Structure-preserving conforming and nonconforming discretizations of mixed problems

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STRUCTURE-PRESERVING CONFORMING AND NONCONFORMING DISCRETIZATIONS OF MIXED PROBLEMS

MARTIN CAMPOS PINTO*

Abstract. We study conforming and nonconforming methods that preserve the Helmholtz structure of mixed problems at the discrete level. On the conforming side we essentially gather classical tools to design numerical approximations that are compatible with an underlying Helmholtz decomposition. We then show that a recent approach developped for the time-dependent Maxwell equations allows to design new nonconforming methods based on fully discontinuous finite element spaces, that share the same stability and compatibility properties, with no need of penalty terms.

Key words. Structure-preserving methods, compatible approximations, Helmholtz decompositions, mixed formulations, discontinuous elements.

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1. Introduction.

For mixed problems of the form

$$\begin{cases} a(u,v) + b(v,p) = \langle f, v\rangle_{V'\times V} & v \in V \\ b(u,q) = \langle g, q\rangle_{Q'\times Q} & q \in Q, \end{cases} \tag{1}$$

conforming finite elements are a reference theory [19, 23, 7]. In several cases of physical interest, such as the Maxwell system or the incompressible Stokes equation

$$\begin{cases} -\nu\Delta u + \nabla p = F \\ \operatorname{div} u = 0, \end{cases} \tag{2}$$

the space V admits a Helmholtz decomposition relative to the differential operators involved. Discretizations that preserve the geometric (de Rham) structure of the underlying functional spaces then provide a vast field of efficient methods with well understood stability and convergence properties, see e.g. [24, 3, 1, 2, 6]. An important asset of structure-preserving discretizations is their ability to approximate the different terms of the equations in a way that is compatible with the Helmholtz decomposition of the continuous problem. In the case of the incompressible Stokes equation (2) for instance, standard schemes which do not enforce a strong divergence constraint typically compute an approximate velocity field u_h that depends on the pressure p, which is unfortunate when p is large or unsmooth, or when the viscosity $\nu > 0$ is small. As explained in [21, 27, 9], this is caused by an improper discretization of the source: at the continuous level indeed, one can use a Helmholtz decomposition associated with the operators involved in (2) to write F as the sum of an irrotational component and a divergence-free one, which respectively determine p and u. To compute a pressure-independent velocity, the numerical scheme should maintain a one-way separation between these components and their discrete counterparts: as u is fully determined by the divergence-free part of F, that should also be the case for the approximate velocity u_h.

In this article we propose a natural criterion to formalize this one-way compatibility, and we use the resulting framework to extend a recent nonconforming discretization method primarily developped for the time-dependent Maxwell equations

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[18, 16, 17]. A key feature of this approach is the use of non-standard exact sequences involving fully discontinuous finite element spaces, based on reference exact sequences of conforming spaces [15]. For the time-dependent Maxwell equations the resulting Conforming/Nonconforming Galerkin (Conga) method has been shown to have long time stability and conservation properties, despite the lack of stabilization (e.g., penalty) terms. By applying the same discretization techniques to abstract problems with a Helmholtz structure, we now prove uniform stability and error estimates for several classes of compatible nonconforming mixed schemes, in a way that naturally extends the ones of conforming methods.

The outline is as follows. In Section 2 we specify two mixed problems with an abstract Helmholtz structure and describe a few motivating applications. In Section 3 we then review the main ingredients of structure-preserving discretizations with conforming spaces and propose a criterion to characterize one-way compatibility properties of the source discretization. Sufficient conditions are given for compatible approximation operators, which rely either on orthogonality or commuting diagram properties. The extension to fully discontinuous spaces is addressed in Sections 4 and 5. The construction of structure-preserving nonconforming discretizations is first recalled in Section 4, and sufficient conditions are given for compatible approximation operators in this nonconforming framework. Finally, structure-preserving nonconforming schemes are derived in Section 5, together with uniform stability and error estimates.

2. Mixed problems with a Helmholtz structure. To describe our problems in a generic form we borrow some notation from finite element exterior calculus [1, 6] and essentially follow the Hilbert complex framework of [2, Sec. 3]. Here it will be sufficient to consider three L^2 spaces (with scalar or vector-valued functions) denoted W^0, W^1, W^2, and two closed operators $d^l : W^l \to W^{l+1}$, $l = 0, 1$, with dense domains $V^l \subset W^l$. The first important poperty is that the sequence

$$V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} W^2 \tag{3}$$

is assumed exact, in the sense that we have

$$d^0 V^0 = \ker d^1. \tag{4}$$

Second, we assume that the V^l's are Hilbert spaces when endowed with the norms

$$\|q\|^2_{V^0} = \|q\|^2 + \|d^0 q\|^2 \quad \text{and} \quad \|v\|^2_{V^1} = \|v\|^2 + \|d^1 v\|^2 \tag{5}$$

where $\|\cdot\|$ denotes the L^2 norm in the corresponding W^l space. This implies that $\ker d^l$ is closed in W^l. In particular the range of d^0 is closed in W^1, and we further assume that $d^1 V^1$ is closed in W^2. Using the $\perp$ exponent to denote an L^2 orthogonal complement in the proper W^l space, we then denote

$$K^0 = V^0 \cap (\ker d^0)^\perp, \quad K^1 = V^1 \cap (\ker d^1)^\perp, \quad Z^1 = d^0 V^0, \quad Z^2 = d^1 V^1 \tag{6}$$

and we observe that for $l = 0, 1$, d^l defines a bounded bijection from K^l to Z^{l+1} which are two Hilbert spaces with the respective V^l and W^{l+1} norms. Banach's bounded inverse theorem then guarantees the existence of two constants c_0, c_1, such that the following Poincaré estimates hold,

$$\begin{aligned} \|q\| &\le c_0 \|d^0 q\|, \qquad && q \in K^0, \\ \|v\| &\le c_1 \|d^1 v\|, \qquad && v \in K^1. \end{aligned} \tag{7}$$

2.1. Helmholtz decompositions. Let $\delta^{l+1} : W^{l+1} \to (V^l)'$, $l = 0, 1$, be respectively defined by

$$\begin{aligned} \langle \delta^1 v, q\rangle_{(V^0)'\times V^0} &= \langle v, d^0 q\rangle, \qquad && v \in W^1,\ q \in V^0 \\ \langle \delta^2 \xi, v\rangle_{(V^1)'\times V^1} &= \langle \xi, d^1 v\rangle, \qquad && \xi \in W^2,\ v \in V^1 \end{aligned} \tag{8}$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 product in the proper W^l space. Then the exact sequence property (4) gives $K^1 = V^1 \cap (d^0 V^0)^\perp = \ker \delta^1|_{V^1}$ and we have a first Helmholtz decomposition

$$V^1 = K^1 \overset{\perp}{\oplus} Z^1 = \ker \delta^1|_{V^1} \overset{\perp}{\oplus} \ker d^1 \tag{9}$$

where the direct sum is orthogonal for both the L^2 and V^1 inner products. Note that since Z^1 is also closed in the larger space W^1, we have another Helmholtz decomposition in this L^2 space, namely

$$W^1 = \bar{K}^1 \overset{\perp}{\oplus} Z^1 = \ker \delta^1 \overset{\perp}{\oplus} \ker d^1 \tag{10}$$

where $\bar{K}^1 = (\ker d^0)^\perp$ coincides with the closure of K^1 in W^1.

By duality, we can decompose $(V^1)'$ as the direct sum of the polar spaces of K^1 and Z^1. Specifically, we can write an arbitrary source $f \in (V^1)'$ as

$$f = f^K + f^Z \tag{11}$$

with respective terms defined by the relations

$$\langle f^K, v\rangle_{(V^1)'\times V^1} = \langle f, v^K\rangle_{(V^1)'\times V^1}, \qquad \langle f^Z, v\rangle_{(V^1)'\times V^1} = \langle f, v^Z\rangle_{(V^1)'\times V^1}, \tag{12}$$

for $v \in V^1$ decomposed as $v = v^K + v^Z \in K^1 \oplus Z^1$. We consider two types of problems.

2.2. A first mixed problem. Our first problem is obtained by setting $V := V^1$, $Q := K^0 = V^0 \cap (\ker d^0)^\perp$, see (6), and

$$\begin{aligned} a(v, w) &= \langle d^1 v, d^1 w\rangle, \qquad && v, w \in V^1, \\ b(v, q) &= \langle v, d^0 q\rangle, \qquad && v \in V^1, \quad q \in K^0. \end{aligned} \tag{13}$$

Assuming that $g \in (V^0)'$, System (1) then leads to the following problem.

Problem 2.1. *Find $u \in V^1$, $p \in K^0$ such that*

$$\begin{cases} \langle d^1 u, d^1 v\rangle + \langle v, d^0 p\rangle = \langle f, v\rangle_{(V^1)'\times V^1} & v \in V^1 \\ \qquad\qquad\qquad \langle u, d^0 q\rangle = \langle g, q\rangle_{(V^0)'\times V^0} & q \in K^0. \end{cases} \tag{14}$$

Using the classical theory of e.g. [12, 7], it is easy to verify that Problem 2.1 is well-posed: clearly a and b are continuous,

$$|a(v, w)| = |\langle d^1 v, d^1 w\rangle| \le \|v\|_{V^1}\|w\|_{V^1}, \qquad |b(v, q)| = |\langle v, d^0 q\rangle| \le \|v\|_{V^1}\|q\|_{V^0}.$$

As for the operator $B : V^1 \to (V^0)'$, $\langle Bv, q\rangle_{(V^0)'\times V^0} = b(v, q)$ from [7, Sec. 4.2.1], it coincides with $\delta^1|_{V^1}$. Estimate (7) then shows that a is coercive on $\ker B$, indeed

$$v \in \ker B = \ker \delta^1|_{V^1} = K^1 \implies a(v, v) = \|d^1 v\|^2 \ge (1 + c_1^2)^{-1}\|v\|^2_{V^1}.$$

The inf-sup condition on b is obtained by taking $v = d^0 q \in V^1$ for $q \in K^0$, and writing

$$\inf_{q\in K^0} \sup_{v\in V^1} \frac{b(v,q)}{\|v\|_{V^1}\|q\|_{V^0}} \geq \inf_{q\in K^0} \frac{\|d^0 q\|^2}{\|d^0 q\|_{V^1}\|q\|_{V^0}} = \inf_{q\in K^0} \frac{\|d^0 q\|}{\|q\|_{V^0}} \geq (1+c_0^2)^{-\frac{1}{2}} \tag{15}$$

where the equality $\|d^0 q\|_{V^1} = \|d^0 q\|$ follows from the fact that $d^1 d^0 = 0$. Hence Theorem 4.2.3 from [7] applies: For all $f \in (V^1)'$ and $g \in (V^0)'$, Problem 2.1 has a unique solution (u,p); moreover there is a constant depending on c_0, c_1, such that

$$\|u\|_{V^1} + \|p\|_{V^0} \lesssim \|f\|_{(V^1)'} + \|g\|_{(V^0)'}.$$

To exhibit the Helmholtz structure of Problem 2.1 we decompose

$$u = u^K + u^Z \in K^1 \oplus Z^1 \tag{16}$$

according to (9), and rewrite (14) in the form

$$\begin{cases} \langle d^1 u^K, d^1 v\rangle = \langle f, v\rangle_{(V^1)'\times V^1} & v \in K^1 \\ \langle w, d^0 p\rangle = \langle f, w\rangle_{(V^1)'\times V^1} & w \in Z^1 \\ \langle u^Z, d^0 q\rangle = \langle g, q\rangle_{(V^0)'\times V^0} & q \in K^0. \end{cases} \tag{17}$$

This formulation clarifies how the different parts of the source contribute to the solution: decomposing f according to (11)-(12), we can rewrite (17) in the form

$$\begin{cases} \delta^2 d^1 u^K = f^K \\ d^0 p = f^Z \\ \delta^1 u^Z = g. \end{cases} \tag{18}$$

2.3. A second mixed problem. Our second model problem corresponds to the case where $V := V^1$, $Q := Z^2 = d^1 V^1 \subset W^2$, and

$$\begin{aligned} a(v,w) &= \langle v, w\rangle, & v, w \in V^1, \\ b(v,\xi) &= \langle d^1 v, \xi\rangle, & v \in V^1, \quad \xi \in Z^2. \end{aligned} \tag{19}$$

Using new notations to distinguish the functions in the space W^2 (which, as an L^2 space, is identified with its dual $(W^2)'$), Problem (1) then becomes:

Problem 2.2. *Given $f \in (V^1)'$ and $\ell \in W^2$, find $u \in V^1$, $\zeta \in Z^2$ such that*

$$\begin{cases} \langle u, v\rangle + \langle d^1 v, \zeta\rangle = \langle f, v\rangle_{(V^1)'\times V^1} & v \in V^1 \\ \langle d^1 u, \xi\rangle = \langle \ell, \xi\rangle & \xi \in Z^2. \end{cases} \tag{20}$$

Again the well-posedness of this problem is easily proven using classical results: the continuity of a and b is clear, and the operator $B : V^1 \to (Z^2)'$ defined by $\langle Bv, \xi\rangle_{(Z^2)'\times Z^2} = b(v,\xi)$ now coincides with d^1. Hence a is obviously coercive on $\ker B$. Letting next $w \in K^1$ be such that $d^1 w = \xi$ for $\xi \in Z^2$ and using (7), we have

$$\inf_{\xi\in Z^2} \sup_{v\in V^1} \frac{b(v,\xi)}{\|v\|_{V^1}\|\xi\|} \geq \inf_{\xi\in Z^2} \frac{\|\xi\|^2}{\|w\|_{V^1}\|\xi\|} = \inf_{\xi\in Z^2} \frac{\|d^1 w\|}{\|w\|_{V^1}} \geq (1+c_1^2)^{-\frac{1}{2}}. \tag{21}$$

