A counterexample to the weak density of smooth maps between manifolds in Sobolev spaces

Fabrice Bethuel

A counterexample to the weak density of smooth maps between manifolds in Sobolev spaces

Fabrice BETHUEL * †

Abstract

The present paper presents a counterexample to the sequential weak density of smooth maps between two manifolds $\mathcal{M}$ and $\mathcal{N}$ in the Sobolev space $W^{1,p}(\mathcal{M},\mathcal{N})$, in the case p is an integer. It has been shown (see e.g. [6]) that, if $p < \dim \mathcal{M}$ is not an integer and the $[p]$-th homotopy group $\pi_{[p]}(\mathcal{N})$ of $\mathcal{N}$ is not trivial, $[p]$ denoting the largest integer less then p, then smooth maps are not sequentially weakly dense in $W^{1,p}(\mathcal{M},\mathcal{N})$. On the other hand, in the case $p < \dim \mathcal{M}$ is an integer, examples have been provided where smooth maps are actually sequentially weakly dense in $W^{1,p}(\mathcal{M},\mathcal{N})$ with $\pi_p(\mathcal{N}) \neq 0$, although they are not dense for the *strong convergence*. This is the case for instance for $\mathcal{M} = \mathbb{B}^m$, the standard ball in $\mathbb{R}^m$, and $\mathcal{N} = \mathbb{S}^p$ the standard sphere of dimension p, for which $\pi_p(\mathcal{N}) = \mathbb{Z}$. The main result of this paper shows however that such a property does not holds for arbitrary manifolds $\mathcal{N}$ and integers p.

Our counterexample deals with the case $p = 3$, $\dim \mathcal{M} \geq 4$ and $\mathcal{N} = \mathbb{S}^2$, for which the homotopy group $\pi_3(\mathbb{S}^2) = \mathbb{Z}$ is related to the Hopf fibration. We construct explicitly a map which is not weakly approximable in $W^{1,3}(\mathcal{M},\mathbb{S}^2)$ by maps in $C^\infty(\mathcal{M},\mathbb{S}^2)$. One of the central ingredients in our argument is related to issues in branched transportation and irrigation theory in the case of the exponent is critical, which are possibly of independent interest. As a byproduct of our method, we also address some questions concerning the $\mathbb{S}^3$-lifting problem for $\mathbb{S}^2$-valued Sobolev maps.

1 Introduction

1.1 Setting and statements

Let $\mathcal{M}$ and $\mathcal{N}$ be two manifolds, with $\mathcal{N}$ isometrically embedded in some euclidean space $\mathbb{R}^\ell$, $\mathcal{M}$ having possibly a boundary. For given numbers $0 < s < \infty$ and $1 \leq p < \infty$, we consider the Sobolev space $W^{s,p}(\mathcal{M},\mathcal{N})$ of maps between $\mathcal{M}$ and $\mathcal{N}$ defined by

$$W^{s,p}(\mathcal{M},\mathcal{N}) = \{u \in W^{s,p}(\mathcal{M},\mathbb{R}^\ell),\ u(x) \in \mathcal{N} \text{ for almost every } x \in \mathcal{M}\}.$$

The study of these spaces is motivated in particular by various problems in physics, as liquid crystal theory, Yang-Mills-Higgs or Ginzburg-Landau models, where singularities of topological nature appear, yielding maps which are hence not continuous but belong to suitable

*Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

†CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

Sobolev spaces, built up in view of the corresponding variational frameworks. Starting with the seminal works of Schoen and Uhlenbeck ([29]), this field of research has grown quite fast in the last decades. A central issue is the approximation of maps in $W^{s,p}(\mathcal{M},\mathcal{N})$ by smooth maps (or maps with singularities of prescribed type) between $\mathcal{M}$ and $\mathcal{N}$. Restricting ourselves to the case $s=1$, is as we will do actually in the rest of the paper, it is easily seen that, if $p>m\equiv\dim\mathcal{M}$ then smooth maps are indeed dense in $W^{1,p}(\mathcal{M},\mathcal{N})$, with no restriction on the target manifold $\mathcal{N}$, since they are already Hölder continuous due to Sobolev embedding: Standard arguments based on convolution by mollifiers and reprojections allow to conclude. The result and the argument extend to the limiting case $p=\dim\mathcal{M}$. It turns out that, when $1\leq p<\dim\mathcal{M}$, the answer to the approximation problem is strongly related to the nature of the $[p]$-th homotopy group $\pi_{[p]}(\mathcal{N})$ of the target manifold $\mathcal{N}$, where $[p]$ denotes the largest integer less or equal to p. Indeed, if $\pi_{[p]}(\mathcal{N})\neq 0$, then as we will recall below, one may construct maps in $W^{1,p}(\mathcal{M},\mathcal{N})$ which cannot be approximated by smooth maps between $\mathcal{M}$ and $\mathcal{N}$ for the *strong topology* (see [6]), whereas the condition $\pi_{[p]}(\mathcal{N})=0$ yields approximability be smooth maps when the domain has a simple topology, for instance a ball (see Sections I to IV in [6]). When the domain $\mathcal{M}$ has a more complicated topology, it was shown in [16, 17] that it might induce some other obstructions to the approximation problem, obstructions which has actually been missed in [6]. [1]

Approximation by sequences of smooth maps at the level of the *weak convergence* is the focus of the present paper. In order to avoid problems with the topology of $\mathcal{M}$ we restrict ourselves *first* to the case $\mathcal{M}=\mathbb{B}^m$, the standard unit ball of $\mathbb{R}^m$ and, motivated by the above discussion, we assume that

$$1\leq p<m \text{ and } \pi_{[p]}(\mathcal{N})\neq 0. \tag{1}$$

Indeed if one of the conditions in (1) is not met, then we already now that $C^\infty(\mathbb{B}^m,\mathcal{N})$ is dense for the strong topology in $W^{1,p}(\mathbb{B}^m,\mathcal{N})$, hence *also* sequentially weakly dense. As a matter of fact, we may even restrict ourselves to the case p is an integer, since the following observation made in [6] settles the case p is not:

Theorem 1. *Assume that* (1) *holds and that p is* not *an integer. Then $C^\infty(\mathbb{B}^m,\mathcal{N})$ is not sequentially weakly dense in $W^{1,p}(\mathbb{B}^m,\mathcal{N})$.*

Sketch of the proof of Theorem 1. The proof relies on a dimension reduction argument together with the fact that homotopy classes are preserved under weak convergence in $W^{1,p}(\mathbb{S}^{m-1},\mathcal{N})$ for $p>m-1$. First, since we assume in view of (1) that $\pi_{m-1}(\mathcal{N})\neq 0$, there exists some smooth map $\varphi:\mathbb{S}^{m-1}\to\mathcal{N}$ such that φ is not homotopic to a constant map and hence cannot be extended continuously to the whole ball $\mathbb{B}^m$. Consider next the map $\mathcal{U}_{\rm sing}$ defined by

$$\mathcal{U}_{\rm sing}(x)=\varphi\left(\frac{x}{|x|}\right), \text{ for } x\in\mathbb{B}^m\setminus\{0\}, \tag{2}$$

which is smooth, except at the origin. Introducing the p-Dirichlet energy E_p defined by

$$\mathrm{E}_p(v,\mathcal{M})=\int_{\mathcal{M}}|\nabla v|^p dx, \text{ for } v:\mathcal{M}\to\mathbb{R}^\ell,$$

[1]the argument in Section V [6], which is aimed to extend the case of a cube to an arbitrary manifolds being erroneous.

we observe that

$$\mathrm{E}_p(\mathcal{U}_{\rm sing}, \mathbb{B}^m) = \int_0^1 r^{m-1-p}\left(\mathrm{E}_p(\varphi, \mathbb{S}^{m-1})\right) dr < +\infty,$$

so that $\mathcal{U}_{\rm sing}$ belongs to $W^{1,p}(\mathbb{B}^m, \mathcal{N})$, provided $1 \leq p < m$. Assume by contradiction that there exist a sequence $(u_n)_{n\in\mathbb{N}}$ of maps in $C^\infty(\mathbb{B}^m, \mathcal{N})$ converging weakly to $\mathcal{U}_{\rm sing}$ in $W^{1,p}(\mathbb{B}^m, \mathbb{R}^\ell)$. Then there exists [2] some radius $0 < r < 1$ such that the restriction of $(u_n)_{n\in\mathbb{N}}$ to the sphere $\mathbb{S}^{m-1}_r$ of radius r and centered at 0 converges up to a subsequence to the restriction of $\mathcal{U}_{\rm sing}$ to $\mathbb{S}^{m-1}_r$ in $W^{1,p}(\mathbb{S}^{m-1}_r)$. By compact Sobolev embedding the convergence is *uniform* and hence $\mathcal{U}_{\rm sing}$ and u_n restricted to $\mathbb{S}^{m-1}_r$ are in the same homotopy class for n large. This however is a contradiction, since u_n can be extended inside the sphere $\mathbb{S}^{m-1}_r$ whereas the restriction of $\mathcal{U}_{\rm sing}$ to $\mathbb{S}^{m-1}_r$ does not possess this property. This contradiction establishes the theorem in the case considered.

When $p = m - 1$ is an integer, the previous arguments *can not be extended*, since weak convergence in $W^{1,m-1}(\mathbb{S}^{m-1}_r)$ does not necessarily yield uniform convergence. As a matter of fact, we have in this case:

Proposition 1. *There exists a sequence of maps $(U_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{B}^m, \mathcal{N})$ converging to $\mathcal{U}_{\rm sing}$ weakly in $W^{1,m-1}(\mathbb{B}^m, \mathbb{R}^\ell)$. Moreover, the sequence $(U_n)_{n\in\mathbb{N}}$ has the following properties:*

- *The sequence $(U_n)_{n\in\mathbb{N}}$ converges uniformly on every compact set of $\mathbb{B}^m \setminus \mathcal{I}_m$ to $\mathcal{U}_{\rm sing}$, where $\mathcal{I}_m$ denotes the segment $\mathcal{I}_m = [0, \mathbb{P}_{\rm north}]$ where $\mathbb{P}_{\rm north}$ denotes the north pole $\mathbb{P}_{\rm north} = (0, \ldots, 0, 1) \in \mathbb{R}^m$.*
- *We have the convergenve*

$$|\nabla U_n|^{m-1} \rightharpoonup |\nabla \mathcal{U}_{\rm sing}|^{m-1} + \nu \mathcal{H}^1 \llcorner [0, \mathbb{P}_{\rm north}] \text{ in the sense of measures on } \mathbb{B}^m, \text{ where}$$
$$\nu = \mathrm{v}_{\mathcal{N},m-1}([\![\varphi]\!]) = \inf\{E_{m-1}(w), w \in C^1(\mathbb{S}^{m-1}, \mathcal{N}) \text{ homotopic to } \varphi\} > 0. \tag{3}$$

Since this type of results is central in the whole discussion, we briefly sketch the argument. The proof of Proposition 1 combines a dimension reduction argument similar to the one we used for Theorem 1 together with the *bubbling phenomenon* occuring in dimension $m - 1$ for which the E_{m-1} energy is scale invariant. We discuss this property first.

The bubbling phenomenon. We recall first the scaling properties of the functional E_p. Consider more generally an arbitrary integer $\mathrm{q} \in \mathbb{N}$, $p > 0$ and an arbitrary map $u : \mathbb{B}^{\mathrm{q}} \to \mathcal{N}$. The scaling transformations yields the formula, for $r > 0$

$$\mathrm{E}_p(u_r, \mathbb{B}^{\mathrm{q}}_r) = r^{\mathrm{q}-p}\mathrm{E}_p(u, \mathbb{B}^{\mathrm{q}}) \text{ where } u_r(x) = u(\frac{x}{r}) \text{ for } x \in \mathbb{B}^{\mathrm{q}}_r \equiv \mathbb{B}^{\mathrm{q}}(0, r), \tag{4}$$

In particular in the critical case where the exponent is equal to the dimension, i.e. when we have $p = \mathrm{q}$, then the energy is scale invariant, namely $\mathrm{E}_p(u_r, \mathbb{B}^{\mathrm{q}}_r) = \mathrm{E}_p(u, \mathbb{B}^{\mathrm{q}})$. Choosing small values for r, this invariance allows for concentration of $q-energy$ at *isolated points* for weakly converging sequences.

[2]similar arguments, based an Fubini's theorem combined with an averaging argument, will be detailed in Section 5.

We next replace the domain $\mathbb{B}^q$ by the sphere $\mathbb{S}^q$ of same dimension and consider now regular maps from $\mathbb{S}^q$ to $\mathcal{N}$ assuming that $\pi_q(\mathcal{N})$ is not trivial. Given $\varphi \in C^\infty(\mathbb{S}^q, \mathcal{N})$ we denote by $[\![\varphi]\!]$ its homotopy class. Homotopy class are not preserved in $W^{1,q}$ under weak convergence as the next result shows.

Lemma 1. *Let $\varphi : \mathbb{S}^q \to \mathcal{N}$ be a given smooth map. Then there exists a sequence of smooth maps $(\varphi_n)_{n\in\mathbb{N}}$ from $\mathbb{S}^q$ to $\mathcal{N}$ such that such that the following holds*

- *φ_n is homotopic to a constant map for any $n \in \mathbb{N}$*
- *$\varphi_n(x) = \varphi(x)$, for any $n \in \mathbb{N}^\star$, for any $x \in \mathbb{S}^q \setminus \mathbb{B}^{q+1}(\mathbb{P}_{\rm north}, (n+1)^{-1})$ where $\mathbb{P}_{\rm north}$ denotes the north pole $\mathbb{P}_{\rm north} = (0, \dots, 0, 1)$*
- *$|\nabla\varphi_n|^q \rightharpoonup |\nabla\varphi|^q + \nu_q\delta_P$ in the sense of measures on $\mathbb{S}^q$* as $n \to +\infty$, *where we have set*

$$\nu_q = \nu_{\mathcal{N},q}([\![\varphi]\!]) = \inf\{E_q(w), w \in C^1(\mathbb{S}^q, \mathcal{N}) \text{ homotopic to } \varphi\} > 0. \tag{5}$$

The idea of the proof of Lemma 1 is to glue a scaled copy of a minimizer or an almost minimizer for (5) at the north pole $\mathbb{P}_{\rm north}$.

Remark 1. There is also a kind of converse to Lemma 1. Indeed, given any sequence $(\psi_n)_{n\in\mathbb{N}}$ of smooth maps from $\mathbb{S}^q$ to $\mathcal{N}$, there exists a subsequence still denoted $(\psi_n)_{n\in\mathbb{N}}$, points $a_1, \dots, a_s$, positive numbers $\mu_1, \dots, \mu_s$ and a positive measure $\omega_\star$ such that

$$|\nabla\varphi_n|^q \rightharpoonup |\nabla\varphi|^q + \sum_{i=1}^{s} \mu_i\delta_{a_i} + \omega_\star \text{ in the sense of measures on } \mathbb{S}^q \textit{ as } n \to +\infty, \tag{6}$$

with $\sum \mu_i \geq \nu_q$. We consider next the minimal energy of weakly approximating sequences namely the number $\tau_\star(\varphi)$ given by

$$\tau_\star(\varphi) \equiv \inf\left\{\liminf_{n\to+\infty} \mathrm{E}_q(w_n), (w_n)_{n\in\mathbb{N}} \text{ s.t. } [\![w_n]\!] = 0 \text{ and } w_n \underset{n\to+\infty}{\rightharpoonup} \varphi\right\}. \tag{7}$$

We may write $\tau_\star(\varphi) = \mathrm{E}_q(\varphi) + \epsilon_\star(\varphi)$. In view of Banach-Steinhaus theorem, we have $\epsilon_\star(\varphi) \geq 0$: The number $\epsilon_\star(\varphi)$ will be called *the defect energy* for approximating sequences. If the sequence $(\psi_n)_{n\in\mathbb{N}}$ fulfills the optimality condition

$$\liminf_{n\to+\infty} \mathrm{E}_q(\psi_n) = \mathrm{E}_q(\varphi) + \epsilon_\star(\varphi)$$

then, one may show that we have $\omega_\star = 0$ and $\sum \mu_i = \nu_q$. Hence one deduces that the defect energy is given by

$$\epsilon_\star(\varphi) = \nu_q, \tag{8}$$

a number which depends only on the homotopy class of φ.

Sketch of the proof of Proposition 1. Proposition 1 is deduced from Lemma 1 for the choice $q = p = m - 1$, constructing the sequence $(U_n)_{n\in\mathbb{N}}$ as

$$U_n(x) = \varphi_n\left(\frac{x}{|x|}\right) \text{ for } \frac{1}{n} \leq |x| \leq 1. \tag{9}$$

and extend U_n inside the small ball $\mathbb{B}(\frac{1}{n})$ is a smooth way: This is possible since the map φ_n is in the trivial homotopy class, and with an energetical cost tending to 0 and n goes to $+\infty$. Since the energy of the map φ_n concentrates at the North Pole $\mathbb{P}_{\rm north}$ in view of Lemma 1, it follows from the construction (9) that the $(m-1)$-energy of the sequence $(U_n)_{n\in\mathbb{N}}$ concentrates on the radial extension of the North Pole, that is the segment $[0, \mathbb{P}_{\rm north}]$.

After this digression, we come back to the general problem of sequential weak density of smooth maps. In view of the previous discussion, the main problem to consider is the case

$$p \text{ is an integer, } 1 \le p < m \text{ and } \pi_{[p]}(\mathcal{N}) \neq 0. \tag{10}$$

So far several results have been obtained, where *sequentially weak density* of smooth maps between $\mathbb{B}^m$ and $\mathcal{N}$ have been established[3]. For instance, when $\mathcal{N} = \mathbb{S}^p$ for which $\pi_p(\mathcal{N}) = \mathbb{Z}$ we have:

Theorem 2 ([10, 5, 6])**.** *Let p be an integer. Then given any manifold $\mathcal{M}$, $C^\infty(\mathcal{M}, \mathbb{S}^p)$ is sequentially weakly dense in $W^{1,p}(\mathcal{M}, \mathbb{S}^p)$.*

In a related direction, a positive answer was given in [15, 26] for $(p-1)$-connected manifolds $\mathcal{N}$ and in [26] in the case $p = 2$, whatever manifold $\mathcal{N}$, similar results involving the H^2 energy are given in [22]. The main result of this paper presents an *obstruction to sequential weak density of smooth maps* when (10) holds and deals with the special case $\mathcal{N} = \mathbb{S}^2$ and $p = 3$, for which $\pi_3(\mathbb{S}^2) = \mathbb{Z}$. More precisely, the main result of this paper is the following:

Theorem 3. *Given any manifold $\mathcal{M}$ of dimension larger or equal to 4, $C^\infty(\mathcal{M}, \mathbb{S}^2)$ is* not sequentially weakly dense *in $W^{1,3}(\mathcal{M}, \mathbb{S}^2)$.*

As a matter of fact, the topology and the nature of the manifold $\mathcal{M}$ is of little importance in the proof. We rely indeed on the construction of a counterexample in the special case $\mathcal{M} = \mathbb{B}^4$, imposing however an additional condition on the boundary $\partial\mathbb{B}^4$.

Theorem 4. *There exists a map $\mathcal{U}$ in $W^{1,3}(\mathbb{B}^4, \mathbb{S}^2)$ which is not the weak limit in $W^{1,3}(\mathbb{B}^4, \mathbb{R}^3)$ of smooth maps between $\mathbb{B}^4$ and $\mathbb{S}^2$. Moreover the restriction of $\mathcal{U}$ to the boundary $\partial\mathbb{B}^4 = \mathbb{S}^3$ is a constant map.*

As far as we are aware of, this is the first case where *an obstruction to sequential weak density* of smooth maps between manifolds has been established when p is an integer. Theorem 3 also answers a question explicitly raised in [19, 20, 21].

