Stabilization of a class of underactuated vehicles with uncertain position measurements and application to visual servoing
Henry de Plinval, Pascal Morin, Philippe Mouyon

To cite this version:
Henry de Plinval, Pascal Morin, Philippe Mouyon. Stabilization of a class of underactuated vehicles with uncertain position measurements and application to visual servoing. Automatica, Elsevier, 2017, 77, pp.155 - 169. 10.1016/j.automatica.2016.11.012. hal-01475915

HAL Id: hal-01475915
https://hal.sorbonne-universite.fr/hal-01475915
Submitted on 24 Feb 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Stabilization of a Class of Underactuated Vehicles with Uncertain Position Measurements and Application to Visual Servoing

H. de Plinval\textsuperscript{a}, P. Morin\textsuperscript{b}, P. Mouyon\textsuperscript{a},

\textsuperscript{a}ONERA - The French Aerospace Lab, Toulouse, France. Henry.de_Plinval@onera.fr, Philippe.Mouyon@onera.fr
\textsuperscript{b}ISIR, Sorbonne Universités, UPMC Univ. Paris 06, CNRS UMR 7222, Paris, France. Corresponding author: morin@isir.upmc.fr

Abstract
Stabilization of a class of underactuated vehicles with uncertain measurements of the position tracking error is addressed. Nonlinear feedback laws ensuring semi-global stability for a large class of uncertainties on these measurements are derived based on properties of saturated controls. Practical relevance of the proposed results is illustrated by two application examples for Vertical Take-Off and Landing aerial vehicles equipped with a mono-camera sensor: point stabilization in front of a planar target and visual way-points navigation based on interpolation of homography measures.

Keywords: Underactuated vehicle, aerial vehicle, stability, robustness, measurement uncertainties

1. Introduction
Underactuated vehicles have long been a source of inspiration for nonlinear control theory. Recent applications with aerial or underwater vehicles have renewed the interest on this topic. This study is motivated by applications with VTOL UAVs (i.e. Vertical Take-Off and Landing Unmanned Aerial Vehicles) but it is relevant to any underactuated vehicle that can be modeled as a rigid body with a body fixed thrust control force and full torque control (so-called "thrust propelled vehicles" Hua et al. (2009)). Stabilization of the vehicle’s pose (i.e. position and orientation) is an important issue in this context. Large stability domains are needed for small vehicles due to their sensitivity to perturbations (wind, sea currents, etc), and several nonlinear control designs have been proposed to address this issue (see, e.g., Hauser et al. (1992); Isidori et al. (2003); Pflimlin et al. (2007); Hua et al. (2009)). Good robustness properties of the closed-loop system is at least as important in practice. Robustness to external perturbations (e.g. wind effects for aerial vehicles or currents for underwater vehicles) has been addressed in Pflimlin et al. (2007); Marconi and Naldi (2007); Aguiar and Pascoal (2007); Hua et al. (2009). Robustness to parameter uncertainties (mass, inertia, etc) has been considered e.g. in Aguiar and Pascoal (2007). Robustness to input disturbances has been considered in Aguiar et al. (2007). This paper concerns robustness w.r.t. (with respect to) uncertainties on the measurement model.

Motion capture systems provide high-quality pose measurements that can yield impressive performance for small aerial vehicles Lupashin et al. (2010); Mellinger et al. (2012). Alternatively, ground based passive visual markers have been used to estimate onboard the pose as in Masselli and Zell (2012). For most applications, however, motion capture systems or ground based localization systems cannot be used and localization must rely exclusively on embarked sensors. Good orientation estimates can be obtained with embarked IMUs (Inertial Measurement Units). Estimation of the vehicle’s position/velocity is more challenging. GPS may be used to this purpose but it is not always available. Furthermore, in many applications (e.g., inspection) a measurement of the relative position of the vehicle w.r.t. its environment is needed, rather than an absolute position measurement (GPS-like). The former is best obtained from embarked exteroceptive sensors (cameras, lasers, etc). With such sensors, however, the relation between the output function (i.e. measurement) and the relative position error is seldom known precisely. Uncertainties may come from calibration errors, lack of depth information with mono-camera sensors, uncertainties on the environment structure, etc... This leads to the control problem addressed in this paper: Given a class of uncertainties on the position measurements, can one design feedback control laws that guarantee stability of the system for any position measurement in this class?

To our knowledge this problem has only been addressed in very specific cases, like when uncertainties reduce to a positive scale factor on the position vector Metni et al. (2004); Le Bras et al. (2010). The results here proposed address a much larger class of uncertainties. They make use of properties of saturated controls. There is a large litterature on this topic, especially for linear systems (see, e.g., Teel (1991); He et al. (2005)). In those works, it is assumed that the system’s dynamics and state are perfectly known. Saturated controls have also been used for UAVs in order to ensure some type of robustness property, e.g., Teel (1996) and López-Araujo et al. (2010) for robustness w.r.t. input saturation, Marconi and Naldi (2007), Hua et al. (2009), and López-Araujo et al. (2010) for robustness w.r.t. unmodelled perturbations.

\textsuperscript{a}This work has been supported by the "Chaire d’excellence en Robotique RTE-UPMC" and the ANR SCAR Project.
dynamics. We show that saturated controls can be instrument-
mental in ensuring robustness properties w.r.t. measurement errors.
More precisely, we propose nonlinear feedback laws that ensure
semi-global stability of the pose tracking error for a large class of
uncertainties on the position measurement. These results are
reminiscent of Robust Stability results for linear systems (Doyle
et al., 1992, Ch. 4), where the objective is to guarantee stability
for a family of plants that satisfy some uncertainty bound w.r.t.
a nominal system. Two application scenarios are addressed. In
the first one we assume that velocity measurements are avail-
able, e.g., a GPS provides these measurements and an extero-
ceptive sensor (e.g., camera) provides position measurements
w.r.t. the environment. In the second scenario no velocity mea-
surement is available, i.e., "GPS-denied" environment. Note
that position control of UAVs without velocity measurements
has already been addressed Abdessameud and Tayebi (2010),
but position was assumed to be perfectly known. Finally, let us
remark that a preliminary version of this paper was presented
in de Pinval et al. (2012).

The results here proposed are illustrated by visual servo-
ing applications with mono-camera measurements. This type of
application has been considered in several works (see, e.g.,
Pebrianti et al. (2010); Saripalli et al. (2003) and additional ref-
erences in Section 6) usually with a camera pointing downward
and observing a flat and horizontal ground, and under the assump-
tion that altitude is measured independently. Our results
provide stability guarantees for much more general application
scenarios.

The paper is organized as follows. Background and prob-
lem statement are presented in Section 2. A preliminary result
is provided in Section 3 for the fully actuated case. The main re-
sults on the underactuated case are provided in Sections 4 and
5: in Section 4 we assume that linear velocity measurements
are available; in Section 5 such measurements are not available.
Application to visual servoing is considered in Section 6, with
simulation results given in Section 7. Proofs are given in the
Appendix.

2. Background and problem statement

The n × n identity matrix is denoted as In. The transpose
of a matrix M is denoted as Mt. For any square matrix M, M1 :=
\frac{M + Mt}{2} and M2 := \frac{M - Mt}{2} respectively denote the symmet-
tric and antisymmetric part of M. The maximum singular value
of a matrix M is denoted as \|M\| and when M is a matrix-valued
time-function, \|M(t)\| := sup |M(t)|. Given a matrix-valued time-
function M : t \rightarrow M(t) \in \mathbb{R}^{m \times n} with M(t) \geq 0 \forall t, we define
\|M\| := sup_{t \in [a,b]} \|M(t)\|, \alpha. Note that \|M\| = \|M^t\|. Given a smooth
function f defined on an open set of \mathbb{R}, its derivative is den-
ted as f’. Throughout the paper, AS, GAS, and LES stand
for Asymptotically Stable, Globally Asymptotically Stable, and
Locally Exponentially Stable respectively. CD stands for Con-
vergence Domain.

Definition 1 Given \delta := [\delta_m; \delta_M] with 0 < \delta_m < \delta_M, sat\delta : \mathbb{R}^n \rightarrow \mathbb{R}^n is called a saturation function if:

i) There exists a class C^1 function \sigma_\delta : \mathbb{R}^n \rightarrow \mathbb{R}^n such that sat\delta(x) = \sigma_\delta(|x|^2)x for all x \in \mathbb{R}^n;
ii) The function defined on \mathbb{R}^+ by \tau \rightarrow \sigma_\delta(\tau^2) is non-
decreasing, upper-bounded by \delta_M, and is equal to the identity
function on [0, \delta_m];
iii) \sigma_\delta(\tau) \leq 0 for all \tau.

From i) \sigma_\delta is fully defined from the associated function \sigma_\delta. From i)–ii) saturation functions in the sense of Def. 1 inherit
the classical properties of a saturation function: \sigma_\delta is upper-
bounded in norm by \delta_M and \sigma_\delta(x) = x for |x| \leq \delta_m (because
\sigma_\delta(\tau^2) = \tau for \tau \in [0, \delta_m]). Also,

\tau \sigma_\delta(\tau) \leq 1, \forall \tau \in \mathbb{R}^+ \quad \tau \rightarrow +\infty \quad \text{when} \quad \tau \rightarrow +\infty \quad (1)

where the first relation follows from ii) and iii) and the second relation from i). Then, ii) implies that the derivative of the function
\tau \rightarrow \sigma_\delta(\tau^2) is non-negative. This property and iii)
implies that 2|\sigma_\delta(\tau)| \leq \sigma_\delta(\tau) \forall \tau \in \mathbb{R}^+. Thus, from (1),

C_\delta := sup_{\tau \in \mathbb{R}^+} (\sigma_\delta(\tau) + 2|\sigma_\delta(\tau)|) \leq 2 < +\infty \quad (2)

An example of a function \sigma_\delta is given by

\sigma_\delta(\tau) := \begin{cases} 1 & \text{if } \tau \leq \delta_m \\ \left(\frac{\tau}{\delta_m}\right)^{\alpha} & \text{if } \tau > \delta_m^2 \end{cases} \quad (3)

2.1. Dynamics of thrust-propelled underactuated vehicles

We focus on the class of so-called "thrust-propelled under-
actuated vehicles" Hua et al. (2009), i.e., rigid bodies moving
in 3D-space under the action of one body-fixed force control
and full torque control. This class contains most VTOL UAVs
(quadrors, ducted-fans, etc). The dynamics of these systems
is described by the following equations, expressed in a "North-
East-Down" (NED) frame:

\begin{align*}
\dot{p} &= -uRe_3 + ge_3 \\
\dot{R} &= RS(\omega) \\
\dot{\omega} &= J_\omega \times \omega + \Gamma
\end{align*} \quad (4)

with p the position vector of the vehicle’s center of mass, ex-
pressed in a reference (inertial) frame, R the rotation matrix
from the body frame to the reference frame, ω the angular ve-
locity vector expressed in the body frame, S(·) the matrix-valued
function associated with the cross product, i.e. S(x) = x × y,
∀x, y ∈ \mathbb{R}^3, u the normalized thrust input, i.e. u = \frac{u}{m}
where m is the mass and T the thrust input, e3 = (0, 0, 1)^T, J the inertia
matrix, Γ the torque vector, and g the gravity constant. In this
paper we mainly focus on the system

\begin{align*}
\dot{p} &= -uRe_3 + ge_3 \\
\dot{R} &= RS(\omega)
\end{align*} \quad (5)

with u and ω as control inputs, i.e., considering ω instead of Γ as
orientation control. Extension of the paper’s results to System
(4) is discussed in Section 4.
2.2. Problem statement

The main objective of this paper is to investigate the stabilization of $p$ to a reference trajectory $p_r$ from some relative position measurements of the following form:

$$\hat{\sigma}(t) := R(t)^T M(t) \hat{p}(t)$$  \hspace{1cm} (6)

with $M(t)$ an unknown matrix and $\hat{p} := p - p_r$ the position error. In other words $\hat{\sigma}$ provides information on the position error in body frame and $M(t)$ accounts for measurements uncertainties. Measurements $\hat{\sigma}$ are typically obtained from embarked exterceptive sensors (cameras, lasers, etc). Examples are provided in Section 6. Due to the system’s underactuation, stabilization of $p$ to $p_r$ fixes two degrees of freedom on the vehicle’s orientation. There remains a degree of freedom (yaw angle). Its control is not addressed here since it does not affect the position control.

Let us introduce some assumptions. The first one is made throughout the paper. The other ones concern particular cases.

