Mini-symposium on automatic differentiation and its applications in the financial industry

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MINI-SYMPOSIUM ON AUTOMATIC DIFFERENTIATION AND ITS APPLICATIONS IN THE FINANCIAL INDUSTRY

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Abstract. Automatic differentiation is involved for long in applied mathematics as an alternative to finite difference to improve the accuracy of numerical computation of derivatives. Each time a numerical minimization is involved, automatic differentiation can be used. In between formal derivation and standard numerical schemes, this approach is based on software solutions applying mechanically the chain rule to obtain an exact value for the desired derivative. It has a cost in memory and cpu consumption.

For participants of financial markets (banks, insurances, financial intermediaries, etc), computing derivatives is needed to obtain the sensitivity of its exposure to well-defined potential market moves. It is a way to understand variations of their balance sheets in specific cases. Since the 2008 crisis, regulation demands to compute this kind of exposure to many different cases, to be sure market participants are aware and ready to face a wide spectrum of market configurations.

This paper shows how automatic differentiation provides a partial answer to this recent explosion of computations to be performed. One part of the answer is a straightforward application of Adjoint Algorithmic Differentiation (AAD), but it is not enough. Since financial sensitivities involve specific functions and mix differentiation with Monte-Carlo simulations, dedicated tools and associated theoretical results are needed. We give here short introductions to typical cases arising when one uses AAD on financial markets.

1. Introduction

Numerous discussions with professionals underlined the recent importance of automatic differentiation in financial institutions. It hence appeared natural to include a mini-symposium on this topic in the "International Conference on Monte Carlo techniques".

One of the roots of the recent success of automatic differentiation on financial markets is the ever demanding regulations on sensitivity computations (see for instance Section B.2 of [2]) questioned the traditional way to estimate such characteristics (i.e. the *Greeks*) of a bank portfolio or book. Since the usual ways to compute them is derived from Euler's schemes, a numerical discretization is needed in each targeted direction to obtain a sensitivity. With automatic differentiation (or adjoint algorithmic differentiation: AAD), you have not to develop a dedicated code for each targeted directions; it is "enough" to embed your overall numerics in a

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framework supporting AAD (see Section 3). As sessions of this mini-symposium (and hence sections of this paper) show: it has an overall cost, but once this cost is paid, the marginal cost of computing more sensitivities is small.

If AAD exists for long and has numerous applications (see [4] for a survey on interactions of AAD and statistical learning), the specificities of financial payoffs need some dedicated developments, as in [26] (see Section 3). Moreover, the reformulation of some financial problems (like Credit Value Adjustement, see Section 5) helps their inclusion in an AAD framework.

The sections of this paper reflect the sessions of the mini-symposium: each Section has been written by a speaker and contains a “take home message” about a specific aspect of AAD in finance.

In the first section, Barak Pearlmutter gives an overview of the most important aspects of AAD, in general. The reader can refer to [4] for details. This section underlines the role of AAD in applied mathematics: progresses in numerical optimization, as a balance between manual derivation of specific functions involved in the criterion to minimize, and a pure numerical scheme via finite differences. Hence domains like non linear statistics, and more recently machine learning[1], took profit of progresses in AAD.

In Section 3, Sébastien Geeraert emphasizes some practical aspects he explored at Murex. He underlines how the specific mix of deterministic and stochastic processes involved in the payoffs of financial contracts can be exploited thanks to ad hoc choices to improve the efficiency of AAD. Typically ADD has to be used inside Monte-Carlo simulations, giving birth to specific constraints in terms of memory and cpu consumption.

In the next section, Olivier Pironneau put the emphasis on ways to apply AAD on non differentiable functions. This section follows the idea of applying the Vibrato method to a non differentiable function via smoothing. It allows to obtain accurate estimates for partial derivatives of order two in some cases.

Adil Reghai exposes in the last section how Natixis uses AAD in production to compute sensitivities of its exposure to potential market moves. It demonstrates the advantage of AAD in the scope of the specific case of sensitivity to the counterparty risk.

2. What Automatic Differentiation Means to Me

by *Barak A. Pearlmutter*

2.1. Introduction

Automatic Differentiation or AD [19] is a subdiscipline of numeric computation dedicated to the study of an enticing idea: the mechanical transformation of a computer program that computes a numeric function into an augmented computer program that also computes some desired derivatives accurately and efficiently. The intuition for how this might be accomplished can be formed by considering a computer program implementing the function f which maps an input vector to an output vector

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

as consisting of a flow graph: a directed acyclic graph leading from inputs to outputs whose edges carry $\mathbb{R}$ values and whose nodes represent primitive arithmetic operations or fan-out.

2.2. Forward AD

Using modern terminology, [39] proposed that each real number v in the flow graph could be replaced by a “dual number” [11], a pair of real numbers $\langle v, v' \rangle$ representing a truncated power series $v + v'\epsilon + O(\epsilon^2)$, with each primitive arithmetic operation g,

$$(v_1, \ldots, v_m) = g(u_1, \ldots, u_n)$$

[1]Well-known machine learning frameworks, like Theano or TensorFlow, use AAD.

suitably modified to Fg,

$$(\langle v_1, v_1' \rangle, \ldots, \langle v_m, v_m' \rangle) = Fg(\langle u_1, u_1' \rangle, \ldots, \langle u_n, u_n' \rangle)$$

where

$$\langle v_i, v_i' \rangle = Fg_i(\langle u_1, u_1' \rangle, \ldots, \langle u_n, u_n' \rangle) = \langle g_i(u_1, \ldots, u_n), \sum_j u_j'(d/du_j) g_i(u_1, \ldots, u_n) \rangle$$

or, gathering values into vectors $\mathbf{u} = (u_1, \ldots, u_n)$, etc, and letting $J_g(\mathbf{u})$ be the Jacobian matrix of g at $\mathbf{u}$, and gathering the primed values into vectors as well,

$$\mathbf{v} = g(\mathbf{u}) \qquad \mathbf{v}' = J_g(\mathbf{u})\mathbf{u}' \qquad \langle \mathbf{v}, \mathbf{v}' \rangle = Fg(\langle \mathbf{u}, \mathbf{u}' \rangle) = \langle g(\mathbf{u}), J_g(\mathbf{u})\mathbf{u}' \rangle$$

Assuming we have replaced each primitive arithmetic operation by an operation which does what the original operation did but also performs a Jacobian-vector product, as above (see Figure 1), the entire computation will now perform a Jacobian-vector product.