Let us emphasize that the map $\mathcal{U}$ constructed in theorem 4 *necessarily must have a infinite number of singularities*, and is hence very different from the example $\mathcal{U}_{\rm sing}$ provided in (2). Indeed, let us recall that, for $m - 1 \le p < m$, the set of maps with a finite number of isolated singularities

$$\mathcal{R}^p(\mathbb{B}^m, \mathcal{N}) = \{u \in W^{1,p}(\mathbb{B}^m, \mathcal{N}), \text{ s.t } \ u \in C^\infty(\mathbb{B}^m \setminus A) \text{ for a finite set } A\}. \tag{11}$$

is not only dense in $W^{1,p}(\mathbb{B}^m, \mathcal{N})$ for the strong topology, but, in the case $p = m - 1$, is also contained in the sequential weak closure of smooth maps with values into $\mathcal{N}$. The proof of this latest fact, given in [5, 6, 10] and which will be sketched in a moment, is actually inspired

[3] In several of these results, an additional boundary condition is imposed.

by a method introduced in the seminal work of Brezis, Coron and Lieb [11], and along the same idea the singularity of $\mathcal{U}_{\text{sing}}$ was removed using concentration of energy along lines connecting the singularity to the boundary, or possibly to other singularities with opposite topological charges. In view of (3), the energy of the constructed approximating maps are controlled in the limit by a term which is of the order of the length of the connecting lines, multiplied by a topological charge. This number, which corresponds to a defect energy, is obviously bounded when the number of singularities is finite, yielding hence the mentioned weak approximability of maps in $\mathcal{R}(\mathbb{B}^4,\mathcal{N}) \equiv \mathcal{R}^3(\mathbb{B}^4,\mathcal{N})$ by smooth maps. We may however not a priori exclude the fact that, when approximating a map in $W^{1,3}(\mathbb{B}^4,\mathcal{N})$ by maps with a finite number of singularities, the defect energy grows when the number of singularities grows. As a matter of fact, our strategy in the proof of Theorem 4 is to produce a map $\mathcal{U}$ for which this phenomenon occurs.

As this stage, it is worthwhile to compare, when the exponent p is equal to 3, the results obtained for the respective cases the target manifolds are $\mathbb{S}^2$ or $\mathbb{S}^3$. In both cases the homotopy groups are similar, since $\pi_3(\mathcal{N}) = \mathbb{Z}$, for $\mathcal{N} = \mathbb{S}^2$ or $\mathcal{N} = \mathbb{S}^3$. However, we obtain, provided $\dim \mathcal{M} \geq 4$, sequentially weak density of smooth maps in the case $\mathcal{N} = \mathbb{S}^3$ thanks to Theorem 2, whereas in the case $\mathcal{N} = \mathbb{S}^2$, we obtain exactly the opposite result, since there are obstructions to weak density of smooth maps in view of Theorem 4. Hence ultimately, not only the nature of the homotopy group matters, but also more subtle issues related to the way its elements behave according to the Sobolev norms and the E_3 energy.

In the next subsection, we review with more details the constructions mentioned above and emphasize its connection with optimal transportation theory.

1.2 Defect measures and optimal transportation of topological charges

As in Remark 1, but now in a higher dimension, given $u \in W^{1,3}(\mathbb{B}^4,\mathcal{N})$, we introduce the *defect energy* $\epsilon_\star(u)$ related to its weak approximability by smooth maps defined by

$$\mathrm{E}_3(u) + \epsilon_\star(u) \equiv \inf \left\{ \liminf_{n\to+\infty} \mathrm{E}_3(w_n), (w_n)_{n\in\mathbb{N}} \text{ s.t. } w_n \in C^\infty(\mathbb{B}^4,\mathcal{N}) \text{ and } w_n \underset{n\to+\infty}{\rightharpoonup} u \right\}, \quad (12)$$

with the convention that $\epsilon_\star(u) = +\infty$ if u cannot be approximated weakly by smooth maps. In this subsection, we specify the discussion to maps u with a finite number of singularities and describe briefly how one may approximate maps in $\mathcal{R}(\mathbb{B}^4,\mathcal{N})$ weakly by smooth maps in $W^{1,3}$-norm and how this leads to upper bounds for the defect energy $\epsilon_\star$. As for identity (8) in Remark 1, the numbers $\nu_{\mathcal{N},3}$ enter directly in these estimates and we describe first some relevant properties of these numbers in the special cases $\pi_3(\mathcal{N}) = \mathbb{Z}$, emphasizing thereafter asymptotic properties in the cases $\mathcal{N} = \mathbb{S}^3$ or $\mathcal{N} = \mathbb{S}^2$.

Infimum of energy in homotopy classes when $\pi_3(\mathcal{N}) = \mathbb{Z}$. When $\pi_3(\mathcal{N}) = \mathbb{Z}$, each homotopy class in $C^0(\mathbb{S}^3,\mathcal{N})$ can be labelled by an integer which will be termed *the topological charge* of the homotopy class or of its elements. Setting in this case, for d given in $\mathbb{Z}$

$$\nu_{\mathcal{N}}(d) \equiv \nu_{\mathcal{N},3}([\![\varphi]\!]) \text{ with } [\![\varphi]\!] = d,$$

We verify that $\nu_{\mathcal{N}}(-d) = \nu_{\mathcal{N}}(d)$ and that concentrating bubbles of topological charge ± 1 at $|d|$ distinct points, we are led, for $d \in \mathbb{Z}$, to the upper bound

$$\nu_{\mathcal{N}}(d) \leq |d|\nu_{\mathcal{N}}(1) \text{ and more generally } \nu_{\mathcal{N}}(kd) \leq k\nu_{\mathcal{N}}(d) \text{ for } k \in \mathbb{N}. \quad (13)$$

A natural question is therefore to determine whether this upper bound on $\nu_{\mathcal{N}}(d)$ *is sharp or not*. It turns out that the answer to the previous question strongly depends on the nature of the target manifold $\mathcal{N}$.

Asymptotic behavior of $\nu_{\mathcal{N}}(d)$ as $|d| \to +\infty$ when $\mathcal{N} = \mathbb{S}^3$ and $\mathcal{N} = \mathbb{S}^2$. When $\mathcal{N} = \mathbb{S}^3$ the topological charge is called the *degree* and denoted $\deg(\varphi)$. It can be proved (see Section 2) that, for any $\varphi : \mathbb{S}^3 \to \mathbb{S}^3$, one has

$$\int_{\mathbb{S}^3} |\nabla \varphi|^3 dx \geq |\mathbb{S}^3||\deg(\varphi)|,$$

so that, setting $\nu_3(d) = \nu_{\mathbb{S}^3}(d)$ we are led to the identity

$$\nu_3(d) = |\mathbb{S}^3||d|. \tag{14}$$

When $\mathcal{N} = \mathbb{S}^2$, the topological charge is usually called the *Hopf* invariant and denoted in this paper $\mathbb{H}(\varphi)$. As we will recall in Section 2 (see (2.11)), one verifies easily that for any map $u : \mathbb{S}^3 \to \mathbb{S}^2$, we have the lower bound

$$\int_{\mathbb{S}^3} |\nabla u|^3 dx \geq \mathrm{C}_\nu |d|^{\frac{3}{4}}, d = \mathbb{H}(u), \tag{15}$$

so that $\nu_2(d) \geq \mathrm{C}_\nu |d|^{\frac{3}{4}}$, where $\mathrm{C}_\nu > 0$ is some universal constant, and where we have set $\nu_2(d) = \nu_{\mathbb{S}^2}(d)$. In [28], Rivière made the remarkable observation that the bound on the left hand side is in fact optimal, that is, there exist a universal constant $\mathrm{K}_\nu > 0$ such that, for any d

$$\nu_2(d) \leq \mathrm{K}_\nu |d|^{\frac{3}{4}} \text{ with } \nu_2(d) = \nu_{\mathbb{S}^2}(d), \tag{16}$$

so that the function ν_2 is actually sublinear on $\mathbb{N}$ and in other words, the minimal energy necessary for creating a map of charge d is no longer proportional to $|d|$, but grows in fact sublinearily as $|d|^{\frac{3}{4}}$. This fact has in turn important consequences on the way to connect optimally defect for maps from $\mathbb{B}^4$ to $\mathbb{S}^2$ having a finite number of singularities and the definition of the corresponding defect measures[4].

Removing singularities of maps in $\mathcal{R}_{\mathrm{ct}}(\mathbb{B}^4, \mathcal{N})$ when $\pi_3(\mathcal{N}) = \mathbb{Z}$. Consider again an arbitrary manifold with $\pi_3(\mathcal{N}) = \mathbb{Z}$ and a map $v \in \mathcal{R}_{\mathrm{ct}}(\mathbb{B}^4, \mathcal{N})$, the subset of $\mathcal{R}(\mathbb{B}^4, \mathcal{N})$ of maps which are constant on the boundary $\partial \mathbb{B}^4$, that is

$$\mathcal{R}_{\mathrm{ct}}(\mathbb{B}^4, \mathcal{N}) = \{u \in \mathcal{R}(\mathbb{B}^4, \mathcal{N}), u \text{ is constant on } \partial \mathbb{B}^4\}.$$

Given a singularity a of v, the homotopy class of the restriction of v to any small sphere centered at a does not depend on the radius, provided the later is sufficiently small. We will denote $[\![a]\!]$ this element in $\pi_3(\mathcal{N})$ and in the case $\pi_3(\mathcal{N}) = \mathbb{Z}$, the number d labeling the homotopy class $[\![a]\!]$ will be referred to as a the *topological charge of the singularity a*. For sake of simplicity, we assume that that all singularities have either topological charges $+1$ or -1[5]. We denote by $P_1, \ldots, P_r$ the singularities of charge $+1$ and $Q_1, \ldots, Q_r$ the singularities of charge -1: Since we assume that the map v is constant on the boundary, there is indeed

[4]in [19], the authors extend this discussion to several other targets.

[5]This is not a true restriction, since the class of maps having this property is also strongly dense

an equal number of singularities of charge $+1$ and -1. In order to approximate weakly v by smooth maps, we adapt the idea of the proof of Proposition 1. For $i = 1, \dots, r$, we consider bounded curves $\mathcal{L}_i$ joining the singularities of opposite charges, for instance P_i with Q_i, and construct a sequence of smooth maps $(\varphi_n)_{n\in\mathbb{N}}$ such that

$$|\nabla\varphi_n|^3 \rightharpoonup |\nabla v|^3 + \mu_* \text{ as } n \to +\infty \quad \text{where } \mu_* = \nu_{\mathcal{N}}(1)\mathcal{H}^1 \llcorner \left(\bigcup_{i=1}^{r} \mathcal{L}_i\right), \tag{17}$$

so that

$$\lim_{n\to\infty} \mathrm{E}_3(\varphi_n) = \mathrm{E}_3(v) + |\mu_*| \text{ with } |\mu_*| = \nu_{\mathcal{N}}(1)\left(\sum_{i=1}^{r}\mathcal{H}^1(\mathcal{L}_i)\right). \tag{18}$$

The measure μ_* represents a defect energy measure for the above convergence, and it follows from the definition of $\epsilon_\star$ that

$$\epsilon_\star(u) \leq |\mu_\star|, \tag{19}$$

so that a good estimate for $|\mu_\star|$ yields an estimate of the defect energy. Notice that the formula for $\mu_\star$ given in (18) depends not only on the position of the singularities but also on the way we choose to connect them. In order to obtain general weak approximation results, we choose therefore optimal connections of the singularities, with the hope that the upper bound (19) can be turned into a related lower bound. It turns out that this program can be completed in the case $\mathcal{N} = \mathbb{S}^3$.

Minimal connections for $\mathcal{N} = \mathbb{S}^3$. This notion has been introduced in the present context in [11]. Consider as above v in $\mathcal{R}_{\mathrm{ct}}(\mathbb{B}^4, \mathbb{S}^3)$, with topological charges ± 1 and constant on the boundary. In order to have the value of energy defect as small as possible, it is natural to connect the singularities with straight segments and to choose the configuration with the smallest total length. This leads to introduce the notion of length of a minimal connection between the points $\{P_i\}_{i\in J}$ and $\{Q_i\}_{i\in J}$ given by

$$\mathrm{L}\left(\{P_i\}, \{Q_i\}\right) = \inf\left\{\sum_{i\in J}|P_i - Q_{\sigma(i)}|, \text{ for } \sigma \in \mathfrak{S}\right\}, \tag{20}$$

where $J = \{1, \dots, r\}$ and $\mathfrak{S}$ denotes the set of permutations of J. In the language of optimal transportation, this can be rephrased as the optimal transportation of the measure $\sum \delta_{P_i}$ to the measure $\sum \delta_{Q_i}$ with cost functional given by the distance function. Going back to (18) we obtain hence

$$|\mu_\star| = |\mathbb{S}^3|\,\mathrm{L}(v) \text{ where } \mathrm{L}(v) \equiv \mathrm{L}(\{P_i\}, \{Q_i\}) \text{ since } \nu_{\mathcal{N}}(1) = |\mathbb{S}^3|. \tag{21}$$

The important observation made in [11] (see also [1] for a different proof) is that the length of a minimal connection can be related to the energy of the map as follows

$$\mathrm{E}_3(v) \geq |\mathbb{S}^3|\mathrm{L}(v) = |\mathbb{S}^3|\,\mathrm{L}(\{P_i\}, \{Q_i\}), \tag{22}$$

so that, in view of (19), the defect energy $\epsilon_*(v)$ is bounded by

$$\epsilon_*(v) \leq \mathrm{E}_3(v). \tag{23}$$

Using the fact that $\mathcal{R}_{\rm ct}(\mathbb{B}^4,\mathbb{S}^3)$ is dense in $W^{1,p}_{\rm ct}(\mathbb{B}^4,\mathbb{S}^3)$ where

$$W^{1,p}_{\rm ct}(\mathbb{B}^4,\mathbb{S}^3)=\{v\in\mathcal{R}(\mathbb{B}^4,\mathbb{S}^3),\text{s.t } v \text{ is constant on } \partial\mathbb{B}^4\}$$

we deduce from (23) that maps in $C^\infty_{\rm ct}(\mathbb{B}^4,\mathbb{S}^3)$ are sequentially weakly dense in $W^{1,p}_{\rm ct}(\mathbb{B}^4,\mathbb{S}^3)$. As a matter of fact, it can even be shown that $\varepsilon_\star=|\mu_\star|$ so that our previous construct is optimal (see [7, 13, 14]). This means that given any sequence $(v_n)_{n\in\mathbb{N}}$ of maps in $C^\infty_{\rm ct}(\mathbb{B}^4,\mathbb{S}^3)$ converging weakly to v one has

$$\liminf_{n\to+\infty}\mathrm{E}(v_n)\geq \mathrm{F}(v)\equiv \mathrm{E}_3(v)+|\mathbb{S}^3|\mathrm{L}(v), \tag{24}$$

and as shown before, there are sequences for which equality holds. Morever, it can be proved (see e.g [13]) that any sequence such that equality holds in (24) behaves according to (17). Both functionals L and F, which is termed the relaxed energy of the problem (see [7]), are continuous in the space $W^{1,3}(\mathbb{B}^4,\mathbb{S}^3)$, F being lower-semicontinuous for the weak convergence.

Remark 2. We have assumed that all singularities have only topological charges of values ±1: This is indeed not a restriction since the subset of $\mathcal{R}_{\rm cte}(\mathbb{B}^4,\mathbb{S}^2)$ maps with topological charges ±1 is also dense. When $\mathcal{N}=\mathbb{S}^3$, multiplicities do not really affect the property of the functional L, it suffices to repeat each singularity in the collection according to its multiplicity.

Removing singularities of maps in $\mathcal{R}_{\rm ct}(\mathbb{B}^4,\mathbb{S}^2)$: Branched transportation. The approximation scheme proposed in (17) is not optimal when the growth of $\nu_{\mathcal{N}}$ is sublinear, that means that the defect energy $\epsilon_\star$ might by much smaller then $|\mu_\star|$ as constructed above. We illustrate this on the case $\mathcal{N}=\mathbb{S}^2$.

Given u in $\mathcal{R}_{\rm ct}(\mathbb{B}^4,\mathbb{S}^2)$ and assuming as before that all topological charges are equal to ±1, we approximate weakly u by smooth maps from $\mathbb{B}^4$ to $\mathbb{S}^2$ connecting again the positive charges $(P_i)_{i\in J}$ to the negative charges $(Q_i)_{i\in J}$. In contrast with the case $\mathcal{N}=\mathbb{S}^3$ however, straight lines joining positive charges to negative charges may however not be the optimal solution. Indeed, it may be energetically more favorable, in view of the subadditivity property (16), that some parts of the connection carry a higher topological charge and we need therefore to introduce branching points. Such a connection with branching points has been modeled by Q. Xia in his pioneering work [30] with the notion of *transport path.* We adapt this notion and term it in our setting *branched connection*, a notion depending only on the distribution of the charges.

A branched connection associated to the distribution of points $A=\{P_i,Q_i\}_{i\in J}$ is given as a directed graph G in $\mathbb{B}^4$ with corresponding source points given by the distribution. It is is represented by the following data:

- a finite vertex set $V(G)\subset\overline{\mathbb{B}^4}$, such that the collection of source points belongs to $V(G)$, thet is $A\subset V(G)$. There may also be other points, called *branching points.*
- A set $E(G)$ of *oriented segments* joining the vertices, possibly with multiplicity d: For $\vec{\mathbf{e}}\in E(G)$, we denote by e^- and e^+ the *endpoints* of e, so that $\vec{\mathbf{e}}=[e^-,e^+]$, with $e^-,e^+\in V(G)$.

For $a \in V(G)$, set $E^{\pm}(a,G) = \{e \in E(G), e^{\pm} = a\}$. We impose for $a \in V(G) \setminus \partial\Omega$ the *Kirchhoff law*

$$\begin{cases} \sharp\left(E^-(a,G)\right) = \sharp\left(E^+(a,G)\right) + 1 \text{ if } a \in \{P_i\}_{i\in J} \\ \sharp\left(E^-(a,G)\right) = \sharp\left(E^+(a,G)\right) - 1 \text{ if } a \in \{Q_i\}_{i\in J} \\ \sharp\left(E^-(a,G)\right) = \sharp\left(E^+(a,G)\right) \text{ if } a \text{ is a branching point,} \end{cases} \tag{25}$$

In our context, the multiplicity or density $d(\vec{\mathbf{e}})$ of a segment represents the topological charge carried through the segment $\vec{\mathbf{e}}$ and relation (25) expresses a conservation of this charge at the vertex points, with a source provided by the topological charges at the point singularities $\{P_i, Q_i\}$. We denote by $\mathcal{G}(\{P_i, Q_i\}_{i\in J})$ the set of all graphs having the previous properties and introduce the quantity

$$\mathrm{L}_{\mathrm{branch}}(\{P_i, Q_i\}_{i\in J}) = \inf\{\mathbf{W}_2(G), G \in \mathcal{G}(\{P_i, Q_i\}_{i\in J})\},$$

where the functional $\mathbf{W}_2(G)$ is the weighted length of the graph connection defined by

$$\mathbf{W}_2(G) = \sum_{e\in E(G)} \nu_2(d(\vec{\mathbf{e}}))\mathcal{H}^1(\vec{\mathbf{e}}) \text{ for } G \in \mathcal{G}(\{P_i, Q_i\}_{i\in J}). \tag{26}$$

As a matter of fact, we may notice at this point that the length of a minimal connection $\mathrm{L}(P_i, Q_i)$ may be defined using the same framework as the infimum of the function $\mathbf{W}_3(G)$ defined according to the formula (26) with ν_2 turned into ν_3: However a optimal connection will not require additional branching points.