**A1:** $M(t)$ is an unknown matrix and $\hat{p} := p - p_r$ the position error. In other words $\hat{\sigma}$ provides information on the position error in body frame and $M(t)$ accounts for measurements uncertainties. Measurements $\hat{\sigma}$ are typically obtained from embarked exterceptive sensors (cameras, lasers, etc). Examples are provided in Section 6. Due to the system’s underactuation, stabilization of $p$ to $p_r$ fixes two degrees of freedom on the vehicle’s orientation. There remains a degree of freedom (yaw angle). Its control is not addressed here since it does not affect the position control.

Let us introduce some assumptions. The first one is made throughout the paper. The other ones concern particular cases.

**A2:** $M$ is a constant function.

**A3:** $\dot{p}_r = 0$ and $|\dot{p}_r| < g$.

**A4:** $p_r$ is a constant vector and $M_r$ and $M_o$ commute.

3. A preliminary result

Before addressing the control of underactuated systems we consider a fully actuated system modelled by:

$$\dot{p} = -Ru$$  \hspace{1cm} (8)

where $u \in \mathbb{R}^3$ denotes the body-fixed thrust input and $R$ satisfies the third kinematic relation in (5). We assume that the following measurements are available:

$$\tilde{\sigma}, \tilde{v} := R^T \dot{p}$$  \hspace{1cm} (9)

with $\tilde{\sigma}$ defined by (6) and $\tilde{v}$ the linear velocity in body frame. A reference trajectory is defined by:

$$p_r, v_r := R^T \hat{p}_r, a_r := R^T \dot{\hat{p}}_r$$  \hspace{1cm} (10)

and the associated tracking error by:

$$\ddot{\hat{p}} := p - p_r, \ddot{v} := v - v_r$$  \hspace{1cm} (11)

**Proposition 1** Let $\text{sats}_g, \overline{\text{sat}}_g$ denote two $\mathbb{R}^3$-valued saturation functions with associated functions $s_\delta, \overline{s}_\delta$. Consider control gains $k_1, k_2 > 0$ such that

$$\begin{cases} 
k_1 |M_{ii}| > k_1 |M_{i3}|, \text{max} \{C_\delta |M_{13}|, |M_{12}|\} 
k_2 \overline{s}_\delta > k_1 s_\delta \end{cases}$$  \hspace{1cm} (12)

and define the control law

$$u := k_1 \text{sats}_g(\tilde{\sigma}) + k_2 \overline{\text{sat}}_g(\tilde{v}) - a_r$$  \hspace{1cm} (13)

Then,

i) if **A1** and **A2** hold, then $(p, \dot{p}) = (p_r, \dot{p}_r)$ is a (uniformly) GAS and LES equilibrium trajectory for the closed-loop system (8)-(13).

ii) if **A1** holds then, for any $\rho > 0$, there exists $\sigma > 0$ such that, for any $M$ with $|M_{ii}| < \sigma$, $(p, \dot{p}) = (p_r, \dot{p}_r)$ is a (uniformly) AS and LES equilibrium trajectory for the closed-loop system (8)-(13) with CD containing $\mathcal{A}_r := \{(p, \dot{p}(0) : |(\dot{p}(0), \tilde{v}(0))| \leq \rho\}$, with $|(\tilde{p}, \tilde{v})| := \sqrt{\tilde{p}^2 + \tilde{v}^2}$.

Proposition 1 provides bounded feedback laws that ensure global (or semi-global) stability in the presence of uncertain measurements for System (8). Condition (12) can be used to specify admissible control gains given upper bounds on the uncertainties (the norm of $M$ and of its skew-symmetric part $M_o$). This kind of result is very similar to classical Robust Stability theory for linear systems (Doyle et al., 1992, Ch. 4), Doyle and Stein (1981), Chen and Desoer (1982), where the objective is to guarantee stability for a set of plants that satisfy some uncertainty bound w.r.t. a nominal system. In our case, uncertainty corresponds to the difference between $M$ and the identity matrix. If the uncertainty is small (which implies in particular that $M_o$ is close to the zero matrix and $|M_{i3}| \approx 1$), then Condition (12) puts little constraints on the control gains. If $M_o$ is large, however, large values of $k_2$ are needed. In summary, Proposition 1 can provide stability guarantees given an a priori bound on uncertainties. Note also that when $M$ is constant, global asymptotic stability can be obtained while only semi-global asymptotic stability is obtained when $M$ varies with time (Case ii)).

4. Underactuated case with velocity measurements

Let us consider the control system (5), and assume that the following measurements are available:

$$\tilde{\sigma}, \tilde{v} := gR^T e_1, \nu, \omega$$  \hspace{1cm} (14)

Compared to the fully actuated case, there are two additional measurements, i.e. $\gamma$ and $\omega$. The latter is typically obtained from the gyrometers of an IMU, while the former is obtained by fusing accelerometer and gyrometer measurements (see, e.g. Mahony et al. (2012)). Let $q := R e_3$ denote the thrust direction, so that the first equality in (5) can also be written as

$$\ddot{\hat{p}} = -u q + g e_3 = R(-u e_3 + \gamma)$$  \hspace{1cm} (15)

If **A3** is satisfied then, along the reference position trajectory $p_r$, the thrust direction is well defined (this is no longer true if $\dot{p}_r = g e_3$ since any thrust direction $q$ is solution to (15) for $u = 0$). More precisely, assuming that $u$ is positive, this reference thrust direction is

$$q_r := \frac{g e_3 - \dot{p}_r}{|g e_3 - \dot{p}_r|}$$  \hspace{1cm} (16)

**Proposition 2** Let $\text{sats}_\delta, \overline{\text{sat}}_\delta$ denote two saturation functions. Consider control gains $k_1, k_2 > 0$ satisfying (12) with $C_\delta$ defined by (2), and the additional condition

$$k_1 \delta_M + k_2 \overline{\delta}_M + |\dot{p}_r| < g$$  \hspace{1cm} (17)
Define a dynamic augmentation

\[ \dot{\eta} = \eta \times \omega - k_3(\eta - \ddot{\varphi}), \quad k_3 > 0 \]  

(18)

together with the control law

\[
\begin{align*}
\omega_1 &= -\frac{k_p}{\mu p^2} - \frac{1}{\mu p} S_1 e_1 \\
\omega_2 &= \frac{q_p}{\mu p^2} - \frac{1}{\mu p} S_2 e_2 \\
u &= \mu_3
\end{align*}
\]

(19)

with \( k_4 > 0 \) and \( \mu \) and \( \varphi \) defined by:

\[
\begin{align*}
\mu &:= \gamma + k_3 \sigma \left( \eta + k_5 \sigma \bar{\varphi} - a_r \right) \\
\varphi &:= -k_3 \left[ \sigma \left( \gamma \right) I_3 + 2 \sigma \left( \gamma \right) S_1 \right] \left( \eta - \ddot{\varphi} \right)
\end{align*}
\]

(20)

with \( \gamma \) and \( a_r \) defined by (10) and (11). Then,

i) if \( A_1, A_2, \) and \( A_3 \) hold then, there exists \( k_3 > 0 \) such that, for any \( k_1 > k_3, (p, \varrho, q, \eta) = (p_r, \varpi, q_r, 0) \) is a (uniformly) AS and LES equilibrium trajectory for the closed-loop system (5)-(18)-(20) with CD containing

\[ \mathcal{A}_1 := \{(p, \varrho, q, \eta, 0) : \mu(0) < -|\mu(0)| \} \]  

(21)

ii) if \( A_1 \) and \( A_3 \) hold then, for any \( \varrho > 0 \), there exist \( \vartheta, k_3 > 0 \) such that, for any \( k_3 > k_3 \) and any \( M \) with \( |M|_i < \vartheta, (p, \varrho, q, \eta) = (p_r, \varpi, q_r, 0) \) is an AS and LES equilibrium trajectory for the closed-loop system (5)-(18)-(20) with CD containing

\[ \mathcal{A}_2 := \{(p, \varrho, q, \eta, 0) : |\varrho(0)| < \varrho, |\varpi(0)| < \varpi, |q(0)| < q \} \]  

(22)

Let us discuss the links between this result and Proposition 1. First, except for the \( \gamma \) term, \( \mu \) in (20) is reminiscent of the control law (13) with \( \varphi \) replaced by \( \eta \). In view of (18), this latter variable can be viewed as a "filtered value" of \( \varphi \). The important point is that \( \eta \) is known, since it is explicitly given by (18), while the time derivative of \( \varphi \) is not, since \( \varphi \) is unknown. Then, the control inputs \( u, \omega_1, \omega_2 \) are defined so that \( \mu \) converges to \( |\mu|e_3 \). This implies, using the second equality in (15), that \( \rho \) converges to \( R(\gamma, \bar{\varpi}, \bar{\varphi}) \). This expression is the same as (8)-(13) with \( \ddot{\varphi} \) being replaced by \( \eta \). This explains the relation between Propositions 1 and 2. Finally,

\[
\begin{align*}
|k_1 \sigma \left( \gamma \right) I_3 + k_5 \sigma \bar{\varphi} - a_r | &\leq k_1 \delta M + k_2 \delta \bar{M} + |\varpi_3| < g = |\gamma|
\end{align*}
\]

(23)

where the second inequality comes from (17). This inequality implies that:

**Lemma 1** \( \mu(0) < -|\mu(0)|e_3 \)

\[
\gamma_1(0) > -\sqrt{g^2 - \left(k_1 \delta M + k_2 \delta \bar{M} + |\varpi_3| \right)^2}
\]

(24)

Since \( \gamma_1 = q g \), Lemma 1 implies that in both cases i) and ii) the CD in roll/pitch contains the upper hemisphere. Note that Condition (24) is conservative. Thus, in both cases i) and ii), a large stability domain in orientation is guaranteed. Global stability is ruled out because \( q \) belongs to a compact set (i.e., the unit sphere). If (24) is satisfied, there is no constraint on the initial values of position, linear velocity, and dynamic augmentation variables in case i) (see (21)). In case ii), initial values can be made arbitrarily large under conditions on \( k_3 \) and \( M \).

4.1. Simplified control law

Another control expression with similar robustness properties is proposed next. It involves a simpler control expression and does not require the dynamic extension (18).

**Proposition 3** With the Notation of Prop. 2, assume that the following extra condition on the gains \( k_1, k_2 \) is satisfied:

\[
k_1 \delta M + k_2 \delta \bar{M} + |\varpi_3| \leq g(1 - \epsilon), \quad 0 < \epsilon < 1
\]

(25)

and define the control law as:

\[
\begin{align*}
\omega_1 &= -k_4 \mu_2, \quad \omega_2 = k_4 \mu_1 \\
u &= \mu_3
\end{align*}
\]

(26)

with

\[
\mu := \gamma + k_4 \sigma \left( \bar{\varphi} \right) + k_2 \delta \bar{M} \left( \bar{\varphi} \right) - a_r
\]

(27)

Then,

i) if \( A_1, A_2, \) and \( A_3 \) hold then, for any \( \varrho > 0 \), there exists \( k_4 > 0 \) such that, for any \( k_3 > k_4, (p, \varrho, q, \eta) = (p_r, \varpi, q_r, 0) \) is an (uniformly) AS and LES equilibrium trajectory for the closed-loop system (5)-(26)-(27), with CD containing

\[ \mathcal{A}_3 := \{(p, \varrho, q, \eta, 0) : |\varrho(0)| < \varrho, |\varpi(0)| < \varpi, |q(0)| < q \} \]

(28)

ii) If \( A_1 \) and \( A_3 \) hold then, for any \( \varrho > 0 \), there exist \( \vartheta, k_3 > 0 \) such that, for any \( k_3 > k_3 \) and any \( M \) with \( |M|_i < \vartheta, (p, \varrho, q, \eta) = (p_r, \varpi, q_r, 0) \) is an AS and LES equilibrium trajectory for the closed-loop system (5)-(26)-(27) with CD containing

\[ \mathcal{A}_4 := \{(p, \varrho, q, \eta, 0) : |\varrho(0)| < \varrho, |\varpi(0)| < \varpi, |q(0)| < q \} \]

(29)

The main assets of Proposition 3 are a large stability domain, robustness to position measurement uncertainties, and the simplicity of the control expression. Concerning the latter aspect, the fact that the control expression is essentially linear (modulo saturation functions) is clearly an asset with respect to the control law of Proposition 2, e.g., when considering effects of measurement noise. Another asset is related to the extension of the present analysis to the full model (4) (i.e., considering \( \Gamma \) as control input instead of \( \omega \)). A classical solution in this case would be a linear torque feedback with feedforward action. Computing the feedforward action requires to differentiate angular velocity inputs proposed above. Differentiating \( \omega_1, \omega_2 \) in (26) is much simpler than for (19) and requires much less information. In addition, one may want in this case to replace \( \ddot{\varphi} \) by \( \eta \) given by (18) since \( \eta \) is known. Additional work is needed for the stability analysis of such a torque control law and this issue is left for future research.