This is "Forward Accumulation Mode Automatic Differentiation", or "Forward AD". Note that since n and m are very small for primitive operations (one or two in general), the extra arithmetic added is a small constant factor overhead. And note that we can allow the new "prime" values to flow through the computation in parallel with the original ones, meaning that the storage overhead is bounded by a factor of two.

The same mathematical transformation as Forward AD has many names in many fields: propagation of perturbations in machine learning, the pushforward in differential geometry, a directional derivative in multivariate calculus, forward error analysis in computer science, etc. And many software tools exist that perform the Forward AD transformation, using a variety of implementation strategies ranging from source-to-source transformation (generally the fastest, but the least flexible or convenient) to overloading of operators in an object-oriented system (easy to implement and use but usually quite slow and sometimes not very robust due to semantic limitations of the overloading mechanisms.)

2.3. **Reverse AD**

Consider an attempt to calculate the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ using Forward AD. This would require running the transformed function $Ff : \langle \mathbf{x}, \mathbf{x}' \rangle \mapsto \langle \mathbf{y}, \mathbf{y}' \rangle$ repeatedly, with $\mathbf{x}'$ being cycled through the n basis vectors $(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$. When n is large (say, 10^6) this overhead is unacceptable. Fortunately there is a different AD transformation which can calculate gradients more efficiently. Each value v in the original flow graph is augmented with an adjoint value $\bar{v}$, which are propagated backwards through the flow graph. Each primitive arithmetic operation is modified (see Figure 2) to also multiply the vector of sensitivities of its *output* by the *transpose* of its Jacobian matrix, yielding the vector of sensitivities of its inputs. Transforming the entire graph in this fashion allows the efficient computation of a Jacobian-transpose-vector product with only a small constant factor increase in operation count. This allows the gradient to be calculated with only a small constant factor overhead!

Although low overhead for computing a gradient, in terms of operation count, makes this technique extremely useful, there are a variety of technical difficulties associated with Reverse AD. Since the derivatives are propagated backwards relative to the original computation, the original values might need to be preserved, increasing the storage burden. Fanout in the original computation corresponds to a linear primitive operation $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}[u]$, implying that the adjoint computation must multiply by the transpose of this tiny matrix, $\bar{u} = [1\ 1]\begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} = \bar{v}_1 + \bar{v}_2$, corresponding to a simple addition. In other words, fanout in the original computation graph results in addition in the reverse-direction adjoint computation graph. These complexities of the transformation makes efficient implementation, via a source-to-source transformation of the code, quite involved. The most popular implementation strategy is to actually store the dataflow graph and later retrace it in reverse order, which imposes considerable constant-factor overhead.

Although this transformation was formulated in the AD community, and automated, by [38] and others [18] it was known, albeit not automated, in the field of control theory where it was known as the Pontryagin Maximum

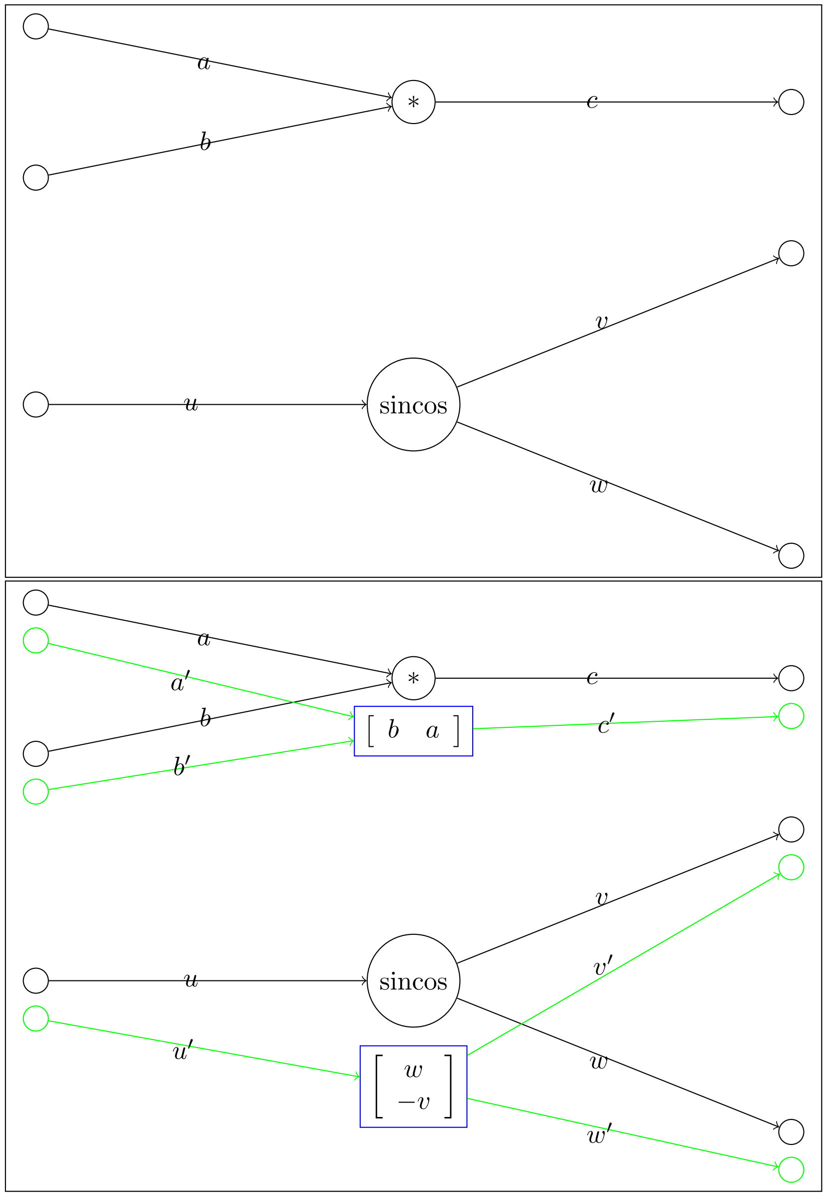

Figure 1. Forward AD augments all primitive arithmetic operations in the original flow graph (top) to also propagate derivatives forward through the computation (bottom).