The functional $\mathrm{L}_{\mathrm{branch}}$ plays now a similar role for $\mathbb{S}^2$ valued maps as did the length of a minimal connection for $\mathbb{S}^3$ valued maps: It yields the defect energy when approximating maps in $\mathcal{R}_{\mathrm{ct}}(\mathbb{B}^4, \mathbb{S}^2)$ by sequences of smooth maps between $\mathbb{B}^4$ and $\mathbb{S}^2$. Indeed, let $u \in \mathcal{R}_{\mathrm{ct}}(\mathbb{B}^4, \mathbb{S}^2)$ an G be a graph in $\mathcal{G}(P_i, Q_i)$, where $\{P_i, Q_i\}_{i\in J}$ denotes the set of singularities of u. Using concentration of maps along the segements composing G, with the corresponding multiplicity, one may construct a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of maps in $C^\infty_{\mathrm{ct}}(\mathbb{B}^4, S^2)$ converging weakly to u such that

$$|\nabla\varphi_n|^3 \rightharpoonup |\nabla u|^3 + \mu_* \text{ as } n \to +\infty \text{ where } \mu_* = \mathcal{H}^1 \llcorner \left(\underset{\vec{\mathbf{e}}\in E(G)}{\cup} \nu_2(d)\vec{\mathbf{e}}\right),$$

so that

$$\lim_{n\to+\infty} \mathrm{E}_3(\varphi_n) = \mathrm{E}_3(u) + |\mu_\star| \text{ with } |\mu_\star| = \mathrm{W}_2(\mathrm{G}).$$

Choosing the graph G as a minimizer for $\mathrm{W}_2(\mathrm{G})$ we obtain $|\mu_\star| = \mathrm{L}_{\mathrm{branch}}(\{P_i, Q_i\}_{i\in J})$ so that

$$\epsilon_\star(u) \leq \mathrm{L}_{\mathrm{branch}}(\{P_i, Q_i\}_{i\in J}).$$

The reverse inequality is also valid. More precisely it has been proved in [19] that, if $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of maps in $C^\infty_{\mathrm{ct}}(\mathbb{B}^4, \mathbb{S}^2)$ such that $\varphi_n \rightharpoonup u$ in $W^{1,3}(\mathbb{B}^4, \mathbb{S}^2)$, then

$$\liminf_{n\to\infty} \mathrm{E}_3(\varphi_n) \geq \mathrm{E}_3(u) + \mathrm{L}_{\mathrm{branch}}(\{P_i, Q_i\}_{i\in J}). \tag{27}$$

So that we finally have

$$\epsilon_\star(u) = \mathrm{L}_{\mathrm{branch}}(\{P_i, Q_i\}_{i\in J}), \forall u \in \mathcal{R}_{\mathrm{ct}}(\mathbb{B}^4, \mathbb{S}^2).$$

The defect energy is hence again described by a quantity involving only the location of the singularities and the sign of their topological charge. For further uses, we will use the notation

$$\mathrm{L}_{\mathrm{branch}}(u) \equiv \mathrm{L}_{\mathrm{branch}}(\{P_i, Q_i\}_{i\in J}) \text{ for } u \in \mathcal{R}_{\mathrm{ct}}(\mathbb{B}^4, \mathbb{S}^2).$$

1.3 How to produce counterexamples

We have seen in the case of $\mathbb{S}^3$-valued that that we may bound the defect energy of a map in $\mathcal{R}_{\rm ct}(\mathbb{B}^4, \mathbb{S}^3)$ by the 3-energy of the map itself (see inequality (23)) and that this upper bound, combined with the strong density of $\mathcal{R}_{\rm ct}(\mathbb{B}^4, \mathbb{S}^3)$ directly leads to the weak density of smooth maps. If an estimate similar to (23) would exist for $\mathbb{S}^2$-valued maps, then the same line of thoughts would yield weak approximability as well. Our next result states precisely that there is no analog for (23) for $\mathbb{S}^2$-valued maps.

Proposition 2. *Given any $k \in \mathbb{N}^*$, there exists a map $\mathfrak{v}_k \in \mathcal{R}_{\rm ct}(\mathbb{B}^4, \mathbb{S}^2)$ such that*

$$\mathrm{E}_3(\mathfrak{v}_k) \leq \mathrm{C}_1 k^3 \tag{28}$$

and

$$\mathrm{L}_{\rm branch}(\mathfrak{v}_k) \geq \mathrm{C}_2 \log(k) k^3, \tag{29}$$

where $\mathrm{C}_1 > 0$ and $\mathrm{C}_2 > 0$ are universal constants.

Notice that inequality (29) shows that

$$\mathrm{L}_{\rm branch}(\mathfrak{v}_k) \geq C(\log k\,)\, \mathrm{E}_3(\mathfrak{v}_k), \text{ so that } \frac{\mathrm{L}_{\rm branch}(\mathfrak{v}_k)}{\mathrm{E}_3(\mathfrak{v}_k)} \to +\infty \text{ as } n \to +\infty.$$

The functional $\mathrm{L}_{\rm branch}$, which, as seen, corresponds also to the defect energy $\epsilon_\star$, is therefore *not controled* by the Dirichlet energy E_3, in contrast to inequality (23) for $\mathcal{N} = \mathbb{S}^3$. This property is the heart of the paper. Indeed, not only does it show that the argument for $\mathbb{S}^2$-valued maps cannot be transposed, it also provides a way to construct counterexamples. The map $\mathcal{U}$ in Theorem 4 is obtained gluing together a infinite countable number of copies of scaled and translated versions of the maps $\mathfrak{v}_k$, for suitable choices of the integer k and the scaling factors. We choose and tune these parameters in such a way the total sum of the energies in finite, whereas the sum of the defect energies diverges.

Remark 3. The fact that the defect energy should grow at most linearly in terms of the energy in order to obtain weak approximability as already been noticed in [18].

The next paragraphs present the main steps of the construction of the sequence $(\mathfrak{v}_k)_{k\in\mathbb{N}}$.

1.4 On the construction of $\mathfrak{v}_k$

The construction of the maps $\mathfrak{v}_k$ faces two, in principle opposite, constraints:

- having the functional $\mathrm{L}_{\rm branch}(\mathfrak{v}_k)$ as large as possible. Since this functional is related to the configuration of singularities, this task requires to have a large number of singularities, and branched transportation teaches us that the best way to to increase the functional is to have singularities well-separated (at least if they have the same sign)
- having an energy as small as possible. An intuitive idea suggest that increasing the number of singularities will increases the energy.

As we will see at the end of the construction, the number of singularities of $\mathfrak{v}_k$ will be of order of k^4, consisting of two well-separated clouds of singularities of the same sign, whereas the energy will be of order k^3.

Related to the energy constraint, the starting point of the construction is to step one dimension below and consider maps from $\mathbb{S}^3$ (or actually $\mathbb{R}^3$ through compactification at infinity, see details in subsection 2.1) to $\mathbb{S}^2$ which are nearly optimal for the energy inequality (16). Such maps have been construction in [28]. These maps from $\mathbb{R}^3$ to $\mathbb{S}^2$, denoted $\mathbf{S}^k_{\mathbf{pag}}$ and termed in this paper *k-spaghettons*, carry a topological charge of order k^4, with an energy of order k^3. For the definition of $\mathbf{S}^k_{\mathbf{pag}}$, we modify somewhat the original construction given in [28], and recast it into a more general framework known as the Pontryagin construction [27], see also [24] for a detailed presentation. In order to describe briefly $\mathbf{S}^k_{\mathbf{pag}}$, let us mention that these maps are constant outside $2k^2$ closed thin tubes of section of order $h = k^{-1}$, of length of order 1. The thin tubes are gathered in two distinct regular bundles which are *linked*: This linking provided the non trivial topology.

The next step is to go to dimension 4: This is provided by a deformation denoted $\mathbf{G}^k_{\mathbf{ord}}$ of $\mathbf{S}^k_{\mathbf{pag}}$ on the strip $\Lambda = \mathbb{R}^3 \times [0, 50]$, which is such that:

$$\begin{cases} \text{the restriction of } \mathbf{G}^k_{\mathbf{ord}} \text{ to the slice } \mathbb{R}^3 \times \{0\} \text{ is equal to } \mathbf{S}^k_{\mathbf{pag}} \\ \text{its restriction to the slice } \mathbb{R}^3 \times \{50\} \text{ is a constant function.} \end{cases} \tag{30}$$

Such a deformation is of course not possible in the continuous class, since the maps on the top and on the botton belong to different homotopy classes. In constrast, it is allowed in the Sobolev class $W^{1,3}$, with an energy of the same order than the energy restricted to the on the bottom, that is the energy of the spaghetton $\mathbf{S}^k_{\mathbf{pag}}$. In particular, one is able to untie the thin linked tubes thanks to crossings. We will term therefore this map $\mathbf{G}^k_{\mathbf{ord}}$ the Gordian cut of order k. Each of the *cuts* creates a singularity of the map $\mathbf{G}^k_{\mathbf{ord}}$.

Finally, the construction is completed deforming $\mathbf{G}^k_{\mathbf{ord}}$ into a map on $\mathbb{B}^4$ with the desired properties, a step which is more elementary than the previous ones.

We next go a little further in our description of the maps $\mathfrak{v}_k$.

1.4.1 The Pontryagin construction and the k-spaghetton map.

The Pontryagin construction we present next provides a beautiful way to produce maps from $\mathbb{R}^{m+\ell}$ a map from $\mathbb{R}^{m+\ell}$ to $\mathbb{S}^\ell$ with non trivial topology. This construction, introduced first in [27], relates to a framed smooth m-dimensional submanifold in $\mathbb{R}^{m+\ell}$ a map from $\mathbb{R}^{m+\ell}$ to $\mathbb{S}^\ell$. By framed submanifold, we mean here that for each point a of the submanifold, we are given an orthonormal basis $\mathfrak{e}^\perp \equiv (\vec{\tau}_1(a), \vec{\tau}_2(a), \dots, \vec{\tau}_\ell(a))$ of the ℓ-dimensional cotangent hyperplane at the point a, which varies continuously with the point a.

We specify the Pontryagin construction to the case $m = 1$ and $\ell = 2$, which is the situation of interest for us. The framed manifold we consider is therefore a framed closed curve $\mathcal{C}$ in $\mathbb{R}^3$, for which we are given a orthonormal basis of its orthogonal plane $\mathfrak{e}^\perp(\cdot) \equiv (\vec{\tau}_1(\cdot), \vec{\tau}_2(\cdot))$. This frame in turn induces a natural *orientation* of the curve, choosing the vector $\vec{\tau}_3(a) = \vec{\tau}_1(a) \times \vec{\tau}_2(a)$ as a unit tangent vector to the curve at the point a, so that *any framed curve is oriented.* Our next task is to map a small annular region around the curve to the sphere $\mathbb{S}^2$.

To that aim, we present first an preliminary ingredient which is the construction of a map from a small disk onto the sphere $\mathbb{S}^2$.

Mapping a disk to the sphere. We consider in the plane $\mathbb{R}^2$ the unit disk $\mathbb{D} = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 \leq 1\}$ and define a map χ from the disk $\mathbb{D}$ onto the standard two-sphere $\mathbb{S}^2$ by setting, for $(x_1, x_2) \in \mathbb{R}^2$,

$$\chi(x_1, x_2) = (x_1 f(r), x_2 f(r), g(r)) \ \text{ with } r = \sqrt{x_1^2 + x_2^2},\ r^2 f^2(r) + g^2(r) = 1, \tag{31}$$

where f and g are smooth given real functions on $[0, 1]$ such that

$$\begin{cases} f(0) = f(1) = 0,\ 0 \leq r f(r) \leq 1 \text{ for any } r \in [0, 1] \\ -1 \leq g \leq 1 \text{ and } g \text{ decreases from } g(0) = 1 \text{ to } g(1) = -1. \end{cases}$$

It follows from this definition what χ maps one to one the interior of the disk $\mathbb{D}$ to the set $\mathbb{S}^2 \setminus \{\mathbb{P}_{\text{south}}\}$, where $\mathbb{P}_{\text{south}}$ denotes the south pole $\mathbb{P}_{\text{south}} = (0, 0 - 1)$. Moreover the boundary $\partial\mathbb{D}$ is mapped onto the south pole $\mathbb{P}_{\text{south}} = (0, 0, -1)$, whereas the origin 0 is mapped to the North pole $\mathbb{P}_{\text{north}} = (0, 0, 1)$. It is possible to choose the functions f and g so that ξ is "almost conformal" in some suitable sense which is not relevant for the rest of the discussion. Given $\varrho > 0$ we then define the scaled function χ_ϱ on $\mathbb{R}^2$ by setting

$$\chi_\varrho(x_1, x_2) = \chi(\frac{x_1}{\varrho}, \frac{x_2}{\varrho}),\ \text{for } (x_1, x_2) \in \mathbb{D}_\varrho,\ \chi_\varrho(x_1, x_2) = \mathbb{P}_{\text{south}} \text{ otherwise,}$$

so that we have the gradient estimate

$$\|\nabla \chi_\varrho\|_{L^\infty(\mathbb{D}_\varrho)} \leq C\varrho^{-1}. \tag{32}$$

Mapping an annular neighborhood of $\mathcal{C}$ to $\mathbb{S}^2$. Let $\mathcal{C}$ be a framed curve in $\mathbb{R}^3$. For $a \in \mathcal{C}$, let $P_a^\perp$ be the plane orthogonal to $\vec{\tau}_a$, and denote $\mathbb{D}_a^\perp(\varrho)$ the disk in $P_a^\perp$ centered at a of radius $\varrho > 0$. We consider the tubular neighborhood $\mathrm{T}_\varrho(\mathcal{C})$ of $\mathcal{C}$ defined by

$$\mathrm{T}_\varrho(\mathcal{C}) = \underset{a \in \mathcal{C}}{\cup} \mathbb{D}_a^\perp(\varrho). \tag{33}$$

Notice that there exists some number $\varrho_0 = \varrho_0(\mathcal{C}) > 0$ depending only on $\mathcal{C}$, such that, if $0 < \varrho \leq \varrho_0$ then all disk $\mathbb{D}_a^\perp(\varrho)$ are mutually disjoint. In particular, for any $x \in \mathrm{T}_\varrho(\mathcal{C})$, there exists a unique point $a \in \mathcal{C}$, and a unique point $(x_1, x_2) \in \mathbb{D}_\varrho$ such that x has the form

$$x = a + x_1 \vec{\tau}_1(a) + x_2 \vec{\tau}_2(a). \tag{34}$$

For given $0 < \varrho < \varrho_0$, we construct a smooth map $\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}^\perp] : \mathrm{T}_\varrho(\mathcal{C}) \to \mathbb{S}^2$ as follows: For given $x \in \mathrm{T}_\varrho(\mathcal{C})$ of the form (34) we set

$$\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}^\perp](x) = \chi_\varrho(x_1, x_2). \tag{35}$$

Since $\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}^\perp]$ is equal to $\mathbb{P}_{\text{south}}$ on $\partial\mathrm{T}_r(\mathcal{C})$ we may extend this map to the whole of $\mathbb{R}^3$ setting

$$\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}^\perp](x) = \mathbb{P}_{\text{south}} \text{ for } x \in \Omega_\varrho(\mathcal{C}) \equiv \mathbb{R}^3 \setminus \mathrm{T}_\varrho(\mathcal{C}),$$

so that $\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}^\perp]$ is now a Lipschitz map from $\mathbb{R}^3$ to $\mathbb{S}^2$. The map $\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}^\perp]$ is called the Pontryagin map related to the framed curve $\mathcal{C}$ of order ϱ. Since $\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}^\perp]$ is equal to $\mathbb{P}_{\text{south}}$ outside a bounded region and in view of (32), we have hence shown:

Lemma 2. *If* $0 < \varrho \leq \varrho_0(\mathcal{C})$ *then the map* $\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}^\perp]$ *belongs to* $\text{Lip} \cap C_0^0(\mathbb{R}^3, \mathbb{S}^2)$*, where we have set*

$$C_0^0(\mathbb{R}^3, \mathbb{S}^2) = \{u \in C^0(\mathbb{R}^3, \mathbb{S}^2) \text{ such that } \lim_{|x| \to +\infty} u(x) \text{ exists}\}. \tag{36}$$

Moreover, we have

$$|\nabla \mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}^\perp](x)| \leq C\varrho^{-1} \text{ for every } x \in \mathbb{R}^3, \tag{37}$$

where $C > 0$ *is some constant depending possibly on the curve* $\mathcal{C}$ *as well as on the choice of frame* $\mathfrak{e}^\perp$ *of the orthonormal plane.*

The case of planar curves. All curves $\mathcal{C}$ that will enter through the Pontryagin construction in our later definition of the spaghetton map $\mathbf{S}_{\mathbf{pag}}^k$ will be planar or be an union of planar curves. Moreover, they will lie in planes either parallel to the plane $P_{1,2}$ or to the plane $P_{2,3}$ where

$$\begin{cases} P_{1,2} = (\mathbb{R}\vec{\mathbf{e}}_3)^\perp = P_{1,2}(0) \text{ where } P_{1,2}(s) \equiv \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ s.t } x_3 = s\}, \forall s \in \mathbb{R} \\ P_{2,3} = (\mathbb{R}\vec{\mathbf{e}}_1)^\perp = P_{2,3}(0) \text{ where } P_{2,3}(s) \equiv \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ s.t. } x_1 = s\}, \forall s \in \mathbb{R}, \end{cases}$$

where we set $\vec{\mathbf{e}}_1 = (1,0,0)$, $\vec{\mathbf{e}}_2 = (0,1,0)$ and $\vec{\mathbf{e}}_3 = (0,0,1)$. For such curves, we define a *reference framing* as follows. We first choose the orientation of the curves: Curves in $P_{1,2}$ and $P_{2,3}$ will be orientated trigonometrically, that is counter-clockwise according to the orthonormal bases $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ and $(\vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3)$ of $P_{1,2}$ and $P_{2,3}$ respectively. With this convention, we will denote by $\vec{\tau}_{\tan}(a)$ a unit tangent vector at the point a of the curve oriented accordingly. Second, we choose the first orthonormal vector $\vec{\tau}_1$ as

$$\begin{cases} \vec{\tau}_1(a) = \vec{\mathbf{e}}_3 \text{ for curves in } P_{1,2} \text{ and} \\ \vec{\tau}_1(a) = \vec{\mathbf{e}}_1 \text{ for curves in } P_{2,3}. \end{cases} \tag{38}$$

Finally, we set $\vec{\tau}_2(a) = \vec{\tau}_{\tan}(a) \times \vec{\tau}_1(a)$, so that $\vec{\tau}_2(a)$ is a unit vector orthogonal to the vector $\tau_{\tan}(a)$ included in the plane $P_{1,2}$ or $P_{2,3}$ respectively, and *exterior to the curve.* We consider the frame of the orthonormal plane given by

$$\mathfrak{e}_{\text{ref}}^\perp(a) = (\vec{\tau}_1(a), \vec{\tau}_2(a)) \text{ for } a \in \mathcal{C}. \tag{39}$$

It has in particular the property that $(\vec{\tau}_1(a), \vec{\tau}_2(a), \vec{\tau}_{\tan}(a))$ is a direct orthonormal basis of $\mathbb{R}^3$. In order to simplify a little notation, we will often use the notation

$$\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}] = \mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}_{\text{ref}}^\perp],$$

in the case $\mathcal{C}$ is a planar curve in affine planes parallel to $P_{1,2}$ or $P_{2,3}$ or a infinite union of such curves.

Homotopy classes of Pontryagin maps. If $\mathcal{C}$ is a planar curve in $P_{1,2}$ or $P_{2,3}$ *framed with the reference frame* $\mathfrak{e}_{\text{ref}}^\perp$ defined above then it turns out that the homotopy class of $\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}_{\text{ref}}^\perp]$ is *trivial.* There are at least two simple ways to make not trivial homotopy classes emerge from the Pontryagin construction:

- Twisting the frame of the orthogonal plane to the curve, *a method which we will not use in this paper.*

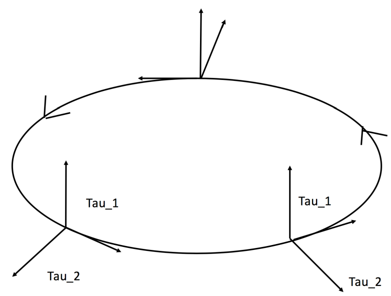

Figure 1: *The reference frame for a planar curve*

- Considering *planar curves as above which are linked.*

The idea of the construction of the Spaghetton map relies on this latest idea.