Another common approach to extend the controller from kinematics to dynamics is to use a high gain controller: \( \Gamma = -J_\omega \times \omega - kJ (\omega - \omega^0) \) with \( k \) chosen large enough and \( \omega^0 \) the kinematic controller Hua et al. (2009); Brescianini and D’Andrea (2016). This simple solution is motivated by a time separation argument. To the authors’ knowledge its stability analysis remains an open issue.
5. Extensions to GPS-denied environments

This section considers extension of the results of Section 4 to velocity-free scenarios, i.e., when velocity \( v \) is not measured. This is a challenging problem and we only consider a special case of the general framework addressed in Section 4. We will show in the application section, however, that this special case shows important scenarios.

Let \( v_M := R^T M \dot{p} \). Assume that \( M \) and \( p_i \) are constant values. Then, \( v_M = R^T M \dot{p} \) and it follows from (6) that

\[
\begin{align*}
\dot{\hat{v}} &= \hat{v}_M \\
\hat{v}_M &= v_M \times \omega + R^T M \ddot{p}
\end{align*}
\]

We want to obtain an estimation of the non-measured variable \( v_M \). To this purpose, consider the following observer:

\[
\begin{align*}
\dot{\hat{v}} &= \hat{v}_M - k_2 \hat{v} \\
\hat{v}_M &= \hat{v}_M - k_1 \hat{v}
\end{align*}
\] (30)

**Proposition 5** Let \( \varepsilon_p := \hat{\varepsilon} - \varepsilon \) and \( \varepsilon_v := \hat{v}_M - v_M - \varepsilon_v - \varepsilon_v \) denote the estimation errors. Assume that there exists a constant \( C \) such that, for any initial condition, \( |\varepsilon_0| \leq C \). Then, for any \( \varepsilon > 0 \) and any \( \alpha > 0 \) there exists \( k_2 \) such that, for any \( k_1 \) and any initial condition \( (\hat{\varepsilon}(0), \hat{v}(0)) \), \( |\varepsilon_\varepsilon| + |\varepsilon_v| \) is ultimately bounded by \( \varepsilon \).

From Proposition 4 a good estimate of \( v_M \) can be built from the measurement \( \hat{v} \) if \( \hat{p} \) is bounded. Note that the control laws derived in Section 4 ensure the boundedness of \( \hat{p} \). However, for the control laws with \( v_M \) as velocity input, we neglect the discrepancy between \( \hat{v}_M \) and \( v_M \). Stability analysis of the couple controller/observer is left for future studies.

The rest of this section will invoke Assumptions 1, 2, and 4. Note that, when \( p_i \) is a constant vector, the control law \( u \) in (13) can be written as \( u := k_1 \hat{\varepsilon}_p \hat{\varepsilon} + k_2 \hat{\varepsilon}_v \hat{v} \). The following propositions show that the results of Section 4 can be extended to the case of velocity measurements \( v_M \) with minor modifications. Due to space limitations, we only address extension of Propositions 1 and 3.

**Proposition 6** Let \( \text{sat}_A \), \( \text{sat}_B \) denote two \( \mathbb{R}^3 \)-valued saturation functions with associated functions \( s_A, s_B \). Consider control gains \( k_1, k_2 > 0 \) such that

\[
\begin{align*}
&k_1^2 |M|_1 > k_1 |M |_1 |M |_1 |M |_1 \\
&k_2^2 |M|_2 > k_1 |M |_1 |M |_2
\end{align*}
\] (32)

and define the control law

\[
u := k_1 \hat{\varepsilon}_p \hat{\varepsilon} + k_2 \hat{\varepsilon}_v \hat{v} \] (33)

If \( A1, A2 \), and \( A4 \) hold then \((p, \hat{p}) = (p, 0)\) is a (uniformly) GAS and LES equilibrium point for the closed-loop system (8)–(33).
This matrix, which transforms the target’s points coordinates from the reference pose to the current pose, is

\[ H := R^T - \frac{1}{d'} R^T p m'^T \]  

(37)

with \( d' \) the distance from the UAV reference position to the target plane and \( n' \) the normal to this plane expressed in the reference frame. Both \( d' \) and \( n' \) are unknown and thus unavailable for the control design. We show next that for any orientation of the visual target one can extract from \( H \) position measurements of the form (6). The case of a non-vertical target is first briefly described. Then, the case of a vertical target is studied in more details.

6.2. Non-vertical target

In this case, \( n'_3 := n'^T e_3 > 0 \) and it follows from (37) that

\[ He_3 = R^T e'_3 - (n'_3/d')R^T p = \frac{\gamma}{\sqrt{n'_3}} R^T p. \]

As recalled in Section 4, \( \gamma \) is usually estimated from the UAV’s IMU. By substracting \( \frac{\gamma}{\sqrt{n'_3}} R^T p \) to \( He_3 \), one obtains the measurement \( \tilde{\gamma} = \frac{n'_3}{\sqrt{n'_3}} R^T p = R^T M p \) with \( M = \frac{n'_3}{\sqrt{n'_3}} I > 0 \). Since \( M \) is constant and diagonal, it satisfies all the conditions in Assumptions 1, 2, and 4. All the results of Sections 4 and 5 apply and yield stability conditions in term of the control gains and the constant number \( n'_3/d' \).

![Figure 1: Problem scheme](image)

6.3. Vertical target, i.e. \( n'_3 = 0 \)

This case is of interest in many inspection applications since many man-made buildings are vertical. Let

\[ \tilde{\sigma} := (He_2) \times (He_3) - He_1, \quad \gamma := gHe_3 \]  

(38)

From (37) and the assumption \( n'_3 = 0 \), one can verify that

\[ \tilde{\sigma} = R^T M(p'_d/d') p, \quad \gamma = gR^T e_3 \]  

(39)

with \( M(\tau) := \tau_1 I + S(\tau_2 e_3) \). Thus, \( \tilde{\sigma} \) and \( \gamma \) satisfy Eq. (6), (14) with \( p_r = 0 \). Note that \( M_r(\tau) = \tau_1 I \) and \( M(\tau) = S(\tau_2 e_3) \) commute for any \( \tau \). Thus, like for a non-vertical target, \( M \) satisfies all the conditions in Assumptions 1, 2, and 4. We detail below application of our results to fixed-point stabilization and way-points navigation.

Vision based point stabilization. Let us first address the stabilization of the UAV at the reference pose. From (39), Proposition 2-ii) applies directly with \( M = M(\tau_3) \) provided the gain conditions (12) and (17) are satisfied. We deduce that the control law (18)-(19) ensures asymptotic stabilization of the reference pose (with global convergence domain in position/velocity) if:

\[ a) \ n'_1, k_1, k_2 > 0 \quad \quad b) \ k_3 \delta M > k_1 \delta M \]
\[ c) \ k_1 \delta M + k_2 \delta M < g \quad \quad d) \ n'_3 d'_3 k_2 > k_1 n'_1 \left[ \left( \frac{n'_2}{n'_1} + \frac{2 n'_1}{3\sqrt{3}} \right) \right] \]

Condition \( n'_3 > 0 \), which ensures that \( M > 0 \), means that the camera is “facing” the target at the reference pose (obvious condition in practice). Given bounds on the uncertain parameters \( d', n' \), i.e., \( d' \in [d'_1, d'_2] \), \( n'_1 \in [n'_1, 1] \). Condition d) can be replaced by: \( n'_3 d'_3 k_2 > k_1 (1 + 2/(3 \sqrt{3})) \). Thus, one obtains stability conditions on the control parameters given bounds on the uncertain parameters \( d', n' \).

Yaw control. The yaw degree of freedom is not involved in the stabilization objective. In practice, it matters to keep the target inside the field of view of the camera. We propose the following yaw control law: \( \omega = k_3 H_3 \). Upon convergence of the position, velocity, roll and pitch errors to zero, the yaw dynamics will be close to \( \dot{\psi} \approx -k_5 \sin \psi \), thus ensuring the convergence of \( \psi \) to zero unless \( \psi(0) = \pi \) (a case contradictory with the visibility assumption).

Visual-based way-points navigation. Consider a sequence of reference images of a planar scene taken from different reference frames (hereafter referred to as way-points). The objective is to make the UAV navigate along this sequence of way-points. Without loss of generality, we consider two way-points. From the two reference images and the current image, one can define two homography matrices, from which are computed two uncertain relative position measurements \( \tilde{\sigma}_1, \tilde{\sigma}_2 \) as defined by Eq. (38). Let \( p_i \) denote the position vector of the current frame w.r.t. the i-th reference frame and \( R_i \) denote the rotation matrix from the current frame to the i-th reference frame. If \( \chi \) (resp. \( \chi'_d \)) denotes the coordinate vector of a point of the scene in the current frame (resp. in the i-th reference frame), then

\[ \chi'_i = R_i \chi + p_i \quad \text{and} \quad \chi'_d = R_d \chi + p_d. \]

The transformation between the two reference frames is defined as \( \chi'_2 = \tilde{R} \chi'_1 + \tilde{p} \) where \( \tilde{R} \) is a constant matrix and \( \tilde{p} \) is a constant vector, and one has \( R_2 = R \tilde{R}_1 \) and \( p_2 = \tilde{R} p_1 + \tilde{p} \). We implicitly define a reference trajectory by considering a time-varying interpolation \( \tilde{\sigma} \) of \( \tilde{\sigma}_1 \) and \( \tilde{\sigma}_2 \) as: \( \tilde{\sigma} := (1 - \lambda(t)) \tilde{\sigma}_1 + \lambda(t) \tilde{\sigma}_2 \) with \( \lambda \) an increasing function ranging over \([0,1]\). Consider the generic case where both parameters \( n', d' \) change between the two reference images. Then, we deduce from the above relations that

\[ \begin{align*}
\tilde{\sigma} & = R^T_2 \tilde{M}(p_1 - p_r) \\
\tilde{M}(t) & := (1 - \lambda(t)) M(p'_d/d') + \lambda(t) R^T \tilde{M}(p'_d/d') \tilde{R} \\
p_r(t) & := -\lambda(t) \tilde{M}(t)^{-1} R^T \tilde{M}(p'_d/d') \tilde{p}
\end{align*} \]  

(40)

Thus, \( \tilde{\sigma} \) is of the form of Eq. (6) with \( M(t) \) replaced by \( \tilde{M}(t) \). Proposition 2-ii) can then be used to ensure semi-global stability of this non-stationary reference trajectory. Note that the
existence of $|\mathcal{M}| > 0, |\mathcal{M}| < +\infty$ and $|\tilde{\mathcal{M}}| < +\infty$ in Assumption A1 for the matrix $\mathcal{M}(\cdot)$ follows from the fact that $M > 0$ and $\dot{M}(t) \in [0, 1]$. Thus, the linear interpolation of $\dot{\sigma}_1$ and $\dot{\sigma}_2$ implicitly defines a reference trajectory with compact image set. Compactness plays a key role in ensuring that Assumption A1 is satisfied.

7. Simulation results

We illustrate the results of this paper for the visual servoing applications of Section 6, in the case of a vertical visual target. We first consider the fixed-point stabilization problem. The initial conditions and scene parameters at the reference pose are:

$$
\begin{align*}
p_0 &= (-5.5m; 1.2m; 1.3m)^T \\
v_0 &= (-2.3m.s^{-1}; -1.6m.s^{-1}; 0.4m.s^{-1})^T \\
\phi_0 &= -0.5^\circ, \theta_0 = 11.4^\circ, \psi_0 = 4.1^\circ \\
n^* &= (0.7; -0.71; 0)^T, \ d^* = 3m
\end{align*}
$$

(41)

For the simulation reported on Fig. 2, the control law of Prop. 2 is used with $\eta(0) = 0$. For the simulation reported on Fig. 3, the simplified controller of Prop. 3 is used. For the simulation reported on Fig. 4, the velocity-free controller of Prop. 6 is used with $v_M$ replaced by $\dot{v}_M$ and $\bar{v}_M$ the output of the observer (31). The control gains are $[k_1, k_2, k_3, k_4, k_5] = [1, 1.5, 1, 1, 1]$ and the saturation functions $\text{sat}_s, \text{sat}_d$ are defined from the expressions (3) of the associated functions $\bar{s}_d, \bar{s}_d$, with $\delta = [9; 1], \delta = [1; 1.1]$. The gains of the observer (31) are $k = 7, \alpha = 0.7$ and $\sigma(0) = \bar{v}_M(0) = 0$. All controllers stabilize the system. Transient behaviors are qualitatively similar but differences can be noticed. With respect to the control law of Prop. 2, the simplified control law of Prop. 3 yields a shorter settling time, less overshoot, and smaller angular velocities. The velocity-free control law of Prop. 6 yields results similar to that of the first simulation but the convergence of the angular velocity to zero is slower. From Fig. 4 one can also conjecture asymptotic stability of the couple observer/controller, i.e., asymptotic convergence of $\dot{v}_M$ to $v_M$ and of the tracking error to zero.