In particular, [7, Th. 4.2.3] applies: Problem 2.2 has a unique solution (u,ζ) and

$$\|u\|_{V^1} + \|\zeta\| \lesssim \|f\|_{(V^1)'} + \|\ell\|$$

holds with a constant that depends only on c_1 from (7). To see the Helmholtz structure we decompose $u = u^K + u^Z \in K^1 \oplus Z^1$ as in (16) and recast Problem 2.2 as

$$\begin{cases} \langle u^K, v\rangle + \langle d^1 v, \zeta\rangle = \langle f, v\rangle_{(V^1)'\times V^1} & v \in K^1 \\ \langle u^Z, w\rangle = \langle f, w\rangle_{(V^1)'\times V^1} & w \in Z^1 \\ \langle d^1 u^K, \xi\rangle = \langle \ell, \xi\rangle & \xi \in Z^2. \end{cases} \tag{22}$$

Here the third, second and first equations define u^K, u^Z and ζ respectively. Specifically, decomposing f as in (11)-(12) we find

$$\begin{cases} u^K + \delta^2 \zeta = f^K \\ u^Z = f^Z \\ d^1 u^K = \ell. \end{cases} \tag{23}$$

2.4. Examples. Problems like 2.1 and 2.2 are ubiquitous in the finite element litterature. We may give a few examples to motivate our study. For instance, it is well-known that on a bounded and simply-connected Lipschitz domain Ω, the sequence

$$V^0 = H_0(\mathrm{curl};\Omega) \xrightarrow{d^0 = \mathrm{curl}} V^1 = H_0(\mathrm{div};\Omega) \xrightarrow{d^1 = \mathrm{div}} W^2 = L^2_0(\Omega) \tag{24}$$

is exact, moreover $Z^2 = d^1 Z^1 = W^2$. See e.g. [10, 23] or [31, Sec. 3.2]. Here the Hilbert spaces are denoted with classical notation, in particular

$$\begin{aligned} &H_0(\mathrm{curl};\Omega) = \{q \in H(\mathrm{curl};\Omega) : n \times q = 0 \text{ on } \partial\Omega\}, \\ &H_0(\mathrm{div};\Omega) = \{v \in H(\mathrm{div};\Omega) : n \cdot q = 0 \text{ on } \partial\Omega\}, \\ &L^2_0(\Omega) = \{\xi \in L^2(\Omega) : \textstyle\int_\Omega \xi = 0\}, \end{aligned}$$

see e.g. [23]. Problem 2.2 with $f = 0$ and $\ell = -F$ reads then

$$\begin{cases} \langle u, v\rangle + \langle \mathrm{div}\, v, \zeta\rangle = 0 & v \in V^1 = H_0(\mathrm{div},\Omega) \\ \langle \mathrm{div}\, u, \xi\rangle = -\langle F, \xi\rangle & \xi \in Z^2 = L^2_0(\Omega), \end{cases}$$

which is a mixed formulation for the Poisson equation $\Delta\zeta = -F$ with Neumann boundary conditions, see e.g. [5, Eqs. (10), (12)]. Another example is the incompressible Stokes equation (2), with the non-standard boundary conditions on the velocity

$$u \cdot n = \phi \quad \text{and} \quad \mathrm{curl}\, u \times n = \omega \times n \qquad \text{on } \partial\Omega \tag{25}$$

used in [11, 9] following [4, 22]. If $u \in H^2(\Omega)^3$ and $p \in H^1$ satisfy (2) and (25), then using Green formulas and the relation $-\Delta = \mathrm{curl}\,\mathrm{curl} - \mathrm{grad}\,\mathrm{div}$, we find that

$$\begin{cases} \nu\langle \mathrm{curl}\, u, \mathrm{curl}\, v\rangle + \langle \nabla p, v\rangle = \langle F, v\rangle - \langle n \times v, \omega\rangle_{\partial\Omega}, & v \in H(\mathrm{curl},\Omega) \\ \langle u, \nabla q\rangle = \langle \phi, q\rangle_{\partial\Omega}, & q \in H^1(\Omega). \end{cases}$$

Up to the viscosity parameter ν, this corresponds to Problem 2.1 with the sequence

$$V^0 = H^1(\Omega) \xrightarrow{d^0 = \nabla} V^1 = H(\mathrm{curl};\Omega) \xrightarrow{d^1 = \mathrm{curl}} W^2 = L^2(\Omega)^3 \tag{26}$$

which is also exact ([31, Sec. 3.2]), and the source terms

$$\langle f, v\rangle_{(V^1)'\times V^1} = \langle F, v\rangle - \langle n \times v, \omega\rangle_{\partial\Omega} \quad \text{and} \quad \langle g, q\rangle_{(V^0)'\times V^0} = \langle \phi, q\rangle_{\partial\Omega}.$$

Note that in Problem 2.1 the orthogonality condition $p \in (\ker \nabla)^\perp$ is meant to ensure the uniquess of the solution. Another application involves eigenproblems of the form

$$\langle d^1 u, d^1 v\rangle = \lambda \langle u, v\rangle \qquad v \in V^1 = H_0(d^1;\Omega) \tag{27}$$

with $d^1 =$ grad, curl or div. Indeed, following [6, Part 4] we can see Problem 2.1 (with $g = 0$) as the source problem associated with a first mixed formulation of (27),

$$\begin{cases} \langle d^1 u, d^1 v\rangle + \langle v, d^0 p\rangle = \lambda\langle u, v\rangle & v \in V^1 = H_0(d^1;\Omega) \\ \langle u, d^0 q\rangle = 0 & q \in V^0 = H_0(d^0;\Omega) \end{cases}$$

and Problem 2.2 (with $f = 0$) as the source problem associated with a second mixed formulation of (27), namely

$$\begin{cases} \langle u, v\rangle + \langle d^1 v, \zeta\rangle = 0 & v \in V^1 = H_0(d^1;\Omega) \\ \langle d^1 u, \xi\rangle = -\lambda\langle \zeta, \xi\rangle & \xi \in Z^2 = d^1 V^1. \end{cases}$$

Finally, another motivation is the study of time dependent wave or Maxwell equations, and more generally evolution problems of the form

$$\begin{cases} \langle \partial_t u(t), v\rangle - \langle d^1 v, \chi(t)\rangle = 0 & v \in V^1 \\ \langle \partial_t \chi(t), \xi\rangle + \langle d^1 u(t), \xi\rangle = \langle F(t), \xi\rangle & \xi \in Z^2. \end{cases} \tag{28}$$

In [8] indeed, it is shown that uniform error estimates for the approximation operator $\Pi_h :\ (V^1 \times Z^2) \to V_h^1 \times Z_h^2\ (u, \zeta) \to (u_\Pi, \zeta_\Pi)$ defined by

$$\begin{cases} \langle u_\Pi, v_h\rangle - \langle d^1 v_h, \zeta_\Pi\rangle = \langle u, v_h\rangle - \langle d^1 v_h, \zeta\rangle & v_h \in V_h^1 \\ \langle d^1 u_\Pi, \xi_h\rangle = \langle d^1 u, \xi_h\rangle & \xi_h \in Z_h^2 \end{cases}$$

lead to uniform estimates for the Galerkin approximation of (28).

3. Conforming discretizations. Many results are known on the conforming approximation of the above problems, see e.g. Th. 5.2.5 in [7] or [6, Sec. 18]. Here we focus on structure-preserving discretizations, following [1, 2]. We consider a conforming sequence of discrete spaces $V_h^l \subset V^l$, $l = 0, 1$, and $W_h^2 \subset W^2$, that preserve the two main properties of the continuous Hilbert complex (3), namely: (i) the sequence

$$V_h^0 \xrightarrow{d_h^0 := d^0|_{V_h^0}} V_h^1 \xrightarrow{d_h^1 := d^1|_{V_h^1}} W_h^2 \tag{29}$$

is exact, in the sense that

$$d^0 V_h^0 = \ker d_h^1, \tag{30}$$

(ii) uniform Poincaré estimates hold with constants denoted as in (7) for simplicity,

$$\begin{aligned} \|q_h\| &\le c_0 \|d^0 q_h\|, & q_h &\in K_h^0 \\ \|v_h\| &\le c_1 \|d^1 v_h\|, & v_h &\in K_h^1. \end{aligned} \tag{31}$$

Here the spaces are defined consistent with (6), by

$$K_h^0 = V_h^0 \cap (\ker d_h^0)^\perp, \qquad K_h^1 = V_h^1 \cap (\ker d_h^1)^\perp, \qquad Z_h^1 = d^0 V_h^0, \qquad Z_h^2 = d^1 V_h^1. \tag{32}$$

Such compatible conforming discretizations are well known: for the approximation of usual de Rham sequences like (24) or (26) one can use standard finite element spaces of Lagrange, Raviart-Thomas [30], Nedelec [28, 29] or Brezzi-Douglas-Marini [13] type, see also [23, 7] and [24, 2], where unified analyses of their construction and stability properties have been carried out using the framework of differential forms and Finite Element Exterior Calculus.

Remark 3.1. Here by *conforming* we mean that the exact operators d^l are well defined (in a strong sense) on the discrete spaces, so that finite element approximations can be obtained with a standard Galerkin projection. However we point out that the resulting discretizations are not necessarily conforming in the more restrictive sense where all the spaces involved should be approximated by discrete subspaces. In particular, the spaces K_h^l are typically *not* subspaces of K^l.

Since d_h^l is merely a restriction of d^l, we will use the latter notation when possible, and reserve the former one to specify the finite-dimensional domain (e.g., write $\ker d_h^l$ rather than $\ker d^l \cap V_h^l$, for conciseness). It will also be convenient to let $\delta_h^1 : V_h^1 \to V_h^0$ and $\delta_h^2 : W_h^2 \to V_h^1$ be the discrete adjoints of d_h^0 and d_h^1, i.e.,

$$\begin{aligned} \langle \delta_h^1 v, q\rangle &= \langle v, d^0 q\rangle, \qquad && v \in V_h^1,\ q \in V_h^0 \\ \langle \delta_h^2 \xi, v\rangle &= \langle \xi, d^1 v\rangle, \qquad && \xi \in W_h^2,\ v \in V_h^1. \end{aligned} \tag{33}$$

Using the discrete exact sequence property (30) we then verify that K_h^1 coincides with the kernel of δ_h^1, and that the decomposition

$$V_h^1 = K_h^1 \overset{\perp}{\oplus} Z_h^1 = \ker \delta_h^1 \overset{\perp}{\oplus} \ker d_h^1 \tag{34}$$

is orthogonal both in the L^2 and V^1 inner products, just as (9).

3.1. Conforming discretizations with arbitrary sources. We first study the stability of conforming discretizations of Problems 2.1 and 2.2 with arbitrary discrete sources.

3.1.1. Conforming discretization of the first problem. Given arbitrary sources g_h, f_h in $(V_h^0)'$ and $(V_h^1)'$ we discretize Problem 2.1 as follows: find $u_h \in V_h^1$, $p_h \in K_h^0$ such that

$$\begin{cases} \langle d^1 u_h, d^1 v_h\rangle + \langle v_h, d^0 p_h\rangle = \langle f_h, v_h\rangle_{(V^1)'\times V^1} & v_h \in V_h^1 \\ \langle u_h, d^0 q_h\rangle = \langle g_h, q_h\rangle_{(V^0)'\times V^0} & q_h \in K_h^0. \end{cases} \tag{35}$$

A priori estimates are available for such discretizations, see e.g. [6, Prop. 18.1]. Here we focus on the discrete Helmholtz structure: Decomposing u_h according to (34),

$$u_h = u_h^K + u_h^Z \in K_h^1 \oplus Z_h^1 \tag{36}$$

we restate Problem (35) as: find $(u_h^K, u_h^Z, p_h) \in K_h^1 \times Z_h^1 \times K_h^0$ such that

$$\begin{cases} \langle d^1 u_h^K, d^1 v_h\rangle = \langle f_h, v_h\rangle_{(V^1)'\times V^1} & v_h \in K_h^1 \\ \langle w_h, d^0 p_h\rangle = \langle f_h, w_h\rangle_{(V^1)'\times V^1} & w_h \in Z_h^1 \\ \langle u_h^Z, d^0 q_h\rangle = \langle g_h, q_h\rangle_{(V^0)'\times V^0} & q_h \in K_h^0. \end{cases} \tag{37}$$

LEMMA 3.2. *The discrete problem* (35) *is well-posed, and its solution satisfies*

$$\begin{aligned} \|u_h\|_{V^1} &\lesssim \|f_h|_{K_h^1}\|_{(V_h^1)'} + \|g_h\|_{(V_h^0)'} \\ \|p_h\|_{V^0} &\lesssim \|f_h|_{Z_h^1}\|_{(V_h^1)'} \end{aligned} \tag{38}$$

where $f_h|_{K_h^1}$ *and* $f_h|_{Z_h^1}$ *denote the restrictions of* f_h *to the respective spaces. Here the constants depend only on* c_0, c_1 *from* (31)*. Specifically, writing* u_h *as in* (36)*, we have*

$$\begin{aligned} \|u_h^K\|_{V^1} &\leq (1+c_1^2)\|f_h|_{K_h^1}\|_{(V_h^1)'} \\ \|u_h^Z\|_{V^1} &\leq (1+c_0^2)^{\frac{1}{2}}\|g_h\|_{(V_h^0)'} \\ \|p_h\|_{V^0} &\leq (1+c_0^2)^{\frac{1}{2}}\|f_h|_{Z_h^1}\|_{(V_h^1)'}. \end{aligned} \tag{39}$$

Although the proof is standard we detail it, as it will readily extend to the non-conforming case.

