sin 2(\varphi-\theta) & -\cos 2(\varphi-\theta)\end{bmatrix} \quad \text{with} \quad \begin{cases} s_m = \frac{1}{2}\operatorname{tr}\mathbf{s} \\ s_{eq} = \sqrt{\frac{1}{2}\mathbf{s}':\mathbf{s}'}\end{cases} \tag{2}$$

with first (mean) and second (2D von Mises) invariants defined as s_m and s_{eq} and where φ is the orientation of principal basis of $\mathbf{s}$ (it is not an invariant of $\mathbf{s}$). The expression of associated quadratic form is:

$$\frac{1}{2}\mathbf{s}:\mathbb{T}:\mathbf{s} = 2t_0 s_{eq}^2 + 4t_1 s_m^2 + 2r_0 s_{eq}^2\cos 4(\varphi_0-\varphi) + 8r_1 s_m s_{eq}\cos 2(\varphi_1-\varphi) \tag{3}$$

Explicit formulae giving polar invariants as a function of components T_{ijkl} can be found in [7].

2. Proposed Tensorial Polar Decomposition

Introducing the two second order deviatoric tensors $\mathbf{R}_0$, $\mathbf{R}_1$,

$$\mathbf{R}_0 = \mathbf{R}_0' = \begin{bmatrix} \cos 2(\varphi_0-\theta) & \sin 2(\varphi_0-\theta) \\ \sin 2(\varphi_0-\theta) & -\cos 2(\varphi_0-\theta)\end{bmatrix} \qquad \mathbf{R}_1 = \mathbf{R}_1' = \begin{bmatrix} \cos 2(\varphi_1-\theta) & \sin 2(\varphi_1-\theta) \\ \sin 2(\varphi_1-\theta) & -\cos 2(\varphi_1-\theta)\end{bmatrix} \tag{4}$$

of 2D von Mises equivalent norm $R_{0eq} = R_{1eq} = 1$, and of principal direction φ_0, φ_1, of course possibly different from principal direction φ of tensor $\mathbf{s}$. One has first equalities concerning r_1-term,

$$(\mathbf{s}:\mathbf{R}_1')\operatorname{tr}\mathbf{s} = \operatorname{tr}(\mathbf{s}\cdot\mathbf{R}_1'\cdot\mathbf{s}) = 4s_m s_{eq}\cos 2(\varphi_1-\varphi) \tag{5}$$

From $(\mathbf{s}:\mathbf{R}_0')^2 = 4s_{eq}^2\cos^2 2(\varphi_0-\varphi) = 2s_{eq}^2(1+\cos 4(\varphi_0-\varphi))$ and $2s_{eq}^2 = \mathbf{s}':\mathbf{s}'$ second equality concerning r_0-term is

$$2r_0 s_{eq}^2\cos 4(\varphi_0-\varphi) = r_0\left[\left(\mathbf{s}:\mathbf{R}_0'\right)^2 - \mathbf{s}':\mathbf{s}'\right] \tag{6}$$

Quadratic form (3) can therefore be rewritten into following intrinsic form

$$\frac{1}{2}\mathbf{s}:\mathbb{T}:\mathbf{s} = t_0\,\mathbf{s}':\mathbf{s}' + t_1(\operatorname{tr}\mathbf{s})^2 + r_0\left[\left(\mathbf{s}:\mathbf{R}_0'\right)^2 - \mathbf{s}':\mathbf{s}'\right] + 2r_1\operatorname{tr}(\mathbf{s}\cdot\mathbf{R}_1'\cdot\mathbf{s}) \tag{7}$$

From last equation the intrinsic form of polar decomposition of a symmetric fourth order tensor $\mathbb{T}$ is obtained in terms of polar invariants t_0, t_1, r_0 and r_1 and of the two second order deviatoric tensors $\mathbf{R}_0'$ and $\mathbf{R}_1'$,

$$\mathbb{T} = 2t_0\mathbb{J} + 2t_1\mathbf{1}\otimes\mathbf{1} + 2r_0\left[\mathbf{R}_0'\otimes\mathbf{R}_0' - \mathbb{J}\right] + 2r_1\left(\mathbf{1}\,\overline{\underline{\otimes}}\,\mathbf{R}_1' + \mathbf{R}_1'\,\overline{\underline{\otimes}}\,\mathbf{1}\right) \tag{8}$$