Validation of the visual way-points navigation, with the control law of Prop. 3 and $\sigma$ defined by (40), is presented on Fig. 5. The interpolation function $\lambda$ is defined as: $\lambda(t) = 0$ for $t \leq t_v$, $\lambda(t) = 1$ for $t \geq t_v$ and $\lambda(t) = \frac{t-t_v}{t_v-t_v}$ for $t_v < t < t_v$ with $t_v = 50s, t_e = 80s$. Initial conditions and scene parameters for the first image are still given by (41). Scene parameters for the second image (i.e., second way-point) are $n^* = (0.87, -0.5, 0)^T, d^* = 1/3m$. The position vector between the two way-points is defined by \( \bar{p} = (-5, 10, 20)^T \). Let us notice the smooth transition and small control input values on the transition interval.

Figure 2: Fixed-point stabilization: control law of Proposition 2.

Figure 3: Fixed-point stabilization: control law of Proposition 3.

Figure 4: Fixed-point stabilization: control law of Proposition 6.

Figure 5: Way-points navigation: control law of Proposition 3.
8. Conclusion

We have proposed a feedback control design and robustness analysis for the stabilization of a class of underactuated vehicles with uncertain position measurements. Strong stability results have been obtained for a large class of position measurements uncertainties and sufficient stability conditions on the control gains have been derived in relation with the norm of the uncertainties. We have shown direct applications of these results to UAVs for two visual servoing problems: fixed-point stabilization and visual way-points navigation w.r.t. a planar scene. Simulation results support the proposed analysis.

Proof of Propositions

Proof of Proposition 1: From (6), \( x := R\hat{\nu} = \hat{M} \hat{p} \). From (9)-(11) \( y := R\hat{\nu} = \hat{p} \). Thus, \( \dot{x} = My + \hat{M} \hat{p} \) and, from (8), (11) and (13), \( \dot{y} = -R(k_1\text{sat}_s(\dot{y}) + k_2\text{sat}_s(\dot{y})) \). From Definition 1-i), \( R\text{sat}_s(\xi) = \text{sat}_s(R\xi) \) for any \( \xi \). Therefore, the closed-loop system in \((x, y)\) coordinates is given by:

\[
\begin{align*}
\dot{x} &= My + \hat{M} \hat{p} \\
\dot{y} &= -k_1\text{sat}_s(x) - k_2\text{sat}_s(y)
\end{align*}
\]

(42)

Lemma 2 Let \( y \) denote a solution of the equation

\[
\dot{y} = -k_1\text{sat}_s(x) - k_2\text{sat}_s(y) + q(t)
\]

with \( x \) any time-function and \( q \) a bounded and continuous time-scan function such that, for \( t \geq t_0 \) and \( c_0 = 0 \),

\[
|q(t)| \leq k_1 \delta_{M} - k_1 \delta_{M} - c_0, \quad \forall t \geq t_1
\]

Then, there exists a continuous function \( T \) such that, for \( t \geq T(y(0)) \), \( \text{sat}_s(y(t)) = y(t) \), i.e., the function \( \text{sat}_s \) desaturates after time \( T(y(0)) \).

Property i): Assumption A2 implies that \( M = 0 \). Applying Lemma 2 to the second equation in (42) with \( q \equiv 0 \) and using the second condition in (12), one deduced that along any solution of System (42) the function \( \text{sat}_s \) desaturates after some time and the solution then satisfies:

\[
\begin{align*}
\dot{x} &= My \\
\dot{y} &= -k_1\text{sat}_s(x) - k_2y = -k_1\text{sat}_s(x) - k_2y
\end{align*}
\]

(45)

Consider the CLF (Candidate Lyapunov Function) \( V \) defined by:

\[
V(x, y) = k_1 \int_0^{\text{sat}_s(x)} s_\text{sat}_s(\tau) d\tau + \frac{1}{\varepsilon} k_2\text{sat}_s(x)^2 M_y + k_3\text{sat}_s(x)^2 y
\]

(46)

where \( \kappa \) is a constant positive number. We show that \( V \) is a Lyapunov function for \( \kappa \) small enough. Let \( \kappa \), \( \kappa_1 \), \( \kappa_2 \), \( \kappa_3 \), and \( 0 < \kappa < \text{min}\{\kappa_1, \kappa_2, \kappa_3\} \) where positivity of \( \kappa, \kappa_1 \) follows from (12) and positivity of \( \kappa_2, \kappa_3 \) is a consequence thereof. We first prove that \( V \) is positive definite and proper. Integrating by part and using the fact that \( s_\text{sat}_s(\tau) \leq 0 \) for due to Def. 1-i), we get

\[
\int_0^{\text{sat}_s(x)} s_\text{sat}_s(\tau) d\tau = -\int_0^{\text{sat}_s(x)} s_\text{sat}_s(\tau) d\tau + |x|^2 s_\text{sat}_s(|x|^2)
\]

From (1), \( s_\text{sat}_s(|x|^2) \leq \sqrt{k_2}/(\kappa^2) \). Therefore, from (46),

\[
V \geq k_1|x|^2 s_\text{sat}_s(|x|^2) - \left(\frac{2k_1}{k_2}|M_\text{d}x| + \kappa\right) s_\text{sat}_s(|x|^2) |x| |y| + |M||y|^2
\]

Therefore, \( V \) is positive definite provided that

\[
b^2 < 4k_1|M||y|, \quad b := \frac{2k_1}{k_2}|M_\text{d}x| + \kappa
\]

which is equivalent to \( \kappa < \kappa_1 \). Since \( s_\text{sat}_s(\tau) > 0 \) for \( \tau \neq 0 \) this ensures that \( V \) is a positive definite function of \( x \) and \( y \), and \( V \) is proper due to (1). Let us now prove that \( V \) is non-increasing along the solutions of System (45). Differentiating \( V \) along these solutions yields

\[
\begin{align*}
\dot{V} &= -2k_2 y^T M_y - 2b y^T M^2 F_\delta(x)y + k_3 y^T \text{sat}_s(x)^2 y \\
\dot{F}_\delta(x) &:= s_\text{sat}_s(x)^2 |x|^2 + 2s_\text{sat}_s(x)^2 x^T y
\end{align*}
\]

(49)

By (2), \( |F_\delta(x)| \leq C_b, \forall x \) and we deduce that

\[
V \leq -C_{11}|y|^2 + C_{12}|y||s_\text{sat}_s(x)|^2 - C_{13}|s_\text{sat}_s(x)|^2
\]

(50)

with

\[
C_{1,1} := 2k_2|M| - b|M|C_b, \quad C_{1,2} := k_2, \quad C_{1,3} := \kappa k_1
\]

The right-hand side of (50) is a quadratic form in \(|y|\) and \(|s_\text{sat}_s(x)|\). Therefore, \( V \) is negative definite provided that

\[
a) \quad C_{1,1} > 0, \quad b) \quad C_{1,3} > 0, \quad c) \quad C_{1,2} < 4C_{1,1}C_{1,3}
\]

Condition a) follows from the fact that \( \kappa < \kappa_2 \). Condition b) holds true for any \( \kappa > 0 \) since \( k_1 > 0 \), and Condition c) follows from the fact that \( \kappa < \kappa_3 \). Thus, \( \lambda \beta > 0 \) such that

\[
V \leq -\beta(|y|^2 + |s_\text{sat}_s(x)|^2)
\]

(53)

This shows global asymptotic stability of \((p, \dot{p}) = (p_t, \dot{p}_t)\). Local exponential stability readily follows by noting that both \( V \) and \( \dot{V} \) are locally quadratic in \( x \) and \( y \) around the origin, i.e., from Def. 1-i), \( s_\text{sat}_s(\tau) = 1 \) for \( \tau \leq \delta_\alpha \).

Property ii): Since \(|y| \leq ||y||, (p(0), \dot{p}(0)) \in \text{rel} \) implies that \(|y(0)| \leq \rho \). Applying Lemma 2 to System (42) with \( q \equiv 0 \) and using the continuity property of the function \( T \) in Lemma 2, one deduced that for any \( t \geq T_p := \text{max}_{\text{sat}_s T(y)} T(y) \) and along any solution with initial condition in \( \text{rel} \), \( \text{sat}_s \) desaturates and the solution satisfies

\[
\begin{align*}
\dot{x} &= My + \hat{M} \hat{p} \\
\dot{y} &= -k_1\text{sat}_s(x) - k_2y
\end{align*}
\]

(54)
From (6) and (7),
\[ ∀t, \quad |M|\|\tilde{p}(t)\| ≤ |x(t)| = |M(t)|\|\tilde{p}(t)\| ≤ |M|\|\tilde{p}(t)\| \] (55)

Therefore, \(|(x(0), y(t))| ≤ ρ\max|1, |M|\| for \((p(t), y(t)) \in \mathcal{A}_p\).

Assumption A1, (54), and (55) imply that \(|(x, y)| ≤ k|t,x|\) for some constant \(k\). As a consequence, there exists a constant \(\rho\) such that, along any solution with initial condition in \(\mathcal{A}_p\), \(|x(T_p), y(T_p)| ≤ \rho\).

Consider \(V\) defined by (46) as a CLF for System (54). Its derivative along the solutions of the system satisfies
\[ \dot{V} = V_{\text{casc}+} + 2k_2s_0(|x|)^2x^TM\dot{p} + 2\omega_1s_1(|x|)x^TM\dot{p} + \kappa_0\dot{p}^TM\dot{p} + \kappa_1\dot{p} + \kappa_2\dot{p} \] (66)

with \(\dot{p}\) given by (49) and \(V_{\text{casc}+}\) the expression (49) of \(V\). Using (55) then implies that for any \(M\) with \(|M| < \theta\),
\[ \dot{V} ≤ V_{\text{casc}+} + \theta \left( \frac{2k_2}{\theta} |s_0(|x|)|x| + |y| \right) + \left( \frac{2\omega_1}{\theta} |s_1(|x|)|x| + |y| \right) \] (67)

\[ < -C_{1,1} - \theta |y| - C_{1,3} |s_0(|x|)|x| \] (68)

\[ + \left( C_{1,2} + 2\frac{k_2}{\theta} |s_0(|x|)|x| + \frac{2\omega_1}{\theta} |s_1(|x|)|x| + \frac{C_1}{\theta} \right) |x||y| \] (69)

Let:
\[ V_M := \max(|x|, |y|) \quad \text{and} \quad s_M := \max(|s_0(|x|)|, |s_1(|x|)|) \] (70)

\[ s_m := \min(s_0(|x|), s_1(|x|)) > 0 \] (71)

\(V_M\) exists because \(V\) is continuous and \(V_M > 0\) because \(V\) is positive definite. \(s_m\) exists because \(V\) is radially unbounded and \(s_M > 0\) because \(V_M > 0\). Finally, \(s_m\) exists because \(s_0\) is continuous and \(s_M > 0\) because otherwise \(s_0\) vanishes at some point, which contradicts ii) of Definition 1. From the definition of \(s_m\), note that
\[ |x| ≤ x_M \implies |x| \leq \frac{|s_0(|x|)|x|}{s_0(|x|)} \leq \frac{1}{s_m} |s_0(|x|)|x| \] (72)

Therefore, as long as \(|x| ≤ x_M\), it follows from (57) that
\[ V ≤ -C_{2,1} |y|^2 + C_{2,2} |y| |s_0(|x|)|x| - C_{2,3} |s_0(|x|)|x|^2 \] (73)

with
\[ C_{2,1} := C_{1,1} - \theta \] (74)
\[ C_{2,2} := C_{1,2} + 2k_2 \frac{k_2}{\theta} + \frac{2\omega_1}{\theta} |s_1(|x|)|x| + \kappa_0 \] (75)
\[ C_{2,3} := C_{1,3} - 2k_2 \frac{k_2}{\theta} |s_0(|x|)|x| \] (76)

From the above expression, the \(C_{2,1}'s\) tend to the \(C_{1,1}'s\) as \(\theta\) tends to zero. Thus, it follows from (52) that for \(\theta > 0\) small enough and \(\kappa\) satisfying (47), the right-hand side of (60) is a negative-definite quadratic form in \(|y|\) and \(|s_0(|x|)|x|\). We thus have shown that for \(\theta > 0\) small enough, \(0 \neq |x| ≤ x_M \implies V < 0\). From the definition of \(x_M\), we deduce that \(0 \neq V ≤ V_M \implies V < 0\). From the definition of \(V_M\) and the fact that \(|x(T_p), y(T_p)| ≤ \tilde{p}\) along any solution with initial condition in \(\mathcal{A}_p\), this implies convergence of \((x, y)\) to zero along these solutions. Local exponential stability is proved for Property i).