Proof. This linear problem has as many equations than unkowns, hence its well-posedness follows from the stability estimates (38). We will prove (39) and use the V^1-orthogonal decomposition (34): For the first bound in (39) we use the second Poincaré estimate (31) and the first equation in (37) with $v_h = u_h^K$. This yields

$$\|u_h^K\|_{V^1}^2 \leq (1+c_1^2)\|d^1 u_h^K\|^2 \leq (1+c_1^2)\|f_h|_{K_h^1}\|_{(V_h^1)'}\|u_h^K\|_{V^1}.$$

For the second estimate we use again (37) with $q_h \in K_h^0$ such that $d^0 q_h = u_h^Z$. Then

$$\|u_h^Z\|^2 \leq \|g_h\|_{(V_h^0)'}\|q_h\|_{V^0} \leq (1+c_0^2)^{\frac{1}{2}}\|g_h\|_{(V_h^0)'}\|d^0 q_h\| = (1+c_0^2)^{\frac{1}{2}}\|g_h\|_{(V_h^0)'}\|u_h^Z\| \tag{40}$$

where the second inequality uses (31), and we conclude by observing that $\|u_h^Z\| = \|u_h^Z\|_{V^1}$ in Z_h^1. For the last estimate we use now (37) with $w_h = d^0 p_h$. It gives

$$\|d^0 p_h\|^2 \leq \|f_h|_{Z_h^1}\|_{(V_h^1)'}\|d^0 p_h\|_{V^1} = \|f_h|_{Z_h^1}\|_{(V_h^1)'}\|d^0 p_h\|$$

where we have used the discrete exact sequence property (30), and the third estimate follows by using again the first Poincaré inequality (31), like in (40). ∎

Remark 3.3. Using the discrete stability (31) and reasonning as in Section 2.2 one also verifies that the bilinear forms (13) satisfy the standard properties of the classical analysis, see [7, Sec. 5.1.1], namely the coercivity of a on the relevant discrete kernel (here, K_h) and a uniform discrete inf-sup condition for b.

3.1.2. Conforming discretization of the second problem. For Problem 2.2 we consider the following discretization: find $u_h \in V_h^1$, $\zeta_h \in Z_h^2$ such that

$$\begin{cases} \langle u_h, v_h\rangle + \langle d^1 v_h, \zeta_h\rangle = \langle f_h, v_h\rangle_{(V^1)'\times V^1} & v_h \in V_h^1 \\ \langle d^1 u_h, \xi_h\rangle = \langle \ell_h, \xi_h\rangle & \xi_h \in Z_h^2 \end{cases} \tag{41}$$

where f_h, ℓ_h are given discrete sources in $(V_h^1)'$ and $(W_h^2)' = W_h^2$. Again, a priori estimates are available for such discretizations, see e.g. [6, Prop. 18.2]. Focusing on the discrete Helmholtz structure we decompose $u_h = u_h^K + u_h^Z$ as in (36) and restate Problem (41) as: find $(u_h^K, u_h^Z, \zeta_h) \in K_h^1 \times Z_h^1 \times Z_h^2$ such that

$$\begin{cases} \langle u_h^K, v_h\rangle + \langle d^1 v_h, \zeta_h\rangle = \langle f_h, v_h\rangle_{(V^1)'\times V^1} & v_h \in K_h^1 \\ \langle u_h^Z, w_h\rangle = \langle f_h, w_h\rangle_{(V^1)'\times V^1} & w_h \in Z_h^1 \\ \langle d^1 u_h^K, \xi_h\rangle = \langle \ell_h, \xi_h\rangle & \xi_h \in Z_h^2. \end{cases} \tag{42}$$

Here the third, second and first equations define u_h^K, u_h^Z and ζ_h respectively.

LEMMA 3.4. *The discrete problem* (41) *is well-posed, and its solution satisfies*

$$\begin{aligned} \|u_h\|_{V^1} &\lesssim \|f_h|_{Z_h^1}\|_{(V_h^1)'} + \|\ell_h\| \\ \|\zeta_h\| &\lesssim \|f_h|_{K_h^1}\|_{(V_h^1)'} + \|\ell_h\| \end{aligned} \tag{43}$$

where $f_h|_{K_h^1}$ *and* $f_h|_{Z_h^1}$ *denote the restrictions of* f_h *to the respective spaces. Here the constants depend only on* c_0, c_1 *from* (31)*. Specifically, decomposing* u_h *as in* (36)*, we have*

$$\begin{aligned} \|u_h^K\|_{V^1} &\le (1+c_1^2)^{\frac12}\|\ell_h\| \\ \|u_h^Z\|_{V^1} &\le \|f_h|_{Z_h^1}\|_{(V_h^1)'} \\ \|\zeta_h\| &\le (1+c_1^2)^{\frac12}\|f_h|_{K_h^1}\|_{(V_h^1)'} + c_1^2\|\ell_h\|. \end{aligned} \tag{44}$$

Again the proof is standard but we recall it because it readily extends to the nonconforming case.

Proof. As above it suffices to prove (44). Using (42) with $\xi_h = d^1 u_h^K \in Z_h^2$ gives

$$\|d^1 u_h^K\| \le \|\ell_h\| \tag{45}$$

and with the second estimate from (31) this shows the first bound in (44). The second one is obtained from the second equation in (42) and $Z_h^1 \subset Z^1$. For the last one we use the first equation from (42) with $v_h \in K_h^1$ such that $d^1 v_h = \zeta_h$. It yields

$$\begin{aligned} \|\zeta_h\|^2 = \|d^1 v_h\|^2 &\le \|f_h|_{K_h^1}\|_{(V_h^1)'}\|v_h\|_{V^1} + \|u_h^K\|\|v_h\| \\ &\le \big((1+c_1^2)^{\frac12}\|f_h|_{K_h^1}\|_{(V_h^1)'} + c_1^2\|\ell_h\|\big)\|d^1 v_h\| \end{aligned}$$

where we have used (45) and the second Poincaré estimate (31). ∎

Remark 3.5. Using the discrete stability (31) and reasonning as in Section 2.3 one also verifies that the bilinear forms (19) satisfy the standard properties of the classical analysis, see [7, Sec. 5.1.1], namely the coercivity of a on the relevant discrete kernel (here, Z_h) and a uniform discrete inf-sup condition for b.

3.2. Error estimates for the standard (Z-compatible) approximation. The standard Galerkin approximation of the sources involves L^2 projections, such as

$$f_h = P_{V_h^1} f \tag{46}$$

(for simplicity we may assume hare that $f \in L^2$, but the discussion readily extends to sources in $(V^1)'$). An important property of this projection is to preserve the orthogonality with respect to functions in the kernel of d^1. Specifically, decomposing $f = f^K + f^Z$ as in (11)-(12) and $f_h = f_h^K + f_h^Z$ according to the discrete Helmholtz decomposition (34), we infer from the embedding $Z_h^1 \subset Z^1$ that

$$\langle f_h^Z, w_h\rangle = \langle f_h, w_h\rangle = \langle f, w_h\rangle = \langle f^Z, w_h\rangle, \qquad w_h \in Z_h^1.$$

Thus, the Z-component of the approximated source only depends on the Z-component of the exact one, which by linearity amounts to say that

$$f^Z = 0 \quad \Longrightarrow \quad f_h^Z = 0. \tag{47}$$

In this article we call *Z-compatible* such an approximation, see Def. 3.10 below.

3.2.1. Z-compatible conforming approximation of the first problem. Using (46) and a similar projection for g, Problem (35) becomes: find $u_h \in V_h^1$, $p_h \in K_h^0$ such that

$$\begin{cases} \langle d^1 u_h, d^1 v_h\rangle + \langle v_h, d^0 p_h\rangle = \langle f, v_h\rangle_{(V^1)'\times V^1} & v_h \in V_h^1 \\ \langle u_h, d^0 q_h\rangle = \langle g, q_h\rangle_{(V^0)'\times V^0} & q_h \in K_h^0. \end{cases} \tag{48}$$

As shown in the classical error estimate below, the resulting discretization enjoys good accuracy properties when the optimal error on u dominates that on p.

Proposition 3.6. *The solutions to problems* (14) *and* (48) *satisfy*

$$\begin{cases} \|u-u_h\|_{V^1} \lesssim \inf_{\bar u_h\in V_h^1}\|u-\bar u_h\|_{V^1} + \inf_{\bar p_h\in K_h^0}\|d^0(p-\bar p_h)\| \\ \|p-p_h\|_{V^0} \lesssim \inf_{\bar p_h\in K_h^0}\|p-\bar p_h\|_{V^0} \end{cases} \tag{49}$$

with constants depending only on c_0 and c_1 from (31).

Proof. The steps are standard. We detail them as our new estimates below will follow from slight variations. Given $(\bar u_h, \bar p_h) \in V_h^1 \times K_h^0$, and using (14) and (48) we have

$$\begin{aligned} \langle d^1(u_h-\bar u_h), d^1 v_h\rangle &= \langle f, v_h\rangle_{(V^1)'\times V^1} - \langle d^1\bar u_h, d^1 v_h\rangle \\ &= \langle d^1(u-\bar u_h), d^1 v_h\rangle + \langle v_h, d^0 p\rangle \\ &= \langle d^1(u-\bar u_h), d^1 v_h\rangle + \langle v_h, d^0(p-\bar p_h)\rangle, \end{aligned}$$

for $v_h \in K_h^1$, where the presence of p is a consequence of the nonembedding of K_h^1 into K^1. For $w_h \in Z_h^1$, due to the embedding $Z_h^1 \subset Z^1 = \ker d^1$, we have

$$\langle w_h, d^0(p_h-\bar p_h)\rangle = \langle f, w_h\rangle_{(V^1)'\times V^1} - \langle w_h, d^0\bar p_h\rangle = \langle w_h, d^0(p-\bar p_h)\rangle$$

and for $q_h \in K_h^0$, using the embeddings $K_h^0 \subset V_h^0 \subset V^0$ we write

$$\langle u_h-\bar u_h, d^0 q_h\rangle = \langle g, q_h\rangle_{(V^0)'\times V^0} - \langle \bar u_h, d^0 q_h\rangle = \langle u-\bar u_h, d^0 q_h\rangle.$$

In particular, Lemma 3.2 applies and yields (with constants depending on c_0 and c_1) $\|u_h-\bar u_h\|_{V^1} \lesssim \|u-\bar u_h\|_{V^1} + \|d^0(p-\bar p_h)\|$ and $\|p_h-\bar p_h\|_{V^0} \lesssim \|d^0(p-\bar p_h)\|$. This gives $\|u-u_h\|_{V^1} \lesssim \|u-\bar u_h\|_{V^1} + \|d^0(p-\bar p_h)\|$ and $\|p-p_h\|_{V^0} \lesssim \|p-\bar p_h\|_{V^0}$, and the bounds (49) follow by taking the infimum over $\bar u_h \in V_h^1$ and $\bar p_h \in K_h^0$. ∎

Remark 3.7. Using $d^1(u-\bar u_h) = d^1(u^K - \bar u_h^K)$ and $\langle u-\bar u_h, d^0 q_h\rangle = \langle u^Z - \bar u_h^Z, d^0 q_h\rangle$ in the above proof we can show a refined result, for the decompositions (16) and (36) of u and u_h. namely that $\|u^K - u_h^K\|_{V^1}$ and $\|u^Z - u_h^Z\|_{V^1}$ are controlled by terms that do not involve u^Z and u^K, respectively.

3.2.2. Z-compatible conforming approximation of the second problem. Using Galerkin (L^2) projections for the sources as in (48), Problem (41) becomes

$$\begin{cases} \langle u_h, v_h\rangle + \langle d^1 v_h, \zeta_h\rangle = \langle f, v_h\rangle_{(V^1)'\times V^1} & v_h \in V_h^1 \\ \langle d^1 u_h, \xi_h\rangle = \langle \ell, \xi_h\rangle & \xi_h \in Z_h^2. \end{cases} \tag{50}$$

The following error estimate shows that the resulting discretization enjoys good accuracy properties when the optimal error on ζ dominates that on u.