It is equivalent in present 2D case to

$$\mathbb{T} = 2t_0\mathbb{J} + 2t_1\mathbf{1}\otimes\mathbf{1} + 2r_0\left[\mathbf{R}_0'\otimes\mathbf{R}_0' - \mathbb{J}\right] + 2r_1(\mathbf{1}\otimes\mathbf{R}_1' + \mathbf{R}_1'\otimes\mathbf{1}) \tag{9}$$

thanks to the mathematical property (5) valid $\forall\mathbf{s}$, which implies

$$\mathbf{1}\,\overline{\underline{\otimes}}\,\mathbf{R}_1' + \mathbf{R}_1'\,\overline{\underline{\otimes}}\,\mathbf{1} = \mathbf{1}\otimes\mathbf{R}_1' + \mathbf{R}_1'\otimes\mathbf{1} \tag{10}$$

Eq.(10) is not intrinsic to tensorial products, it stands only in 2D. The tensor $\mathbb{J} = \mathbb{I} - \frac{1}{2}\mathbf{1}\otimes\mathbf{1}$ (defined here in 2D) takes the deviatoric part of any second order tensor $\mathbf{X}$ (*i.e.* $\mathbb{J}:\mathbf{X} = \mathbf{X}'$).

Equation (8)-(9) define Tensorial Polar Decomposition of any 2D tensor $\mathbb{T}$ having both minor and major symmetries. As both r_0- and r_1-terms are found rari-constant, rari-constancy $\mathrm{T}_{ijkl} = \mathrm{T}_{ikjl}$ resumes to $t_0 = t_1$.

Note that joint invariant $\mathbf{R}'_0 : \mathbf{R}'_1$ reads

$$\mathbf{R}'_0 : \mathbf{R}'_1 = 2\cos 2(\varphi_0 - \varphi_1) \tag{11}$$

It is an invariant of tensor $\mathbb{T}$, as is the polar angular invariant $\varphi_0 - \varphi_1$.

The intrinsic form of the polar decomposition makes explicitly appear polar moduli and angles, therefore the material symmetries, including ordinary orthotropies $\varphi_0 - \varphi_1 = k\frac{\pi}{4}$, $k \in \{0, 1\}$ [7]. For instance if $\mathbb{T}$ is 2D elasticity tensor: isotropy is $r_0 = r_1 = 0$, square symmetry is $r_1 = 0$, r_0−orthotropy is $r_0 = 0$, ordinary orthotropy with $k = 0$ is $\mathbf{R}'_0 = \mathbf{R}'_1$ and ordinary orthotropy with $k = 1$ is $\mathbf{R}'_0 : \mathbf{R}'_1 = 0$.

3. Orthogonality of generators

Tensorial Polar Decomposition (9) can be recast as the sum of polar moduli $2g_n$ times generators $\mathbb{G}^{(n)}$ which are fourth order tensors (factors 2 appear for consistency with original Verchery work, polar moduli g_n standing either for t_n or for r_n),

$$\mathbb{T} = \sum_0^3 2g_n\mathbb{G}^{(n)} = \sum_1^2 2t_n\mathbb{G}_t^{(n)} + \sum_1^2 2r_n\mathbb{G}_r^{(n)} \tag{12}$$

Fourth order generator tensors $\mathbb{G}^{(n)}$ are of two kinds: the $\mathbb{G}_t^{(n)}$ are definite positive and do not depend upon frame orientation θ, while the $\mathbb{G}_r^{(n)} = \mathbb{G}_r^{(n)}(\theta)$ are frame dependent

$$\mathbb{G}_t^{(0)} = \mathbb{J}\,, \qquad \mathbb{G}_t^{(1)} = \mathbf{1}\otimes\mathbf{1}\,, \qquad \mathbb{G}_r^{(0)} = \mathbf{R}'_0\otimes\mathbf{R}'_0 - \mathbb{J}\,, \qquad \mathbb{G}_r^{(1)} = \mathbf{1}\otimes\mathbf{R}'_1 + \mathbf{R}'_1\otimes\mathbf{1} \tag{13}$$

The generators are orthogonal with respect to scalar product :: as

$$\mathbb{G}^{(n)} :: \mathbb{G}^{(m)} = \sum_{i=1}^2\sum_{j=1}^2\sum_{k=1}^2\sum_{l=1}^2 G_{ijkl}^{(n)}G_{ijkl}^{(m)} = 0 \quad \forall\, m \neq n \tag{14}$$