Principle [31], in Mathematical Physics [13] where it was known as an adjoint calculation, in Machine Learning where it is known as backpropagation [35], and in differential geometry where it is known as the pull-back. The history of the discovery of both Forward AD and Reverse AD is fascinating, with many brilliant and colorful players. This former is discussed by [23], and the latter, at some length, by [18].

2.4. **Practical AD Systems and Applications**

Most useful complicated numeric routines are not so simple as a static dataflow graph, and part of the complexity of AD is handling control. Other "modes" of AD are also available, to compute higher-order

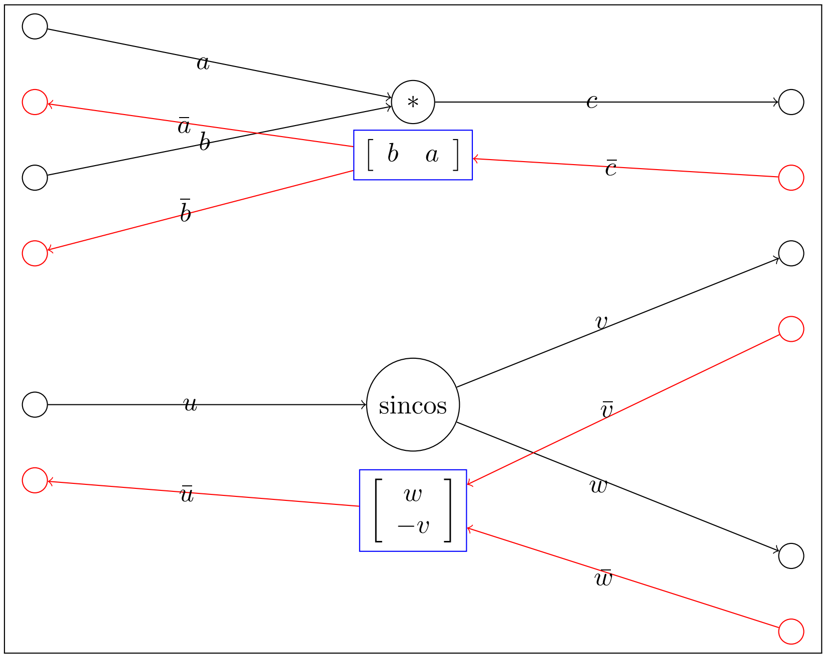

FIGURE 2. Reverse AD augments all primitive arithmetic operations in the original flow graph to also propagate derivatives backwards through the computation.

derivatives [24, 29], to vectorize multiple similar derivative calculations with stacked tangents or cotangents sometimes called vector mode, to conserve storage while still calculating gradients efficiently [17, 36], and many many others. Another substantial body of work has arisen as a consequence of the fact that taking exact derivatives and numeric approximation for implementation do not commute—a particularly difficult issue with loops that iterate to convergence. And yet more work has gone into aggressive implementation technologies, with enormous effort put into aggressive systems like TAPENADE [20, 27, 28], ADIFOR [7], and others. Currently, AD is enjoying substantial applications not just in finance [6] and scientific computation like climate science [21], but also in machine learning [3, 12, 14]. The field of AD is active, and welcoming to newcomers. A catalog of AD implementations and literature is maintained at a community web portal, `http://www.autodiff.org/`.

2.5. An AD Dream

In considering AD, it may be helpful to distinguish a hierarchy of notions, spanning a spectrum of automation and generality.

- **AD Type I:** A calculus for efficiently calculating derivatives of functions specified by a set of equations.
- **AD Type II:** Rules applied manually which mechanically transform a computer program implementing a numeric function to also efficiently calculate some derivatives.
- **AD Type III:** A computer program which automatically transforms an input computer program specifying a numeric function into one that also efficiently calculates derivatives.
- **AD Type IV:** AD-taking operators included in language in which numeric algorithms are written, and they can be applied to procedure which use them.

The present author's AD research program, done in collaboration with Jeffrey Mark Siskind, has been to bring Type IV into existence, and to make it fast and robust and general and convenient [30]. These efforts have therefore focused not just on formalizations of the AD process suitable for mathematical analysis and integration into compilers, but also in making AD more general and robust, allowing the AD operators to be used as first-class operators in a nested fashion, thus expanding the scope of programs involving derivatives that can be written succinctly and conveniently [4, 5, 32, 33]. We have also explored the limits of speed that Type IV AD integrated into an aggressive compiler can attain [37].

It is our hope that AD will become as ubiquitous for all numeric programmers as libraries of linear algebraic routines are now, and that optimizing compilers will in the future support AD with the same nonchalance with which they currently support trigonometric functions or loops.