PROPOSITION 3.8. *The solution to the discrete problem* (50) *satisfies*

$$\begin{cases} \|u-u_h\|_{V^1} \lesssim \inf\limits_{\bar{u}_h\in V_h^1} \|u-\bar{u}_h\|_{V^1} \\ \|\zeta-\zeta_h\| \lesssim \inf\limits_{\bar{\zeta}_h\in Z_h^2} \|\zeta-\bar{\zeta}_h\| + \inf\limits_{\bar{u}_h\in V_h^1} \|u-\bar{u}_h\|_{V^1}. \end{cases} \tag{51}$$

Proof. Again we consider $(\bar{u}_h,\bar{\zeta}_h)\in V_h^1\times Z_h^2$, and for $w_h\in Z_h^1$ we compute

$$\langle u_h-\bar{u}_h,w_h\rangle = \langle f,w_h\rangle_{(V^1)'\times V^1} - \langle \bar{u}_h,w_h\rangle = \langle u-\bar{u}_h,w_h\rangle$$

where we have used (20) and the embedding $Z_h^1\subset Z^1$. For $\xi_h\in Z_h^2$ we have

$$\langle d^1(u_h-\bar{u}_h),\xi_h\rangle = \langle \ell,\xi_h\rangle - \langle d^1\bar{u}_h,\xi_h\rangle = \langle d^1(u-\bar{u}_h),\xi_h\rangle$$

using again (20) and now the embedding $Z_h^2\subset Z^2$. Finally for $v_h\in K_h^1$, we have

$$\begin{aligned} \langle u_h-\bar{u}_h,v_h\rangle + \langle d^1v_h,\zeta_h-\bar{\zeta}_h\rangle &= \langle f,v_h\rangle_{(V^1)'\times V^1} - \langle \bar{u}_h,v_h\rangle - \langle d^1v_h,\bar{\zeta}_h\rangle \\ &= \langle u-\bar{u}_h,v_h\rangle + \langle d^1v_h,\zeta-\bar{\zeta}_h\rangle \end{aligned}$$

where we have used the first equation from (20) and the embedding $K_h^1\subset V_h^1\subset V^1$. Here the non-conformity $K_h^1\not\subset K^1$ prevents us to use the first equation from (22), which explains the presence of u instead of just u^K. Applying Lemma 3.4 yields then $\|u_h-\bar{u}_h\|_{V^1}\lesssim \|u-\bar{u}_h\|_{V^1}$ and $\|\zeta_h-\bar{\zeta}_h\|\lesssim \|u-\bar{u}_h\|_{V^1}+\|\zeta-\bar{\zeta}_h\|$ with constants depending on c_1, and this leads to (51) by reasonning as above. ∎

Remark 3.9. Writing $\langle u-\bar{u}_h,w_h\rangle = \langle u^Z-\bar{u}_h^Z,w_h\rangle$ and $d^1(u-\bar{u}_h)=d^1(u^K-\bar{u}_h^K)$ in the proof above we can show again a refined estimate, namely that $\|u^K-u_h^K\|_{V^1}$ and $\|u^Z-u_h^Z\|_{V^1}$ are controlled by terms that do not involve u^Z and u^K, respectively.

3.3. Compatibility of the source approximation. In the previous section we have seen that a standard Galerkin discretization of Problem 2.1 yields an a priori error estimate for $u-u_h$ that depends on p, and in the proof of Proposition 3.6 we have pointed out that this was caused by the non conformity $K_h^1\not\subset K^1$. On the level of principles the reason for this fact stems from a poor preservation of the Helmholtz structure as observed e.g. in [21, 27, 9], and becomes transparent when we consider the form (18) of Problem 2.1. Decomposing $f_h = f_h^K + f_h^Z$ according to (34) and using the discrete adjoints (33), the discrete problem (37) can then be put in a similar form. Writing both systems side by side, we find

$$\begin{cases} \delta^2 d^1 u^K = f^K \\ d^0 p = f^Z \\ \delta^1 u^Z = g \end{cases} \quad \text{and} \quad \begin{cases} \delta_h^2 d^1 u_h^K = f_h^K \\ d^0 p_h = f_h^Z \\ \delta_h^1 u_h^Z = g_h. \end{cases} \tag{52}$$

Here the Z-compatibility (47) of the L^2 projection (46) results in f_h^Z to only depend on f^Z. From (52) this implies that p_h only depends on p, which explains why the error estimate on p does not involve u in Proposition 3.6. If one rather wishes an error estimate on u that does not involve p, then the converse should be required. Specifically, the operator $f\mapsto f_h$ should be such that f_h^K only depends on f^K, i.e.,

$$f^K = 0 \quad \Longrightarrow \quad f_h^K = 0. \tag{53}$$

To distinguish this property from (47), we introduce the following definition.

Definition 3.10. *We say that an operator $f \mapsto f_h \in V_h^1$ is Z-compatible if* (47) *holds, and we shall say that it is K-compatible if* (53) *holds.*

Remark 3.11. A related B-compatibilty property is defined in [7, Sec. 5.1.2] to establish uniform inf-sup conditions for the discrete spaces. This property is somewhat stronger that ours in that it is restricted to specific approximation properties, and it corresponds to different relations depending on the problems: with the first problem it implies a Z-compatibility in the sense defined above, and for the second problem it leads to K-compatibility.

A similar discussion applies to Problem 2.2 based on its decomposed form (23), which we may write together with its conforming discretization (42) in a similar form, i.e.,

$$\begin{cases} u^K + \delta^2\zeta = f^K \\ u^Z = f^Z \\ d^1 u^K = \ell \end{cases} \quad \text{and} \quad \begin{cases} u_h^K + \delta_h^2\zeta_h = f_h^K \\ u_h^Z = f_h^Z \\ d^1 u_h^K = \ell_h \end{cases} \tag{54}$$

again with $f_h = f_h^K + f_h^Z$ according to (34). Here a Z-compatible discretization allows $u_h^Z = f_h^Z$ to only depend on f^Z and hence on u^Z. However f_h^K may also depend on f^Z, which results in ζ_h depending not only on ζ but also on u^Z (and u^K). This effect is visible in the associated error estimate (51) which is satisfactory when the optimal error on ζ dominates that on u. When that is not the case, one may resort to K-compatible discretizations: then f_h^K would not depend on f^Z anymore and the resulting ζ_h would not depend on u^Z. However, it could still depend on u^K. To achieve the stronger property that ζ_h only depends on ζ, one would need to consider a different formulation of the problem that does not involve u, such as $d^1\delta^2\zeta = d^1 f - \ell$. In this article we shall restrict ourselves to the original formulation.

It is known that the compatibility of approximation operators is closely linked with the existence of commuting diagrams see e.g. [7, 18]. The following lemma expresses two sufficient conditions for K-compatibility, both involving commuting diagrams.

Lemma 3.12. *If the operator $\pi_h^1 : V^1 \to V_h^1$ satisfies a commuting diagram either on d^0 or on d^1, in the sense that (i) there exists an operator $\pi_h^0 : V^0 \to V_h^0$, such that*

$$\pi_h^1 d^0 = d^0 \pi_h^0 \tag{55}$$

holds on V^0, or (ii) there exists an operator $\pi_h^2 : W^2 \to W_h^2$, such that

$$d^1 \pi_h^1 = \pi_h^2 d^1 \tag{56}$$

holds on V^1, then π_h^1 is K-compatible on V^1, in the sense that property (53)*, namely $f^K = 0 \implies f_h^K = 0$, holds for all $f \in V^1$.*

Remark 3.13. Here we have considered operators defined on the domains V^0, V^1 and W^2 for simplicity. If they are defined on larger spaces, e.g. if π_h^1 is defined on W^1, then the proof below shows that (53) holds for all $f \in W^1$. If on the other hand they are defined on smaller spaces (of smooth functions for instance), then a refined study may be required to determine the validity domain of (53).

Proof. According to the Helmholtz decomposition (9), any $f \in V^1$ such that $f^K = 0$ is in Z^1. The result is then clear if π_h^1 satisfies (55): by definition any $f \in Z^1$ is of the form $f = d^0\phi$ with $\phi \in V^0$, hence $\pi_h^1 f = \pi_h^1 d^0 \phi = d^0 \pi_h^0 \phi \in d^0 V_h^0 = Z_h^1$ and

hence $f_h^K = 0$. To handle the second case we begin by verifying the standard relation

$$K_h^1 = \delta_h^2 W_h^2 \tag{57}$$

in two steps: by writing that $\langle \delta_h^2 \xi_h, d^0 q_h \rangle = \langle \xi_h, d^1 d^0 q_h \rangle = 0$ holds for all $q_h \in V_h^0$ and $\xi_h \in W_h^2$, we first see that $\delta_h^2 W_h^2 \subset K_h^1$. Next we observe that any $v_h \in K_h^1 \cap (\delta_h^2 W_h^2)^\perp$ satisfies $\|d^1 v_h\|^2 = \langle v_h, \delta_h^2 d^1 v_h \rangle = 0$ and hence is zero due to the discrete Poincaré estimate (31), which shows (57). Thus, writing any $v_h \in K_h^1$ as $v_h = \delta_h^2 \xi_h$ for some $\xi_h \in W_h^2$, we have $\langle \pi_h^1 f, v_h \rangle = \langle \pi_h^1 f, \delta_h^2 \xi_h \rangle = \langle d^1 \pi_h^1 f, \xi_h \rangle = \langle \pi_h^2 d^1 f, \xi_h \rangle = 0$ for all $f \in Z^1 = \ker d^1$, see (4). Again this shows that $f_h^K = 0$ and ends the proof. ∎

3.4. Error estimates for K-compatible discretizations. In this Section we consider a K-compatible approximation operator π_h^1 defined on some domain D^1, and for simplicity we assume that $Z^1 \subset D^1 \subset W^1$. In particular, if $f \in D^1$ then both components f^K and f^Z of its L^2 Helmholtz decomposition (10) also belong to D^1.

3.4.1. K-compatible conforming approximation of the first problem. If f belongs to the domain D^1 of π_h^1, then we can investigate the following modification of Problem (48): find $u_h \in V_h^1$ and $p_h \in K_h^0$ such that

$$\begin{cases} \langle d^1 u_h, d^1 v_h \rangle + \langle v_h, d^0 p_h \rangle = \langle \pi_h^1 f, v_h \rangle & v_h \in V_h^1 \\ \langle u_h, d^0 q_h \rangle = \langle g, q_h \rangle_{(V^0)' \times V^0} & q_h \in K_h^0. \end{cases} \tag{58}$$

As expected, it yields an error estimate for u that no longer involves p.

THEOREM 3.14. *If π_h^1 is a K-compatible operator in the sense of (53), then the solution of Problem (58) satisfies the error estimate*

$$\begin{cases} \|u - u_h\|_{V^1} \lesssim \|(\pi_h^1 - I) f^K\|_{(V_h^1)'} + \inf\limits_{\bar{u}_h \in V_h^1} \|u - \bar{u}_h\|_{V^1} \\ \|p - p_h\|_{V^0} \lesssim \|(\pi_h^1 - I) f\|_{(V_h^1)'} + \inf\limits_{\bar{p}_h \in K_h^0} \|p - \bar{p}_h\|_{V^0}. \end{cases} \tag{59}$$

Remark 3.15. Using (18) we can rewrite (59) in terms of the solution only,

$$\begin{cases} \|u - u_h\|_{V^1} \lesssim \|(\pi_h^1 - I) \delta^2 d^1 u\|_{(V_h^1)'} + \inf\limits_{\bar{u}_h \in V_h^1} \|u - \bar{u}_h\|_{V^1} \\ \|p - p_h\|_{V^0} \lesssim \|(\pi_h^1 - I)(\delta^2 d^1 u + d^0 p)\|_{(V_h^1)'} + \inf\limits_{\bar{p}_h \in K_h^0} \|p - \bar{p}_h\|_{V^0}. \end{cases}$$

Proof. As above we consider an arbitrary field $(\bar{u}_h, \bar{p}_h) \in V_h^1 \times K_h^0$. For $v_h \in K_h^1$, the K-compatibility of π_h^1 yields $\pi_h^1 f^Z \in Z_h^1$, hence $\langle \pi_h^1 f^Z, v_h \rangle = 0$. It follows that

$$\begin{aligned} \langle d^1 (u_h - \bar{u}_h), d^1 v_h \rangle &= \langle \pi_h^1 f, v_h \rangle - \langle d^1 \bar{u}_h, d^1 v_h \rangle \\ &= \langle \pi_h^1 f^K, v_h \rangle - \langle d^1 \bar{u}_h, d^1 v_h \rangle \\ &= \langle (\pi_h^1 - I) f^K, v_h \rangle + \langle f^K, v_h \rangle - \langle d^1 \bar{u}_h, d^1 v_h \rangle \\ &= \langle (\pi_h^1 - I) f^K, v_h \rangle + \langle d^1 (u - \bar{u}_h), d^1 v_h \rangle \end{aligned}$$

where we have used $\langle f^K, v_h \rangle = \langle \delta^2 d^1 u, v_h \rangle = \langle d^1 u, d^1 v_h \rangle$ in the last equality, see (18) and (8). For $w_h \in Z_h^1$, due to the embedding $Z_h^1 \subset Z^1 = \ker d^1$, we have

$$\langle w_h, d^0 (p_h - \bar{p}_h) \rangle = \langle (\pi_h^1 - I) f, w_h \rangle + \langle w_h, d^0 (p - \bar{p}_h) \rangle$$

and for $q_h \in K_h^0$, using the embeddings $K_h^0 \subset V_h^0 \subset V^0$ we write

$$\langle u_h - \bar{u}_h, d^0 q_h \rangle = \langle g, q_h \rangle_{(V^0)' \times V^0} - \langle \bar{u}_h, d^0 q_h \rangle = \langle u - \bar{u}_h, d^0 q_h \rangle.$$

The proof is then completed by arguing as for Estimate (49). ∎

3.4.2. K-compatible conforming approximation of the second problem. Similarly, we may replace (50) with the problem: find $u_h \in V_h^1$ and $\zeta_h \in Z_h^2$ such that

$$\begin{cases} \langle u_h, v_h\rangle + \langle d^1 v_h, \zeta_h\rangle = \langle \pi_h^1 f, v_h\rangle & v_h \in V_h^1 \\ \langle d^1 u_h, \xi_h\rangle = \langle \ell, \xi_h\rangle & \xi_h \in Z_h^2. \end{cases} \tag{60}$$

As announced above, it yields an error estimate for ζ that no longer involves u^Z.