On the construction of the k-spaghetton. The construction of the k-spaghetton maps involves two sheaves of planar curves which will be denoted $\mathfrak{L}^k$ and $\mathfrak{L}^{k,\perp}$ respectivement. Each of the sheaves contains exactly k^2 *stadium shaped* connected curves, included in k parallel planes, each of the planes containing k such curves, which are concentric, so that the general idea is that each of these sheaves consist of parallel [6] planar curve. Our construction yields actually

$$\mathfrak{L}^k \subset \bigcup_{q=1}^{k} P_{1,2}(qh) \text{ and } \mathfrak{L}^{k,\perp} \subset \bigcup_{q=1}^{k} P_{2,3}(qh) \text{ where } h = k^{-1}, \tag{40}$$

so that the distance between neighboring parallel planes is exacly $h = k^{-1}$. The curves in each of these planes are deduced from the other by translation that is

$$\begin{cases} \mathfrak{L}^k \cap P_{1,2}(qh) = \mathfrak{L}^k \cap P_{1,2}(h) + (q-1)h\,\vec{\mathbf{e}}_3 \text{ for } q = 1, \dots, k \\ \mathfrak{L}^{k,\perp} \cap P_{2,3}(qh) = \mathfrak{L}^{k,\perp} \cap P_{2,3}(h) + (q-1)h\,\vec{\mathbf{e}}_1 \text{ for } q = 1, \dots, k. \end{cases} \tag{41}$$

Finally, the curves in each sheaves are organized in a quite regular way. For instance the intersection of $\mathfrak{L}^k$, with the plane $P_{2,3}$ is given by a set of $2k^2$ points organized in two two-dimensional grids, namely

$$\mathfrak{L}^k \cap P_{2,3} = \{0\} \times \left(\left(\boxplus_k^2(h)\right) \cup \left((13,0) + \boxplus_k^2(h)\right)\right) \tag{42}$$

where the symbol $\boxplus_k^2(h)$ represents the discrete sets of points located on the regular two dimensional grid given, for $k \in \mathbb{N}^*$ and $h > 0$ by

$$\boxplus_k^2(h) = h\boxplus_k^2 = \{hI, I \in \{1, \dots, k\}^2\}$$

Similarly, we have, for $P_{1,3} = (\mathbb{R}\vec{\mathbf{e}}_2)^\perp$

$$\mathfrak{L}^{k,\perp} \cap P_{1,3} = \left\{(x_1, 0, x_3), \text{ with } (x_1, x_3) \in \left((0,-7) + \tilde{\boxplus}_k^2(h)\right) \cup \left((0,6) + \boxplus_k^2(h)\right)\right\}.$$

[6]parallel has to be taken here in an intuitive meaning and not in a rigorous mathematical sense

Finally, the two sheaves do not intersect and each curve is linked to all curves of the other sheaves, but with none of its own sheave. On each of the planar curves, we choose the reference frame, set $\mathcal{S}^k = \mathfrak{L}^k \cup \mathfrak{L}^{k,\perp}$ and define the k-spaghetton map $\mathfrak{S}_k$ as the Pontryagin map related to $\mathcal{S}^k$

$$\mathfrak{S}_k = \mathbf{P}^{\text{ontya}}_{\varrho_k}[\mathcal{S}^k, \mathfrak{e}^{\perp}_{\text{ref}}], \tag{43}$$

where the parameter parameter ϱ_k is choosen suitably of order h. Details are provided in Section 3. We summarize some its main properties in the next proposition.

Proposition 3. *The k-spaghetton $\mathbf{S}^k_{\mathbf{pag}}$ is a smooth map from $\mathbb{R}^3$ to $\mathbb{S}^2$ with the following properties:*

- $\mathbf{S}^k_{\mathbf{pag}}(x) = \mathbb{P}_{\text{south}}$ *if* $|x| \geq 17$
- $|\nabla \mathbf{S}^k_{\mathbf{pag}}(x)| \leq \mathbf{C}_{\text{spg}} k$*, for any* $x \in \mathbb{R}^3$*, where* $\mathbf{C}_{\text{spg}} > 0$ *is a universal constant.*
- *The Hopf invariant of* $\mathbf{S}^k_{\mathbf{pag}}$ *is* $\mathbb{H}(\mathbf{S}^k_{\mathbf{pag}}) = 2k^4$
- *The 3-energy verifies the energy bound* $\mathrm{E}_3(\mathbf{S}^k_{\mathbf{pag}}) \leq \mathbf{K}_{\text{spg}} k^3$*, where* $\mathbf{K}_{\text{spg}} > 0$ *is a universal constant.*

1.4.2 The Gordian cut

This construction represents the second step of the construction and yields now a map from a subset $\Lambda \subset \mathbb{R}^4 \to \mathbb{S}^2$, where Λ is the strip of $\mathbb{R}^4$ given by

$$\Lambda = \mathbb{R}^3 \times [0, 50] = \{\mathbf{x} = (x, x_4),\ x \in \mathbb{R}^3,\ 0 \leq x_4 \leq 50\}.$$

The gordian cut $\mathbf{G}^k_{\mathbf{ord}}$ corresponds actually to a deformation of the k-spaghetton to a constant map which belongs to the Sobolev class $W^{1,3}$, the fourth coordinate standing for the defomation coordinates, similar to the time variable in usual deformations. The map $\mathbf{G}^k_{\mathbf{ord}}$ belongs to the class of maps $w : \Lambda \to \mathbb{S}^2$ such that the following four conditions are met:

$$\begin{cases} w \in \mathcal{R}(\Lambda, \mathbb{S}^2) \text{ and } \mathrm{E}_3(w, \Lambda) \equiv \displaystyle\int_\Lambda |\nabla w|^3 < \infty \\ w(x, 0) = \mathbf{S}^k_{\mathbf{pag}}(x, 0) \ \text{ for almost every } x \in \mathbb{R}^3 \\ w(x, 50) = \mathbb{P}_{\text{south}} \ \text{ for almost every } x \in \mathbb{R}^3 \\ w(x, s) = \mathbb{P}_{\text{south}} \text{ for every } x \in \mathbb{R}^3 \text{ such that } |x| \geq 40 \text{ and } 0 \leq s \leq 50. \end{cases} \tag{44}$$

The second and third conditions in (44) have already been encountered in a slightly weaker form in (30). They make sense in view of the trace theorem and the boundedness of the energy stated in the first condition.

Proposition 4. *There exists a map* $\mathbf{G}^k_{\mathbf{ord}} : \Lambda \to \mathbb{S}^2$ *verifying* (44) *such that* $\mathbf{G}^k_{\mathbf{ord}}$ *has exactly* k^4 *topological singularities of charge* $+2$ *and such that*

$$\mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}) \leq \mathbf{K}_{\text{Gord}}\, k^3, \tag{45}$$

where $\mathbf{K}_{\text{Gord}} > 0$ *is some universal constant. Let* $\mathbb{A}^k_{\text{sing}}$ *denotes the set of singularities of* $\mathbf{G}^k_{\mathbf{ord}}$*. We have*

$$\mathbb{A}^k_{\text{sing}} = \boxplus^2_k(h) \times \mathbb{T}_k\left(\boxplus^2_k(h)\right), \tag{46}$$

where $\mathbb{T}^k$ is an affine one to one mapping from $\mathbb{R}^2$ into itself which is given by

$$\mathbb{T}_k(x_3, x_4) = (x_3, -2x_3 + x_4) + M_h \text{ where } M_h = (0, 7 - \frac{h}{8}). \tag{47}$$

Introducing the sets of points on a uniform grid of dimension 4 given by

$$\boxplus_k^4 = \{1, \dots, k\}^4 \text{ and } \boxplus_k^4(h) = h\boxplus_k^4 = \{hI, I \in \{1, \dots, k\}^4\}, \tag{48}$$

we observe that a consequence of (46) is that

$$\mathbb{A}^k_{\text{sing}} = \Phi_k\left(\boxplus_k^4(h)\right) \text{ with } \Phi_k(\mathbf{x}) = (x_1, x_2, \mathbb{T}_k(x_3, x_4)) \text{ for } \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4, \tag{49}$$

so that Φ_k is an affine one to one mapping on $\mathbb{R}^3$. It follows that $\mathbb{A}^k_{\text{sing}}$ a regular grid of singularities: This observation is crucial, in particular in relation to the minimal branched connection and the result described in Appendix A.

Although the detailed argument of the proof of Proposition 4 involves some technicalities, The heuristic idea is rather simple: We consider x_4 as a time variable, and push down along the x_3-axis the sheaf $\mathfrak{L}^{k,\perp}$, keeping however its shape essentially unchanged, whereas the sheaf $\mathfrak{L}^k$ does not move. This process presents no major difficulty as long as the sheaf $\mathfrak{L}^{k,\perp}$ does not encounter the sheaf $\mathfrak{L}^k$. When some fibers touch, we are no longer able to define the corresponding Pontryagin map $\mathbf{P}^{\text{ontya}}_{\varrho_k}$. To overcome this difficulty, we take advantage of the fact that we are working in a Sobolev class were singularities are allowed: Using such singularities, the fiber in contact are able to cross, that is the sheaf $\mathfrak{L}^k_\perp$ is able to pass through the fibers of $\mathfrak{L}^k$. Each time fibers cross a singularities of topological charge 2 is created. These singularities form a cloud of uniformly distributed points as stated in Proposition 4.

1.4.3 Construction of the sequences of map $(\mathfrak{v}_k)_{n\in\mathbb{N}}$

The construction of the sequences of map $(\mathfrak{v}_k)_{k\in\mathbb{N}}$ described in Proposition 2 is then deduced rather directly modifying the maps $\mathbf{G}^k_{\mathbf{ord}}$ constructed in Proposition 4 using some elementary transformations as affine mappings or reflections, in such a way that we have $\mathfrak{v}_k \in C^0_{\text{ct}}(\mathbb{B}^4 \setminus \Sigma^k_{\text{sing}}, \mathbb{S}^2)$, where the set Σ^k_{sing} of singularities of $\mathfrak{v}_k$ is given by

$$\Sigma^k_{\text{sing}} = \boxplus_k^4(\text{h}_{\text{scal}}) \cup \mathbb{S}_{\text{ym}}\left(\boxplus_k^4(\text{h}_{\text{scal}})\right) \text{ where } \text{h}_{\text{scal}} = \frac{h}{400} = \frac{1}{400k}, \tag{50}$$

and where $\mathbb{S}_{\text{ym}}$ stands for the reflection symmetry through the hyperplane $x_4 = 0$, i.e.

$$\mathbb{S}_{\text{ym}}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4), \text{ for any } (x_1, x_2, x_3, x_4) \in \mathbb{R}^4. \tag{51}$$

The singumarities in $\boxplus_k^4(h)$ have Hopf invariant $+2$ whereas the singularities in $\mathbb{S}_{\text{ym}}\left(\boxplus_k^4(h)\right)$ have Hopf invariant -2, the total charge being equal to 0. The energy estimate (28) for the map $\mathfrak{v}_k$ follows from the corresponding energy estimate for $\mathbf{G}^k_{\mathbf{ord}}$.

1.4.4 Irrigability of a cloud of points

To complete the proof of Proposition 2, it remains to establish estimate (29) for the branched transportation of the map $\mathfrak{v}_k$, whose value involves only the location of singularities of $\mathfrak{v}_k$. The main property which we will use in the proof of (29) is expressed in property (50), which shows that the singularities are located on a regular grid. It follows from (50) that

$$\frac{1}{k^4}\left(\sum_{a\in\Sigma_{\rm sing}} \mathbb{H}(a)\delta_a\right) \to f\,{\rm d}x \text{ as k } \to +\infty, \text{ where } f = 2\left[\mathbb{1}_{[0,a]^4} - \mathbb{1}_{[0,a]^3\times[-a,0]}\right], \qquad (52)$$

where $a = 1/400$. It turns out, in view of (52), that the behavior of the functional $\mathrm{L}_{\rm branch}$ as k growths is related to the irrigation problem for the Lebesgue measure, a central question in the theory of branched transportation. It has been proven in [12] (see also Devillanova's thesis or the general description in [4], in particular Chap 10) that the Lebesgue measure *is not irrigable* for the critical exponent $\alpha_c = \frac{3}{4}$. This result can be interpretated directly as the fact that the functional $\mathrm{L}_{\rm branch}$ growth more rapidly that the number of points at the power α_c, hence more rapidly then k^3. A lower bound for this divergence then directly yields (29), completing hence the proof of Proposition 2. As a matter of fact, we will rely on a precise lower bound of logarithmic form for this divergence which is established in a separate Appendix at the end of this paper.

1.5 On the proof of the main theorems

Concerning Theorem 3, the proof consist in adding additional dimensions to the previous constructing and is rather standard.

1.6 The lifting problem

As a by product of our method, in particular the construction of the spaghetton maps, we are able to address some questions related to the lifting problem of $\mathbb{S}^2$-valued maps within the Sobolev context. Such question have already been raised and partially solved in [8, 19, 20]. The main additional remark we wish to provide in the present paper is that the question is *not related in an essential way* to topological singularities, since our counterexamples do not have such singularities.

Recall that maps into $\mathbb{S}^2$ and maps into $\mathbb{S}^3$ are connected through a projection map $\Pi : \mathbb{S}^3 \to \mathbb{S}^2$ termed the Hopf map and which we describe briefly. To start with an intuitive picture (but as we will see in a moment, this picture is not completely correct) the sphere $\mathbb{S}^3$ is very close, at least from the point of view of topology, to the group of rotations $SO(3)$ of the three dimensional space $\mathbb{R}^3$, the sphere $\mathbb{S}^3$ may be in fact identified with its universal cover. Any rotation R in $SO(3)$ yields an element on $\mathbb{S}^2$ considering the image by R of an arbitrary fixed point of the sphere, for instance the North pole $\mathbb{P}_{\rm north} = (0,0,1)$, so that we obtain a projection from $SO(3)$ to $\mathbb{S}^2$ considering the correspondance $R \mapsto R(P)$. The construction of the projection Π from $\mathbb{S}^3$ onto $\mathbb{S}^2$ is in the same spirit, but requires to introduce some preliminary objects.

Identifying $\mathbb{S}^3$ *with* $SU(2)$. Here $SU(2)$ denotes the Lie group of two dimensional complex unitary matrices of determinant one, i.e. $SU(2) = \{U \in \mathbb{M}_2(\mathbb{C}),\ UU^* = I_2 \text{ and } \det(U) = 1\}$

or

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad a \in \mathbb{C}, b \in \mathbb{C} \text{ with } |a|^2 + |b|^2 = 1 \right\} \simeq \mathbb{S}^3. \tag{53}$$

The Lie algebra $su(2) = \{X \in \mathcal{M}_2(\mathbb{C}),\ X + X^* = 0 \text{ and } \mathrm{tr}(X) = 0\}$ of $SU(2)$ consists in traceless anti-hermitian matrices. A canonical basis of this 3-dimensional space, which is actually orthonormal for the euclidean norm $|X|^2 \equiv \det(X)$ is provided by the Pauli matrices

$$\sigma_1 \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We identify the 2-sphere $\mathbb{S}^2$ with the unit sphere of $su(2)$ for the previous scalar product:

$$\mathbb{S}^2 \simeq \{X \in su(2),\ |X|^2 = \det(X) = 1\}.$$

The Hopf map. The group $SU(2)$ acts naturally on $su(2)$ by conjugation: If $g \in SU(2)$, then

$$su(2) \ni X \mapsto Ad_g(X) \equiv gXg^{-1} \in su(2) \text{ is an isometry of determinant 1.}$$

Definition 1. *The map* $\Pi : SU(2) \simeq \mathbb{S}^3 \to \mathbb{S}^2 \subset su(2)$ *defined by*

$$\Pi(g) \equiv Ad_g(\sigma_1) = g\sigma_1 g^{-1} \text{ for } g \in SU(2)$$

is called the Hopf map.

Notice that $\Pi(g) = \sigma_1$ if and only if g is of the form

$$g = \exp(\sigma_1 t) = \begin{pmatrix} \exp it & 0 \\ 0 & \exp -it \end{pmatrix}$$

with $t \in [0, 2\pi]$. More generally, if g and g' are such that $\Pi(g) = \Pi(g')$, then $g' = g \exp \sigma_1 t$ for some $t \in [0, 2\pi]$, so that the fiber $\Pi^{-1}(u)$ is diffeomorphic to the circle $\mathbb{S}^1$ for every $u \in \mathbb{S}^2$. By the Hopf map, $SU(2)$ appears hence as a fiber bundle with base space $\mathbb{S}^2$ and fiber $\mathbb{S}^1$. Moreover, this bundle is not trivial, but twisted since $Id_{\mathbb{S}^2}$ *does not* admit a continuous lifting $\Phi : \mathbb{S}^2 \to \mathbb{S}^3$ such that $Id_{\mathbb{S}^2} = \Pi \circ \Phi$. Indeed, Φ is homotopic to a constant, but not $Id_{\mathbb{S}^2}$.

Projecting maps onto $\mathbb{S}^2$. Given any domain $\mathcal{M}$ and a map $\mathrm{U} : \mathcal{M} \to \mathbb{S}^3$, we may associate to this map the map $u : \mathcal{M} \to \mathbb{S}^2$ obtained through to the composition with the map Π, that is setting $u = \Pi \circ \mathrm{U}$. This construction works for a rather general class of maps, with mild regularity assumptions, for instance measurability. In particular, since Π is smooth, if U belongs to $W^{1,3}(\mathcal{M}, \mathbb{S}^3)$, then the same Sobolev regularity holds for $u = \Pi \circ \mathrm{U}$ and the correspondence $U \mapsto u$ is smooth. This correspondence is of course not one to one. Indeed, given any scalar function $\Theta : \mathcal{M} \to \mathbb{R}$, then we have

$$u = \Pi \circ \mathrm{U} = \Pi \circ (\mathrm{U} \exp(\sigma_1 \Theta(\cdot))).$$

Conversely, given two maps U_1 and U_2 such that $u = \Pi \circ \mathrm{U}_1 = \Pi \circ \mathrm{U}_2$ then there exists a map $\Theta : \mathcal{M} \to \mathbb{R}$ such that $\mathrm{U}_2 = \mathrm{U}_1 \exp(\sigma_1 \Theta(\cdot))$. The map Θ is often referred to as the gauge freedom.

Lifting maps to $\mathbb{S}^2$ *as maps to* $\mathbb{S}^3$. The lifting problem corresponds to invert the projection Π, which means that, given a map u from $\mathcal{M}$ to $\mathbb{S}^2$ in a prescribed regularity class, one seeks for a map U from $\mathcal{M}$ to $\mathbb{S}^3$, if possible in the same regularity class, such that $u = \Pi \circ \mathrm{U}$. The map U is then called a *lifting* of u. As seen in the previous paragraph, if a lifting exists, then there is no-uniqueness, since if U is a solution, then the same holds for the map $U \exp(\sigma_1 \Theta)$, where Θ is arbitrary scalar functions $\Theta : \mathcal{M} \to \mathbb{R}$ in the appropriate regularity class.

If $\mathcal{M}$ is simply connected with $\pi_2(\mathcal{M}) = \{0\}$, it can be shown that the lifting problem has always a solution in the continuous class, i.e. for any continuous maps u from $\mathcal{M}$ to $\mathbb{S}^2$ there exists a continuous lifting U from $\mathcal{M}$ to $\mathbb{S}^3$ of u such that $u = \Pi \circ \mathrm{U}$. As an example in the case $\mathcal{M} = \mathbb{S}^3$, the identity from $\mathbb{S}^3$ into itself is a lifting of the Hopf map.