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3. Practical implementation of adjoint algorithmic differentiation (AAD)

by *Sébastien Geeraert*

3.1. Aim of the study

In this study, we show a way of implementing adjoint algorithmic differentiation (AAD). AAD is a method producing the exact derivatives of a computation, in an automatic and efficient way. By using pathwise differentiation with AAD, we can compute Monte Carlo sensitivities of a price: if a price is given by $p(\theta) = \mathbb{E}\left[f(\theta, W)\right]$ where θ is a parameter and W is a random variable, by swapping the differentiation and the expectation, we can estimate its derivative by $p'(\theta) = \mathbb{E}\left[\frac{\partial f}{\partial \theta}(\theta, W)\right] \approx \frac{1}{M}\sum_{i=1}^{M} \frac{\partial f}{\partial \theta}(\theta, W_i)$. The advantage of AAD is that the computation time does not depend on the number of sensitivities. In the existing automatic differentiation libraries, there are two methods used: operator overloading (e.g. ADOL-C, Adept) and source code transformation (e.g. Tapenade).

Source code transformation consists in automatically adding in the source code, before compilation, the instructions to compute the derivatives. It requires a tool able to read an existing source code and to understand it (like a compiler does). Therefore, this method is quite hard to implement. It is often unable to handle advanced features of a language (such as object-oriented programming).

Operator overloading is a much easier method. It consists in redefining basic operators (such as $+$, $*$, $/$) to also handle the computation of derivatives. It allows to quickly apply AAD to an existing code, with very few modifications to the code. In comparison with source code transformation, operator overloading cannot be as well optimized by the compiler, and may add some overhead during execution. Thus, it is considered as less efficient.

For the sake of simplicity, we choose to use operator overloading in C++. The specificities of our implementation are that it focuses on Monte Carlo computations, and that it is efficiently parallelized, whether on CPU (using OpenMP) or on GPU (using CUDA).

3.2. Basic implementation

We model a computation as a sequence of n intermediate results x_i produced by elementary functions f_i: $x_i = f_i(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, n$. To apply AAD, we must first do a forward sweep, where we compute and store every intermediate result x_i from the beginning to the end. We can then do the reverse sweep, where we compute the adjoints $\overline{x_i}$ from the end to the beginning. Here is the resulting algorithm:

for i from 1 to n

  $x_i = f_i(x_1, \ldots, x_{i-1})$

$\overline{x_i} = 0$ for $i < n$, $\overline{x_n} = 1$

for i from n to 1

  for j from 1 to $i-1$

    $\overline{x_j} \mathrel{+}= \overline{x_i}\frac{\partial f_i}{\partial x_j}$

This is very easy to implement using operator overloading: the functions f_i are elementary operators which are overloaded to compute and store the relevant quantities during the forward and reverse sweep. To apply AAD on a Monte Carlo computation, we must differentiate the same computation a large number of times with different inputs (the random variables). Therefore, we apply the algorithm above to vectors: each component of the vectors corresponds to one path of the Monte Carlo. At the end of the computation, we can average the components of any adjoint to get the corresponding sensitivity. We propose two versions of the algorithm that are very close to each other. In the CPU version, the overloaded operators use for loops, parallelized with OpenMP. In the GPU version, they call CUDA kernels.

We need to be careful to avoid useless computations. Some computations are identical across all the Monte Carlo paths: if a computation does not depend on random variables (directly or indirectly), it always has the same inputs and outputs. Therefore, to be efficient, we need to do these computations in a scalar way.

3.3. **Optimisations**

Problems of efficiency arise when a big part of the original computation is deterministic. In that case, the forward sweep is cheap, because almost all the computations are scalar. However, the reverse sweep is very expensive, because all the adjoints computations are vectorial (all the intermediate adjoints indirectly depend on the random variables). We can solve this problem by computing intermediate means in the middle of the reverse sweep. If we divide the computation as $f(\theta, W) = g(h(\theta), W)$ where $h = (h_1, \dots, h_q)$ is the deterministic part and g the non-deterministic part, the sensitivity is given by

$$\frac{1}{M}\sum_{i=1}^{M}\frac{\partial f}{\partial \theta}(\theta, W_i) = \frac{1}{M}\sum_{i=1}^{M}\sum_{j=1}^{q}\frac{\partial g}{\partial h_j}(h(\theta), W_i)\frac{\partial h_j}{\partial \theta}(\theta) = \sum_{j=1}^{q}\left(\frac{1}{M}\sum_{i=1}^{M}\frac{\partial g}{\partial h_j}(h(\theta), W_i)\right)\frac{\partial h_j}{\partial \theta}(\theta).$$

In practice, during the reverse sweep, we pause the computation at the limit between the deterministic and non-deterministic parts. We replace every (vectorial) adjoint by its (scalar) mean. Thus, the subsequent computations are scalar.

Note however that this trick works only because taking the mean of the derivatives is a linear operation. It cannot be applied to a variance computation. But if we want to estimate the variance in addition to the sensitivities, we can divide the Monte Carlo paths into a small number of groups and apply the trick on each group to compute an estimate of the sensitivity; the variance can then be estimated using the different samples of sensitivity obtained on the different groups.

To increase the parallelization, we do independent computations in parallel. Two computations are independent if the inputs of one do not depend on the output of the other. Thus, the computations are divided into layers, which are chronologically ordered: each layer can only begin when the previous layer is finished. Inside a layer, everything can be computed in parallel.

When an intermediate result is used as input by many computations, the corresponding adjoint has to be updated many times during the reverse sweep. If these updates are done naively (i.e. by adding the updates successively one by one), the number of layers will grow linearly with respect to the number of adjoint updates, which breaks the parallelization. Therefore, we aggregate the adjoint updates by powers of two. This way, the number of layers grows only logarithmically.

Finally, we can also take advantage of the specificities of the model and the payoff. For example, for a CVA computation, the payoff can be seen as a sum of much simpler payoffs. Therefore, we can replace one big AAD with many small ones, which improves parallelization. Furthermore, we can in the GPU version make the work on CPU (the analysis of dependencies to divide the computational graph into layers) overlap with the work on GPU (the actual computation of intermediate results and adjoints): while the GPU computes the results on one AAD, the CPU works on the next AAD.