THEOREM 3.16. *If $\pi_h^1 : V^1 \to V_h^1$ is a K-compatible approximation operator in the sense of (53), then for $f \in V^1$ the solution of Problem (60) satisfies the error estimate*

$$\begin{cases} \|u - u_h\|_{V^1} \lesssim \|(\pi_h^1 - I)f\|_{(V_h^1)'} + \inf\limits_{\bar{u}_h \in V_h^1} \|u - \bar{u}_h\|_{V^1}. \\ \|\zeta - \zeta_h\| \lesssim \|(\pi_h^1 - I)f^K\|_{(V_h^1)'} + \inf\limits_{\bar{u}_h \in V_h^1} \|u^K - \bar{u}_h\|_{V^1} + \inf\limits_{\bar{\zeta}_h \in Z_h^2} \|\zeta - \bar{\zeta}_h\|. \end{cases} \tag{61}$$

Remark 3.17. Using (23) we can rewrite (61) in terms of the solution only,

$$\|u - u_h\|_{V^1} \lesssim \|(\pi_h^1 - I)(u + \delta^2\zeta)\|_{(V_h^1)'} + \inf_{\bar{u}_h \in V_h^1} \|u - \bar{u}_h\|_{V^1}.$$
$$\|\zeta - \zeta_h\| \lesssim \|(\pi_h^1 - I)(u^K + \delta^2\zeta)\|_{(V_h^1)'} + \inf_{\bar{u}_h \in V_h^1} \|u^K - \bar{u}_h\|_{V^1} + \inf_{\bar{\zeta}_h \in Z_h^2} \|\zeta - \bar{\zeta}_h\|.$$

Proof. We repeat the proof of Proposition 3.8 with minor changes similar to the proof of Theorem 3.14: given an arbitrary $(\bar{u}_h, \bar{\zeta}_h) \in V_h^1 \times Z_h^2$, we compute for $w_h \in Z_h^1$

$$\langle u_h - \bar{u}_h, w_h\rangle = \langle \pi_h^1 f, w_h\rangle - \langle \bar{u}_h, w_h\rangle = \langle (\pi_h^1 - I)f, w_h\rangle + \langle u - \bar{u}_h, w_h\rangle$$

by using (20) and the embedding $Z_h^1 \subset Z^1$. For $\xi_h \in Z_h^2$ we have

$$\langle d^1(u_h - \bar{u}_h), \xi_h\rangle = \langle \ell, \xi_h\rangle - \langle d^1 \bar{u}_h, \xi_h\rangle = \langle d^1(u - \bar{u}_h), \xi_h\rangle$$

by using again (20) and the embedding $Z_h^2 \subset Z^2$. We finally consider $v_h \in K_h^1$ and observe that the K-compatibility of π_h^1 yields $\langle \pi_h^1 f^Z, v_h\rangle = 0$. We thus compute

$$\begin{aligned} \langle u_h - \bar{u}_h, v_h\rangle + \langle d^1 v_h, \zeta_h - \bar{\zeta}_h\rangle &= \langle \pi_h^1 f, v_h\rangle - \langle \bar{u}_h, v_h\rangle - \langle d^1 v_h, \bar{\zeta}_h\rangle \\ &= \langle \pi_h^1 f^K, v_h\rangle - \langle \bar{u}_h, v_h\rangle - \langle d^1 v_h, \bar{\zeta}_h\rangle \\ &= \langle (\pi_h^1 - I)f^K, v_h\rangle + \langle u^K - \bar{u}_h, v_h\rangle + \langle d^1 v_h, \zeta - \bar{\zeta}_h\rangle \end{aligned}$$

where we have used the first equation from (23) in the last equality, with the fact that $\langle v_h, \delta^2\zeta\rangle = \langle d^1 v_h, \zeta\rangle$, see (8). The proof is then completed by arguing as for (51). ∎

4. Structure-preserving nonconforming discretizations. We now discuss an extension of the above methods to spaces $\tilde{V}_h^1$ which are no longer subspaces of V^1, such as discontinuous Galerkin spaces. Following the Conforming/Nonconforming Galerkin (Conga) approach developped in [18, 15, 16], our construction relies on a non-standard exact sequence involving $\tilde{V}_h^1$, derived from a reference sequence of conforming spaces (29) that is structure-preserving in the sense of (30)-(31). The resulting nonconforming discretization is then shown to be stable without it being necessary to introduce penalty parameters as is usual in DG methods, see e.g., [26, 25, 14]. As in [16] it is convenient to consider that $\tilde{V}_h^1$ is larger than V_h^1, thus

$$\tilde{V}_h^1 \not\subset V^1, \qquad V_h^1 \subset \tilde{V}_h^1 \subset W^1.$$

The main ingredient of the nonconforming extension is a conforming projection,

$$\mathcal{P}_h^1 : W^1 \to V_h^1$$

that we assume L^2-stable in the sense that there is a uniform constant $c_{\mathcal{P}}$ such that

$$\|\mathcal{P}_h^1 v\| \leq c_{\mathcal{P}} \|v\|, \qquad v \in W^1. \tag{62}$$

For the convergence of the subsequent methods we further require that these conforming projections preserve spaces of moments $M_h^1 \subset \tilde{V}_h^1$ (typically piecewise polynomials of degree lower than the functions in $\tilde{V}_h^1$),

$$\langle \mathcal{P}_h^1 v, w_h \rangle = \langle v, w_h \rangle, \qquad w_h \in M_h^1. \tag{63}$$

One may think of defining $\mathcal{P}_h^1$ as the L^2 projection on V_h^1, as it satisfies the above properties with $M_h^1 = V_h^1$. However its application involves the inversion of a V_h^1 mass matrix (a global operation on the underlying mesh, due to the V^1-conformity that prevents the matrix to be block diagonal), which questions the practical interest of the resulting nonconforming method. We believe that a more interesting choice consists of constructing $\mathcal{P}_h^1$ by locally averaging piecewise degrees of freedom of V_h^1 type, first conveniently extended to L^2 in a stable way, see [20, 16]. The spaces of preserved moments M_h^1 are then typically of lower order than V_h^1 but the resulting $\mathcal{P}_h^1$ becomes a local operator, which should considerably reduce the computational complexity of numerical methods.

4.1. A discrete exact sequence with nonconforming spaces. As in [18] we first define a natural extension of $d_h^1 : V_h^1 \to W_h^2$ to the nonconforming space,

$$\tilde{d}_h^1 : \tilde{V}_h^1 \to W_h^2, \quad \tilde{v}_h \mapsto d^1 \mathcal{P}_h^1 \tilde{v}_h. \tag{64}$$

A new exact sequence involving this operator can then be constructed following [15], by introducing: (i) a nonstandard discretization of V^0,

$$\tilde{V}_h^0 := V_h^0 \times \tilde{V}_h^1$$

and (ii) an extension of $d_h^0 : V_h^0 \to V_h^1$ to this product space,

$$\tilde{d}_h^0 : \tilde{V}_h^0 \to \tilde{V}_h^1, \quad (q_h, \tilde{v}_h) \mapsto d^0 q_h + (I - \mathcal{P}_h^1)\tilde{v}_h. \tag{65}$$

Consistent with the notation (6) and (32) we then let

$$\tilde{K}_h^0 = \tilde{V}_h^0 \cap (\ker \tilde{d}_h^0)^\perp, \qquad \tilde{K}_h^1 = \tilde{V}_h^1 \cap (\ker \tilde{d}_h^1)^\perp, \qquad \tilde{Z}_h^1 = \tilde{d}^0 \tilde{V}_h^0 \tag{66}$$

and we observe that $\tilde{d}^1 \tilde{V}_h^1 = d^1 V_h^1 = Z_h^2$, indeed $V_h^1 \subset \tilde{V}_h^1$ yields $\mathcal{P}_h^1 \tilde{V}_h^1 = V_h^1$. As shown below, this construction yields a new structure-preserving discretization.

THEOREM 4.1. *If the conforming discrete sequence* (29) *is structure-preserving in the sense of* (30) *and* (31)*, then the nonconforming discrete sequence*

$$\tilde{V}_h^0 \xrightarrow{\tilde{d}_h^0} \tilde{V}_h^1 \xrightarrow{\tilde{d}_h^1} W_h^2 \tag{67}$$

is also structure-preserving, in the sense where: (i) it is exact, and specifically

$$\tilde{d}_h^0 \tilde{V}_h^0 = d^0 V_h^0 \oplus (I - \mathcal{P}_h^1)\tilde{V}_h^1 = \ker \tilde{d}_h^1, \tag{68}$$

(ii) the following Poincaré estimates hold

$$\begin{aligned} &\|\tilde q_h\| \le \tilde c_0 \|\tilde d_h^0 \tilde q_h\| && \tilde q_h \in \tilde K_h^0 = \tilde V_h^0 \cap (\ker \tilde d_h^0)^\perp, \\ &\|\tilde v_h\| \le \tilde c_1 \|\tilde d_h^1 \tilde v_h\| && \tilde v_h \in \tilde K_h^1 = \tilde V_h^1 \cap (\ker \tilde d_h^1)^\perp, \end{aligned} \tag{69}$$

with $\tilde c_0 = (2c_0^2 c_{\mathcal P}^2 + 1)^{\frac12}$ *and* $\tilde c_1 = c_1$.

Proof. For the exactness of (67) we recall the proof of [15]: using (30) we have

$$\tilde v_h \in \ker \tilde d_h^1 \implies \mathcal P_h^1 \tilde v_h \in \ker d_h^1 = d^0 V_h^0 \implies \tilde v_h \in d^0 V_h^0 \oplus (I - \mathcal P_h^1)\tilde V_h^1$$

with a direct sum checked by noting that any $\tilde w_h \in d^0 V_h^0 \cap (I - \mathcal P_h^1)\tilde V_h^1 \subset V_h^1$ satisfies $\tilde w_h = \mathcal P_h^1 \tilde w_h \in \mathcal P_h^1 (I - \mathcal P_h^1)\tilde V_h^1 = \{0\}$. The reverse inclusion is readily verified, hence the second equality in (68). The first one holds by construction of $\tilde d_h^0$ and $\tilde V_h^0$.

For the first stability estimate in (69) we infer from $d^0 V_h^0 \cap (I - \mathcal P_h^1)\tilde V_h^1 = \{0\}$ that $\ker \tilde d_h^0 = \ker d_h^0 \times (\tilde V_h^1 \cap \ker(I - \mathcal P_h^1)) = \ker d_h^0 \times V_h^1$, hence

$$\tilde K_h^0 = \tilde V_h^0 \cap (\ker \tilde d_h^0)^\perp = (V_h^0 \cap (\ker d^0)^\perp) \times (\tilde V_h^1 \cap (V_h^1)^\perp). \tag{70}$$

Taking $\tilde q_h = (q_h, \tilde v_h)$ in the latter space we then compute

$$\begin{aligned} \|\tilde q_h\|^2 = \|q_h\|^2 + \|\tilde v_h\|^2 &\le c_0^2 \|d^0 q_h\|^2 + \|\tilde v_h\|^2 \\ &\le c_0^2 (\|d^0 q_h - \mathcal P_h^1 \tilde v_h\| + \|\mathcal P_h^1 \tilde v_h\|)^2 + \|\tilde v_h\|^2 \\ &\le 2c_0^2 \|d^0 q_h - \mathcal P_h^1 \tilde v_h\|^2 + (2c_0^2 c_{\mathcal P}^2 + 1)\|\tilde v_h\|^2 \\ &\le \tilde c_0^2 (\|d^0 q_h - \mathcal P_h^1 \tilde v_h\|^2 + \|\tilde v_h\|^2) \\ &= \tilde c_0^2 \|d^0 q_h + (I - \mathcal P_h^1)\tilde v_h\|^2 = \tilde c_0^2 \|\tilde d_h^0 \tilde q_h\|^2. \end{aligned}$$

Here the first inequality uses the conforming stability (31), the third one uses the uniform L^2 bound (62) for $\mathcal P_h^1$ (with $c_{\mathcal P} \ge 1$), and the next to last equality follows from the observation that $d^0 q_h - \mathcal P_h^1 \tilde v_h \in V_h^1$ is orthogonal to $\tilde v_h$, see (70). This shows the first stability estimate. For the second one we consider $\tilde v_h \in \tilde K_h^1$. By construction $\tilde d_h^1 \tilde v_h$ is in $d^1 V_h^1$, hence there is a (unique) conforming $v_h \in K_h^1$ such that $d^1 v_h = \tilde d_h^1 \tilde v_h$. Using the conforming stability (31) this gives

$$\|v_h\| \le c_1 \|d^1 v_h\| = c_1 \|\tilde d_h^1 \tilde v_h\| \tag{71}$$

so we are left to control $\tilde v_h$ by its conforming counterpart. For this we observe that the difference $v_h - \mathcal P_h^1 \tilde v_h$ is in $V_h^1 \cap \ker d^1$, hence it is orthogonal to $\tilde v_h$. We thus have $0 = \langle \tilde v_h, v_h - \mathcal P_h^1 \tilde v_h\rangle = \langle \tilde v_h, v_h - \tilde v_h\rangle$ where the second equality follows from the fact that $\tilde v_h$ is also orthonal to the functions in $(I - \mathcal P_h^1)\tilde V_h^1$. As a consequence, $\|\tilde v_h\|^2 \le \|\tilde v_h\|^2 + \|v_h - \tilde v_h\|^2 = \|v_h\|^2$ and the desired estimate follows by combining this bound with (71). ∎