The fact that the lifting property holds in the continuous class allows to provide a one to one correspondance between homotopy classes in $C^0(\mathcal{M}, \mathbb{S}^3)$ and $C^0(\mathcal{M}, \mathbb{S}^2)$. Indeed, two maps u_1 and u_2 from $\mathcal{M}$ to $\mathbb{S}^2$ are homotopic if and only if their respective liftings U_1 and U_2 are in the same homotopy class. Specifying this property to the case $\mathcal{M} = \mathbb{S}^3$, we obtain as already mentioned an identification of $\pi_3(\mathbb{S}^2)$ and $\pi_3(\mathbb{S}^3)$. On the level of Sobolev regularity, the picture is quite different. We will prove in this paper:

Theorem 5. *Let* $\mathcal{M}$ *be a smooth compact manifold. For any* $2 \leq p < m = \dim \mathcal{M}$ *there exist a map* $\mathcal{V}$ *in* $W^{1,p}(\mathcal{M}, \mathbb{S}^2)$ *such there exist no map* $\mathrm{V} \in W^{1,p}(\mathcal{M}, \mathbb{S}^3)$ *satisfying* $\mathcal{V} = \Pi \circ \mathrm{V}$. *Moreover* $\mathcal{V}$ *belong to the strong closure of smooth maps in* $W^{1,p}(\mathcal{M}, \mathbb{S}^2)$.

This results supplements earlier results obtained on this question in [8, 19]. It is proved in [8] that, if $1 \leq p < 2 \leq \dim \mathcal{M}$, $p \geq \dim \mathcal{M} \geq 3$ or $p > \dim \mathcal{M} = 2$, then any map $\mathcal{V}$ in $W^{1,p}(\mathcal{M}, \mathbb{S}^2)$ admits a lifting V in $W^{1,p}(\mathcal{M}, \mathbb{S}^3)$, whereas a map was produced there in the cases $2 \leq p < 3 \leq \dim \mathcal{M}$ or $p = 3 < \dim \mathcal{M}$, which possesses no lifting in $W^{1,p}(\mathcal{M}, \mathbb{S}^2)$. In the later case, however, the example produced in [8] is not in the strong closure of smooth maps, in contrast with the map constructed in Theorem 5. Notice that as a matter of fact, Theorem 5 gives a negative answer to Open Question 4 in [8], and that the only case left open is the case $p = \dim \mathcal{M} = 2$ corresponding to the Open Question 3 in [8].

1.7 Concluding remarks and open questions

As perhaps the previous presentation shows, the construction of our counterexample relies on several specific properties of the Hopf invariant, a topological invariant which combines in an appealing way various aspects of topology in the three dimensional space. Our proof is built on the fact that the related branched transportation involves precisely the critical exponent, yielding a divergence in some estimates which are crucial. An analog for this exponent for more general target manifolds with infinite homotopy group $\pi_p(\mathcal{N})$ has been provided and worked out in [21], based on more sophisticated notions in topology. It is likely that this exponent plays an important role in issues related to weak density of smooth maps. In the case the exponent provided in [21] is larger then the critical exponent of the related branched transportation, as described above, one may reasonably conjecture that there should exist some obstruction the sequential weak density of smooth maps. However the *effective* constructions of such obstructions, perhaps similar to the ones proposed in this work remain unclear. In particular the Pontryagin construction used here seems at first sight somehow restricted to the case the target is a sphere.

In another direction, let us also notice that most if not all results related to the weak closure of smooth maps between manifolds for integer exponents deal with manifolds having infinite homotopy group $\pi_p(\mathcal{N})$. The case when $\pi_p(\mathcal{N})$ is finite seems widely open and raises interesting questions also on the level of the related notions of minimal connections. As a first example, one may start again with $\mathcal{M} = \mathbb{S}^2$, $p = 4$, for with we have $\pi_4(\mathbb{S}^2) = \mathbb{Z}^2$. In this case a nice description of the homotopy classes in terms of the Pontragyin construction is also available.

This paper is organized as follows. In the next section, we recall some notion of topology which are used in the course of paper. Section 3 is devoted to the construction of the k-spaghetton map, whereas in Section 4, we construct the Gordian cut $\mathbf{G}^k_{\mathbf{ord}}$, providing the proof to Proposition 4, which is the central part of the paper. In Section 5, we provide the proofs of the main results, relying also on some results provided in Appendix A, in particular Theorem A.1, which, beside Proposition A.3, is the main result there.

Acknowledgements. The author wishes to thank the referees for their careful reading of the first version of this paper, pointing out several mistakes and indicating several lines of improvements.

2 Some topological background

We review in this section some basic properties of maps from $\mathbb{S}^3$ into $\mathbb{S}^2$ or $\mathbb{S}^3$.

2.1 Compactification at infinity of maps from $\mathbb{R}^3$ into $\mathcal{N}$

Whereas the emphasis was put in several places of the previous discussion on maps defined on the 3-sphere $\mathbb{S}^3$, it turns out that it is sometimes easier to work on the space $\mathbb{R}^3$ instead of $\mathbb{S}^3$. Since our maps will have some limits at infinity or even are constant outside a large ball, we are led to introduce the space

$$C^0_0(\mathbb{R}^3, \mathbb{R}^\ell) = \{u \in C^0(\mathbb{R}^3, \mathbb{R}^\ell) \text{ s.t } \lim_{|x|\to\infty} u \text{ exists}\}$$

and define accordingly the space $C^0_0(\mathbb{R}^3, \mathcal{N})$. The space $C^0_0(\mathbb{R}^3, \mathbb{R}^\ell)$ may be put in one to one correspondance with the space $C^0(\mathbb{S}^3, \mathbb{R}^\ell)$ thanks to the stereographic projection St_3 which is a smooth map from $\mathbb{S}^3 \setminus \{\mathbb{P}_{\rm south}\}$ onto $\mathbb{R}^3$ and is defined by

$$\mathrm{St}_3(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{1+x_4}, \frac{x_2}{1+x_4}, \frac{x_3}{1+x_4}\right) \text{ for } (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \text{ s.t } \sum_{i=1}^4 x_i^2 = 1.$$

For any map $u \in C^0(\mathbb{S}^3, \mathcal{N})$ we may define $u \circ \mathrm{St}_3 \in C^0_0(\mathbb{R}^3, \mathcal{N})$ and conversely given any map v in $C^0_0(\mathbb{R}^3, \mathcal{N})$ the map $v \circ \mathrm{St}_3^{-1}$ belongs to $C^0(\mathbb{S}^3, \mathcal{N})$. This allows to handle maps in $C^0_0(\mathbb{R}^3, \mathcal{N})$ as maps in $C^0(\mathbb{S}^3, \mathcal{N})$ and yields a one to one correspondance of homotopy classes. In particular, when $\mathcal{N} = \mathbb{S}^3$ or $\mathcal{N} = \mathbb{S}^2$ we may define the degree in the first case or the Hopf invariant in the second for maps in $C^0_0(\mathbb{R}^3, \mathcal{N})$.

2.2 Degree theory

Degree theory yields a topological invariant which classifies homotopy class for maps from $\mathbb{S}^3$ to $\mathbb{S}^3$. For a smooth U from $\mathbb{S}^3$ to $\mathbb{S}^3$, its analytical definition is given by

$$\deg \mathrm{U} = \frac{1}{|\mathbb{S}^3|}\int_{\mathbb{S}^3} \mathrm{U}^\star(\omega_{\mathbb{S}^3}) = \frac{1}{(2\pi^2)}\int_{\mathbb{S}^3} \det(\nabla \mathrm{U})\,\mathrm{d}x, \tag{2.1}$$

where $\omega_{\mathbb{S}^3}$ stands for a standard volume form on $\mathbb{S}^3$ and U^* stands for pullback. It turns out that $\deg \mathrm{U}$ is an integer which is a homotopy invariant, that is, two maps in $C^1(\mathbb{S}^3,\mathbb{S}^3)$ which are homotopic have same degrees and conversely, two maps with the same degree are homotopic, leading as mentioned to a complete classification of homotopy classes. Notice that the degree of the identity map of $\mathbb{S}^3$ whose homotopy class is the generator of $\pi_3(\mathbb{S}^3)$ is 1. The area formula yields a more geometrical interpretation, namely

$$\deg u = \sum_{a\in u^{-1}(z_0)} \mathrm{sign}(\det(\nabla u)), \tag{2.2}$$

where $z_0 \in \mathbb{S}^3$ is any regular point, so that $u^{-1}(z_0)$ is a finite set. Finally, an important property, which is quite immediately deduced from (2.1) is the lower bound

$$\mathrm{E}_3(\mathrm{U}) \equiv \int_{\mathbb{S}^3} |\nabla \mathrm{U}|^3 \geq |\mathbb{S}^3|\,|d| = 2\pi^2|d| \ \text{ provided } \deg(\mathrm{U}) = d. \tag{2.3}$$

This bounds is optimal. Indeed, as a consequence of the scale invariance of the energy E_3 in dimension 3, one may prove for any $d \in \mathbb{Z}$, gluing $|d|$ copies of degree one maps that

$$\nu_3(d) \equiv \inf\left\{E_3(u), u \in W^{1,3}(\mathbb{S}^3,\mathbb{S}^3), \deg(u) = d\right\} = |\mathbb{S}^3||d| = 2\pi^2|d|. \tag{2.4}$$

2.2.1 The Hopf invariant

We next turn to maps u from $\mathbb{S}^3$ into $\mathbb{S}^2$ which are assumed to have sufficient regularity. Since in this case, there is a lifting $\mathrm{U} : \mathbb{S}^3 \to \mathbb{S}^3$ such that $u = \Pi \circ \mathrm{U}$ the degree theory for $\mathbb{S}^3$ valued maps allows to classify also the homotopy classes of maps from $\mathbb{S}^3$ to $\mathbb{S}^2$. Set

$$\mathbb{H}(u) = \deg(\mathrm{U}).$$

This number, is called the Hopf invariant of u and as seen before classifies homotopy classes in $C^0(\mathbb{S}^3,\mathbb{S}^2)$. Notice that, since $\Pi = \Pi \circ \mathrm{Id}_{\mathbb{S}^3}$, the Hopf invariant of the Hopf map Π is $\mathbb{H}(\Pi) = 1$, so that its homotopy class $[\Pi]$ is a generator of $\pi_3(\mathbb{S}^2)$.

Integral formulations. Let $\mathcal{M}$ be a simply connected manifold, $\mathrm{U} : \mathcal{M} \to SU(2)$ sufficiently smooth and set $u \equiv \Pi \circ \mathrm{U}$. We construct a 1-form A with values into the Lie algebra $su(2)$ setting $A \equiv \mathrm{U}^{-1}d\mathrm{U}$. Conversely, given any sufficiently smooth $su(2)$ valued 1-form A on $\mathcal{M}$, on object also called a *connection*, one may find a map $\mathrm{U} : \mathcal{M} \to SU(2)$ such that $A = \mathrm{U}^{-1}d\mathrm{U}$, provided the zero curvature equation for connections holds, that is provided

$$dA + \frac{1}{2}[A, A] = 0. \tag{2.5}$$

Decomposing A on the canonical basis of $su(2)$ as $A = A_1\sigma_1 + A_2\sigma_2 + A_3\sigma_3$, where A_1, A_2 and A_3 denote scalar 1-forms on $\mathcal{M}$ we are led to the relations

$$du = U[A, \sigma_1]U^{-1} = A_3\sigma_2 - A_2\sigma_3,$$

so that the component A_2 and A_3 of A are completely determined by the projected map $u = \Pi\circ\mathrm{U}$. On the other hand, A_1 is not, a consequence of the *gauge freedom* mentioned before. Indeed, for any sufficiently smooth function $\Theta : \mathcal{M} \to \mathbb{R}$, let $\mathrm{U}_\Theta(x) \equiv \exp(\Theta(x)\sigma_1)\mathrm{U}(x)$, so that $u = \Pi \circ \mathrm{U}_\Theta$ and $\mathrm{U}_\Theta^{-1}d\mathrm{U}_\Theta = \mathrm{U}^{-1}d\mathrm{U} + (d\Theta)\sigma_1 = A + d\Theta\sigma_1$.The values of A_2 and A_3 are left unchanged by the gauge transformation, and A_1 is changed into $A_1^\Theta = A_1 + d\Theta$. We notice also the relations

$$\begin{cases} u^\star(\omega_{\mathbb{S}^2}) = A_2 \wedge A_3, \ \mathrm{U}^*(\omega_{\mathbb{S}^3}) = A_1 \wedge A_2 \wedge A_3, \\ |dU|^2 = |A_1|^2 + |A_2|^2 + |A_3|^2 \text{ and } |du|^2 = (|A_2|^2 + |A_3|^2), \end{cases} \tag{2.6}$$

where $\omega_{\mathbb{S}^2}$ stands for the standard volume form on $\mathbb{S}^2$. The curvature equation (2.5) yields the relation

$$2dA_1 = A_2 \wedge A_3 = u^\star(\omega_{\mathbb{S}^2}), \tag{2.7}$$

so that dA_1 is also completely determined by the projected map u. Going back to (2.6) we may write

$$\mathrm{U}^*(\omega_{\mathbb{S}^3}) = A_1 \wedge u^\star(\omega_{\mathbb{S}^2}).$$

Specifying the discussion to the case $\mathcal{M} = \mathbb{S}^3$, the integral formula for the degree yields in turn an integral formula for the Hopf invariant namely, for any map $u : \mathbb{S}^3 \to \mathbb{S}^2$, we have

$$\mathbb{H}(u) = \frac{1}{4\pi^2}\int_{\mathbb{S}^3} \alpha \wedge u^\star(\omega_{\mathbb{S}^2}), \text{ with } d\alpha = u^\star(\omega_{\mathbb{S}^2}), \tag{2.8}$$

where actually α corresponds to the one form $\alpha = 2A_1^\Theta$, whatever choice of gauge Θ.

Choosing a good gauge. Recall that at this stage $d\alpha = dA_1^\Theta$ is completely determined by (2.7). To remove the gauge freedom we may supplement this condition imposing another one in order to obtain an elliptic system. Hence are led a impose a condition on $d^\star\alpha$, for instance

$$d^\star\alpha = 0, \text{ and hence } \alpha = d^\star\Phi, \tag{2.9}$$

where Φ is some two form verifying $d\Phi = 0$. In view of (2.7), (2.9) and the definition $\Delta = dd^\star + d^\star d$ of the Laplacian, we have the identity

$$\Delta_{\mathbb{S}^3}\Phi = u^\star(\omega_{\mathbb{S}^2}). \tag{2.10}$$

Hence Φ is determined up to some additive constant form.

Energy estimates and the Hopf invariant. By standard elliptic theory, we obtain the estimates

$$\|\alpha\|_{L^3(\mathbb{S}^3)} \le C\|\nabla\Phi\|_{L^3(\mathbb{S}^3)} \le C\|\nabla u\|^2_{L^3(\mathbb{S}^3)} \tag{2.11}$$

so that, going back to formula (2.8), we deduce that $\mathbb{H}(u) \le C\|\nabla u\|_3^4$ and hence, as mentioned the lower bound (15) is readily an immediat consequence of the integral formula for the Hopf invariant. The fact that this lower bound is optimal is proved [28] and stated here as (16)(see also [3] for related ideas). It is far more subtle and relies on the identity

$$\mathbb{H}(\omega \circ u) = (\deg \omega)^2 \mathbb{H}(u). \tag{2.12}$$

for any $\omega : \mathbb{S}^2 \to \mathbb{S}^2$. Since this fact is somewhat central in our later arguments, we briefly indicate how (2.12) may lead to the lower bound (16). A first elementary observation is that, given any integer $\ell \in \mathbb{Z}$ one may construct a smooth map $\omega_\ell : \mathbb{S}^2 \to \mathbb{S}^2$ such that

$$\deg(\omega_\ell) = \ell \ \text{ and } |\nabla \omega_\ell|_{L^\infty(\mathbb{S}^2)} \leq C\sqrt{|\ell|}, \tag{2.13}$$

the idea being to glue together $|\ell|$ copies of degree ± 1 maps scaled down to cover disks of radii of order $\sqrt{|\ell|}$. Set $\mathrm{u}_\ell = \omega_\ell \circ \Pi$. It follows from (2.12) and (2.13) that

$$\mathbb{H}(\mathrm{u}_\ell) = \ell^2 \text{ and } |\nabla \mathrm{u}_\ell|_{L^\infty(\mathbb{S}^2)} \leq C\sqrt{|\ell|}$$

so that

$$\mathrm{E}_3(\mathrm{u}_\ell) \leq C|\ell|^{\frac{3}{2}} \ \leq C|\mathbb{H}(\mathrm{u}_\ell)|^{\frac{3}{4}},$$

yielding hence the proof of (16), at least when the hopf invariant $d = \ell^2$ is a square. The spaghetton map which we will construct later corresponds actually to a modification of the map u_ℓ and enjoys essentially the same properties, as it will be seen at the light of the next paragraph.

2.3 Linking numbers for preimages and the Pontryagin construction

Properties of the preimages of regular points yield another, very appealing, geometrical interpretation of the Hopf invariant which is parallel to (2.2) for the degree. Given a smooth map $u : \mathbb{R}^3 \to \mathbb{S}^2$ in $C^0_0(\mathbb{R}^3, \mathbb{S}^2)$ and a regular point M of $\mathbb{S}^2$, its preimage $L_M \equiv u^{-1}(M)$ is a smooth bounded curve in $\mathbb{S}^3$. The curve L_M inherits also from the original map u a normal framing and hence an orientation. Indeed, consider an arbitrary point $a \in L_M$, that is such that $u(a) = M$. Since M is supposed to be a regular point, the differential $Du(a)$ induces an isomorphism of the normal plane $P(a) \equiv (\mathbb{R}\vec{\tau}_{\tan}(a))^\perp$ onto the tangent space $T_M(\mathbb{S}^2)$. If $(\overrightarrow{W}_{1,M}, \overrightarrow{W}_{2,M})$ is an orthonormal basis of $T_M(\mathbb{S}^2)$ such that $(\overrightarrow{W}_{1,M}, \overrightarrow{W}_{2,M}, \overrightarrow{OM})$ is a direct orthonormal basis of $\mathbb{R}^3$, then its image $\mathfrak{f}^\perp$ by the inverse $T = (Du(a)_{|P(a)})^{-1}$ is a frame of $P(a)$ which is however not necessarily orthonormal. We define a framing on L_M, choosing the first vector $\vec{\tau}_1(a)$ of the frame as

$$\vec{\tau}_1(a) = \frac{T(\overrightarrow{W}_{1,M})}{|T(\overrightarrow{W}_{1,M})|}$$

and then $\vec{\tau}_2(a)$ as the unique unit vector orthogonal to $\vec{\tau}_1(a)$ such that $\mathfrak{e}^\perp_u \equiv (\vec{\tau}_1(a), \vec{\tau}_2(a))$ has the same orientation as $\mathfrak{f}^\perp$. A first remarkable observation (see [27] and [24], chapter XI, section 3) is that $(L_M, \mathfrak{e}^\perp_u)$ completely determines the homotopy class of u: Indeed, if $\varrho > 0$ is sufficiently small, then

$$\mathbb{H}(u) = \mathbb{H}\left(\mathbf{P}^{\text{ontya}}_\varrho[L_M, \mathfrak{e}^\perp_u]\right).$$

A second important property is that the linking number $\mathfrak{m}(L_{M_1}, L_{M_2})$ of the preimages of any two regular points M_1 and M_2 on $\mathbb{S}^2$ is independent of the choice of the two points and is equal to the Hopf invariant, that is

$$\mathfrak{m}(L_{M_1}, L_{M_2}) = \mathbb{H}(u). \tag{2.14}$$

Recall that the linking number of two *oriented curves* $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbb{R}^3$ is given by the Gauss integral formula

$$\mathfrak{m}(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{\overrightarrow{a_1 - a_2}}{|a_1 - a_2|^3} \, . \, \overrightarrow{da}_1 \times \overrightarrow{da}_2. \tag{2.15}$$

Notice in particular that the linking number is always an integer, that it is symmetric, i.e.

$$\mathfrak{m}(\mathcal{C}_1, \mathcal{C}_2) = \mathfrak{m}(\mathcal{C}_2, \mathcal{C}_1), \tag{2.16}$$

that its sign changes when the orientation of one of the curves is reversed and that $\mathfrak{m}(\mathcal{C}_2, \mathcal{C}_1) = 0$ if the two curves are not linked. Moreover, in case of several connected components, we have the rule

$$\mathfrak{m}(\mathcal{C}_{1,1} \cup \mathcal{C}_{1,2}, \mathcal{C}_2) = \mathfrak{m}(\mathcal{C}_{1,1}, \mathcal{C}_2) + \mathfrak{m}(\mathcal{C}_{1,2}, \mathcal{C}_2). \tag{2.17}$$

In practice, as we will do, the linking number of two given curves can be computed as the half sum of the *signed crossing number* of a projection on a two dimensional plane.