3.4. **Results**

On a CVA example where we compute 200 sensitivities with 5000 paths, we get the following timings:

	CPU	GPU
Finite differences	271 s	29 s
AAD	9.1 s	2.4 s

We see that when there are many sensitivities to compute, AAD allows interesting gains in comparison with finite differences: a factor of 30 on CPU and of 12 on GPU. It shows that the method can be efficiently parallelized, both on CPU and on GPU, especially if we take advantage of the specificities of the model and payoff. However, AAD cannot be directly applied to non-smooth payoffs. That is why [15] proposed the Vibrato Monte Carlo method, which consists in applying the Likelihood Ratio Method to the conditional expectation at the last step of an Euler scheme.

4. Second Sensitivities in Quantitative Finance

By *Olivier Pironneau*, based on a joint work with Gilles Pagès and Guillaume Sall

This short summary presents our work aimed at the computation of higher order sensitivities in quantitative finance; in particular Vibrato and automatic differentiation are compared with other methods. We show that this combined technique is more stable than automatic differentiation of second order derivatives by finite differences and more general than Malliavin Calculus. We also extend automatic differentiation for second order derivatives of options with non-twice differentiable payoff.

Sensitivities of large portfolios are time consuming to compute. Vibrato [15], [16] is an efficient method for the differentiation of mean value of functions involving the solution of a stochastic ODE. In many cases, double sensitivities, i.e. second derivatives with respect to parameters, are needed (e.g. gamma hedging). Then one can either apply an automatic differentiation module to Vibrato or try Vibrato of Vibrato, or even do a second order automatic differentiation of the computer program. But in finance the payoff is never twice differentiable and so generalized derivatives have to be used requiring approximations of Dirac functions of which the precision is also doubtful.

Consider a financial asset X_t modeled by

$$\mathrm{d}X_t = X_t\Big(r(t)\mathrm{d}t + \sigma(X_t,t)\mathrm{d}W_t\Big), \quad X_0 \text{ given, } W_t \text{ Brownian}$$

A European put option $V_t = e^{-r(T-t)}\mathbb{E}(K-X_T)^+$. How can we compute the 1^{st} and 2^{nd} derivatives of V_0 w.r. to K,T,X_0, r,σ. The most straightforward approach is to use the following approximation

$$\Gamma = \frac{\partial^2 V_0}{\partial X_0^2} \approx \frac{1}{h^2}\Big(V_0|_{X_0+h} - 2V_0|_{X_0} + V_0|_{X_0-h}\Big)$$

It costs 3 times the computation of the option but it is unstable with respect to h: if h is too small round-off errors polute the result because both the numerator and the denominator are small while the result of the division is not.

One trick is to use a complex h. With $\mathbf{i} := \sqrt{-1}$,

$$Re[\frac{f(a+\mathbf{i}\delta a) - f(a)}{\mathbf{i}\delta a}] = Im\frac{f(a+\mathbf{i}\delta a)}{\delta a} = f'(a) - f^{(3)}\frac{\delta a^2}{6} + o(\delta a^3)$$

and it is seen that the expression in the middle is not cursed by a 0 over 0 indeterminacy.

Does it work on non-differentiable functions? For example take $f(x) = (1-x)^+$ leading to $f'(x) = -\mathbf{1}_{x<1}$ and $f"(x) = \delta(1-x)$. Using Maple, Figure 3 shows clearly that complex finite difference to compute the second derivative is superior because it gives a correct result even with a very small complex increment. The conclusion is that unless something is done the results will be unreliable.

Alternatively Malliavin calculus can be used in some but not all cases. For instance, if the payoff V of a vanilla future with spot price $X_t(x)$ where X_t satisfies the Black-Scholes SDE with constant σ and r, with

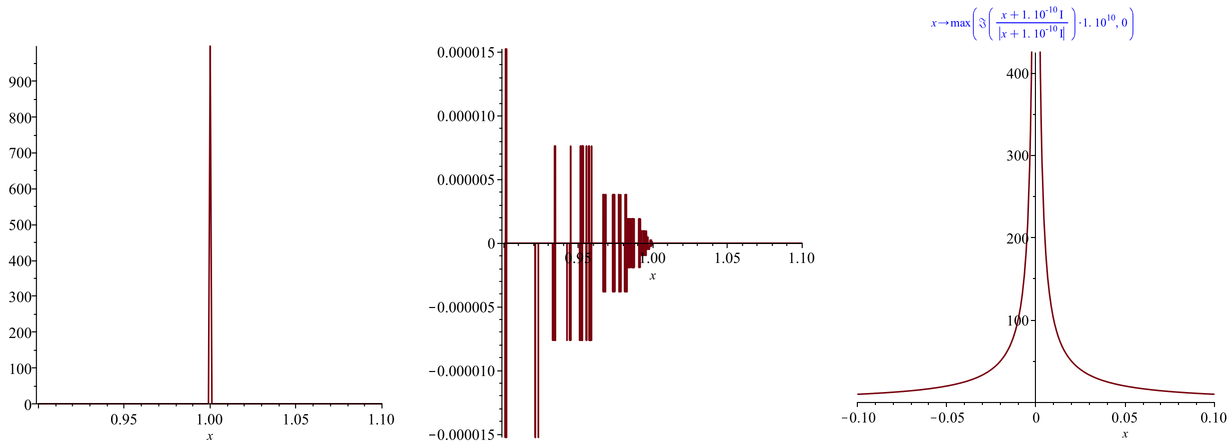

FIGURE 3. using Maple to plot the 2^{nd} derivative of $(1-x)^+$ with $\delta x = 10^{-3}$ (left), $\delta x = 3.510^{-6}$ (middle) and $\delta x = \mathbf{i}10^{-10}$ (right).

$X_0 = x$, then $\Gamma = \mathbb{E}[\frac{e^{-rT}V(X_T)}{x^2T\sigma}(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma})]$, It works so long as the weight functions are known and T not too small, but when applicable it is clearly faster than anything else.