For the nonconforming error analysis we introduce natural energy norms,

$$\|\tilde q_h\|_{\tilde V_h^0}^2 := \|\tilde q_h\|^2 + \|\tilde d_h^0 \tilde q_h\|^2, \qquad \|\tilde v_h\|_{\tilde V_h^1}^2 := \|\tilde v_h\|^2 + \|\tilde d_h^1 \tilde v_h\|^2. \tag{72}$$

Notice that the $\tilde V_h^1$ norm is defined over W^1 (an L^2 space) and on V_h^1 it coincides with the V^1 norm. Introducing a canonical conforming projection on V_h^0,

$$\mathcal P_h^0 : \tilde V_h^0 \to V_h^0, \quad (q_h, \tilde w_h) \mapsto q_h, \tag{73}$$

and using the definition of $\tilde{d}_h^1$, $\tilde{d}_h^0$ and the stability (62) of $\mathcal{P}_h^1$, we further observe that

$$\|\tilde{q}_h\|_{\tilde{V}_h^0} \sim \|(I-\mathcal{P}_h^0)\tilde{q}_h\| + \|\mathcal{P}_h^0\tilde{q}_h\|_{V^0} = \|\tilde{w}_h\| + \|q_h\|_{V^0}, \qquad \tilde{q}_h = (q_h, \tilde{w}_h) \in \tilde{V}_h^0,$$
$$\|\tilde{v}_h\|_{\tilde{V}_h^1} \sim \|(I-\mathcal{P}_h^1)\tilde{v}_h\| + \|\mathcal{P}_h^1\tilde{v}_h\|_{V^1}, \qquad \tilde{v}_h \in \tilde{V}_h^1,$$

hold with constants independent of h. In particular, the conforming projections are stable in the proper energy norms,

$$\|\mathcal{P}_h^0\tilde{q}_h\|_{V^0} \lesssim \|\tilde{q}_h\|_{\tilde{V}_h^0} \qquad \text{and} \qquad \|\mathcal{P}_h^1\tilde{v}_h\|_{V^1} \lesssim \|\tilde{v}_h\|_{\tilde{V}_h^1}. \tag{74}$$

We also let $\tilde{\delta}_h^1 : \tilde{V}_h^1 \to \tilde{V}_h^0$ and $\tilde{\delta}_h^2 : W_h^2 \to \tilde{V}_h^1$ be the discrete adjoints of $\tilde{d}_h^0$ and $\tilde{d}_h^1$,

$$\begin{aligned} \langle \tilde{\delta}_h^1 v, q\rangle &= \langle v, \tilde{d}_h^0 q\rangle, \qquad && v \in \tilde{V}_h^1,\ q \in \tilde{V}_h^0 \\ \langle \tilde{\delta}_h^2 \xi, v\rangle &= \langle \xi, \tilde{d}_h^1 v\rangle, \qquad && \xi \in \tilde{W}_h^2,\ v \in \tilde{V}_h^1. \end{aligned} \tag{75}$$

Using the discrete exact sequence property (68) we then observe that $\tilde{Z}_h^1$ coincides with the kernel of $\tilde{d}_h^1$ and $\tilde{K}_h^1$ coincides with the kernel of $\tilde{\delta}_h^1$. Hence the decomposition

$$\tilde{V}_h^1 = \tilde{K}_h^1 \overset{\perp}{\oplus} \tilde{Z}_h^1 = \ker \tilde{\delta}_h^1 \overset{\perp}{\oplus} \ker \tilde{d}_h^1 \tag{76}$$

is orthogonal both in the L^2 and $\tilde{V}_h^1$ inner products. Before discretizing the mixed problems we discuss the compatibility of nonconforming approximation operators.

4.2. Z- and K-compatible nonconforming operators. In the nonconforming case we extend Def. 3.10 as follows.

Definition 4.2. *We say that an operator $f \mapsto \tilde{f}_h \in \tilde{V}_h^1$ is Z-compatible (for the nonconforming discretization) if*

$$f^Z = 0 \quad \Longrightarrow \quad \tilde{f}_h^Z = 0 \tag{77}$$

holds for the Helmholtz decompositions $f = f^K + f^Z$ and $\tilde{f}_h = \tilde{f}_h^K + \tilde{f}_h^Z$ corresponding to (11)-(12) and (76), respectively. If

$$f^K = 0 \quad \Longrightarrow \quad \tilde{f}_h^K = 0 \tag{78}$$

then we shall say that it is K-compatible (for the nonconforming discretization).

In other terms, $f \mapsto \tilde{f}_h$ is Z (resp. K)-compatible if $\tilde{f}_h^Z$ (resp. $\tilde{f}_h^K$) only depends on f^Z (resp. f^K). In the conforming case we have seen that the L^2 projection on V_h^1 is a Z-compatible operator. With nonconforming spaces this is no longer true: indeed the L^2 projection on $\tilde{V}_h^1$ does not satisfy (77) and we must use another approximation operator (although not a projection).

Lemma 4.3. *The operator $(\mathcal{P}_h^1)^* : W^1 \to \tilde{V}_h^1$, $f \mapsto \tilde{f}_h$, defined by*

$$\langle \tilde{f}_h, \tilde{v}_h\rangle = \langle f, \mathcal{P}_h^1\tilde{v}_h\rangle, \qquad \tilde{v}_h \in \tilde{V}_h^1, \tag{79}$$

is Z-compatible in the sense of Def. 4.2.

Proof. From the relation (68) we infer that

$$\mathcal{P}_h^1(\tilde{Z}_h^1) \subset Z_h^1 \subset Z^1. \tag{80}$$

For $\tilde{v}_h^1 \in \tilde{Z}_h^1$ this yields $\langle \tilde{f}_h^Z, \tilde{v}_h\rangle = \langle \tilde{f}_h, \tilde{v}_h\rangle = \langle f, \mathcal{P}_h^1\tilde{v}_h\rangle = \langle f^Z, \mathcal{P}_h^1\tilde{v}_h\rangle$, hence (77). ∎

In Section 3.3 we have identified two K-compatibility criteria involving commuting diagrams. For the nonconforming case, we first make an easy observation.

LEMMA 4.4. *If π_h^1 is an operator mapping on V_h^1 that is K-compatible for the conforming discretization, then it is also K-compatible for the nonconforming one.*

Proof. Using the embedding $V_h^1 \subset \tilde{V}_h^1$ we see that π_h^1 maps on $\tilde{V}_h^1$, and from the embedding $Z_h^1 \subset \tilde{Z}_h^1$ we infer that if $f_h^K = 0$, then $f_h \in \tilde{Z}_h^1$ and hence $\tilde{f}_h^K = 0$. ∎

Due to the geometric structure we also check that the commuting diagram criteria of Lemma 3.12 naturally extend to the nonconforming case.

LEMMA 4.5. *If the operator $\tilde{\pi}_h^1 : V^1 \to \tilde{V}_h^1$ satisfies a commuting diagram either on d^0 or on d^1, in the sense that (i) there exists an operator $\tilde{\pi}_h^0 : V^0 \to \tilde{V}_h^0$, such that*

$$\tilde{\pi}_h^1 d^0 = \tilde{d}_h^0 \tilde{\pi}_h^0 \tag{81}$$

holds on V^0, or (ii) there exists an operator $\pi_h^2 : W^2 \to W_h^2$, such that

$$\tilde{d}_h^1 \tilde{\pi}_h^1 = \pi_h^2 d^1 \tag{82}$$

holds on V^1, then $\tilde{\pi}_h^1$ is K-compatible on V^1, in the sense that property (78), namely $f^K = 0 \implies \tilde{f}_h^K = 0$, holds for all $f \in V^1$.

Proof. The proof is formally the same than for Lemma 3.12. ∎

5. Nonconforming discretizations of the source problems.

5.1. Discretizations with arbitrary sources. As in the conforming case we first study the stability of nonconforming discretizations with arbitrary sources.

5.1.1. Nonconforming discretization of the first problem. Based on the nonconforming sequence (68) the discretization of Problem 2.1 reads: Given $\tilde{g}_h$, $\tilde{f}_h$ in the nonconforming spaces $(\tilde{V}_h^0)'$ and $(\tilde{V}_h^1)'$, find $\tilde{u}_h \in \tilde{V}_h^1$, $\tilde{p}_h \in \tilde{K}_h^0$ such that

$$\begin{cases} \langle \tilde{d}_h^1 \tilde{u}_h, \tilde{d}_h^1 \tilde{v}_h \rangle + \langle \tilde{v}_h, \tilde{d}_h^0 \tilde{p}_h \rangle = \langle \tilde{f}_h, \tilde{v}_h \rangle_{(\tilde{V}_h^1)' \times \tilde{V}_h^1} & \tilde{v}_h \in \tilde{V}_h^1 \\ \langle \tilde{u}_h, \tilde{d}_h^0 \tilde{q}_h \rangle = \langle \tilde{g}_h, \tilde{q}_h \rangle_{(\tilde{V}_h^0)' \times \tilde{V}_h^0} & \tilde{q}_h \in \tilde{K}_h^0. \end{cases} \tag{83}$$

From the definition of the operators $\tilde{d}_h^0$ and $\tilde{d}_h^1$, system (83) amounts to

$$\begin{cases} \langle d^1 \mathcal{P}_h^1 \tilde{u}_h, d^1 \mathcal{P}_h^1 \tilde{v}_h \rangle + \langle \tilde{v}_h, d^0 p_h + (I - \mathcal{P}_h^1) \tilde{x}_h \rangle = \langle \tilde{f}_h, \tilde{v}_h \rangle_{(\tilde{V}_h^1)' \times \tilde{V}_h^1} \\ \langle \tilde{u}_h, d^0 q_h + (I - \mathcal{P}_h^1) \tilde{y}_h \rangle = \langle \tilde{g}_h, \tilde{q}_h \rangle_{(\tilde{V}_h^0)' \times \tilde{V}_h^0} \end{cases}$$

where we have written $\tilde{p}_h = (p_h, \tilde{x}_h) \in \tilde{V}_h^0 = V_h^0 \times \tilde{V}_h^1$ and similarly $\tilde{q}_h = (q_h, \tilde{y}_h)$. We decompose this problem as the previous ones. According to (76) we write

$$\tilde{u}_h = \tilde{u}_h^K + \tilde{u}_h^Z \in \tilde{K}_h^1 \oplus \tilde{Z}_h^1 \tag{84}$$

and restate Problem (83) as: find $(\tilde{u}_h^K, \tilde{u}_h^Z, \tilde{p}_h) \in \tilde{K}_h^1 \times \tilde{Z}_h^1 \times \tilde{K}_h^0$ such that

$$\begin{cases} \langle \tilde{d}_h^1 \tilde{u}_h^K, \tilde{d}_h^1 \tilde{v}_h \rangle = \langle \tilde{f}_h, \tilde{v}_h \rangle_{(\tilde{V}_h^1)' \times \tilde{V}_h^1} & \tilde{v}_h \in \tilde{K}_h^1 \\ \langle \tilde{w}_h, \tilde{d}_h^0 \tilde{p}_h \rangle = \langle \tilde{f}_h, \tilde{w}_h \rangle_{(\tilde{V}_h^1)' \times \tilde{V}_h^1} & \tilde{w}_h \in \tilde{Z}_h^1 \\ \langle \tilde{u}_h^Z, \tilde{d}_h^0 \tilde{q}_h \rangle = \langle \tilde{g}_h, \tilde{q}_h \rangle_{(\tilde{V}_h^0)' \times \tilde{V}_h^0} & \tilde{q}_h \in \tilde{K}_h^0. \end{cases} \tag{85}$$

LEMMA 5.1. *The discrete problem* (83) *is well-posed, and its solution satisfies*

$$\begin{aligned} \|\tilde{u}_h\|_{\tilde{V}_h^1} &\lesssim \|\tilde{f}_h|_{\tilde{K}_h^1}\|_{(\tilde{V}_h^1)'} + \|\tilde{g}_h\|_{(\tilde{V}_h^0)'} \\ \|\tilde{p}_h\|_{\tilde{V}_h^0} &\lesssim \|\tilde{f}_h|_{\tilde{Z}_h^1}\|_{(\tilde{V}_h^1)'} \end{aligned} \tag{86}$$

where $\tilde{f}_h|_{\tilde{K}_h^1}$ *and* $\tilde{f}_h|_{\tilde{Z}_h^1}$ *denote the restrictions of* $\tilde{f}_h$ *to the respective spaces. Here the constants depending on* $\tilde{c}^0$, $\tilde{c}^1$ *from* (69). *Specifically, writing* $\tilde{u}_h$ *as in* (84) *we have*

$$\begin{aligned} \|\tilde{u}_h^K\|_{\tilde{V}_h^1} &\leq (1+\tilde{c}_1^2)\|\tilde{f}_h|_{\tilde{K}_h^1}\|_{(\tilde{V}_h^1)'} \\ \|\tilde{u}_h^Z\|_{\tilde{V}_h^1} &\leq (1+\tilde{c}_0^2)^{\frac{1}{2}}\|\tilde{g}_h\|_{(\tilde{V}_h^0)'} \\ \|\tilde{p}_h\|_{\tilde{V}_h^0} &\leq (1+\tilde{c}_0^2)^{\frac{1}{2}}\|\tilde{f}_h|_{\tilde{Z}_h^1}\|_{(\tilde{V}_h^1)'}. \end{aligned} \tag{87}$$

Proof. The proof is formally the same than for Lemma 3.2, using the nonconforming Poincaré estimates (69) and the formulation (85) of Problem (83). ∎

Remark 5.2. Using the discrete stability (69) and reasonning as in Section 2.2 one verifies that the bilinear forms involved in (83), i.e. $\tilde{a}_h(\tilde{v}_h, \tilde{w}_h) = \langle \tilde{d}_h^1 \tilde{v}_h, \tilde{d}_h^1 \tilde{w}_h \rangle$ and $\tilde{b}_h(\tilde{v}_h, \tilde{q}_h) = \langle \tilde{v}_h, \tilde{d}_h^0 \tilde{q}_h \rangle$, satisfy standard stability properties, namely the coercivity of $\tilde{a}_h$ on the discrete kernel (here, $\tilde{K}_h$) and a uniform discrete inf-sup condition for $\tilde{b}_h$.