Remark 2.1. Chapter IX of [24] offers a good general background to the topics in this section and their extensions. The book [25] offers a more elementary and intuitive presentation.

2.4 The Hopf invariant of an elementary spaghetto

We go back to the Pontryagin construction and consider here the case the curve $\mathcal{C}$ is planar and connected. We may assume without loss of generally that $\mathcal{C}$ is included in the plane $P_{1,2}$. We assume moreover that it is framed with the reference frame $\mathfrak{e}_{\rm ref}^{\perp}$. In that case, the map $\mathbf{P}_{\varrho}^{\rm ontya}[\mathcal{C}]$ will be called an *elementary spaghetto*. We first observe:

Lemma 2.1. *If* $0 < \varrho < \varrho_0(\mathcal{C})$ *then* $\mathbb{H}(\mathbf{P}_{\varrho}^{\rm ontya}[\mathcal{C}, \mathfrak{e}_{\rm ref}^{\perp}]) = 0$.

Proof. The most direct proof is to use formula (2.14) and to consider the linking number of preimages of any two regular points. As a matter of fact, for the Pontryagin construction, all points are regular points, except the south pole $\mathbb{P}_{\rm south}$ whose preimage is the boundary of $T_{\varrho}(\mathcal{C})$, so that we may consider as regular points the North pole $\mathbb{P}_{\rm north}$ and the point M on the equator given by $M = (1, 0, 0)$. We have

$$L_{(\mathbb{P}_{\rm north})} = \mathcal{C} \;\text{ whereas }\; L_M = \mathcal{C} + g^{-1}(0)\varrho\vec{\mathbf{e}}_3,$$

where the function g is defined in (33). It follows that the two curves are parallel and hence not linked so that in particular

$$\mathfrak{m}\left(L_{(\mathbb{P}_{\rm north})}, L_M\right) = 0.$$

The conclusion then follows directly from (2.14). □

Remark 2.2. An alternate, perhaps more direct and more illuminating though also longer proof would be to construct *explicitely* a continuous deformation with values into $\mathbb{S}^2$ of $\mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}_{\text{ref}}^\perp]$ to a constant map. The main step in this construction is to show that there exists a continuous map Φ from the exterior domain $\mathbb{R}^3 \setminus \mathcal{C}$ to the circle $\mathbb{S}^1$ such that

$$\Phi\left(a + x_1\vec{\tau}_1(a) + x_2\vec{\tau}_2(a)\right) = \frac{(x_1, x_2)}{\sqrt{x_1^2 + x_2^2}} \text{ for any } a \in \mathcal{C} \text{ and } 0 < x_1^2 + x_2^2 \leq \varrho^2. \tag{2.18}$$

Assume for the moment that Φ is constructed and let us define the deformation. We set

$$F(x,t) = \left((1-t)\frac{\mathfrak{r}(x)}{\varrho} f\left(\frac{\mathfrak{r}(x)}{\varrho}(1-t)\right)\Phi(x),\ g\left(\frac{\mathfrak{r}(x)}{\varrho}(1-t)\right)\right) \text{ for } x \in \mathbb{R}^3 \text{ and } t \in [0,1],$$

where the functions f and g have been defined in (31) and where the function $\mathfrak{r}$ is defined as

$$\begin{cases} \mathfrak{r}(x) = \sqrt{x_1^2 + x_2^2} \text{ for any } x = a + x_1\vec{\tau}_1(a) + x_2\vec{\tau}_2(a) \text{ with } a \in \mathcal{C} \text{ and } 0 < x_1^2 + x_2^2 \leq \varrho^2, \\ \mathfrak{r}(x) = \varrho \text{ otherwise.} \end{cases}$$

It follows from the properties of f and g that F is continuous from $\mathbb{R}^3 \times [0,1]$ to $\mathbb{S}^2$, that $F(\cdot, t)$ belongs to $C^0_{\text{ct}}(\mathbb{R}^3, \mathbb{S}^2)$ for any $t \in [0,1]$ and that

$$F(\cdot, 0) = \mathbf{P}_\varrho^{\text{ontya}}[\mathcal{C}, \mathfrak{e}_{\text{ref}}^\perp] \text{ whereas } F(\cdot, 1) = \mathbb{P}_{\text{north}},$$

yielding hence the desired deformation. The construction of the map Φ is obtained adapting the Biot and Savart formula, as done for instance in [2].

Remark 2.3. A first possible way to obtain not trivial homotopy classes through the Pontryagin constructing with planar curves, is to twist the frame. Consider a map $\gamma : \mathcal{C} \to SO(2) \simeq \mathbb{S}^1$, and consider the twisted frame

$$e_\gamma^\perp = \gamma(\mathfrak{e}_{\text{ref}}^\perp) \equiv \left(\gamma(\cdot)(\vec{\tau}_1(\cdot)), \gamma(\cdot)(\vec{\tau}_2(\cdot))\right),$$

where, for $a \in \mathcal{C}$, the map $\gamma(a)$ is considered as a rotation of the plane $(\tau_{\tan}(a))^\perp$. Since $\mathcal{C}$ is topologically equivalent to a circle, one may define a winding number of γ and prove, for instance using the crossing numbers, that

$$\mathbb{H}\left(\mathbf{P}_\varrho^{\text{ontya}}\left[\mathcal{C}, e_\gamma^\perp\right]\right) = \deg(\gamma).$$

In some places, we will denote, for given $d \in \mathbb{Z}$, by $e_{\text{twist}=d}^\perp$ a framing which corresponds to a planer curves whose reference framing is twisted by a degree d map. As an exercise, the reader may construct a deformation showing that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two planar curves which do not intersect and which are not linked then we may merge them into a single curve with a frame twisted by the sum of the twists so that

$$\mathbb{H}\left(\mathbf{P}_\varrho^{\text{ontya}}\left[(\mathcal{C}_1, e_{\text{twist}=d_1}^\perp) \cup (\mathcal{C}_2, e_{\text{twist}=d_2}^\perp)\right]\right) = d_1 + d_2. \tag{2.19}$$

2.5 The Hopf invariant of two linked spaghetti

Another simple way to obtain non trivial homotopy classes is to consider two linked planar curves, yielding what is often called a *Hopf link*. Consider therefore two planar curves without self-intersection, a curve $\mathcal{C}_1$ included in the plane $P_{1,2}$ of equation $x_3 = 0$ and a curve $\mathcal{C}_2$ included in the plane $P_{2,3}$ of equation $x_1 = 0$. To fixe ideas, on may take for $\mathcal{C}_1$ and $\mathcal{C}_2$ the circles

$$\mathcal{C}_1 = \{(x_1, x_2, 0) \in \mathbb{R}^3, x_1^1 + x_2^2 = 1\} \text{ and } \mathcal{C}_2 = \{(0, x_2, x_3) \in \mathbb{R}^3, (x_2+1)^2 + x^3 = 1\}$$

so that the center of $\mathcal{C}_1$ is the origin, the center of $\mathcal{C}_2$ is the point $\mathrm{O}_2 = (0, -1, 0)$, both circles having radius 1. We choose for both circles the reference frames $\mathfrak{e}^{\perp}_{\text{ref}}$ defined before and the corresponding orientation. They are obviously linked, and using the crossing numbers, we verify easily that

$$\mathfrak{m}(\mathcal{C}_1, \mathcal{C}_2) = 1.$$

We then set

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2.$$

Lemma 2.2. *We have, for sufficiently small $\varrho > 0$, $\mathbb{H}(\mathbf{P}^{\text{ontya}}_{\varrho}[\mathcal{C}, \mathfrak{e}^{\perp}_{\text{ref}}]) = 2$.*

Proof. We argue as in the proof of Lemma 2.1 an consider the pre-images $L_{\mathbb{P}_{\text{north}}} = \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and $L_M = \mathcal{C}'_1 \cup \mathcal{C}'_2$ of the North pole and the point $M = (1, 0, 0)$ of the equator respectively, where we have set $\mathcal{C}'_1 = \mathcal{C}_1 + g^{-1}(0)\varrho\vec{\mathbf{e}}_3$ and $\mathcal{C}'_2 = \mathcal{C}_2 + g^{-1}(0)\varrho\vec{\mathbf{e}}_1$. It follows that

$$\begin{aligned} \mathfrak{m}(L_{\mathbb{P}_{\text{north}}}, L_M) &= \mathfrak{m}(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}'_1 \cup \mathcal{C}'_2) \\ &= \mathfrak{m}(\mathcal{C}_1, \mathcal{C}'_1) + \mathfrak{m}(\mathcal{C}_1, \mathcal{C}'_2) + \mathfrak{m}(\mathcal{C}_2, \mathcal{C}'_1) + \mathfrak{m}(\mathcal{C}_2, \mathcal{C}'_2). \end{aligned} \tag{2.20}$$

Since the curves $\mathcal{C}_1$ and $\mathcal{C}'_1$ are parallel and hence not linked $\mathfrak{m}(\mathcal{C}_1, \mathcal{C}'_1) = 0$ and likewise $\mathfrak{m}(\mathcal{C}_2, \mathcal{C}'_2) = 0$. On the other hand $\mathfrak{m}(\mathcal{C}_1, \mathcal{C}'_2) = \mathfrak{m}(\mathcal{C}'_1, \mathcal{C}_2) = \mathfrak{m}(\mathcal{C}_1, \mathcal{C}_2) = 1$ so that we obtain $\mathfrak{m}(L_{\mathbb{P}_{\text{north}}}, L_M) = 2$. Invoking (2.14) the conclusion follows. □

3 Linked k-spaghetton map

We provide in this section a precise definition of the spaghetton map $\mathbf{S}^k_{\mathbf{pag}}$, which has already been described more vaguely in the introduction. The general idea is to extend the construction performed in Subsection 2.5 when each planar curve is replaced by a sheaf of such curves which are parallel. The spaghetton is then obtained by the Pontryagin construction with the corresponding reference frame.

As mentioned in the introduction, each of the curves with which we will perform the Pontragyin construction is stadium shaped. Let us recall that a stadium is a closed curve whose interior consists of the interior of a rectangle, with two parallel ends capped off with semidisks. Given an integer $k \in \mathbb{N}^{\star}$, the total number of curves will be k^2 in each of the two sheaves $\mathfrak{L}^k$ and $\mathfrak{L}^{k,\perp}$ of our construction. Each of the sheaves of $\mathfrak{L}^k$ and $\mathfrak{L}^{k,\perp}$ is composed of parallel segments on the straight part on the stadium in the direction of $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ respectively, and nearly parallel on the round parts.

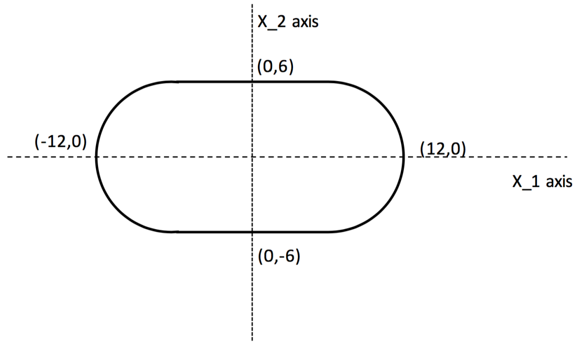

Figure 2: *The reference stadium* $\mathbb{L}_0$

3.1 The sheaf $\mathfrak{L}^k$ of the k^2 stadium shaped curves $\mathfrak{L}^k_{j,q}$

We describe here the curves $\mathfrak{L}^k_{j,q}$, $j, q = 1, \dots, k$ composing the sheave $\mathfrak{L}^k$. These curves are modeled on a standard stadium $\mathbb{L}_0 \subset \mathbb{R}^2$, centered at the origin $O = (0,0)$ we present first.

Construction of the reference stadium $\mathbb{L}_0$ *in the plane* $\mathbb{R}^2$. Working in this paragraph on the plane $\mathbb{R}^2$, we consider first two straight segments $\mathcal{D}^1_0$ and $\mathcal{D}^2_0$ parallel to $\vec{\mathbf{e}}_1 = (1,0)$ each of length 12, given by

$$\mathcal{D}^1_0 = [-6,6] \times \{-6\} \text{ and } \mathcal{D}^2_0 = [-6,6] \times \{6\}.$$

We complete these two parallel segments as a stadium $\mathbb{L}_0$ contained in the plane $\mathbb{R}^2$ adding two half circles so that

$$\mathbb{L}_0 = \mathcal{D}^1_0 \cup \mathcal{D}^2_0 \cup \mathbb{S}^{1,+}_6(O^+_0) \cup \mathbb{S}^{1,-}_6(O^-_0) \subset \mathbb{R}^2,$$

where $\mathbb{S}^{1,+}_5(O^+_0)$ and $\mathbb{S}^{1,-}_5(O^-_0)$ are two half circles of radius $r = 6$ in the plane $\mathbb{R}^2$ of centers $O^+_0 \equiv ((6,0)$ and $O^-_0 \equiv (-6,0)$ respectively, where we have set, for given $r > 0$ and $A = (a_1, a_2) \in \mathbb{R}^2$

$$\begin{cases} \mathbb{S}^{1,+}_r(A) = \{(x_1,x_2) \in \mathbb{R}^2, (x_1-a_1)^2 + (x_2-a_2)^2 = r^2, x_1 \geq a_1\} \\ \mathbb{S}^{1,-}_r(A) = \{(x_1,x_2) \in \mathbb{R}^2, (x_1-a_1)^2 + (x_2-a_2)^2 = r^2, x_1 \leq a_1\}. \end{cases}$$

Notice that

$$\mathbb{L}_0 \subset [-12,12] \times [6,6].$$

Construction of concentric stadia $\mathbb{L}^k_{\ell,0}$. Given $k \in \mathbb{N}^*$, we construct a family of concentric stadia which are deduce from the reference stadium by homothety as

$$\mathbb{L}^k_{\ell,0} = \left(1 + \frac{h(k-\ell)}{6}\right) \mathbb{L}_0, \text{ for } \ell = 0, \dots, k, \text{where } h = k^{-1}.$$

It follows from this definition that $\mathbb{L}^k_{k,0} = \mathbb{L}_0$ and that the domains of $\mathbb{R}^2$ bounded by the curves $\mathbb{L}^k_{\ell,0}$ are decreasing as ℓ increases. Moreover one may verify that

$$\text{dist}\left(\mathbb{L}^k_{\ell,0}, \mathbb{L}^k_{\ell+1,0}\right) = h \text{ for } \ell = 0, \dots, k-1.$$

We set

$$\mathbb{L}^k = \overset{k}{\underset{\ell=1}{\cup}} \mathbb{L}^k_{\ell,0}.$$

Notice that the straight segments in $\mathbb{L}^k_{\ell,0}$ are all parallel to $\vec{\mathbf{e}}_1$, of lengths varying between 12 and 14, that

$$\left(\overset{k}{\underset{\ell=1}{\cup}} \mathbb{L}^k_{\ell,0}\right) \cap ([-6,6]\times\mathbb{R}) = [-6,6]\times[\{-7+h\ell, \ell=1,\dots,k\}\cup\{7-h\ell,\ell=1,\dots,k\}] \quad (3.1)$$

and that, for any $\ell = 0,\dots,k$ $\mathbb{L}^k_{\ell,0} \subset [-14,14]\times[7,7]$.

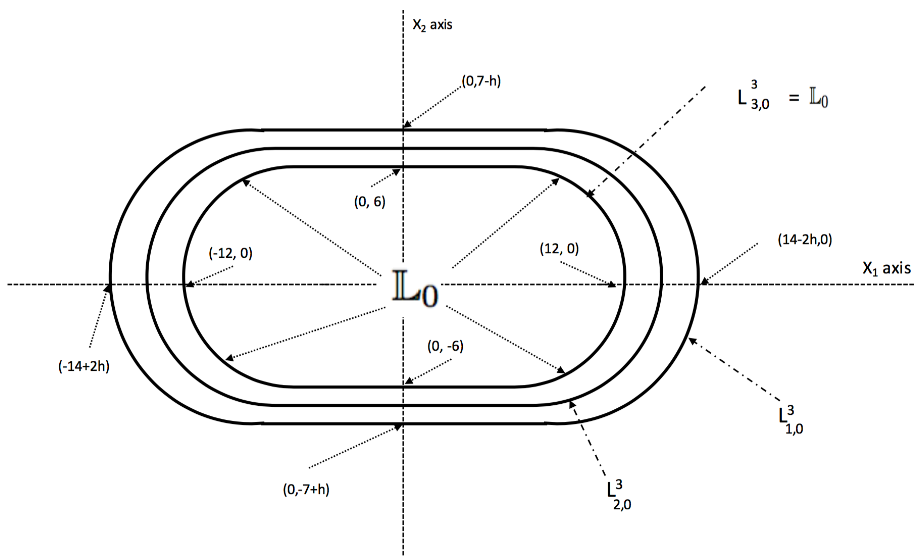

Figure 3: *The 3 stadia* $\mathbb{L}^3_{\ell,0}$, $\ell = 1,2,3$ *and the reference stadium* $\mathbb{L}_0$

Construction of the stadia $\mathbb{L}^k_{j,q}$. For $q = 1,\dots,k$ we consider the k parallel planes $P_{1,2}(hk)$: Identitying these planes with $\mathbb{R}^2$, we construct in each of them the lines $\mathbb{L}^k_{j,q}$ corresponding to the stadia $\mathbb{L}^k_{j,0}$ setting for $j = 0,\dots,k$

$$\mathbb{L}^k_{j,q} = \mathbb{L}^k_{j,0} + qh\vec{\mathbf{e}}_3 \text{ for } q = 1,\dots,k.$$

Notice that curves which distinct set of indices do not intersect. We finally consider the union of the k^2 curves $\mathbb{L}^k_{j,q}$ obtained before, each contained in planes orthogonal to $\vec{\mathbf{e}}_3 = (0,0,1)$, yielding the sheave

$$\mathbb{L}^k = \overset{k}{\underset{j,q=1}{\cup}} \mathbb{L}^k_{j,q} \subset [-14,14]\times[-7,7]\times[0,1]$$

Construction of the curves $\mathfrak{L}^k_{j,q}$ *and of* $\mathfrak{L}^k$. They are deduced from $\mathbb{L}^k_{j,q}$ and $\mathbb{L}^k$ by a simple

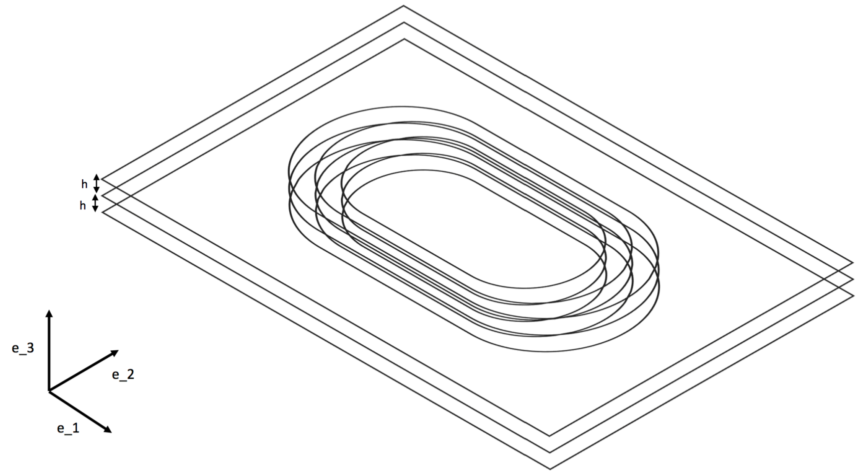

Figure 4: *The set* $\mathbb{L}^3$ *and the three planes* $P_{1,2}(h)$, $P_{1,2}(2h)$ *and* $P_{1,2}(3h) = P_{1,2}(1)$ *containing each three connected curves of the sheave* $\mathbb{L}^3$.

translation in the direction of $\vec{\mathbf{e}}_2$. We set, for $j = 1, \dots, k$ and $q = 1, \dots, k$

$$\mathfrak{L}^k_{j,q} = \mathbb{L}^k_{j,q} + 7\,\vec{\mathbf{e}}_2 \text{ and } \mathfrak{L}^k = \mathbb{L}^k + 7\vec{\mathbf{e}}_2$$

so that

$$\mathfrak{L}^k = \bigcup_{j,q}^{k} \mathfrak{L}^k_{j,q} \subset [-14, 14] \times [h, 14] \times [0, 1]$$

Property (42) presented in the introduction then follows from (3.1) and the above constructions (see in particular figure 6).