4.1. **Vibrato**

Let us now turn to Vibrato, as introduced by Giles in [15]. Let θ be a parameter and let us focus on $\partial_\theta \mathbb{E}[V(X_T)]$ with $dX_t = b\,(\theta, X_t)\,dt + \sigma(\theta, X_t)dW_t, \quad X_0 = x$. With Z a normal vector valued random variable, an Euler explicit scheme leads to

$$\bar{X}_k = \bar{X}_{k-1} + b(\theta, \bar{X}_{k-1})h + \sigma(\theta, \bar{X}_{k-1})\sqrt{h}Z_k, \quad \bar{X}_0 = x, \quad k = 1, \ldots, n.$$

Now write $\mathbb{E}\left[V(\bar{X}_n)\right] = \mathbb{E}\left[\mathbb{E}\left[V(\bar{X}_n) \mid \bar{X}_{n-1}\right]\right]$ and note that, with $\sigma_{n-1}(\theta) = \sigma(\theta, \bar{X}_{n-1}(\theta))$,

$$\begin{aligned}
&\bar{X}_n = \mu_{n-1}(\theta) + \sigma_{n-1}(\theta)Z_n\sqrt{h} \text{ with } \mu_{n-1}(\theta) = \bar{X}_{n-1}(\theta) + b(\theta, \bar{X}_{n-1}(\theta))h, \\
&\frac{\partial}{\partial \theta_i}\mathbb{E}[V(\bar{X}_n(\theta))] = \mathbb{E}_z\left[\frac{\partial}{\partial \theta_i}\left\{\mathbb{E}[V(\mu + \sigma Z\sqrt{h})]\right\}_{\substack{\mu = \mu_{n-1}(\theta) \\ \sigma = \sigma_{n-1}(\theta)}}\right]
\end{aligned} \tag{1}$$

The last time step sees constant coefficient, so we have an explicit solution

Proposition 1.

$$\begin{aligned}
\frac{\partial}{\partial \theta}\mathbb{E}[V(\bar{X}_n(\theta))] = \mathbb{E}\Bigg[&\frac{1}{\sqrt{h}}\frac{\partial \mu}{\partial \theta} \cdot \mathbb{E}_z\left[V(\mu + \sigma Z\sqrt{h})\sigma^{-T}Z\right] \\
&+ \frac{1}{2}\ \frac{\partial(\sigma\sigma^T)}{\partial \theta} : \mathbb{E}_z\left[V(\mu + \sigma Z\sqrt{h})\sigma^{-T}(ZZ^T - I)\sigma^{-1}\right]\Bigg|_{\substack{\mu = \mu_{n-1}(\theta) \\ \sigma = \sigma_{n-1}(\theta)}}\Bigg]
\end{aligned}$$

Note that in the non constant case the tangent process $Y_t = \partial_\theta X_t$ is involved. Note also that V is not differentiated! The computing time in the non constant case is twice the evaluation of $V(X_0)$, similar to finite difference but much more precise. Note finally that antithetic variance reduction is easy to apply. More details can be found in [26].

4.2. Higher Derivatives with Automatic Differentiation

Once the computer program for the vibrato is written, automatic differentiation can be applied. It is exact because every line of the program is differentiated exactly (teaching the compiler for instance by adding in the AD library that the derivative of $\sin x$ is $\cos x$).

Automatic differentiation can be extended to handle the difficulty raised above for non-differentiable pay-off, to some degree, by approximating the Dirac mass at 0 by $\delta^a(x) = \frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}}$. Now, suppose f is discontinuous at $x = z$ and smooth elsewhere; then $f(x) = f^+(x)H(x-z) + f^-(x)(1 - H(x-z))$; hence

$$f'_z(x) = (f^+)'_z(x)H(x-z) + (f^-)'_z(x)(1 - H(x-z)) - (f^+(z) - f^-(z))\delta(x-z)$$

Unless this last term is added, the computation of the second order sensitivities will be wrong.

If in the Automatic library the ramp function x^+ is defined as $xH(x)$ with its derivative to be $H(x)$, if H is defined with its derivative equal to δ^a and if in the program which computes the financial asset these new functions are used explicitly, like $(x-K)^+ = \mathrm{ramp}(x-K)$ and not $\max(x-K, 0)$, then the second derivative in K computed by the AD library will be $\delta^a(x-K)$. Moreover, it will also compute

$$\int_0^\infty f(x)(x-K)^+ dx \approx \frac{1}{N}\sum_{i=1}^{N} f(\xi_i)\delta^a(\xi_i - K)$$

where ξ_i are the N quadrature points of the integral or the Monte−Carlo points used by the programmer to approximate the integral.

4.3. Conceptual algorithm for VAD

In Algorithm 1, we summarize here what needs to be done to compute a second order sensitivity by VAD with antithetic variance reduction in the general case. To generate M simulation paths with time step $h = \frac{T}{n}$ of the underlying asset X and its tangent process $Y = \dfrac{\partial X}{\partial \theta}$ we need Mn realisations of a normal random variable Z; we need also M_Z realisations of Z_n^m, called $\mathbb{Z}_i$, for the last time step for each m. Then given $\bar{X}_0, \bar{Y}_0$:

Algorithm 1 VAD for second derivatives using antithetic variance reduction

for $m = 1, .., M$ **do**
  for $k = 0, \ldots, n-2$ **do**
    $\bar{X}^m_{k+1} = \bar{X}^m_k + rh\bar{X}^m_k + \bar{X}_k\sigma\sqrt{h}Z^m_{k+1}$
    $\bar{Y}^m_{k+1} = \bar{Y}^m_k + rh\bar{Y}^m_k + r'_\theta h\bar{X}^m_k + \left(\bar{Y}^m_k\sigma + \sigma'_\theta\bar{X}^m_k\right)\sqrt{h}Z^m_{k+1}$
  end for
  for $i = 1, .., M_Z$ **do**
    $X^m_{T_\pm} = \bar{X}^m_{n-1} + rh\bar{X}^m_{n-1} \pm \sigma^m\bar{X}_{n-1}\sqrt{h}\mathbb{Z}_i,\ \bar{X}_{T_o} = \bar{X}_{n-1} + rh\bar{X}_{n-1}$
    $V^m_{T_\pm,} = (\bar{X}^m_{T_\pm} - K)^+,\ V^m_{T_o} = (\bar{X}^m_{T_o} - K)^+.$
    $R = \bar{Y}^m_{n-1}(1 + rh) + \bar{X}^m_{n-1}r'_\theta h, \quad S = \bar{Y}^m_{n-1}\sigma + \bar{X}^m_{n-1}\sigma'_\theta$
    $\left(\dfrac{\partial V_T}{\partial\theta}\right)^m_i = R\dfrac{(V^m_{T_+} - V^m_{T_-})\mathbb{Z}_i}{2\bar{X}^m_{n-1}\sigma\sqrt{h}} + S(V^m_{T_+} - 2V^m_{T_o} + V^m_{T_-})\dfrac{{\mathbb{Z}_i}^2 - 1}{2\bar{X}^m_{n-1}\sigma}$
  end for
  Compute $\left(\frac{\partial^2 V_T}{\partial\theta^2}\right)^m_i$ by AD on the program that implements $\left(\frac{\partial V_T}{\partial\theta}\right)^m_i$
  $\left(\frac{\partial^2 V_T}{\partial\theta^2}\right)^m = \frac{1}{M_Z}\sum_{i=1}^{M_Z}\left(\frac{\partial^2 V_T}{\partial\theta^2}\right)^m_i$
end for
$\frac{\partial^2 V_T}{\partial\theta^2} = \frac{1}{M}\sum_{m=1}^{M}\left(\frac{\partial^2 V_T}{\partial\theta^2}\right)^m.$

American Option

The value V_t of an American option requires the best exercise strategy. Let φ be the payoff; the dynamic programming approximation of the option is

$$\bar{V}_{t_n} = e^{-rT}\varphi(\bar{X}_T), \quad C_{t_k} = \mathbb{E}[e^{-rh}\bar{V}_{t_{k+1}} \mid \bar{X}_{t_k}], \quad \bar{V}_{t_k} = \max\left\{e^{-rt_k}\varphi(\bar{X}_{t_k}), C_{t_k}\right\}. \tag{2}$$

Following Longstaff et al. [25] the continuation value is approximated by a least square projection on I real valued basis functions, t_k, $\{\psi_i\}_{i=1}^I$:

$$C_k \simeq \sum_{i=1}^I \alpha_{k,i}\psi_i(\bar{X}_{t_k}), \ \alpha_{k,\cdot} = \operatorname{argmin}_\alpha \mathbb{E}\left[\left(e^{-rh}\bar{V}_{t_{k+1}} - \sum_{i=1}^I \alpha_{k,i}\psi_i(\bar{X}_k)\right)^2\right]. \tag{3}$$

Once the optimal stopping time τ^* is known, the differentiation with respect to θ can be done as for a European contract. The dependency of the τ^* on θ is neglected; arguably this dependency is second order but this point needs to be validated.

We consider the following parameters : $\sigma = 20\%$ or $\sigma = 40\%$, X_0 varying from 36 to 44, $T = 1$ or $T = 2$, $K = 40$ and $r = 6\%$. The Monte Carlo parameters are: $M = 5\ 10^4$ simulation paths and $T/h = 50$ time steps. The basis in the Longstaff-Scharwtz algorithm is $\psi_i(x) = x^{i-1}, i = 1, 2, 3$, i.e. $I = 3$.

We compare with the solution of the Black-Scholes partial differential inequation discretized by an implicit Euler scheme in time, finite element in space and semi-smooth Newton for the inequalities [1]. A large number of grid points are used, 10^4 to make it a reference solution.

A second order finite Difference approximation is also used to compute the Gamma for comparison.

In Table 1, the results are shown for different sets of parameters taken from Longstaff et al. [25]. The method provides a good precision when variance reduction is used, except when the underlying asset price is small with a small volatility. As for the computation time, the method is faster than Finite Difference.

Table 1. Results of the price, the Delta and the Gamma of an American option. The reference values are obtained via the Semi-Newton method plus Finite Difference, they are compared to Vibrato plus Automatic Differentiation on the Longstaff-Schwartz algorithm. We compute the standard error for each American Monte Carlo results. The settings of the American Monte Carlo are 50 time steps and 50,000 simulation paths.

S	σ	T	Price (AMC)	Standard Error	Delta Vibrato (AMC)	Standard Error	Gamma Ref. Value	Gamma VAD (AMC)	Standard Error
36	0.2	1	4.46289	0.013	0.68123	1.820e−3	0.08732	0.06745	6.947e−5
36	0.2	2	4.81523	0.016	0.59934	1.813e−3	0.07381	0.06398	6.846e−5
36	0.4	1	7.07985	0.016	0.51187	1.674e−3	0.03305	0.03546	4.852e−5
36	0.4	2	8.45612	0.024	0.44102	1.488e−3	0.02510	0.02591	5.023e−5
38	0.2	1	3.23324	0.013	0.53063	1.821e−3	0.07349	0.07219	1.198e−4
38	0.2	2	3.72705	0.015	0.46732	1.669e−3	0.05907	0.05789	1.111e−4
38	0.4	1	6.11209	0.016	0.45079	1.453e−3	0.02989	0.03081	5.465e−5
38	0.4	2	7.61031	0.025	0.39503	1.922e−3	0.02233	0.02342	4.827e−5
40	0.2	1	2.30565	0.012	0.40780	1.880e−3	0.06014	0.05954	1.213e−4
40	0.2	2	2.86072	0.014	0.39266	1.747e−3	0.04717	0.04567	5.175e−4
40	0.4	1	5.28741	0.015	0.39485	1.629e−3	0.02689	0.02798	1.249e−5
40	0.4	2	6.85873	0.026	0.35446	1.416e−3	0.01987	0.02050	3.989e−5
42	0.2	1	1.60788	0.011	0.29712	1.734e−3	0.04764	0.04563	4.797e−5
42	0.2	2	2.19079	0.014	0.28175	1.601e−3	0.03749	0.03601	5.560e−5
42	0.4	1	4.57191	0.015	0.34385	1.517e−3	0.02391	0.02426	3.194e−5
42	0.4	2	6.18424	0.023	0.29943	1.347e−3	0.01768	0.01748	2.961e−5
44	0.2	1	1.09648	0.009	0.20571	1.503e−3	0.03653	0.03438	1.486e−4
44	0.2	2	1.66903	0.012	0.21972	1.487e−3	0.02960	0.02765	2.363e−4
44	0.4	1	3.90838	0.015	0.29764	1.403e−3	0.02116	0.02086	1.274e−4
44	0.4	2	5.58252	0.028	0.28447	1.325e−3	0.01574	0.01520	2.162e−4