They all have a constant norm, frame independent, as

$$\mathbb{G}_t^{(0)} :: \mathbb{G}_t^{(0)} = 2 \qquad \mathbb{G}_t^{(1)} :: \mathbb{G}_t^{(1)} = 4 \qquad \mathbb{G}_r^{(0)} :: \mathbb{G}_r^{(0)} = 2 \qquad \mathbb{G}_r^{(1)} :: \mathbb{G}_r^{(1)} = 8 \tag{15}$$

4. Link with harmonic decomposition

In 3D there are only two independent traces $\mathbf{d} = \mathrm{tr}_{12}\mathbb{T} = \mathrm{tr}_{34}\mathbb{T}$ (of components $\sum_{k=1}^3 \mathrm{T}_{kkij}$) and $\mathbf{v} = \mathrm{tr}_{13}\mathbb{T} = \mathrm{tr}_{23}\mathbb{T} = \mathrm{tr}_{14}\mathbb{T} = \mathrm{tr}_{24}\mathbb{T}$ (of components $\sum_{k=1}^3 \mathrm{T}_{kikj}$) for symmetric tensor $\mathbb{T}$. Symmetric second order tensor $\mathbf{d}$ is dilatation tensor, of deviatoric part $\mathbf{d}'$, symmetric second order tensor $\mathbf{v}$ is Voigt tensor, of deviatoric part $\mathbf{v}'$. 3D Harmonic decomposition $2\mathbb{H}^0 \oplus 2\mathbb{H}^2 \oplus \mathbb{H}^4$ of fourth order tensors vector space reads then [2, 3, 4]

$$\mathbb{T} = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\,\mathbf{1}\underline{\overline{\otimes}}\mathbf{1} + \mathbf{1}\otimes\mathbf{h}_1 + \mathbf{h}_1\otimes\mathbf{1} + \mathbf{1}\underline{\otimes}\mathbf{h}_2 + \mathbf{h}_2\underline{\otimes}\mathbf{1} + \mathbf{1}\overline{\otimes}\mathbf{h}_2 + \mathbf{h}_2\overline{\otimes}\mathbf{1} + \mathbb{H} \tag{16}$$

or in an equivalent manner

$$\mathbb{T} = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\,\mathbb{I} + \mathbf{1}\otimes\mathbf{h}_1 + \mathbf{h}_1\otimes\mathbf{1} + 2\left(\mathbf{1}\underline{\overline{\otimes}}\mathbf{h}_2 + \mathbf{h}_2\underline{\overline{\otimes}}\mathbf{1}\right) + \mathbb{H} \tag{17}$$

with as constants $\lambda = \frac{1}{30}(4\,\mathrm{tr}\,\mathbf{d} - 2\,\mathrm{tr}\,\mathbf{v})$ and $\mu = \frac{1}{30}(3\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d})$, as traceless symmetric second order tensors $\mathbf{h}_1 = \mathbf{h}'_1 = \frac{1}{7}(5\mathbf{d}' - 4\mathbf{v}') \in \mathbb{H}^2$ and $\mathbf{h}_2 = \mathbf{h}'_2 = \frac{1}{7}(3\mathbf{v}' - \mathbf{d}') \in \mathbb{H}^2$ and as traceless rari-constant tensor $\mathbb{H} \in \mathbb{H}^4$.

In 2D (see [10]), in a consistent manner with mathematical property (10) and 2D equality $\mathbf{v}' = \mathbf{d}'$ if one still sets $\mathbf{d} = \mathrm{tr}_{12}\mathbb{T}$, $\mathbf{v} = \mathrm{tr}_{13}\mathbb{T}$ of components $d_{ij} = \sum_{k=1}^2 \mathrm{T}_{kkij}$ and $v_{ij} = \sum_{k=1}^2 \mathrm{T}_{kikj}$, harmonic decomposition of fourth order tensors vector space reads $2\mathbb{H}^0 \oplus \mathbb{H}^{2(2\mathrm{D})} \oplus \mathbb{H}^{4(2\mathrm{D})}$ or

$$\mathbb{T} = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\,\mathbb{I} + \mathbf{1}\otimes\mathbf{h} + \mathbf{h}\otimes\mathbf{1} + \mathbb{H} \tag{18}$$