The mutual distant between the individual spaghetti is bounded below by

$$\operatorname{dist}(\mathfrak{L}^k_{j,q}, \mathfrak{L}^k_{j',q'}) \geq h = \frac{1}{k} \text{ for } (j, q) \neq (j', q'). \tag{3.2}$$

Moreover, going back to (33) we may observe also that, at least for large k we have

$$\varrho_0(\mathfrak{L}^k) \geq \frac{1}{3k}. \tag{3.3}$$

At this stage, the total linking number of $\mathfrak{L}^k$ is still equal to zero. In order to produce topology, we need to to define a second sheaf.

3.2 The sheaf $\mathfrak{L}^{k,\perp}$

We first construct as above a sheave $\mathbb{L}^{k,\perp}$ deduced from the sheave $\mathbb{L}^k$ as

$$\mathbb{L}^{k,\perp} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ s.t. } (x_2, x_3, x_1) \in \mathbb{L}^k \right\}.$$

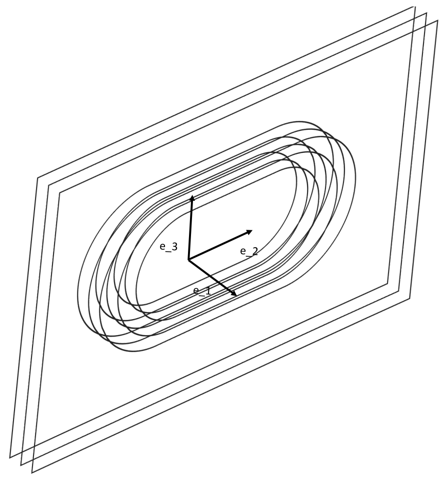

Figure 5: *The set* $\mathbb{L}^{3,\perp}$, $k=3$ *and the three hyperplanes* $P_{2,3}(h), P_{2,3}(2h)$ *and* $P_{2,3}(3h)$ *with* $h=1/3$.

Alternatively, we may define $\mathbb{L}^{k,\perp}$ as the image of $\mathbb{L}^k$ by the rotations $\mathfrak{R}_0$ of $\mathbb{R}^3$ which sends $\vec{\mathbf{e}}_1$ onto $\vec{\mathbf{e}}_2$, $\vec{\mathbf{e}}_2$ onto $\vec{\mathbf{e}}_3$ and $\vec{\mathbf{e}}_3$ onto $\vec{\mathbf{e}}_1$. We have

$$\begin{cases} \mathbb{L}^{k,\perp} = \bigcup\limits_{i,q=1}^{k} \mathbb{L}^{k,\perp}_{i,q} \text{ where} \\ \mathbb{L}^{k,\perp}_{i,q} = \mathcal{R}_0(\mathbb{L}^k_{k-q+1,i}) = \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 \text{s.t. } (x_2,x_3,x_1) \in \mathbb{L}^k_{q-k+1,i}\right\}, \end{cases} \tag{3.4}$$

so that, for $q=1,\dots,k$, the connected curves $\mathbb{L}^{k,\perp}_{i,q}$ are included in the plane $P_{2,3}(ih)$. Notice an important difference in the way we label the curves $\mathbb{L}^{k,\perp}_{i,q}$ with the way we label the curves $\mathbb{L}^{j,\perp}_{i,q}$: The domain included in the plane $P_{2,3}(ih)$ bounded by the curves $\mathbb{L}^{k,\perp}_{i,q}$ are increasing with q, for fixed i. As a matter of fact, we may also write

$$\mathbb{L}^{k,\perp}_{i,q} = \left(1+\frac{h\ell}{6}\right)\left(\mathbb{L}^{\perp}_0 + ih\vec{\mathbf{e}}_1\right), \text{ for } \ell = 0,\dots,k, \text{where } \mathbb{L}^{\perp}_0 = \mathcal{R}_0(\mathbb{L}_0). \tag{3.5}$$

We notice

$$\mathbb{L}^{k,\perp} \subset ([0,1]\times[-14,14]\times[-7,7]) \tag{3.6}$$

And that $\mathbb{L}^{k,\perp}$ is composed of segments in the direction $\vec{\mathbf{e}}_2$ in its central part, of lengths between 12 and 14. More precisely, we have

$$\mathbb{L}^{k,\perp} \cap (\mathbb{R}\times[-6,6]\times\mathbb{R}) = h\mathbb{I}_k \times [-6,6] \times [(h\mathbb{I}_k - \{7\}) \cup (h\mathbb{I}_k + \{6-h\})],$$

where we have set $\mathbb{I}_k = \{1,\dots,k\}$.

The set $\mathfrak{L}^{k,\perp}$ is deduced from the set $\mathbb{L}^{k,\perp}$ by a translation in the direction of $\vec{\mathbf{e}}_2$. We set

$$\mathfrak{L}^{k,\perp} = \mathbb{L}^{k,\perp} - 3\,\vec{\mathbf{e}}_2 \text{ and } \mathfrak{L}^{k,\perp}_{i,q} = \mathbb{L}^{k,\perp}_{i,q} - 3\,\vec{\mathbf{e}}_2 \text{ for } i,q = 1,\dots,k.$$

Inclusion (3.6) then yields

$$\begin{cases} \mathfrak{L}^{k,\perp} \subset [0,1] \times [-17,11] \times [-7,7] \\ \mathfrak{L}^{k,\perp} \cap (\mathbb{R} \times [-2,2] \times \mathbb{R}) = h\mathbb{I}_k \times [-2,2] \times [(h\mathbb{I}_k - \{7\} \cup h\mathbb{I}_k + \{6-h\})]\,. \end{cases} \tag{3.7}$$

Remark 3.1. The labeling (3.4) and (3.5) of the curves $\mathfrak{L}^{k,\perp}_{i,q}$ which is different from the labeling of the curves $\mathfrak{L}^{k}_{j,q}$ is motivated by the fact that

$$\mathfrak{L}^{k,\perp}_{i,q} \cap \mathbb{R} \times [-2,2] \times \mathbb{R}^+ = \{ih\} \times [-2,2] \times \{6+qh\} \tag{3.8}$$

so that that the q index labels the upper straight part of the fibers with increasing height x_3.

3.3 First properties of the sheaves $\mathfrak{L}^k$ and $\mathfrak{L}^{k,\perp}$

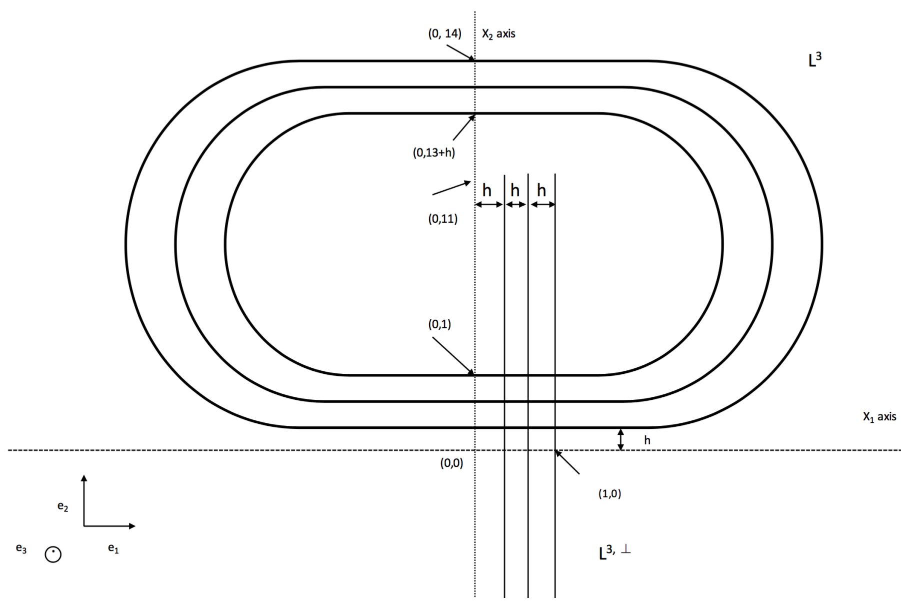

Figure 6: *The set $\mathcal{S}^3$ seen from above. The intersection of the orthogonal projection onto $P_{1,2}$ of $\mathfrak{L}^{3,\perp}$ with $\mathfrak{L}^3$ is the grid $\boxplus^2_3(\frac{1}{3})$.*

Notice first (see figure 6 and 7) that the intersection of two sheaves $\mathfrak{L}^k$ and $\mathfrak{L}^{k,\perp}$ is empty and that moreover

$$\operatorname{dist}(\mathfrak{L}^k, \mathfrak{L}^{k,\perp}) = 2. \tag{3.9}$$

Since each of the curves $\mathfrak{L}^{k}_{j,q}$ and $\mathfrak{L}^{k,\perp}_{i,q}$ are planar curves which are either included in affine planes parallel to $P_{1,2}$ or to $P_{2,3}$ we may frame them with the reference frames $\mathfrak{e}^{\perp}_{\mathrm{ref}}$ which have been defined in Subsection 2.3. This yields, as we have already seen, a natural orientation of the curves. For instance, the curves $\mathfrak{L}^{k}_{i,j}$ are oriented counter-clokwise with respect to the frame $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ and similarily the curves $\mathfrak{L}^{k,\perp}_{i,q}$ are oriented counter-clokwise with respect to the frame $(\vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3)$.

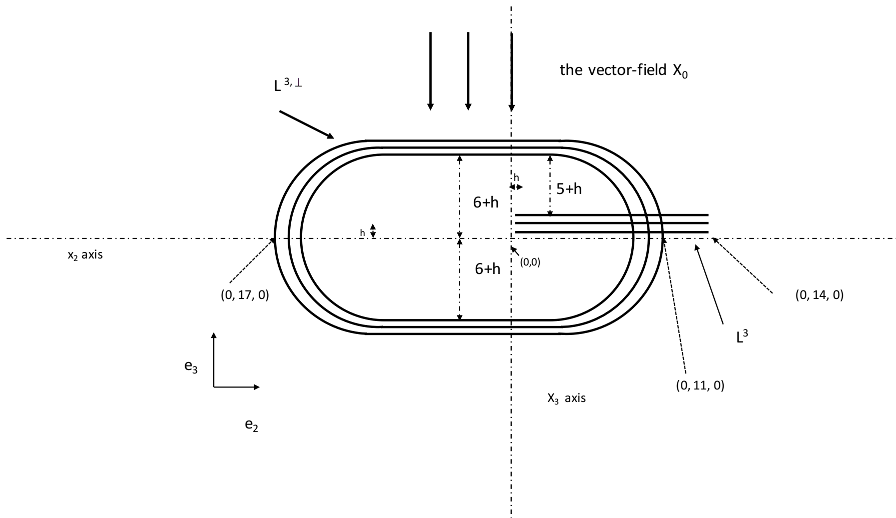

Figure 7: *The set $\mathcal{S}^3$ seen from the $\vec{\mathbf{e}}_1$ direction. The intersection of the orthogonal projection onto $P_{1,2}$ of $\mathfrak{L}^{3,\perp}$ with $\mathfrak{L}^3$ is the grid $\boxplus_3^2(\frac{1}{3})$. The vector filed $\vec{X}_0$ pushes $\mathfrak{L}^{3,\perp}$ onto $\mathfrak{L}^3$ until they meet.*

Concerning *topology*, each curve $\mathfrak{L}^k_{i_0,j_0}$ is linked to the k^2 curves $\mathfrak{L}^{k,\perp}_{i,q}$ with linking number 1 and each curve $\mathfrak{L}^{k,\perp}_{i_0 q_0}$ is linked with the k^2 curves $\mathfrak{L}^k_{i,j}$ with linking number 1. Hence, we obtain for the total linking number:

Lemma 3.1. *We have $\mathfrak{m}(\mathfrak{L}^k, \mathfrak{L}^{k,\perp}) = k^4$ for any $k \in \mathbb{N}$.*

Proof. We have

$$\mathfrak{m}(\mathfrak{L}^k, \mathfrak{L}^{k,\perp}) = \mathfrak{m}\left(\underset{j,q}{\cup}\mathfrak{L}^k_{j,q}, \underset{i,q'}{\cup}\mathfrak{L}^{k,\perp}_{i,q'}\right) = \sum_{j,q}\sum_{j,q'}\mathfrak{m}\left(\mathfrak{L}^k_{j,q}, \mathfrak{L}^{k,\perp}_{i,q'}\right) = k^4\mathfrak{m}\left(\mathfrak{L}_0, \mathfrak{L}_0^\perp\right),$$

and the conclusion follows from the identities (2.16) and (2.17). □

Finally, as mentioned in the introduction, we consider the one-dimensional set

$$\mathcal{S}^k = \mathfrak{L}^k \cup \mathfrak{L}^{k,\perp}. \tag{3.10}$$

It follows from (3.3) and (3.9) that

$$\varrho_0(\mathcal{S}^k) \geq \frac{1}{3k} \text{ and } \mathcal{S}^k \subset \mathbb{B}(17). \tag{3.11}$$

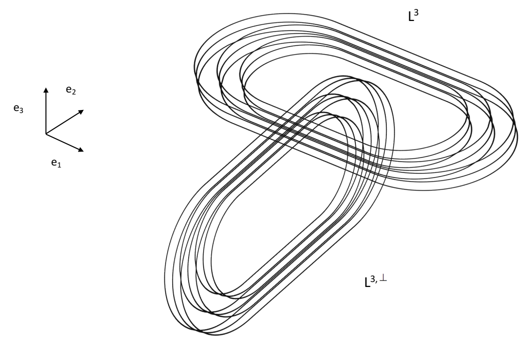

Figure 8: *The sets $\mathfrak{L}^k$ and $\mathfrak{L}^{k,\perp}$ are linked*

3.4 The k-spaghetton map $\mathbf{S}^k_{\mathbf{pag}}$ and its properties

Choosing $\varrho_k = 10^{-3}\varrho_0(\mathcal{S}^k)$ we define the k-spaghetton map $\mathfrak{S}_k$ as

$$\mathbf{S}^k_{\mathbf{pag}} = \mathbf{P}^{\text{ontya}}_{\varrho_k}[\mathcal{S}^k, \mathfrak{e}^{\perp}_{\text{ref}}]. \tag{3.12}$$

Some of its most relevant properties have been summarized in Proposition 3 in the introduction, which we prove next.

Proof of Proposition 3. The first assertion concerning the support of $\mathbf{S}^k_{\mathbf{pag}}$ follows from the inclusion given in (3.11), whereas the second, the bound on the gradient, is an immediate consequence of (32). Since all fibers have the same shape, which does not depend on k, the constant $\mathbf{C}_{\text{spg}}$ involved in the gradient estimate does not depend on k either. Turning to the third assertion, the computation of $\mathbb{H}(\mathbf{S}^k_{\mathbf{pag}})$ follows the same lines as the proof of Lemma 2.2, considering the pre-images

$$L_{\mathbb{P}_{\text{north}}} = \mathcal{S}^k = \mathfrak{L}^k \cup \mathfrak{L}^{k,\perp} \text{ and } L_M = (\mathfrak{L}^k + g^{-1}(0)\varrho_k\vec{\mathbf{e}}_3) \cup (\mathfrak{L}^{k,\perp} + g^{-1}(0)\varrho_k\vec{\mathbf{e}}_1)$$

of the North pole $\mathbb{P}_{\text{north}}$ and the point $M = (1, 0, 0)$ of the equator respectively. Arguing as in (2.20), we obtain

$$\begin{aligned}\mathfrak{m}(L_{\mathbb{P}_{\text{north}}}, L_M) &= \mathfrak{m}(\mathfrak{L}_k \cup \mathfrak{L}^{k,\perp}, \mathfrak{L}^k + g^{-1}(0)\varrho\vec{\mathbf{e}}_3) \cup (\mathfrak{L}^{k,\perp} + g^{-1}(0)\varrho\vec{\mathbf{e}}_1)\\ &= 2\mathfrak{m}(\mathfrak{L}^k, \mathfrak{L}^{k,\perp}) = 2k^4,\end{aligned}$$

where the last identity follows from Lemma 3.1. For the estimate on the energy in the statement of Proposition 3, we observe that, since the support of $|\nabla \mathbf{S}^k_{\mathbf{pag}}|$ is included in the ball $\mathbb{B}(17)$ independently of k, it suffices to integrate the uniform bound of the gradient, which is of order k to obtain the result for the 3-energy. In particulat, we may choose the constant as $\mathbf{K}_{\text{spg}} = 17^3\mathbf{C}^3_{\text{spg}}$.

□

4 Untying the spaghetton map $\mathbf{S}^k_{\mathbf{pag}}$: the gordian cut $\mathbf{G}^k_{\mathbf{ord}}$

The proof of Proposition 4 is somewhat technical, its completion will be given at the end of this Section. The heuristic idea is however rather simple: we push down along the x_3-axis the sheaf $\mathfrak{L}^{k,\perp}$, keeping however its shape essentially unchanged, whereas the sheaf $\mathfrak{L}^k$ does not move. This presents no major difficulty, pushing along a constant vector-field as long as the sheaf $\mathfrak{L}^{k,\perp}$ does not encounter the sheaf $\mathfrak{L}^k$. When the two sheafs touch, we take advantage of the fact that we are working in a Sobolev class were singularities are allowed: Using such singularities, the sheaf $\mathfrak{L}^{k,\perp}$ is enabled to follow his way down and to pass through the fibers of $\mathfrak{L}^k$, creating on the way point singularities. These singularities form a cloud of uniformly distributed points, at least in the center of the cloud.