Conclusion. Faced with the problem of writing a general software for the computations of sensitivites in connection with Basel III directives, we found that Automatic Differentiation alone is not up to the task, but applied to Vibrato it is dependable and fairly general.

5. CVA with Greeks and AAD

by *Adil Reghai*

5.1. Framework of application

In [34], we proposed a Monte Carlo approach for pricing CVA using the duality relationship between parameter and hedging sensitivities as well as Ito integral calculus.

CVA is the market-value of counterparty credit risk. It is computed as the difference between the risk-free value of the portfolio and its true value that takes into account counterparties default. When both parties to a bilateral contract may default, the price adjustment is called BCVA (Bilateral Credit Valuation Adjustment).

Under the Monte Carlo approach, CVA is rewritten as a weighted sum of expected discounted exposures from time of calculation t to the maturity of the portfolio T. A number of paths for portfolio underlying market variables are simulated and exposures distributions are obtained at each time between t and T. Expected exposure is then given by averaging exposures along simulated paths. Although, Monte Carlo simulations are quite easy to implement, for some complex portfolios this tool could easily lead to heavy computational requirements, thus quickly proving unworkable. Recently, [9] introduced a PDE representation for the value of financial derivatives under counterparty credit risk. Using the Feynman-Kac theorem, the total value of the derivative is decomposed into a default free value plus a second term corresponding to the CVA adjustment. However, for dimensions higher than 3, the PDE can no longer be solved with a finite-difference scheme. Building on this work, [22] introduces a new method based on branching diffusions using a Galton-Watson random tree. Although this method drastically reduces computational costs, it is still complex to implement.

Summarizing, we see that both existing approaches for CVA computation suffer from serious drawbacks. Hence we introduced a new approach for CVA valuation in a Monte Carlo setting. Our approach is innovative in that it combines both the martingale representation theorem and Adjoint Algorithmic Differentiation (AAD) techniques to retrieve future prices in a highly efficient manner.

The first result of [34, Section 4] links input sensitivities to hedging sensitivities as follows:

$$\frac{\partial V_i}{\partial S}(u, S_u^i) = \frac{1}{S_u^i\, f_S(u, S_u^i)} \cdot \frac{\partial V_i}{\partial \phi(u, S_u^i)}. \tag{4}$$

Then, AAD techniques come into play in order to compute hedging sensitivities by applying the chain rule of instructions in a reversed order with regards to its original formulation. As shown in [10], the execution time of the AAD code is bounded by approximately four times the cost of execution of the original function.

Our approach to compute CVA could be summarized by the following four steps procedure:

(1) Generate a first Monte Carlo consisting of $N^{(1)}_{\text{paths}}$ for the stock process. In this first Monte Carlo we determine the initial price $V(t)$ (price at time of CVA valuation) as well as all input sensitivities $\bar{\phi}(t, S)$ using the chain rule of AAD.

(2) Generate a second Monte Carlo consisting of $N^{(2)}_{\text{paths}}$ for the stock process. At each node (path i, time u) of the stock price:
- Get corresponding input sensitivities $\bar{\phi}(u, S_u^i)$ by linear interpolation using results in step 1.
- Calculate corresponding hedging sensitivities using (4)
- Calculate the price along path at time u

(3) Compute CVA along path

(4) Average all CVA contributions in step 3 to get the CVA adjustment.

We displayed the main results of [34] for some equity payoffs. Furthermore, we showed how to extend the methodology for some path dependent payoffs such as barrier options.

5.2. Practical relevance of our research paper

The method we presented has potentially a considerable impact on the way banks calculate and manage their CVA. Indeed, not only AAD methods enhance the computational burden of CVA, but they also allow one to obtain sensitivities with regards to all input parameters (in a row). This turns out very crucial for all banks that need to risk manage their derivatives portfolios.

As far as CVA is concerned, we tested our approach for a number of equity payoffs and compared it to more standard approaches (basically MC of MC to calculate exposures at future times). As a matter of fact, we compared the computing time required to value CVA for a double barrier option (2 barriers: up and down) where no closed form solution for neither price nor CVA exists. For this option, the price is computed using Monte Carlo and CVA using a Monte Carlo of Monte Carlo (simulation of simulation). The main results are summarized in Table 2.

Number of paths for MC-MC	Number of paths for AAD	Computing time: CVA with AAD	Computing time: CVA with MC-MC
MC for prices: 10,000 MC for CVA: 10,000	Step 1: 1,000,000 Step 2: 10,000	29 seconds	5 hours

Table 2

Results have been obtained with R code with a Processor Intel(R) Core(TM) i7-2600 CPU @ 3.40 GHz and 16 GB of installed RAM. So, we see that the speed up in computing time is over 620 times. Moreover, we have directly the surface of hedging sensitivities using the AAD method (with no other simulations required), whereas simulations would be needed to value hedging sensitivities in the standard MC-MC framework.

Disclaimer. Views expressed by authors are the authors' own and do not necessarily reflect the view of the companies they work at.

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