with as 2D constants $\lambda = \frac{1}{2}(\mathrm{tr}\,\mathbf{d} - \mathrm{tr}\,\mathbf{v})$ and $\mu = \frac{1}{2}(2\,\mathrm{tr}\,\mathbf{v} - \mathrm{tr}\,\mathbf{d})$, as 2D harmonic tensors $\mathbf{h} = \mathbf{d}'/2 = \mathbf{v}'/2 \in \mathbb{H}^{2(2\mathrm{D})}$ and $\mathbb{H} \in \mathbb{H}^{4(2\mathrm{D})}$. One easily recognizes constant and linear terms of Tensorial Polar Decomposition (9), using $\mathbb{I} = \mathbb{J} + \frac{1}{2}\mathbf{1}\otimes\mathbf{1}$, with

$$t_0 = \mu\,, \qquad t_1 = \frac{\lambda+\mu}{2}\,, \qquad 2r_1 = h_{eq}\,, \qquad 2r_1\mathbf{R}'_1 = \mathbf{h}\,. \tag{19}$$

The harmonic $\mathbb{H}^{4(2\mathrm{D})}$-term is explicited in section 2 thanks to polar decomposition by means of an extra traceless second order tensor $\mathbf{h}_0 = \mathbf{h}'_0 = \sqrt{2r_0}\,\mathbf{R}'_0 \in \mathbb{H}^{2(2\mathrm{D})}$ as

$$\mathbb{H} = 2r_0\left[\mathbf{R}'_0\otimes\mathbf{R}'_0 - \mathbb{J}\right] = \mathbf{h}_0\otimes\mathbf{h}_0 - \frac{1}{2}\mathbf{h}_0 : \mathbf{h}_0\,\mathbb{J} \qquad \mathrm{tr}_{12}\mathbb{H} = \mathrm{tr}_{13}\mathbb{H} = 0 \tag{20}$$

This shows that Tensorial Polar Decomposition of 2D symmetric fourth order tensors is direct sum $2\mathbb{H}^0 \oplus 2\mathbb{H}^{2(2\mathrm{D})}$.

We propose in next section to use Kelvin decomposition in order to derive the explicit r_0−form of $\mathbb{H}$ and to prove that $r_0 \geq 0$, as needed.

5. Retrieving the explicit r_0-form of $\mathbb{H} \in \mathbb{H}^{4(2D)}$

Harmonic fourth order tensor $\mathbb{H}$ introduced in previous section is

$$\mathbb{H} = \mathbb{T} - \lambda \mathbf{1} \otimes \mathbf{1} - 2\mu \, \mathbb{I} - \mathbf{1} \otimes \mathbf{h} - \mathbf{h} \otimes \mathbf{1} \tag{21}$$

Let us use its harmonic properties $\mathrm{tr}_{12}\mathbb{H} = \mathrm{tr}_{13}\mathbb{H} = 0$ and the remark that they correspond to the orthogonality of generator $\mathbb{G}_r^{(0)}$ with respect to both constant generators $\mathbb{G}_t^{(0)} = \mathbb{J}$ and $\mathbb{G}_t^{(1)} = \mathbf{1} \otimes \mathbf{1}$.

Kelvin (spectral) decomposition of $\mathbb{H}$ [11, 12, 13, 14, 8], gives, here in 2D,

$$\mathbb{H} = \sum_{I=0}^{2} \Lambda_I \mathbf{e}^I \otimes \mathbf{e}^I \qquad \mathbf{e}^I : \mathbf{e}^J = \delta_{IJ} \tag{22}$$

with $\mathbb{H} : \mathbf{e}^I = \Lambda_I \mathbf{e}^I$ (no sum) defining Kevin moduli Λ_I and modes $\mathbf{e}^I$. Traceless condition $\mathbb{H} : \mathbf{1} = \mathbf{1} : \mathbb{H} = \mathrm{tr}_{12}\, \mathbb{H} = 0$ implies that $\mathbf{1}$ is a eigentensor (a Kelvin mode) of $\mathbb{H}$, associated to Kelvin modulus $\Lambda_2 = 0$ and that the first two eigentensors $\mathbf{e}^I$ are deviatoric, $\mathbf{e}^I = \mathbf{e}^{I\prime}$. The mathematical property that Kelvin projectors give a partition of unit tensor reads then

$$\mathbf{e}^{1\prime} \otimes \mathbf{e}^{1\prime} = \mathbb{I} - \frac{1}{2} \mathbf{1} \otimes \mathbf{1} - \mathbf{e}^{0\prime} \otimes \mathbf{e}^{0\prime} = \mathbb{J} - \mathbf{e}^{0\prime} \otimes \mathbf{e}^{0\prime} \tag{23}$$

so that Kelvin decomposition (22) becomes

$$\mathbb{H} = (\Lambda_0 - \Lambda_1)\, \mathbf{e}^{0\prime} \otimes \mathbf{e}^{0\prime} + \Lambda_1 \mathbb{J} \tag{24}$$