5.1.2. Nonconforming discretization of the second problem. Based on the nonconforming sequence (68) the discretization of Problem 2.2 reads: Given discrete sources $\tilde{f}_h, \ell_h$ in $(\tilde{V}_h^1)'$ and W_h^2, find $\tilde{u}_h \in \tilde{V}_h^1$, $\zeta_h \in Z_h^2$ such that

$$\begin{cases} \langle \tilde{u}_h, \tilde{v}_h \rangle + \langle \tilde{d}_h^1 \tilde{v}_h, \zeta_h \rangle = \langle \tilde{f}_h, \tilde{v}_h \rangle_{(\tilde{V}_h^1)' \times \tilde{V}_h^1} & \tilde{v}_h \in \tilde{V}_h^1 \\ \langle \tilde{d}_h^1 \tilde{u}_h, \xi_h \rangle = \langle \ell_h, \xi_h \rangle & \xi_h \in Z_h^2. \end{cases} \tag{88}$$

From the definition of $\tilde{d}_h^1$ this problem amounts to

$$\begin{cases} \langle \tilde{u}_h, \tilde{v}_h \rangle + \langle d^1 \mathcal{P}_h^1 \tilde{v}_h, \zeta_h \rangle = \langle \tilde{f}_h, \tilde{v}_h \rangle_{(\tilde{V}_h^1)' \times \tilde{V}_h^1} & \tilde{v}_h \in \tilde{V}_h^1 \\ \langle d^1 \mathcal{P}_h^1 \tilde{u}_h, \xi_h \rangle = \langle \ell_h, \xi_h \rangle & \xi_h \in Z_h^2 \end{cases} \tag{89}$$

and writing $\tilde{u}_h$ as in (84) we recast it as: find $(\tilde{u}_h^K, \tilde{u}_h^Z, \zeta_h) \in \tilde{K}_h^1 \times \tilde{Z}_h^1 \times Z_h^2$ such that

$$\begin{cases} \langle \tilde{u}_h^K, \tilde{v}_h \rangle + \langle \tilde{d}_h^1 \tilde{v}_h, \zeta_h \rangle = \langle \tilde{f}_h, \tilde{v}_h \rangle_{(\tilde{V}_h^1)' \times \tilde{V}_h^1} & \tilde{v}_h \in \tilde{K}_h^1 \\ \langle \tilde{u}_h^Z, \tilde{w}_h \rangle = \langle \tilde{f}_h, \tilde{w}_h \rangle_{(\tilde{V}_h^1)' \times \tilde{V}_h^1} & \tilde{w}_h \in \tilde{Z}_h^1 \\ \langle \tilde{d}_h^1 \tilde{u}_h^K, \xi_h \rangle = \langle \ell_h, \xi_h \rangle & \xi_h \in Z_h^2. \end{cases} \tag{90}$$

Here the third, second and first equations define $\tilde{u}_h^K$, $\tilde{u}_h^Z$ and ζ_h, respectively.

LEMMA 5.3. *The discrete problem* (88) *is well-posed, and its solution satisfies*

$$\begin{aligned} \|\tilde{u}_h\|_{\tilde{V}_h^1} &\lesssim \|\tilde{f}_h|_{\tilde{Z}_h^1}\|_{(\tilde{V}_h^1)'} + \|\ell_h\| \\ \|\zeta_h\| &\lesssim \|\tilde{f}_h|_{\tilde{K}_h^1}\|_{(\tilde{V}_h^1)'} + \|\ell_h\| \end{aligned} \tag{91}$$

where $\tilde{f}_h|_{\tilde{K}_h^1}$ *and* $\tilde{f}_h|_{\tilde{Z}_h^1}$ *denote the restrictions of* $\tilde{f}_h$ *to the respective spaces. Here the constants depend only on* $\tilde{c}_1$ *from* (69). *Specifically, writing* $\tilde{u}_h$ *as in* (84) *we have*

$$\begin{aligned} \|\tilde{u}_h^K\|_{\tilde{V}_h^1} &\leq (1+\tilde{c}_1^2)^{\frac{1}{2}}\|\ell_h\| \\ \|\tilde{u}_h^Z\|_{\tilde{V}_h^1} &\leq \|\tilde{f}_h|_{\tilde{Z}_h^1}\|_{(\tilde{V}_h^1)'} \\ \|\zeta_h\| &\leq (1+\tilde{c}_1^2)^{\frac{1}{2}}\|\tilde{f}_h|_{\tilde{K}_h^1}\|_{(\tilde{V}_h^1)'} + \tilde{c}_1^2\|\ell_h\|. \end{aligned} \tag{92}$$

Proof. The proof is formally the same than for Lemma 3.4, using the nonconforming Poincaré estimates (69) and the formulation (90) of Problem (88). ∎

Remark 5.4. Using the discrete stability (69) and reasonning as in Section 2.3 one verifies that the bilinear forms involved in (88), namely $\tilde{a}_h(\tilde{v}_h, \tilde{w}_h) = \langle \tilde{v}_h, \tilde{w}_h \rangle$ and $\tilde{b}_h(\tilde{v}_h, \xi_h) = \langle \tilde{d}^1_h \tilde{v}_h, \xi_h \rangle$, satisfy standard stability properties: the coercivity of $\tilde{a}_h$ on the relevant discrete kernel (here, $\tilde{Z}_h$) and a uniform discrete inf-sup condition for $\tilde{b}_h$.

5.2. Z-compatible nonconforming approximation. Following Lemma 4.3, we may use $(\mathcal{P}^1_h)^*$ as a natural extension of the L^2 source projection (46) in the nonconforming case.

5.2.1. Z-compatible nonconforming approximation of the first problem. Using $(\mathcal{P}^1_h)^*$ for f and the canonical extension $(\mathcal{P}^0_h)^*$ for g, we obtain the following nonconforming method for Problem 2.1: find $\tilde{u}_h \in \tilde{V}^1_h$, $\tilde{p}_h = (p_h, \tilde{x}_h) \in \tilde{K}^0_h$ such that

$$\left\{ \begin{array}{rl} \langle \tilde{d}^1_h \tilde{u}_h, \tilde{d}^1_h \tilde{v}_h \rangle + \langle \tilde{v}_h, \tilde{d}^0_h \tilde{p}_h \rangle = \langle f, \mathcal{P}^1_h \tilde{v}_h \rangle_{(V^1)' \times V^1} & \tilde{v}_h \in \tilde{V}^1_h \\ \langle \tilde{u}_h, \tilde{d}^0_h \tilde{q}_h \rangle = \langle g, \mathcal{P}^0_h \tilde{q}_h \rangle_{(V^0)' \times V^0} & \tilde{q}_h \in \tilde{K}^0_h. \end{array} \right. \tag{93}$$

The accuracy of the resulting method involves that of $\mathcal{P}^1_h$ and its adjoint $(\mathcal{P}^1_h)^*$.

THEOREM 5.5. *The solution to the discrete problem* (93) *satisfies*

$$\left\{ \begin{array}{l} \|u - \tilde{u}_h\|_{\tilde{V}^1_h} \lesssim \|d^1(I - \mathcal{P}^1_h)u\| + \inf\limits_{\bar{u}_h \in V^1_h \cap M^1_h} \|u - \bar{u}_h\|_{V^1} + \inf\limits_{\substack{\bar{p}_h \in K^0_h \\ d^0 \bar{p}_h \in M^1_h}} \|d^0(p - \bar{p}_h)\| \\ \|(p, 0) - \tilde{p}_h\|_{\tilde{V}^0_h} \lesssim \inf\limits_{\substack{\bar{p}_h \in K^0_h \\ d^0 \bar{p}_h \in M^1_h}} \|p - \bar{p}_h\|_{V^0} \end{array} \right.$$

with constants depending on $\tilde{c}_0$, $\tilde{c}_1$, $c_{\mathcal{P}}$, and M^1_h the preserved moments (63) *of $\mathcal{P}^1_h$.*

Proof. Consider $\bar{u}_h \in V^1_h \cap M^1_h$ and $\bar{p}_h \in K^0_h$ with $d^0 \bar{p}_h \in M^1_h$. Using (93), (14) and the fact that $\tilde{d}^1_h = d^1 \mathcal{P}^1_h$ coincides with d^1 on V^1_h, we write for $\tilde{v}_h \in \tilde{K}^1_h$,

$$\begin{aligned} \langle \tilde{d}^1_h (\tilde{u}_h - \bar{u}_h), \tilde{d}^1_h \tilde{v}_h \rangle &= \langle f, \mathcal{P}^1_h \tilde{v}_h \rangle_{(V^1)' \times V^1} - \langle d^1 \bar{u}_h, d^1 \mathcal{P}^1_h \tilde{v}_h \rangle \\ &= \langle d^1(u - \bar{u}_h), d^1 \mathcal{P}^1_h \tilde{v}_h \rangle + \langle \mathcal{P}^1_h \tilde{v}_h, d^0 p \rangle \\ &= \langle d^1(u - \bar{u}_h), d^1 \mathcal{P}^1_h \tilde{v}_h \rangle + \langle \mathcal{P}^1_h \tilde{v}_h, d^0(p - \bar{p}_h) \rangle, \end{aligned}$$

where we used that $\langle \mathcal{P}^1_h \tilde{v}_h, d^0 \bar{p}_h \rangle = \langle \tilde{v}_h, d^0 \bar{p}_h \rangle = 0$ according to (63) and the embedding $Z^1_h \subset \tilde{Z}^1_h$. For $\tilde{w}_h \in \tilde{Z}^1_h$, observing that $\mathcal{P}^1_h \tilde{w}_h \in Z^1$, see (80), and using the form (17) of the exact problem, we write (using again $d^0 \bar{p}_h \in M^1_h$)

$$\langle \tilde{w}_h, \tilde{d}^0_h (\tilde{p}_h - (\bar{p}_h, 0)) \rangle = \langle f, \mathcal{P}^1_h \tilde{w}_h \rangle_{(V^1)' \times V^1} - \langle \tilde{w}_h, d^0 \bar{p}_h \rangle = \langle \mathcal{P}^1_h \tilde{w}_h, d^0(p - \bar{p}_h) \rangle.$$

Next for $\tilde{q}_h = (q_h, \tilde{y}_h) \in \tilde{K}^0_h$, we infer from $\bar{u}_h \in M^1_h$ that $\langle \bar{u}_h, (\mathcal{P}^1_h - I) \tilde{y}_h \rangle = 0$, hence $\langle \bar{u}_h, \tilde{d}^0_h \tilde{q}_h \rangle = \langle \bar{u}_h, d^0 q_h \rangle$. Using (17) with $q_h = \mathcal{P}^0_h \tilde{q}_h \in V^0_h$ gives then

$$\langle \tilde{u}_h - \bar{u}_h, \tilde{d}^0_h \tilde{q}_h \rangle = \langle g, q_h \rangle_{(V^0)' \times V^0} - \langle \bar{u}_h, d^0 q_h \rangle = \langle u - \bar{u}_h, d^0 \mathcal{P}^0_h \tilde{q}_h \rangle.$$

Since $(\bar{p}_h, 0) \in \tilde{K}^0_h$, Lemma 5.1 applies: Using the stability (74) of $\mathcal{P}^0_h$, $\mathcal{P}^1_h$ this gives $\|\tilde{u}_h - \bar{u}_h\|_{\tilde{V}^1_h} \lesssim \|u - \bar{u}_h\|_{V^1} + \|d^0(p - \bar{p}_h)\|$ and $\|\tilde{p}_h - (\bar{p}_h, 0)\|_{\tilde{V}^0_h} \lesssim \|d^0(p - \bar{p}_h)\|$ with constants involving $\tilde{c}_0$, $\tilde{c}_1$, $c_{\mathcal{P}}$. As $\|u - \bar{u}_h\|_{\tilde{V}^1_h} \leq \|u - \bar{u}_h\|_{V^1} + \|d^1(I - \mathcal{P}^1_h)u\|$ this yields $\|u - \tilde{u}_h\|_{\tilde{V}^1_h} \lesssim \|u - \bar{u}_h\|_{V^1} + \|d^1(I - \mathcal{P}^1_h)u\| + \|d^0(p - \bar{p}_h)\|$ and $\|(p, 0) - \tilde{p}_h\|_{\tilde{V}^0_h} \lesssim \|p - \bar{p}_h\|_{V^0}$ and the result follows by taking the infimum over $\bar{u}_h$ and $\bar{p}_h$. ∎

5.2.2. Z-compatible nonconforming approximation of the second problem. Following (93), a Z-compatible nonconforming approximation analogous to Problem 2.2 is: find $\tilde{u}_h \in \tilde{V}_h^1$, $\zeta_h \in Z_h^2$ such that

$$\begin{cases} \langle \tilde{u}_h, \tilde{v}_h \rangle + \langle \tilde{d}_h^1 \tilde{v}_h, \zeta_h \rangle = \langle f, \mathcal{P}_h^1 \tilde{v}_h \rangle_{(V^1)' \times V^1} & \tilde{v}_h \in \tilde{V}_h^1 \\ \langle \tilde{d}_h^1 \tilde{u}_h, \xi_h \rangle = \langle \ell, \xi_h \rangle & \xi_h \in Z_h^2. \end{cases} \tag{94}$$

The accuracy of the resulting method essentially involves that of the adjoint $(\mathcal{P}_h^1)^*$.