**Proof of Proposition 2:** It builds on the proof of Prop. 1. Recall from Prop. 1 that \(x = M\dot{p}\) and \(y = \dot{p}\). In addition, let \(z := R\eta\) and \(\tilde{\mu} := \mu - \mu_0\). Using the expression (15) of \(\tilde{\mu}\), the expression of \(\tilde{\mu}(z)\) in (19), and (18), one obtains (compare with (42)):
\[ \begin{cases} \dot{\hat{x}} = My + M\dot{p} \\ \dot{\tilde{\mu}} = -k_1|s_0(z)| - k_2|s_1(z)| + R\tilde{\mu} \end{cases} \] (62)

\[ \begin{cases} z = -k_3(z - x) \end{cases} \] (63)

**Lemma 3** With \(\omega_1, \omega_2\) defined by (19), \(\tilde{\mu} = 0\) is LER. More precisely, \(\exists \omega, \alpha > 0 : |\tilde{\mu}(z)| ≤ \omega|\hat{y}|e^{-\alpha t}\). \(\forall t, \text{for any initial condition such that } \mu(0) \neq -|\mu(0)|e_3\).

**Property i:** Assumption A2 implies that \(\dot{M} = 0\). From Lemma 3 \(\tilde{\mu}\) tends to zero and from (12) \(k_1\delta y - k_2\dot{\delta} y < 0\). Then, Lemma 2 applies to the second equation in (62) and along any solution, \(\tilde{s}_0\) desaturates after some time and the solution then satisfies:
\[ \begin{cases} \dot{x} = My \\ \dot{y} = -k_1|s_0(z)| - k_2|s_3(x)| + R\tilde{\mu} \end{cases} \] (64)

\[ \begin{cases} z = -k_3(z - x) - M\dot{y} \end{cases} \] (65)

We deduce from the mean-value inequality and (2) that \(|s_0(z)| = |s_0(z)| \leq |C_\delta|z \leq |V(z, x)|. Therefore, the time-derivative \(V_1\) of \(V_1\) along the solutions of (63) satisfies:
\[ V_1 ≤ -C_{3,1} |y|^2 + C_{3,2} |z - x|^2 - C_{3,3} |s_0(|x|)|x|^2 \] (66)
\[ + C_{3,4} |z - x|^2 + C_{3,5} |s_0(|x|)|x|^2 \] (67)
\[ + C_{3,6} |z - x|^2 \] (68)

\[ \text{with the } C_{3,1}'s, C_{3,2}, C_{3,3}, C_{3,4}, C_{3,5}, C_{3,6}, \text{and } k_1, k_2 \text{ as given by (51).} \]

Let us first assume that \(\tilde{\mu} = 0\). We claim that \(V_1\) is a Lyapunov function for a proper choice of \(\kappa\) and \(k_3\). Let \(0 < \kappa < \min\{k_1, k_2\} \) with \(k_1, k_2 \) defined by (47). From the proof of Prop. 1, \(\kappa < k_1\) implies that \(V\) is positive definite and proper, so that \(V_1\) is positive definite and proper too. Then, \(0 < \kappa < k_2\) implies that \(C_{3,1} > 0\). Since \(C_{3,2}, C_{3,3} > 0\) and \(V_1\) is a quadratic form in \(|y|, |z - x|, \text{ and } s_0^2(|x|)|x|\), \(V_1\) is negative definite provided that
\[ 4C_{3,1}C_{3,2}C_{3,3} > C_{3,1}C_{3,4}C_{3,5} + C_{3,2}C_{3,5} + C_{3,3}C_{3,6} \] (69)

This condition is satisfied by a proper choice of \(\kappa\) and \(k_3\). Indeed, the only term depending on \(k_3\) in (66) is \(C_{3,2}\). Thus, (66) is satisfied for \(k_3\) large enough provided that \(4C_{3,1}C_{3,2} > C_{3,4} \).
From (65), this is equivalent to $4C_{1.1}C_{1.3} > C_{1.2}^2$. This inequality, which corresponds to Condition $c$ in (52), is satisfied for $\kappa > 0$ small enough.

Let us now take into account the additive perturbation $\tilde{\mu}$. It follows from (65) and (66) that for some $\beta > 0$,

$$V_1 \leq -\beta |y|^2 + |z - x|^2 + |s_{at}(x)|^2 + \frac{1}{\delta} \frac{\partial V}{\partial y} |\tilde{\mu}|$$

(67)

and from (46), there exists $c > 0$ such that

$$\left| \frac{\partial V}{\partial y}(x, y) \right| \leq c \left| y \right| s_{at}(x) \quad \forall (x, y)$$

(68)

Therefore, by the triangular inequality,

$$V_1 \leq -\frac{\beta}{2} |y|^2 + |z - x|^2 + |s_{at}(x)|^2 + \frac{c^2 |\tilde{\mu}|^2}{2\beta}$$

Convergence to zero of $V_1$ then follows from the convergence of $\tilde{\mu}$ to zero. This, together with Lemma 3 implies the convergence of $(p, p^t, q, \eta)$ to $(p_0, p^t_0, q_0, 0)$ from any initial condition in $\mathcal{A}$. Finally, local exponential stability of solutions is equivalent to local exponential stability of $\tilde{\mu} = 0$ (Lemma 3) and the fact that saturation functions are identity functions around the origin (i.e., System (62) is locally linear).

**Property ii:** Lemma 3 still implies $\tilde{\mu}$ exponentially converges to zero. Proceeding as in the proof of Prop. 1-ii), one deduces from Lemma 2 that for any $t \geq T_{\rho} := \max(T_p, T_1)$ and along any solution starting from $\mathcal{A}_p$, sat$\delta$ desaturates and the solution satisfies

$$\begin{cases}
\dot{x} &= My + M\tilde{\mu} \\
\dot{y} &= -k_1s_{at}(x) - k_2y - k_1\left| s_{at}(x) - s_{at}(x) \right| + \tilde{R}\tilde{\mu} \\
\dot{z} &= \dot{x} - \tilde{z} - k_3(z - M\tilde{\mu})
\end{cases}$$

(69)

Note that System (69) is equivalent to System (54) when $z = x$ and $\tilde{\mu} = 0$. Since $\tilde{\mu}$ exponentially converges to zero and $|\tilde{\mu}(0)| \leq |\mu|_0 \leq 2\kappa + k_1\delta_M + k_2\delta_M$, there exists $T_1 > 0$ such that, for any initial condition in $\mathcal{A}_p$,

$$|\tilde{\mu}(t)| \leq \frac{B_\delta M}{\kappa \sqrt{6}} \quad \forall t \geq T_1$$

(70)

with $\beta$ satisfying (67). Thus, for $t \geq T_{\rho} := \max(T_p, T_1)$, both (69) and (70) are satisfied. Proceeding as in the proof of Prop. 1-ii), one also deduces from (55) that for some $\bar{\rho}$ and along any solution with initial condition in $\mathcal{A}_p$, $|x(T_p), y(T_p), (z - x)(T_p)| \leq \rho_1$. Let $\bar{\rho} := \max(\rho_1, \delta_m)$ so that $|x(T_p), y(T_p), (z - x)(T_p)| \leq \bar{\rho}$. We consider again the CLF $V_1$ defined by (64) and define $s_m, x_M, V_M$ as follows (compare with (58)):

$$V_M := \max(1, \kappa > 0) V_1(x, y, z) \leq 0$$

(71)

$$x_M := \max_{x \leq x_M} V_1(x, y, z) \geq 0$$

$$s_m := \min_{x \leq s_m} \left| s_{at}(x) \right| > 0$$

By using (59), one deduces from (69) that for $|x| \leq x_M$ and for $M$ such that $|M|_1 < \bar{\theta}$,

$$V_1 \leq -C_{4.1}|y|^2 - C_{4.2}|z - x|^2 - C_{4.3}s_{at}(x)^2 |x|^2 + C_{4.4}H_{1\rho}(x, y) + C_{4.5}|s_{at}(x)|^2 |x|$$

$$+ C_{4.6}|z - s_{at}(x)|^2 |x| + \frac{\delta}{\delta} \tilde{R}\tilde{\mu}$$

with:

$$\begin{aligned}
C_{4.1} &:= C_{3.1} - \theta, C_{4.2} := C_{3.2} \\
C_{4.3} &:= C_{3.3} - \frac{2k_1}{\delta} \theta, C_{4.4} := C_{3.4} \\
C_{4.5} &:= C_{3.5} + \frac{2k_1}{\delta} x \left| M_{at} + k \frac{C_{3.4}}{\delta} \right| \theta \\
C_{4.6} &:= C_{3.6} + \frac{2k_1}{\delta} x \left| M_{at} + k \frac{C_{3.4}}{\delta} \right| \theta
\end{aligned}$$

Let us choose $k$ and $k_3$ as in the proof of case i) above, so that (66) is satisfied. Since the $C_{4.1}$’s tend to zero as $\theta$ tends to zero, for $\theta > 0$ small enough the following inequality is satisfied (compare with (67)):

$$V_1 \leq -\frac{\beta}{2} |y|^2 + |z - x|^2 + |s_{at}(x)|^2 + \frac{c^2 |\tilde{\mu}|^2}{2\beta}$$

(72)

Therefore, by (68) and the triangular inequality,

$$V_1 \leq -\frac{\beta}{6} |y|^2 + |z - x|^2 + |s_{at}(x)|^2 + \frac{c^2 |\tilde{\mu}|^2}{2\beta}$$

(73)

Recall that this relation is true as long as $|x(t)| \leq x_M$ and $t \geq T_{\rho}$. In particular, it is true at $t = T_{\rho}$ because $|x(T_{\rho}), y(T_{\rho}), (z - x)(T_{\rho})| \leq \bar{\rho}$ (see above) and from (71),

$$|x, y, z - x| \leq \bar{\rho} \Rightarrow V_1(x, y, z) \leq V_M \Rightarrow |x| \leq x_M$$

From (73) and (70), $|x(t)| \leq x_M$ and $t \geq T_{\rho}$ imply that

$$V_1 \leq -\frac{\beta}{6} \left( |y|^2 + |z - x|^2 + |s_{at}(x)|^2 - \frac{\delta^2}{2} \right)$$

(74)

We claim that

$$V_1(x, y, z) = V_M \Rightarrow |x, y, z - x| \geq \delta_m$$

(75)

Indeed, otherwise, from the definition of $V_M$ in (71) and the fact that $\bar{\rho} \geq \delta_m$, on the set $\{x, y, z : |x, y, z - x| \leq \bar{\rho}\}$ $V_1$ reaches its maximum in the interior of this set, which implies that $V_1$ has a critical point. This contradicts (72) that implies $V_1$ is a Lyapunov function for $\tilde{\mu}$.

From (71), $V_1 = V_M$ implies that $|x| \leq x_M$ and thus, (74) holds true. We thus deduce from (74), (75), and the properties of the function $s_{at}$ that for any $t \geq T_{\rho}$, $V_1 = V_M \Rightarrow V_1 \leq -\frac{\beta}{6} \frac{\delta^2}{2} < 0$. Since $V_1 \leq V_M$ at $t = T_{\rho}$, $V_1 \leq V_M$ everafter and thus, $|x| \leq x_M$ everafter. Then, (73) is satisfied for any $t \geq T_{\rho}$ and convergence to zero of $x, y, z$ follows from the convergence of $\tilde{\mu}$ to zero. Local exponential stability is deduced as for Property i).

**Proof of Proposition 3:** Let $x = R\bar{x}, y = R\bar{y}, Y = (\mu_1, \mu_2)^T, \epsilon = 1/\epsilon$. One obtains in closed-loop, after some calculations:

$$\begin{cases}
\dot{x} &= My + M\tilde{\mu} \\
\dot{y} &= -k_1s_{at}(x) - k_2s_{at}(y) + \tilde{R}\tilde{\mu} \\
\dot{\epsilon}Y &= -\mu_1Y + \epsilon c_{at}(t)Y^T + e\tilde{R}_{\delta}(k_3)F_{\delta}(x, y) \dot{z} + k_2F_{\delta}(y)
\end{cases}$$

(76)

with $Y = (Y_2, -Y_1)^T, R_{\delta}^T$ the first two lines of $R^T$, and

$$\begin{cases}
F_{\delta}(x) := s_{at}(x)^2 I + 2s_{at}(x) xx^T \\
F_{\delta}(y) := s_{at}(y)^2 I + 2s_{at}(y) yy^T
\end{cases}$$

(77)

**Property i:** It relies on the following lemma.
Lemma 4 Assume A1, A2, and A3. Then, for any $\rho > 0$ there exists $\varepsilon_\rho, T > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\rho)$, any $t \geq T$, and along any solution of the system with initial condition in $\mathcal{A}_\rho$:

a) $|y(t)| \leq \bar{\delta}_m$ (i.e. the function $\bar{\delta}_m$ desaturates);

b) $\mu_3(t) > gx(t)/\sqrt{2}$.