By construction, $\mathbb{H}$ is orthogonal to generator $\mathbb{G}_t^{(0)} = \mathbb{J} = \mathbb{I} - \frac{1}{2}\mathbf{1} \otimes \mathbf{1}$. This gives

$$\begin{aligned} \mathbb{H} :: \mathbb{J} &= (\Lambda_0 - \Lambda_1)\, \mathbf{e}^{0\prime} \otimes \mathbf{e}^{0\prime} :: \mathbb{J} + \Lambda_1 \mathbb{J} :: \mathbb{J} = 0 \\ &= (\Lambda_0 - \Lambda_1)\, \mathbf{e}^{0\prime} : \mathbf{e}^{0\prime} + 2\Lambda_1 = (\Lambda_0 - \Lambda_1) + 2\Lambda_1 = 0 \end{aligned} \tag{25}$$

This shows that $\Lambda_1 = -\Lambda_0$ so that one just has proven that any $\mathbb{H} \in \mathbb{H}^{4(2D)}$ has for expression

$$\mathbb{H} = \Lambda_0 \left[2\mathbf{e}^{0\prime} \otimes \mathbf{e}^{0\prime} - \mathbb{J} \right] \tag{26}$$

Setting $\mathbf{R}_0' = \sqrt{2}\, \mathbf{e}^{0\prime}$ as deviatoric second order tensor of equivalent norm $R_{0eq} = 1$, ends up to

$$\mathbb{H} = 2r_0 \left[\mathbf{R}_0' \otimes \mathbf{R}_0' - \mathbb{J} \right] \qquad r_0 = \frac{\Lambda_0}{2} \tag{27}$$

There are two possibilities for the definition of tensor $\mathbf{R}_0'$ and of modulus r_0 as there are two Kelvin modes $I = 0$ and $I = 1$ orthogonal to Kelvin mode $\mathbf{e}^2 = \mathbf{1}/\sqrt{2}$. Only the one at positive eigenvalue, set as $I = 0$, $\Lambda_0 \geq 0$ (leaving then $\Lambda_1 \leq 0$ for $I = 1$) gives a positive r_0 as retained in standard polar decomposition of 2D symmetric tensors and as needed at the end of previous section. Polar modulus $r_0 = \Lambda_0/2 \geq 0$ is therefore shown to be half positive eigenvalue of harmonic fourth order tensor $\mathbb{H}$ and $\mathbf{R}_0'$ is associated Kelvin mode multiplied by $\sqrt{2}$.

Altogether with expression (21) due to harmonic decomposition, present derivations (and key Eq. (27)) are an alternate proof of Verchery polar decomposition, using tensorial mathematical tools instead of a complex variable method in case of original proof.

6. Conclusion

We have proposed a tensorial intrinsic form for Verchery polar decomposition of any 2D fourth order symmetric tensor $\mathbb{T}$. Two proofs are given, a first one from the rewriting of quadratic form (3) associated with tensor $\mathbb{T}$, a second one combining both harmonic and Kelvin decompositions.

Compared to harmonic decomposition the main results are:

- the generators obtained are found orthogonal to each other (in sense of scalar product :: for fourth order tensors) and of constant norm, independent from frame angle,
- the polar invariants of tensor $\mathbb{T}$ explicitly appear, making easy the study of symmetry classes and sub-classes,
- the structure of harmonic fourth order tensor $\mathbb{H} \in \mathbb{H}^{4(2D)}$ is given: any traceless rari-constant (harmonic) tensor $\mathbb{H}$ is shown to be expressed thanks to a single deviatoric (harmonic) second order tensor $\mathbf{h}_0$ or in an equivalent manner in polar formalism thanks to polar invariant r_0 and to deviatoric tensor $\mathbf{R}_0'$ of unit 2D von Mises norm.

As a conclusion any 2D symmetric fourth order tensor can be expressed thanks to 2 scalars and to 2 symmetric second order deviatoric tensors in a decomposition that makes explicitly appear invariants and symmetry classes.

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