THEOREM 5.6. *The solution to the nonconforming approximation* (94) *satisfies*

$$\begin{cases} \|u - \tilde{u}_h\|_{\tilde{V}_h^1} \lesssim \inf\limits_{\bar{u}_h \in V_h^1 \cap M_h^1} \|u - \bar{u}_h\|_{V^1} \\ \|\zeta - \zeta_h\| \lesssim \inf\limits_{\bar{\zeta}_h \in Z_h^2} \|\zeta - \bar{\zeta}_h\| + \inf\limits_{\bar{u}_h \in V_h^1 \cap M_h^1} \|u - \bar{u}_h\|_{V^1}. \end{cases}$$

Proof. We consider $\bar{u}_h \in V_h^1 \cap M_h^1$ and $\bar{\zeta}_h \in Z_h^2$. For $\tilde{w}_h \in \tilde{Z}_h^1$, using (20) we write

$$\langle \tilde{u}_h - \bar{u}_h, \tilde{w}_h \rangle = \langle f, \mathcal{P}_h^1 \tilde{w}_h \rangle_{(V^1)' \times V^1} - \langle \bar{u}_h, \tilde{w}_h \rangle = \langle u, \mathcal{P}_h^1 \tilde{w}_h \rangle - \langle \bar{u}_h, \tilde{w}_h \rangle = \langle u - \bar{u}_h, \mathcal{P}_h^1 \tilde{w}_h \rangle$$

by using $\mathcal{P}_h^1 \tilde{w}_h \in Z^1$, and the moment preserving property (63). For $\xi_h \in Z_h^2 \subset Z^2$,

$$\langle \tilde{d}_h^1 (\tilde{u}_h - \bar{u}_h), \xi_h \rangle = \langle \ell, \xi_h \rangle - \langle d^1 \bar{u}_h, \xi_h \rangle = \langle d^1 (u - \bar{u}_h), \xi_h \rangle$$

follows by using (20) again and $\tilde{d}_h^1 = d^1$ on V_h^1. Finally for $\tilde{v}_h \in \tilde{K}_h^1$, we have

$$\begin{aligned} \langle \tilde{u}_h - \bar{u}_h, \tilde{v}_h \rangle + \langle \tilde{d}_h^1 \tilde{v}_h, \zeta_h - \bar{\zeta}_h \rangle &= \langle f, \mathcal{P}_h^1 v_h \rangle_{(V^1)' \times V^1} - \langle \bar{u}_h, \tilde{v}_h \rangle - \langle \tilde{d}_h^1 \tilde{v}_h, \bar{\zeta}_h \rangle \\ &= \langle u - \bar{u}_h, \mathcal{P}_h^1 \tilde{v}_h \rangle + \langle \tilde{d}_h^1 \tilde{v}_h, \zeta - \bar{\zeta}_h \rangle \end{aligned}$$

by using again (20) and (63). Applying Lemma 5.3 and the stability (74) of $\mathcal{P}_h^1$ yields then $\|\tilde{u}_h - \bar{u}_h\|_{\tilde{V}_h^1} \lesssim \|u - \bar{u}_h\|_{V^1}$ and $\|\zeta_h - \bar{\zeta}_h\| \lesssim \|u - \bar{u}_h\|_{V^1} + \|\zeta - \bar{\zeta}_h\|$ with constants depending on $\tilde{c}_1$, $c_{\mathcal{P}}$. Taking the infimum over $\bar{u}_h$ and $\bar{p}_h$ ends the proof. ∎

5.3. K-compatible nonconforming approximation. As in Section 3.4 we consider a K-compatible approximation operator $\tilde{\pi}_h^1$ defined on some domain D^1, see Def. 4.2. Again we assume that $Z^1 \subset D^1 \subset W^1$, so that if $f \in D^1$ then both components f^K and f^Z of its L^2 Helmholtz decomposition (10) belong to D^1.

5.3.1. K-compatible nonconforming approximation of the first problem. If f belongs to the domain D^1 of $\tilde{\pi}_h^1$, we can investigate a K-compatible version of Problem (93): find $\tilde{u}_h \in \tilde{V}_h^1$, $\tilde{p}_h \in \tilde{K}_h^0$ such that

$$\begin{cases} \langle \tilde{d}_h^1 \tilde{u}_h, \tilde{d}_h^1 \tilde{v}_h \rangle + \langle \tilde{v}_h, \tilde{d}_h^0 \tilde{p}_h \rangle = \langle \tilde{\pi}_h^1 f, \tilde{v}_h \rangle & \tilde{v}_h \in \tilde{V}_h^1 \\ \langle \tilde{u}_h, \tilde{d}_h^0 \tilde{q}_h \rangle = \langle g, \mathcal{P}_h^0 \tilde{q}_h \rangle_{(V^0)' \times V^0} & \tilde{q}_h \in \tilde{K}_h^0. \end{cases} \tag{95}$$

THEOREM 5.7. *If $\tilde{\pi}_h^1 : D^1 \to \tilde{V}_h^1$ is a K-compatible operator in the sense of* (78), *then for $f \in D^1$ the solution to* (95) *satisfies the error estimate*

$$\begin{cases} \|u - \tilde{u}_h\|_{\tilde{V}_h^1} \lesssim \|d^1 (I - \mathcal{P}_h^1) u\| + \|(\tilde{\pi}_h^1 - (\mathcal{P}_h^1)^*) f^K\|_{(\tilde{V}_h^1)'} + \inf\limits_{\bar{u}_h \in V_h^1 \cap M_h^1} \|u - \bar{u}_h\|_{V^1} \\ \|(p, 0) - \tilde{p}_h\|_{\tilde{V}_h^0} \lesssim \|(\tilde{\pi}_h^1 - (\mathcal{P}_h^1)^*) f\|_{(\tilde{V}_h^1)'} + \inf\limits_{\substack{\bar{p}_h \in K_h^0 \\ d^0 \bar{p}_h \in M_h^1}} \|p - \bar{p}_h\|_{V^0} \end{cases}$$

with constants independent of h.

Remark 5.8. As in Remark 3.15 we can eliminate f from these estimates.

Proof. We combine the proofs of Theorems 3.14 and 5.5. In particular, we write

$$\begin{aligned}\langle \tilde{d}^1_h(\tilde{u}_h - \bar{u}_h), \tilde{d}^1_h \tilde{v}_h\rangle &= \langle \tilde{\pi}^1_h f, \tilde{v}_h\rangle - \langle d^1 \bar{u}_h, \tilde{d}^1_h v_h\rangle \\ &= \langle \tilde{\pi}^1_h f^K, \tilde{v}_h\rangle - \langle d^1 \bar{u}_h, \tilde{d}^1_h v_h\rangle \\ &= \langle (\tilde{\pi}^1_h - (\mathcal{P}^1_h)^*) f^K, \tilde{v}_h\rangle + \langle f^K, \mathcal{P}^1_h \tilde{v}_h\rangle - \langle d^1 \bar{u}_h, \tilde{d}^1_h \tilde{v}_h\rangle \\ &= \langle (\tilde{\pi}^1_h - (\mathcal{P}^1_h)^*) f^K, \tilde{v}_h\rangle + \langle d^1 (u - \bar{u}_h), \tilde{d}^1_h \tilde{v}_h\rangle\end{aligned}$$

for $\tilde{v}_h \in \tilde{K}^1_h$, and for $\tilde{w}_h \in \tilde{Z}^1_h$ we compute

$$\begin{aligned}\langle \tilde{w}_h, \tilde{d}^0_h(\tilde{p}_h - (\bar{p}_h, 0))\rangle &= \langle \tilde{\pi}^1_h f, \tilde{w}_h\rangle - \langle \tilde{w}_h, d^0 \bar{p}_h\rangle \\ &= \langle (\tilde{\pi}^1_h - (\mathcal{P}^1_h)^*) f, \tilde{w}_h\rangle + \langle f, \mathcal{P}^1_h \tilde{w}_h\rangle - \langle \tilde{w}_h, d^0 \bar{p}_h\rangle \\ &= \langle (\tilde{\pi}^1_h - (\mathcal{P}^1_h)^*) f, \tilde{w}_h\rangle + \langle \mathcal{P}^1_h \tilde{w}_h, d^0 (p - \bar{p}_h)\rangle.\end{aligned}$$

The end of the proof is the same than for Theorem 5.5. ∎

5.3.2. K-compatible nonconforming approximation of the second problem. For the second problem we consider: find $\tilde{u}_h \in \tilde{V}^1_h$ and $\zeta_h \in Z^2_h$ such that

$$\begin{cases} \langle \tilde{u}_h, \tilde{v}_h\rangle + \langle \tilde{d}^1_h \tilde{v}_h, \zeta_h\rangle = \langle \tilde{\pi}^1_h f, \tilde{v}_h\rangle & \tilde{v}_h \in \tilde{V}^1_h \\ \langle \tilde{d}^1_h \tilde{u}_h, \xi_h\rangle = \langle \ell, \xi_h\rangle & \xi_h \in Z^2_h. \end{cases} \tag{96}$$

As in the conforming case, it has the effect that u^Z is no longer involved in the error estimate for ζ, see Theorem 3.16.

THEOREM 5.9. *If $\tilde{\pi}^1_h : D^1 \to \tilde{V}^1_h$ is a K-compatible operator in the sense of* (78), *then for $f \in D^1$ the solution to* (96) *satisfies the error estimate*

$$\begin{cases} \|u - \tilde{u}_h\|_{\tilde{V}^1_h} \lesssim \|(\tilde{\pi}^1_h - (\mathcal{P}^1_h)^*) f\|_{(\tilde{V}^1_h)'} + \inf\limits_{\bar{u}_h \in V^1_h \cap M^1_h} \|u - \bar{u}_h\|_{V^1}. \\ \|\zeta - \zeta_h\| \lesssim \|(\tilde{\pi}^1_h - (\mathcal{P}^1_h)^*) f^K\|_{(\tilde{V}^1_h)'} + \inf\limits_{\bar{u}_h \in V^1_h \cap M^1_h} \|u^K - \bar{u}_h\|_{V^1} + \inf\limits_{\bar{\zeta}_h \in Z^2_h} \|\zeta - \bar{\zeta}_h\|. \end{cases}$$

Remark 5.10. As in Remark 3.17 we can eliminate f from these estimates.

Proof. Here we combine the proofs of Theorems 3.16 and 5.6. Thus, we write

$$\langle \tilde{u}_h - \bar{u}_h, \tilde{w}_h\rangle = \langle \tilde{\pi}^1_h f, \tilde{w}_h\rangle - \langle \bar{u}_h, \tilde{w}_h\rangle = \langle (\tilde{\pi}^1_h - (\mathcal{P}^1_h)^*) f, \tilde{w}_h\rangle + \langle u - \bar{u}_h, \mathcal{P}^1_h \tilde{w}_h\rangle$$

for $\tilde{w}_h \in \tilde{Z}^1_h$. For $\xi_h \in Z^2_h$ the relation is unchanged and for $\tilde{v}_h \in \tilde{K}^1_h$ we compute

$$\begin{aligned}\langle \tilde{u}_h - \bar{u}_h, \tilde{v}_h\rangle + \langle \tilde{d}^1_h \tilde{v}_h, \zeta_h - \bar{\zeta}_h\rangle &= \langle \tilde{\pi}^1_h f, \tilde{v}_h\rangle - \langle \bar{u}_h, \tilde{v}_h\rangle - \langle \tilde{d}^1_h \tilde{v}_h, \bar{\zeta}_h\rangle \\ &= \langle \tilde{\pi}^1_h f^K, \tilde{v}_h\rangle - \langle \bar{u}_h, \mathcal{P}^1_h \tilde{v}_h\rangle - \langle \tilde{d}^1_h \tilde{v}_h, \bar{\zeta}_h\rangle \\ &= \langle (\tilde{\pi}^1_h - (\mathcal{P}^1_h)^*) f^K, \tilde{v}_h\rangle + \langle u^K - \bar{u}_h, \mathcal{P}^1_h \tilde{v}_h\rangle \\ &\qquad + \langle \tilde{d}^1_h \tilde{v}_h, \zeta - \bar{\zeta}_h\rangle\end{aligned}$$

The proof ends like the one of Theorem 5.6. ∎

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