Lemma 4 implies that by choosing $\varepsilon$ small enough, along any trajectory with initial condition in $\mathcal{A}_\rho$, the function $\bar{\delta}_m$ desaturates after some time, so that the trajectory becomes solution to the system

$$
\begin{align*}
\dot{x} &= My + M\bar{p} \\
\dot{y} &= -k_1\bar{\delta}_m(x) - k_2y + R(Y_1, Y_2, 0)^T \\
\dot{\sigma}Y &= -\mu_3 Y + \varepsilon_0\alpha(t)Y^2 + \varepsilon_1 R_2^T(F_{\alpha}(\hat{x} + k_2\bar{F}_\beta(y))
\end{align*}
$$

(78)

The first two equations of this system correspond to (45), modulo the additional term $R(Y_1, Y_2, 0)^T$. Then, using (53), property b) of Lemma 4, and (68), one deduces that $V_2(x, y, Y) = V(x, y) + |Y|^2$ is a Lyapunov function for System (78) for $\varepsilon$ small enough, with $V$ upper-bounded by a negative definite quadratic function of $s\sigma(x), y,$ and $Y$.

Property ii): It relies on the following lemma.

Lemma 5 Assume A1 and A2. Then, for any $\rho > 0$ there exist $\varepsilon_\rho, T_0, \varepsilon_3, \bar{c}, c, \theta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\rho)$, any $t \geq T_0$, any $M$ such that $|M| < \theta$, and along any solution of the system with initial condition in $\mathcal{A}_\rho$:

a) $|x(t)| \leq \bar{c}_5$;

b) $|y(t)| \leq \bar{\delta}_m$ (i.e. the function $\bar{\delta}_m$ desaturates);

c) $\mu_3(t) > g x(t)/\sqrt{2}$;

d) $\dot{V} \leq -\frac{1}{2}[(y, \sigma(x))]^2 + c|(y, \sigma(x))| |Y|$, with $V$ defined by (46).

Lemma 5 implies that by choosing $\varepsilon$ and $\theta$ small enough, along any trajectory with initial condition in $\mathcal{A}_\rho$, the function $\bar{\delta}_m$ desaturates after some time $T_0$, so that the trajectory becomes solution to the system

$$
\begin{align*}
\dot{x} &= My + M\bar{p} \\
\dot{y} &= -k_1\bar{\delta}_m(x) - k_2y + R(Y_1, Y_2, 0)^T \\
\dot{\sigma}Y &= -\mu_3 Y + \varepsilon_0\alpha(t)Y^2 + \varepsilon_1 R_2^T(k_1F_{\sigma}(\hat{x} + k_2\bar{F}_\beta(y))
\end{align*}
$$

(79)

The first two equations of this system correspond to (54), modulo the term $R(Y_1, Y_2, 0)^T$. Property a) of Lemma 5 implies that $x$ is bounded, so that $|(\dot{x}, \dot{y})|$ is bounded by a linear function of $|y|$, $|\sigma(x)|$, and $|Y|$ (see (99)). From here, Properties c) and d) of Lemma 4 and boundedness of $F_\beta, \bar{F}_\beta$ imply that $V_2(x, y, Y) = V(x, y) + |Y|^2$ is a Lyapunov function for System (79) for $\varepsilon$ small enough.

Proof of Proposition 4: Let $\hat{\varepsilon}_r := R\varepsilon_r - M\bar{p}, \tilde{\varepsilon}_r := R\varepsilon_r - M\bar{p}$. From (30)-(31), $\dot{\varepsilon}_r = \varepsilon_r - 2a_k\hat{\varepsilon}_r$ and $\ddot{\varepsilon}_r = -k_2\dot{\varepsilon}_r - M\bar{p}$. Thus, $\varepsilon_r$ is solution to a Hurwitz-stable second order linear equation with bounded input $u_\theta := -M\bar{p}$, i.e., $\dot{\varepsilon}_r + 2a_k\hat{\varepsilon}_r + k^2\ddot{\varepsilon}_r = u_\theta$. Let $\tau := kt$ and $'$ denote the derivative w.r.t. $\tau$. Then, the previous equation becomes $\ddot{\varepsilon}_r' + 2a_k\dot{\varepsilon}_r' + \ddot{\varepsilon}_r = \frac{u_\theta}{k^2}$. Since $u_\theta$ is bounded, this implies the ultimate boundedness of $|\dot{\varepsilon}_r|$ and $|\ddot{\varepsilon}_r|$ by a value proportional to $1/k^2$. Hence, $|\dot{\varepsilon}_r|$ and $|\ddot{\varepsilon}_r|$ are ultimately bounded by a value proportional to $1/k^2$ and $1/k$ respectively. The same property holds for $|e_r|$ and $|\dot{e}_r|$.

Proof of Proposition 5: Let $x := R\bar{p} = M\bar{p}$ and $y := M^{-1}Rv_M = \rho$. Then, $x, y$ satisfy the following equations:

$$
\begin{align*}
\dot{x} &= My \\
\dot{y} &= -k_1\bar{\delta}_m(x) - k_2\bar{\delta}_m(My)
\end{align*}
$$

(80)

This is the same as (42), except for the presence of the matrix $M$ in the term $\bar{\delta}_m(My)$. This similitude allows one to duplicate the proof of Prop. 1 modulo minor adaptations detailed below.

Lemma 2, with $\bar{\delta}_m(y)$ in (43) replaced by $\bar{\delta}_m(My)$ and the desaturation condition (44) replaced by

$$
|M\bar{p}(t)| \leq k_2\bar{\delta}_m(|M|) - k_1\bar{\delta}_m(My) - c_\varepsilon, \forall t \geq t_1
$$

(81)

still holds true to show that $\bar{\delta}_m(My)$ desaturates. Indeed, $\dot{y} := -k_1M\bar{\delta}_m(x) - k_2M\bar{\delta}_m(y) + M\bar{p}(t)$

and the proof proceeds like for Lemma 2, by considering the Lyapunov function $V_0$ defined by $V_0(y) = \frac{1}{2}|y|^2$. The second inequality in (32) ensures that (81) is satisfied with $\rho \equiv 0$ and some $c_\varepsilon > 0$. After desaturation, solutions to (80) satisfy the following equations (compare with (45)):

$$
\begin{align*}
\dot{x} &= My \\
\dot{y} &= -k_1\bar{\delta}_m(x) - k_2My
\end{align*}
$$

(82)

The Lyapunov function in (46) is modified as follows:

$$
V(x, y) = k_1 \int_0^t s_\alpha(\tau) d\tau + y^T My + \frac{2k^2}{k_2} |\sigma(x)|^2 |\sigma(x)|^T y
$$

(83)

i.e., $V$ so defined only differs from (46) by the matrix term $M^{-1}M$ (in place of $M_\theta$). We show that $V$ is a Lyapunov function for $k$ small enough. Let (compare with (47))

$$
\begin{align*}
\tilde{k} &:= k_2 |M_\theta|^{-1} - k_1 |M_\theta^{-1}| M_\theta - C_\rho > 0 \\
k_1 &:= 2\sqrt{|k_1| |\sigma(x)|^2} |M_\theta^{-1}| M_\theta - C_\rho > 0 \\
k_2 &:= \frac{2k_2}{k_2 |\sigma(x)|^2} > 0 \\
k_3 &:= \frac{k_2 |\sigma(x)|^2 |\sigma(x)|^T |\sigma(x)|^2}{k_2} > 0
\end{align*}
$$

(84)

where positivity of $\tilde{k}, k_1$ follows from (32) and positivity of $k_2, k_3$ is a consequence thereof. Positive definiteness of $V$ is still ensured by (48), with $b$ now defined by $b := \frac{2k_2}{k_2 |\sigma(x)|^2} + \tilde{k} + \kappa$. This yields the condition $\kappa < k_1$. Differentiating $V$ along the solutions of System (82) yields

$$
\begin{align*}
\dot{V} &= -2k_2y^TM_\theta My + 2\frac{k^2}{k_2} |M_\theta^{-1}|^2 F_\theta(x)My \\
&\quad -sk_1s_\alpha(\xi)^2 |\sigma(x)|^2 + k^2 F_\theta(My) \\
&\quad -sk_2s_\alpha(\xi)^2 x^TM_\theta y - 2k_2 |\sigma(x)|^T M_\theta^{-1} |\sigma(x)|
\end{align*}
$$

(85)

We now use the assumption that $M_\theta$ and $M_\theta^{-1}$ commute. This implies that, for any $\xi \in \mathbb{R}^3$, $\xi^TM_\theta M_\theta \xi = 0$ and

$$
\xi^TM_\theta^{-1} \xi = (|M_\theta + M_\theta^{-1}|^2)^{1/2} |M_\theta^{-1}|^2 |\sigma(x)|^T |\sigma(x)|
$$

(86)
Therefore, one deduces from (85) that
\[
\begin{align*}
V & \leq -2k_2|M_y|^2 + 2\frac{k_3}{k_4}Y^T M_2 M^{-1} Y F_3(x) M y \\
& \quad -k_5 s_2(x) |x|^2 + k y F_3(x) M y \\
& \quad -k_6 s_3(x) T M y
\end{align*}
\]
This implies that (50) is still satisfied with the $C_{1,j}$'s defined by (compare with (51))
\[
C_{1,1} := 2k_2|M_y|^2 - b|M_1||C_{o1}, C_{1,2} := k_6|M_1|, C_{1,3} := k_1
\]
Thus, $V$ is negative definite provided that (52) is satisfied with the above-defined $C_{1,j}$'s. Condition a) follows from the fact that $\kappa < \kappa_2$. Condition b) holds true for any $\kappa > 0$ since $k_1 > 0$, and Condition c) follows from the fact that $\kappa < \kappa_3$. Thus, (53) is satisfied for some $\beta > 0$ and the end of the proof follows like for Proposition 1.

**Proof of Proposition 6:** Let $x := R\hat{r}, y := R\hat{v}, Y := (\mu_1, \mu_2)^T, \epsilon := \frac{1}{M_1}$ One obtains in closed-loop, after some calculations (compare with (76) for $M = 0$):
\[
\begin{align*}
\dot{x} &= M y \\
\dot{y} &= -k_1 s_4(x) - k_2 s_3(M y) + R(Y_1, Y_2, 0)^T \\
e\dot{y}Y &= -\mu_1 y + \epsilon \omega_1(t) Y^T \\
&\quad + eR^T_{1,2}(k_1 F_3(x) \dot{x} + k_2 \bar{F}_3(y) M y)
\end{align*}
\]
with $Y^T := (Y_2, -Y_1)^T, R_{1,2}$ the first two lines of $R^T$, and $F_3, \bar{F}_3$ defined by (77). By setting, like in the proof of Prop. 5, $\bar{y} := M y = R^T y$ the above equations can also be written as
\[
\begin{align*}
\dot{x} &= \bar{y} \\
\dot{y} &= -k_1 M s_4(x) - k_2 s_3(M \bar{y}) + MR(Y_1, Y_2, 0)^T \\
e\dot{y}Y &= -\mu_1 \bar{y} + \epsilon \omega_1(t) \bar{y}Y^T \\
&\quad + eR^T_{1,2}(k_1 F_3(x) \dot{x} + k_2 \bar{F}_3(\bar{y}) \bar{y})
\end{align*}
\]
We claim that the conclusion of Lemma 4 is still valid with statement a) replaced by:

a) $|y(t)| \leq \tilde{\delta}_m$ (i.e. the function $\bar{s}_3$ desaturates)

Indeed, based on (87) the proof follows exactly as that of Lemma 4 with $y$ replaced everywhere by $\bar{y}$, and the constant $k_2 \tilde{\delta}_m - k_1 \delta_M$ in (102) and subsequent equations replaced everywhere by $k_2 \tilde{\delta}_m|M_1| - k_1 \delta_M|M_1|$ (compare with (81)). Note also that the value of the constants $c_1, c_2$ must be changed as follows: $c_1 := k_1 \delta_M + k_2 \delta_M + g$ (from (35)) and therefore, from (88), $c_2 := |M_1|(2k_1 \delta_M + 2k_2 \delta_M + g)$.

After desaturation of the function $\bar{s}_3$, the trajectories of System (87) become solution to the system
\[
\begin{align*}
\dot{x} &= M y \\
\dot{y} &= -k_1 s_4(x) - k_2 M \bar{s}_3(Y_1, Y_2, 0)^T \\
e\dot{y}Y &= -\mu_1 \bar{y} + \epsilon \omega_1(t) \bar{y}Y^T \\
&\quad + eR^T_{1,2}(k_1 F_3(x) \dot{x} + k_2 \bar{F}_3(M y) \bar{y})
\end{align*}
\]
The first two equations in (89) correspond to (82) modulo the additional term $R(Y_1, Y_2, 0)^T$. The end of proof follows as for Prop. 3, using the fact that $V$ in (83) satisfies (53).

**Proof of Lemmas**

**Proof of Lemma 1:** We proceed by contradiction. Assume that $\mu(0) = -|\mu(0)| e_3$. From (20) and (23) $\mu(0) \neq 0$. From (20),
\[
\gamma(0) + |\mu(0)| e_3 = (k_1 s_4(\eta) + k_2 s_3(\bar{y} - 2) e_3).
\]
Thus,
\[
\begin{align*}
&\left|k_1 s_4(\eta) + k_2 s_3(\bar{y} - 2) e_3\right|^2 = |\gamma(0)|^2 + |\mu(0)|^2 + 2\mu(0)|\gamma(0)| \geq 0 \\
&\quad + g^2 + 2\mu(0)|\gamma(0)| \geq g^2 + 2\mu(0)|\gamma(0)| \\
&\quad > g^2 + 2\mu(0)|\gamma(0)| \sqrt{g^2 - \left(k_1 \delta_M + k_2 \bar{\delta}_M + |\bar{\pi}| \right)^2} \\
&\quad > \left(k_1 \delta_M + k_2 \bar{\delta}_M + |\bar{\pi}| \right)^2
\end{align*}
\]
where the first inequality comes from (24) and the fact that $\mu(0) \neq 0$. This contradicts (23).

**Proof of Lemma 2:** Consider the function $V_0$ defined by $V_0(y) = \frac{1}{2}|y|^2$. Its derivative along the solutions of Eq. (43) is given by
\[
\dot{V}_0 = -k_1 y^T s_4(x) - k_2 y^T s_3(M y) + y^T \dot{\varphi}(t) \\
\quad \leq k_1 |y|\delta_M - k_2 y^T \bar{s}_3(M y) + |\varphi(t)| \leq 0.
\]

Case 1: $t \in [0, \tau]$. Let $g_\theta$ denote the max of $\varphi$ on this time interval. From the above inequality,
\[
\dot{V}_0 \leq k_1 |y|\delta_M + |g_\theta| \leq (k_1 \delta_M + |g_\theta|) \sqrt{2V_0}
\]
The comparison lemma Khalil (2002) then yields
\[
|y(\tau)| \leq |y(0)| + (k_1 \delta_M + |g_\theta|) \tau
\]

Case 2: $t \geq \tau$. From Def. 1,
\[
|y| \geq \tilde{\delta}_m \Rightarrow y^T \bar{s}_3(y) = |y| \tilde{\delta}_m \geq |\tilde{\delta}_m| \geq \tilde{\delta}_m
\]
Therefore, from (90),
\[
|y(t)| \geq \tilde{\delta}_m \Rightarrow \dot{V}_0(t) \leq -(k_1 \tilde{\delta}_m - k_1 \delta_M - |\varphi(t)|) |y| \\
\quad \Rightarrow \dot{V}_0(t) \leq -c_2 \sqrt{2V_0}
\]
where the last inequality comes from (44). We deduce from this inequality that

1. If $|y(t)| \leq \tilde{\delta}_m$ then $|y(t)| \leq \tilde{\delta}_m$ for any $t \geq \tau$.
2. If $|y(t)| > \tilde{\delta}_m$ then, by application of the comparison lemma to the above inequality, $|y(t)| \leq \tilde{\delta}_m$ for any $t \geq \tau + \frac{|y(t)| - \tilde{\delta}_m}{c_2 \sqrt{2V_0}}$.

Then, it follows from (91) that $|y(t)| \leq \tilde{\delta}_m$ for any $t \geq T(y(0)) := \tau + \max[0, \frac{|y(t)| - \tilde{\delta}_m}{c_2 \sqrt{2V_0}}]$. In other words, $\bar{s}_3(\bar{y}(t)) = y(t)$ for $t \geq T(y(0))$.

**Proof of Lemma 3:** It relies on the following result (Hua et al., 2009, Prop. 1)
Proposition 7. Consider a smooth function $\zeta$ with $|\zeta| = 1$ and $\zeta$ independent of $\omega$. Let $\zeta := R^T \zeta$ and
\[
\begin{align*}
\omega_1 &= -k_1 \frac{\zeta}{|\zeta + (1+c_3)|} - \zeta^T (R \xi_1)^T - \zeta^T (R \zeta) \zeta, \\
\omega_2 &= k_2 \frac{\zeta}{|\zeta + (1+c_3)|} - \zeta^T (R \xi_2)^T, \quad k_2 > 0
\end{align*}
\] (92)

Then, on the unit sphere, $R \xi_3 = \zeta$ is exponentially stable with convergence domain $\{(0) : \zeta(0) R(0) = -1\}$.

We apply the above proposition with $\zeta := \frac{\rho Y}{\bar{\mu}}$. First, $\zeta$ is well defined because, from (20) and (23), $\mu$ never vanishes. Then, $\zeta$ is a smooth function since both $R$ and $\mu$ are smooth. Let us check that $\zeta$ is independent of $\omega$. First, recall that by Def. 1-i), $\text{Rat}(\zeta) = \text{sat}(R \xi_3)$ for any $R, \zeta$. Therefore, from (20) and the definitions of $\gamma, \tilde{\gamma}$, and $\alpha_r$, we have
\[
\zeta = \gamma_3 + k_3 \text{sat}(R \xi_3) \tilde{\gamma} - \bar{\mu}_r
\]
\[
[\gamma_3 + k_3 \text{sat}(R \xi_3) \tilde{\gamma} - \bar{\mu}_r, \gamma_1 + k_1 \text{sat}(R \xi_3) \tilde{\gamma} - \bar{\mu}_1]
\]
The derivatives of $\gamma_3, k_3 \text{sat}(\tilde{\gamma})$ and $\bar{\mu}_r$ do not depend on $\omega$. As for $R \xi_3$, it follows from (6) and (18) that $\frac{d}{dt} (R \xi_3) = -k_3 (R \xi_3 - M \bar{\mu})$. This term is thus also independent of $\omega$. Thus, $\zeta$ is independent of $\omega$.

By replacing $\xi_3$ in (92) by the expression $\xi_3 := \frac{\rho Y}{\bar{\mu}}$, one obtains after a few calculation of the expression (19). From application of Prop. 7, we deduce that $R \xi_3 = \zeta$ is exponentially stable with convergence domain $\{(0) : \zeta(0) R(0) = -1\}$. Thus, $\exists_c, \alpha > 0 : |R \xi_3 - \zeta| \leq c \zeta(0) r(0) e^{-\alpha t}$, $\forall t$. This is equivalent to $|\mu| = \epsilon(0) e^{-\alpha t}$, $\forall t$. Since, from A1, (20), and (23), $|\mu|$ is lower and upper-bound by strictly positive constants independent of the initial conditions, the above inequality is equivalent to $|\zeta(t)| \leq \tilde{c} e^{0} e^{-\alpha t}$ for some constant $\tilde{c}$.

Proof of Lemma 4: Since $\mathcal{A}_y \subset \mathcal{A}_y$, when $\rho < \rho$, it is sufficient to prove the existence of $\epsilon_0, > 0$ for any $\rho > \rho$, where $\rho$ is any strictly positive value. Thus, we assume from now on that
\[
\rho > \tilde{\delta}_m
\] (93)

Let us first establish a few inequalities. From (25) and (27), $|T|^2 + \mu_3^2 = |T|^2 > g^2 x^2$. Thus, $\forall t \geq 0, \mu_3(t) > g^2 x^2 - |T|^2$ (94)

From (27) and Assumption A3,
\[
|\mu| < |c_1| := k_3 \delta_3 + k_2 \delta_3 + 2g
\] (95)

Therefore, $|T| = |(\mu_3, \mu_2)| \leq |c_1$ and from (76),
\[
|\mu| < |c_1| = 2(k_2 \delta_3 + k_3 \delta_3 + g)
\] (96)

Recalling that $y = R \hat{v}$, it follows from (28) and (96) that
\[
\forall t > 0, \quad |y(t)| < \rho + c_2 t
\] (97)

Since $M$ is constant, (76) implies that $\hat{v} = My$. Recalling, from (2), that $F_3$ and $\hat{F}_3$ are bounded by $C_0$, it follows from (76) that
\[
e^t \hat{y} = -\mu_1 + \epsilon \omega_3(t) Y + \epsilon (\xi_1 + \xi_2 Y)
\] (98)

where $\xi_1, \xi_2$ are functions bounded by a constant $c_3$ independent of $\epsilon$. Thus, using the triangular inequality,
\[
\frac{d}{dt} |T|^2 \leq |T|^2 \left(\frac{2 \mu_3(t)}{\epsilon} + 1\right) + c_4 (1 + |T|^2)
\] (99)

Let $T_0$ denote any strictly positive constant. It follows from (97) and (99) that
\[
\forall t \in [0, T_0], \quad \frac{d}{dt} |Y|^2(t) \leq |Y|^2(t) \left(\frac{2 \mu_3(t)}{\epsilon} + 1\right) + c_4
\] (100)

with $c_4 := c_3(1 + \rho + c_2 T_0^2)$ (101)

We show that there exists $\epsilon_0 > 0$ such that, for any $\epsilon \leq \epsilon_0$ and any initial condition in $\mathcal{A}_y$,
\[
|Y(T_0)| < \tilde{\delta}_1 := \min \left(\frac{9(k_2 \delta_3 - k_1 \delta M)}{10}, \frac{g x}{\sqrt{2}}\right)
\] (102)

We claim that there exists $\hat{\delta}_1 > 0$ such that, $\forall \epsilon \leq \hat{\delta}_1$,
\[
\forall t \in [0, T_0], \quad |Y(t)|^2 < g^2 x^2 / 2
\] (103)

Suppose on the contrary that there exists a sequence $\epsilon_n \to 0$ converging to zero such that, for any $\epsilon = \epsilon_n$ there exists a time $t_n \in [0, T_0]$ such that $|Y(t_n)|^2 \geq g^2 x^2 / 2$. Since, from (28), $|Y(t)|^2 < g^2 x^2 / 2$, we can assume without loss of generality that $|Y(t)|^2 \leq |Y(t_n)|^2 = g^2 x^2 / 2$ for $t \leq t_n$. Therefore, $\frac{d}{dt} |Y(t)|^2 \geq 0$ (104)

On the other hand, from (94), the fact that $\mu_3(t) > 0$ (cf. (28)), and the fact that $|Y(t)|^2 \geq g^2 x^2 / 2$ for $t \leq t_n$, we deduce that $\mu_3(t) > g x / \sqrt{2}$. Thus, we deduce from (100) and (104) that
\[
0 \leq g^2 x^2 / 2 - \left(\frac{2 \mu_3(t)}{\epsilon} + 1\right) + c_4 < g^2 x^2 / 2 - \left(\frac{\sqrt{2} g x}{\epsilon} + 1\right) + c_4
\] (105)

This is impossible if $\epsilon < \hat{\delta}_1 := g^3 x^3 \sqrt{2}(g^2 x^2 / 2 + 2c_4)$, which shows (103) for $\epsilon \leq \hat{\delta}_1$. From (103), (94), and the fact that $\mu_3(0) > 0$, it follows that $\mu_3(t) > g x / \sqrt{2}$ for all $t \in [0, T_0]$. In other words, for $\epsilon \leq \hat{\delta}_1$,
\[
\forall t \in [0, T_0], \quad |Y(t)|^2 < g^2 x^2 / 2, \quad \mu_3(t) > g x / \sqrt{2}
\] (106)

Therefore, from (100),
\[
\forall t \in [0, T_0], \quad \frac{d}{dt} |Y|^2(t) \leq |Y|^2(t) \left(\frac{g x \sqrt{2}}{\epsilon} + 1\right) + c_4
\] (107)

Applying the comparison lemma yields
\[
\forall t \in [0, T_0], \quad |Y|^2(t) \leq e^{-at} \left(|Y|^2(0) - \frac{c_4}{a}\right) + \frac{c_4}{a}
\] (108)

with $a = \frac{g x \sqrt{2}}{\epsilon} - 1$. Since $a$ tends to infinity as $\epsilon$ tends to zero and $|Y(0)| < g x / \sqrt{2}$, there exists $\tilde{\delta}_2$ such that, $\forall \epsilon \leq \tilde{\delta}_2$,
\[
e^{-at} \left(|Y|^2(0) - \frac{c_4}{a}\right) + \frac{c_4}{a} < \left(\frac{9(k_2 \delta_3 - k_1 \delta M)}{10}\right)^2
\] (109)

13
This inequality, together with (106) and (103) imply (102) for 

\[ \varepsilon \leq \varepsilon_0 := \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\}. \]

We now assume that

\[ \varepsilon \in (0, \varepsilon_0) \] with 

\[ \varepsilon_0 = \min\left\{ \varepsilon_0, \frac{\sqrt{2g \varepsilon^2 \bar{c}_2}}{\bar{c}_1 + c_4} \right\} \quad (107) \]

We claim that

\[ \forall t \geq T_0, \quad \left\{ \begin{array}{l} |y(t)| < \rho + c_2 T_0 \\ |Y(t)| < \bar{\varepsilon} \\ \mu_3(t) > g_{s} \sqrt{\bar{\varepsilon}} \end{array} \right\} \quad (108) \]

Since \( \varepsilon < \varepsilon_0 \), it follows from (97), (102), and (105) that the three inequalities in (108) are satisfied at \( t = T_0 \). Suppose by contradiction that there exists \( T > T_0 \) such that at least one of the three inequalities in (108) is not satisfied at \( t = T \) and these inequalities are satisfied for any \( t \in [T_0, T) \). We distinguish three cases:

**Case 1:** \( |y(T)| = \rho + c_2 T_0 \). We claim that

\[ \forall t \in [T_0, T), \quad |y(t)| \leq \max\{\delta_m, |y(T_0)|\} \quad (109) \]

Indeed, from (108) for \( t \in [T_0, T) \), \( |y(t)| \leq \bar{\varepsilon} < k_2 \delta_m - k_1 \delta_M \) for any \( t \in [T_0, T) \). Therefore, by (76),

\[ \forall t \in [T_0, T), \quad |y(t)| \geq \delta_m \Rightarrow \frac{d}{dt} |y(t)| < 0 \quad (110) \]

Thus, if \( \max\{\delta_m, |y(T_0)|\} = |y(T)| \) then \( |y(t)| > |y(T)| \) for all \( t \in [T_0, T) \), and if \( \max\{\delta_m, |y(T_0)|\} = \delta_m \) then \( |y(t)| \leq \delta_m \) for all \( t \in [T_0, T) \). This proves (109). By continuity, \( |y(T)| \leq \max\{\delta_m, |y(T_0)|\} \). This contradicts the assumption \( |y(T)| = \rho + c_2 T_0 \) since \( |y(T_0)| < \rho + c_2 T_0 \) by (97), and \( \delta_m < \rho \) by (93).

**Case 2:** \( |Y(T)| = \bar{\varepsilon} \). Since (108) is satisfied for \( t \in [T_0, T) \),

\[ \frac{d}{dt} |Y|^2(t) \geq 0 \quad (111) \]

It follows from (99), (101), and (108) for \( t < T \), that

\[ \frac{d}{dt} |Y|^2(t) \leq |Y|^2(t) \left( -\frac{g_{s} \sqrt{\bar{\varepsilon}}}{\varepsilon} + 1 \right) + c_4 \]

\[ \leq \bar{\varepsilon}^2 \left( -\frac{g_{s} \sqrt{\bar{\varepsilon}}}{\varepsilon} + 1 \right) + c_4 < 0 \quad (112) \]

where the last inequality follows from (107). This contradicts (111).

**Case 3:** \( \mu_3(T) = g_{s} \varepsilon \sqrt{\bar{\varepsilon}} \). Since (108) is satisfied for \( t < T \), it follows by continuity that \( |Y(T)|^2 \leq g^2 \varepsilon^2 / 2 \). Thus, \( \mu_3(T) + |Y(T)|^2 \leq g^2 \varepsilon^2 \), which contradicts (94).

This concludes the proof of (108). Lemma 4 follows from (108) and Lemma 2 applied to the second equation in (76) with \( g(t) = R(t)(Y_1(t), Y_2(t), 0)^T \).

**Proof of Lemma 5:** It is inspired by the proof of Lemma 4. Let us first define some constant numbers. With \( c_2 \) defined by (96) and \( V \) given by (46) the Lyapunov function of Prop. 1, let us define

\[ T_0 := \frac{\bar{\varepsilon}}{2}, \quad c_5 := \left( \rho + \frac{g_{s} \sqrt{\bar{\varepsilon}}}{2} + \frac{\bar{c}_2}{\sqrt{2}} \right) |M|, \]

\[ \bar{V} := \max\{|V(x, y)| : |x| \leq c_5, |y| \leq \delta_m\} \]

\[ \bar{c}_2 := \min\{|x| : V(x, y) \leq 2\bar{V}\} \]

\[ \bar{c}_6 := \min\{|y, s_{a_0}(x)| : V(x, y) = 2\bar{V}\} \]

\[ \bar{\varepsilon} := \min\left\{ \frac{|\bar{c}_5|}{2}, \frac{g_{s} \sqrt{\bar{\varepsilon}}}{2} \right\} \]

with \( c \) any constant satisfying (68). Note that \( \bar{c}_5 \) is well defined because \( \bar{V} \) is proper. Note also, from the definition of \( \bar{V} \) and \( \bar{c}_5 \), that

\[ c_5 \leq \bar{c}_5 \quad (114) \]

Relations (94) to (96) are still valid when \( M \) is not constant. As for (97), using (29) it becomes

\[ \forall t > 0, \quad |y(t)| < \frac{\delta_m}{2} + c_2 t \quad (115) \]

Due to the term \( \bar{M} \bar{p} = MM^{-1}x \) in the expression (76) of \( \dot{x} \), relation (98) becomes

\[ eY = -\mu_3 Y + \varepsilon \omega_1(t) Y^1 + \varepsilon (\bar{c}_1 \bar{x} + \bar{c}_2 \bar{y} + \bar{c}_3 x) \]

with \( \bar{c}_1, \bar{c}_2, \bar{c}_3 \) functions bounded by a constant \( c_4 \) independent of \( \varepsilon \). Thus, by the triangular inequality (99) becomes

\[ \frac{d}{dt} |Y|^2(t) \leq |Y|^2 \left( -\frac{2 \mu_3}{\varepsilon} + 1 \right) + c_4^2 (1 + |y|^2 + |x|^2) \]

\[ (117) \]

Since \( \dot{p} = Rv = y, |\dot{p}(t) - \dot{p}(0)| \leq \int_0^t |v(s)| ds \) for all \( t \). Thus, from (29) and (115), \( |\dot{p}(t)| < \rho + \frac{\bar{c}_1 \bar{x}}{\sqrt{2}} + \frac{\bar{c}_2 \bar{y}}{\sqrt{2}} \) for any \( t > 0 \), which implies by the definition (113) of \( c_5 \) that

\[ \forall t \in (0, T_0), \quad |x(t)| < c_5 \leq \bar{c}_5 \quad (118) \]

From here, one can proceed as in the proof of Lemma 4 to show that for \( \varepsilon \) smaller than some \( \varepsilon_0 > 0 \) (102) and (105) hold true with \( \bar{\varepsilon} \) defined by (113).

Recall from (113) that \( T_0 := \frac{\bar{\varepsilon}}{2\sqrt{2}} \). Thus, from (115), \( |y(T_0)| < \bar{\varepsilon} \). We impose on \( \varepsilon \) the condition (107), with \( c_4 \) defined by (119). We claim that there exists \( \theta > 0 \) such that,

\[ |M| \leq \theta, \quad \forall t \in (0, T_0), \quad |y(t)| < \frac{\delta_m}{2} \]

\[ (116) \]

The four properties in (120) are satisfied at \( t = T_0 \). Indeed, by (113), (115), and (118), \( |V(x(T_0), y(T_0))| \leq \bar{V} \). Property ii), at \( t = T_0 \), follows from (115). Properties iii-iv) follow from (102) and (105), which are satisfied because \( \varepsilon < \varepsilon_0 \). To prove (120) we proceed by contradiction. Suppose that for any \( \theta > 0 \) there exists some \( M \) with \( |M| \leq \theta \) and some \( T > T_0 \) such that at least one of the properties in (108) is not satisfied at \( t = T \) and these properties are satisfied \( \forall t \in [T_0, T) \). We distinguish four cases.

**Case 1:** \( V(x(T), y(T)) = 2 \bar{V} \). Since (120) is satisfied for \( t \in [T_0, T) \), one has

\[ \bar{V}(x(T), y(T)) \geq 0 \quad (121) \]

One also deduces from (120) on the time-interval \([T_0, T)\) that on the time-interval \([T_0, T)\), the function sat \( \bar{c}_5 \) desaturates. Thus, from (76), the solution satisfies on \([T_0, T)\):

\[ \left\{ \begin{array}{l} \dot{x} = My + \bar{M} \bar{p} \\ \dot{y} = -\bar{c}_5 \text{sat}_g(x) - \bar{c}_2 y + R(Y_1, Y_2, 0)^T \end{array} \right\} \quad (122) \]
This system corresponds to (54) modulo the additive "perturbation" $\mathcal{R}(Y_1, Y_2, 0)^T$. It follows from (120) for $t \in [T_0, T]$ that $|V(x(T), y(T))| \leq 2\bar{V}$. From (113), this implies that $|\lambda(t)| \leq \tilde{c}_5 \forall t \in [T_0, T]$. Using the same argument as for (59), one deduces that $\forall t \in [T_0, T]$, $|\lambda(t)| \leq \frac{\lambda_{sat}(\bar{s}) m}{\bar{s}_{max}}$ with $\bar{s}_{max} := \min_{\bar{s}_{min} \leq \bar{s} \leq \bar{s}} |\lambda(t)|^2$. With this definition of $\bar{s}_{max}$, it follows from (60) that the derivative of $V$ along the solutions of (122) satisfies the time-interval $[T_0, T]$:

$$V \leq -C_{21} |y|^2 + C_{22} |y| |sat_d(x)| - C_{23} |sat_d(x)|^2 + \frac{\partial}{\partial y} \mathcal{R}(Y_1, Y_2, 0)^T$$

(123)

with $C_{21}, C_{22}, C_{23}$ defined by (61). From (61), the coefficients $C_{2,j}$ tend to $C_{\theta,j}$ as $\theta \to 0$. Therefore, by (53) there exists $\theta > 0$ such that, for $|\lambda(t)| \leq \theta$, $V \leq -\frac{\theta}{2} |y|^2 + |sat_d(x)|^2 + |\delta_y| \mathcal{R}(Y_1, Y_2, 0)^T$ (124)

Using (68), we then deduce that

$$V \leq -\frac{\theta}{2} |y|^2 + |sat_d(x)|^2 + c_i(y, sat_d(x))$$

(125)

It thus follows from (120) for $t \in [T_0, T]$ that, on the time-interval $[T_0, T]$,

$$V \leq -\frac{\theta}{2} |y|^2 + |sat_d(x)|^2 + c_2 \bar{s}(y, sat_d(x))$$

(126)

From (113) and the assumption $V(x(T), y(T)) = 2\bar{V}$, one deduces that $|V(x(T), y(T))| \geq \bar{c}_6$. Then, it follows from (120) and (113) that $V(x(T), y(T)) < 0$. This contradicts (121).

**Case 2:** $|y(T)| = \delta_m$. Since (120) is satisfied for $t \in [T_0, T)$, it follows that $\frac{\partial}{\partial t} |y|^2 \geq 0$. Since $|Y(t)| \leq \bar{s}$ on $[T_0, T)$, relation (110) still holds, which implies that $\frac{\partial}{\partial t} |y|^2 < 0$, thus a contradiction.

**Case 3:** $|Y(T)| = \bar{s}$. Since (120) is satisfied for $t \in [T_0, T)$, it follows that $\frac{\partial}{\partial t} |y|^2 \geq 0$. Furthermore, (117) for $t < T$ and (113) imply that $x(t) \leq \bar{s}_5$. This implies, from (117) and (120) for $t < T$, that (112) is satisfied with $c_3$ defined by (119). We thus obtain a contradiction.

**Case 4:** $\mu_3(T) = g \sqrt{\bar{s}^2}$. One proceeds as in Case 3 of Lemma 4 to show that this case also yields a contradiction.

We thus have proved that (120) is satisfied if $\theta > 0$ is chosen small enough. Property a) of Lemma 5 follows from i) in (120) and (113). Properties b-c) correspond to ii-iii) in (120). Finally, since $|\lambda(t)| \leq \tilde{c}_5$, (125) is satisfied, which corresponds to Property d) of Lemma 5.

References