In order to provide a sound mathematical meaning to the previous construction, in particular the crossing of fibers, we single out a a few elementary tools which are used extensively in the proof of Proposition 4 and gather them in a *Sobolev deformation toolbox.*

4.1 Sobolev deformation and surgery toolbox

4.1.1 Gluing maps

This is the most elementary operation. Assume first that we are given two subdomains Ω_1 and Ω_2 of a domain Ω_0 of $\mathbb{R}^3$ such that $\bar\Omega_1 \cap \bar\Omega_2 = \emptyset$ and let v_1 and v_2 be two functions with values into $\mathbb{S}^2$ such that $v_1(x) = \mathbb{P}_{\rm south}$ for $x \in \Omega_0 \setminus \Omega_1$ and $v_2(x) = \mathbb{P}_{\rm south}$ for $x \in \Omega_0 \setminus \Omega_2$. Then we define the function $v_1 \vee_3 v_2$ on Ω_0 by

$$\begin{cases} v_1 \vee_3 v_2(x) = v_1(x) \text{ for } x \in \Omega_1, \ v_1 \vee_3 v_2(x) = v_2(x) \text{ for } x \in \Omega_2 \\ v_1 \vee_3 v_2(x) = \mathbb{P}_{\rm south} \text{ for } x \in \Omega_0 \setminus (\Omega_1 \cup \Omega_2), \end{cases} \tag{4.1}$$

an alternative and even simpler definition being

$$v_1 \vee_3 v_2(x) = v_1(x) + v_2(x) - \mathbb{P}_{\rm south} \ \text{ for any } x \in \mathbb{R}^3.$$

In the case both v_1 and v_2 have bounded E_3 energy, then the same holds for $v_1 \vee_3 v_2$ with

$$\mathrm{E}_3(v_1 \vee_3 v_2) = \mathrm{E}_3(v_1) + \mathrm{E}_3(v_2). \tag{4.2}$$

A related situation is encountered in the case $\Omega_2 = \Omega_0 \setminus \Omega_1$, when both v_1 and v_2 have bounded E_3 energy. If the domains are sufficiently smooth, then one may define thanks to the trace Theorem the restrictions $v_{i|_{\partial\Omega_i}}$ for $i = 1, 2$ and if

$$v_1(x) = v_2(x) \text{ for } x \in \partial\Omega_1 \subset \partial\Omega_2 = \partial\Omega_1 \cup \partial\Omega_0$$

then we may define again $v_1 \vee_3 v_2$ according to (4.1) and relation (4.2) still holds. Given two disjoint oriented compact framed curves in $\mathbb{R}^3$, we have, provided $\varrho > 0$ is sufficiently small

$$\mathbf{P}^{\rm ontya}_\varrho[(\mathcal{C}_1, \mathfrak{e}^\perp) \cup (\mathcal{C}_2, \mathfrak{e}^\perp)] = \mathbf{P}^{\rm ontya}_\varrho[\mathcal{C}_1, \mathfrak{e}^\perp] \vee_3 \mathbf{P}^{\rm ontya}_\varrho[\mathcal{C}_2, \mathfrak{e}^\perp].$$

so that in particular

$$\mathbf{S}^k_{\mathbf{pag}} = \left(\mathop{\vee_3}_{j,q=1}^{k} \mathbf{P}^{\rm ontya}_{\varrho_k}[\mathcal{L}^k_{j,q}, \mathfrak{e}^\perp_{\rm ref}] \right) \vee_3 \left(\mathop{\vee_3}_{i,q=1}^{k} \mathbf{P}^{\rm ontya}_{\varrho_k}[\mathfrak{L}^{k,\perp}_{i,q}, \mathfrak{e}^\perp_{\rm ref}] \right)$$

Finally, we refer to similar gluing in $\mathbb{R}^4$, replacing the symbol $\vee_3$ by the symbol $\vee_4$.

4.1.2 Deformations of the domain

Definitions. We consider deformations of *maps* generated by deformations of the *domain* $\mathbb{R}^3$ induced by the integration of a vector field. Given a smooth vector field $\vec{X}$ on $\mathbb{R}^3$, we consider the flow Φ generated by the vector field $\vec{X}$ defined by

$$\frac{d}{dt}\Phi(\cdot,t) = \vec{X}[\Phi(\cdot,t)] \text{ with } \Phi(\cdot,0) = \mathrm{Id}_{\mathbb{R}^3}, \tag{4.3}$$

so that, for each fixed time $t \geq 0$ the map $\Phi(\cdot,t) : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism of $\mathbb{R}^3$. We denote by $\Phi^{-1}(\cdot,t)$ its inverse at time t, so that $\Phi^{-1}(\cdot,t) = \Phi(\cdot,-t)$. The deformation of the domain gives rise also to corresponding deformations of general functions: To each function v defined on $\mathbb{R}^3$ and given $t \geq 0$, we may relate a function $v_t(\cdot)$ defined by

$$v_t(x) = v\left(\Phi^{-1}(x,t)\right) \text{ for } x \in \mathbb{R}^3.$$

The curve $t \mapsto v_t$ is now a continuous deformation of the initial function v, since $v_0 = v$. We will also consider the transportation of subsets of $\mathbb{R}^3$ by the flow Φ. We set accordingly for a subset $A \subset \mathbb{R}^3$ and $t \geq 0$

$$\Phi(A,t) = \{x \in \mathbb{R}^3, \Phi^{-1}(x,t) \in A\}. \tag{4.4}$$

Notice that, if $\mathcal{C}$ is a framed closed curve of $\mathbb{R}^3$, then in general

$$\left(\mathbf{P}_\varrho^{\text{ontya}}\left[\mathcal{C}, \mathfrak{e}_{\text{ref}}^\perp\right]\right)_t(x) \neq \mathbf{P}_\varrho^{\text{ontya}}\left[\Phi(\mathcal{C},t), \mathfrak{e}_{\text{ref}}^\perp\right](x) \text{ for } x \in \mathbb{R}^3 \text{ and } t \geq 0, \tag{4.5}$$

where the frame has been transported accordingly. However equality holds in case $\vec{X}$ is a constant function, since in that case $\Phi(\mathcal{C},t)$ is a translate of $\mathcal{C}$. This observation leads us to introduce a variant of the Pontryagin construction for non-constant vector fields.

Vertical vector fields. We implement the previous construction with a very specific choice of vector fields $\vec{X}$. Since our aim is to push the sheave $\mathfrak{L}^{k,\perp}$ down according to x_3-direction we restrict ourselves to vector fields $\vec{X}$ of the form

$$\vec{X}(x_1,x_2,x_3) = -\zeta(x_1,x_2,x_3)\,\vec{\mathbf{e}}_3, \tag{4.6}$$

where $\zeta : \mathbb{R}^3 \to \mathbb{R}$ is a given non-negative function on $\mathbb{R}^3$. The related flow Φ can then be integrated as

$$\begin{cases} \Phi(x_1,x_2,x_3,t) = (x_1,x_2,\Psi(x_1,x_2,x_3,t)) \text{ where } \Psi \text{ solves the ODE with respect } t \\ \dfrac{d}{dt}\Psi(x_1,x_2,x_3,t) = -\zeta(x_1,x_2,\Psi(x_1,x_2,x_3,t),t). \end{cases} \tag{4.7}$$

It follows directly from (4.3) that

$$|\frac{\partial\Phi}{\partial t}| \leq \|\vec{X}\|_{L^\infty(\mathbb{R}^3)} = \|\zeta\|_{L^\infty(\mathbb{R}^3)}. \tag{4.8}$$

Differentating (4.3) with respect to the variable x_i, we are led to a relation of the form

$$|\frac{\partial}{\partial t}[|\nabla_3\Phi|^2]| \leq C\|\nabla_3\zeta_3\|_{L^\infty(\mathbb{R}^3)}|\nabla\Phi|^2, \tag{4.9}$$

where $C > 0$ is some universal constant. Integrating (4.9), we obtain the exponential bound

$$|\nabla_3 \Phi|^2(\cdot, t) \leq C \exp\left(C \|\nabla_3 \zeta\|_{L^\infty(\mathbb{R}^3)} t\right) \text{ for } t \geq 0. \tag{4.10}$$

In the case ζ does not depend on x_3, the integration of the vector field $\vec{X}$ given by (4.6) is straightforward and yields $\Phi(x_1, x_2, x_3, t) = (x_1, x_2, x_3 - \zeta(x_1, x_2)t)$.

We introduce a deformation operator $\mathcal{P}_\zeta$ which relates to an arbitrary map $v : \mathbb{R}^3 \to \mathbb{R}^\ell$ and $t \geq 0$ the map $P_\zeta(v)(t)$ defined on $\mathbb{R}^3$ by the formula, for $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $t \in \mathbb{R}$

$$\mathcal{P}_\zeta(v)(t)(x_1, x_2, x_3) = v(\Phi^{-1}(x_1, x_2, x_3, t)) = v(\Phi(x_1, x_2, x_3, -t)). \tag{4.11}$$

In some places, we will use the simpler notation $v_t(\cdot) = \mathcal{P}_\zeta(v)(t)(\cdot)$, when this is not ambiguous. In the special case the function ζ does not depend on x_3, we have $v_t = v(x + \zeta(x_1, x_2)t\,\vec{\mathbf{e}}_3)$. As a direct consequence of the chain rule and estimates (4.8) and (4.10), we obtain:

Lemma 4.1. *Assume that v and ζ are differentiable. Then we have for $x \in \mathbb{R}^3$ and $t \geq 0$*

$$\begin{cases} |\dfrac{\partial}{\partial t}\mathcal{P}_\zeta(v)(t)(x)| \leq C\|\nabla_3 v\|_{L^\infty(\mathbb{R}^3)}\|\zeta\|_{L^\infty(\mathbb{R}^3)} \\ |\nabla_3 \mathcal{P}_\zeta(v)(t)(x)| \leq C\|\nabla_3 v\|_{L^\infty(\mathbb{R}^3)} \exp\left(C\|\nabla_3 \zeta\|_{L^\infty(\mathbb{R}^3)} t\right), \end{cases}$$

where $C > 0$ is some universal constant.

Two kinds of vertical fields. We will be even more specific and describe next the two different kinds of vertical vector fields which are used in the construction of the Gordian cut.

Constant vertical fields. We consider here the vector field $\vec{X}_0$ related by (4.6) to the constant function $\zeta_0 = 1$

$$\zeta_0(x_1, x_2, x_3) = 1, \ \ \forall (x_1, x_2, x_3) \in \mathbb{R}^3. \tag{4.12}$$

Since $|\nabla \zeta_0| = 0$, we obtain, if Φ_0 is the flow related to $\vec{X}_0$, $\Phi_0(x_1, x_2, x_3, t) = (x_1, x_2, x_3 - t)$ so that

$$|\partial_t \Phi_0| \leq 1 \text{ and } |\nabla_3 \Phi_0(\cdot, t)| \leq \mathrm{C}_0,$$

where C_0 is some constant. In this case, the map $P_{\zeta_0}(v)(t)$ has a simple form, since

$$P_{\zeta_0}(v)(t)(x) = v(x_1, x_2, x_3 + t), \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

It follows from Lemma 4.1 or computing directly using the chain rule that, for any $t \geq 0$

$$\|\frac{\partial}{\partial t}\mathcal{P}_{\zeta_0}(v)(t)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla_3 \mathcal{P}_{\zeta_0}(v)(t)\|_{L^\infty(\mathbb{R}^3)} \leq \mathrm{K}_0\|\nabla_3 v\|_{L^\infty(\mathbb{R}^3)}. \tag{4.13}$$

where K_0 is some absolute constant.

Remark 4.1. In the course of the proof of Proposition 4, we will be led to transport the Pontryagin maps of the fibers $\mathfrak{L}^{k,\perp}_{i,q}$. We have, for $t \geq 0$ and $x \in \mathbb{R}^3$

$$P_{\zeta_0}(\mathbf{P}^{\mathrm{ontya}}_\varrho[\mathfrak{L}^{k,\perp}_{i,q}], \mathfrak{e}^\perp_{\mathrm{ref}})(t)(x) = \mathbf{P}^{\mathrm{ontya}}_\varrho[\mathfrak{L}^{k,\perp}_{i,q} - t\vec{\mathbf{e}}_3, \mathfrak{e}^\perp_{\mathrm{ref}}](x) \text{ and hence}$$

$$\|\frac{\partial}{\partial t}\mathbf{P}^{\mathrm{ontya}}_\varrho[\mathfrak{L}^{k,\perp}_{i,q} - t\vec{\mathbf{e}}_3, \mathfrak{e}^\perp_{\mathrm{ref}}\|_{L^\infty(\mathbb{R}^3)} + \|\nabla_3 \mathbf{P}^{\mathrm{ontya}}_\varrho[\mathfrak{L}^{k,\perp}_{i,q} - t\vec{\mathbf{e}}_3, \mathfrak{e}^\perp_{\mathrm{ref}}\|_{L^\infty(\mathbb{R}^3)} \leq \mathrm{C}^0_{\mathrm{flow}} k.$$

where $\mathrm{C}^0_{\mathrm{flow}}$ is some universal constant

The vector field $\vec{X}_1^k$. Let $k \in \mathbb{N}^*$ and set $h = k^{-1}$. We consider the numbers x_2^ℓ defined by

$$x_2^\ell = \ell h \text{ for } \ell = 1, \dots, k, \text{ so that } 0 < x_2^1 = h < \dots < x_2^k = 1.$$

The definition of these numbers is motivated by the fact that the collection of segments $[-6, 6] \times \{x_2^\ell\}$, $\ell = 1, \dots, k$ correspond to the straight segments, in the spaghetton construction, of the sheave $\mathfrak{L}^k$ which lie below $\mathfrak{L}^{k,\perp}$ (see figure 6). We construct a vector field $\vec{X}_1^k$ related by (4.6) to a push function ζ_1^k depending only on the last two variables, that is

$$\zeta_1^k(x_1, x_2, x_3) = \zeta_1^k(x_2, x_3),\ \forall (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{4.14}$$

so that we might possibly restrict ourselves to the plane $P_{2,3}$, which actually depends only on the second variable x_2 in the region $x_3 \geq 0$, and such that

$$\begin{cases} \zeta_1^k(x_2, x_3) = \dfrac{1}{4} \text{ for } x_2 \in \bigcup\limits_{\ell=1}^{k} [x_2^\ell - \dfrac{h}{8}, x_2^\ell + \dfrac{h}{8}] \text{ and for } x_3 \geq 0, \\ \zeta_1^k(x_2, x_3) = 1 \text{ for } x_2 \notin \bigcup\limits_{\ell=1}^{k} [x_2^\ell - \dfrac{h}{4}, x_2^\ell + \dfrac{h}{4}] \text{ and } x_3 \geq 0, \\ \zeta_1^k(x_2, x_3) = 1 \text{ for } x_3 \leq -\dfrac{h}{2} \\ \dfrac{1}{4} \leq \zeta_1^k(x_2, x_3) \leq 1 \text{ for } x_3 \geq 0. \end{cases} \tag{4.15}$$

It follows from the above conditions that

$$\zeta_1^k(x) = 1 \text{ except possibly if } x \in \mathcal{O}_h \equiv \mathbb{R} \times [0, 1 + \frac{h}{4}] \times [-\frac{h}{2}, +\infty). \tag{4.16}$$

To construct the function ζ_1^k, we proceed as follows: We choose ζ_1^k of the form

$$\zeta_1^k(x_2, x_3) = 1 - \mathfrak{f}^k(x_2) g_3(x_3) \text{ with } \mathfrak{f}^k(x_2) \equiv \sum_{\ell=1}^{k} g_2\left(k\left(x_2 - x_2^\ell\right)\right), \tag{4.17}$$

where $g_2 : \mathbb{R} \to \mathbb{R}$ denotes a given smooth non-negative function on $\mathbb{R}$ such that

$$g_2(s) = 0 \text{ for } s \in \mathbb{R} \setminus [-\frac{1}{4}, \frac{1}{4}],\ g_2(s) = \frac{3}{4} \text{ for } s \in [-\frac{1}{8}, \frac{1}{8}], 0 \leq g_2(s) \leq \frac{3}{4} \text{ otherwise}, \tag{4.18}$$

and where the function $g_3 : \mathbb{R} \to \mathbb{R}$ denotes a smooth non-negative function such that $0 \leq g_3 \leq 1$ and

$$g_3(s) = 1 \text{ for } s \geq 0 \text{ and } g_3(s) = 0 \text{ for } s \leq -\frac{1}{2}. \tag{4.19}$$

Notice that, in view of (4.18), we have

$$\mathfrak{f}^k(s) = \sum_{\ell=1}^{k} g_2\left(k\left(s - x_2^\ell\right)\right) = 0 \text{ for } s \in \mathbb{R} \setminus \bigcup_{\ell=1}^{k} [x_2^\ell - \frac{h}{4}, x_2^\ell + \frac{h}{4}],$$

So that the conclusion (4.15) follows. The definition (4.17) yields the estimate

$$\|\zeta_1\|_\infty \leq C \text{ and } \|\nabla \zeta_1\|_\infty \leq Ck.$$

for some universal constant $C > 0$ and we have therefore

$$\exp\left(s\,\|\nabla\zeta_1^k\|_{L^\infty(\mathbb{R}^3)}\right) \leq K, \text{ for } s \in [0,h],$$

where $K > 0$ is some universal constant. Hence, it follows from Lemma 4.1 that for any $x \in \mathbb{R}^3$ and any $s \in [0,h]$, we have

$$|\frac{\partial}{\partial t}\mathcal{P}_{\zeta_1^k}(v)(s)(x)| + |\nabla_3\mathcal{P}_{\zeta_1^k}(v)(s)(x)| \leq \mathrm{K}_1\|\nabla v\|_{L^\infty(\mathbb{R}^3)}, \tag{4.20}$$

where $\mathrm{K}_1 \geq \mathrm{K}_0 > 0$ is some universal constant. In view of the simple form (4.6)-(4.17) of the vector field $\vec{X}_1^k$, its integration reduced to the integration of the scalar differential equation in (4.7) which can be solved by *separation of variables.* Going back to (4.7) and writing $\Psi(x_1,x_2,x_3,s) = \Psi_1^k(x_2,x_3,s)$ for our specific choice (4.15) of vector-field we verify that the function Ψ_1^k is given as the solution of the integral equation

$$\int_{\Psi_1^k(x_2,x_3,s)}^{x_3} \frac{\mathrm{d}u}{1-\mathfrak{f}_1^k(x_2)g_3(u)} = s.$$

It follows from this formula that $\Psi_k(x_2,x_3) \leq x_3$, and that

$$\Psi_1^k(x_2,x_3,s) = x_3 - \left[s - s\,\mathfrak{f}_1^k(x_2)\right] \text{ provided } 0 \leq s \leq x_3. \tag{4.21}$$

Transportation of curves by the flow of $\vec{X}_1^k$. We next take a look at the fate of a curve when transported by the flow Φ_1^k of the vector field $\vec{X}_1^k$. Of special interest is the fate of the fibers of the sheave $\mathfrak{L}^{k,\perp}$. In view of statement (4.16), all part of the fibers which are not in $\mathcal{O}_h$ are transported downwards along the direction $\vec{\mathbf{e}}_3$ with constant speed 1. Since the restrictions of the fibers of $\mathfrak{L}^{k,\perp}$ to $\mathcal{O}_h$ are segments parallel to $\vec{\mathbf{e}}_2$, let us first consider the line $D = M + \mathbb{R}\vec{\mathbf{e}}_2$, where $M = (m_1, 0, m_3)$ is given, with $m_3 \geq 0$. Thanks to (4.7) and (4.21), we obtain, for $0 \leq s \leq m_3$

$$\Phi_1^k(D,s) = \{(m_1, x_2, m_3 - \left[s - s\,\mathfrak{f}_1^k(x_2)\right])) \text{ for } x_2 \in \mathbb{R}\}. \tag{4.22}$$

Hence, if $m_1 = 0$ and restricting ourselves to the plane $P_{2,3}$, the curve $\Phi_1^k(D,s)$ corresponds to the graph of the function $x_2 \mapsto m_3 - \left[s - s\,\mathfrak{f}_1^k(x_2)\right]$ (see figure 9). In the course of the proof of Proposition 4 we will use formula (4.22) for the special choice $m_3 \geq s = h$, so that

$$\Phi_1^k(D,h) = \{(m_1, x_2, m_3 - \left[h - h\,\mathfrak{f}_1^k(x_2)\right])) \text{ for } x_2 \in \mathbb{R}\}.$$

and hence

$$\begin{cases} \Phi_1^k(m_1,x_2,m_3,h) = (m_1,x_2,m_3 - \frac{h}{4}) \text{ for } x_2 \in \bigcup_{\ell=1}^{k}[x_2^\ell - \frac{h}{8}, x_2\ell + \frac{h}{8})] \\ \Phi_1^k(m_1,x_2,m_3,h) = (m_1,x_2,m_3 - h) \text{ for } x_2 \in \mathbb{R} \setminus \bigcup_{\ell=1}^{k}[x_2^\ell - \frac{h}{8}, x_2\ell + \frac{h}{8})]. \end{cases} \tag{4.23}$$

Transportation of translates of stadia $\mathfrak{L}_{i,q}^{k,\perp}$ by the flow of $\vec{X}_1^k$. As mentioned, the vector field $\vec{X}_1^k$ will be used to transport verticale translates of the stadia $\mathfrak{L}_{i,q}^{k,\perp}$, so that, for arbitrary

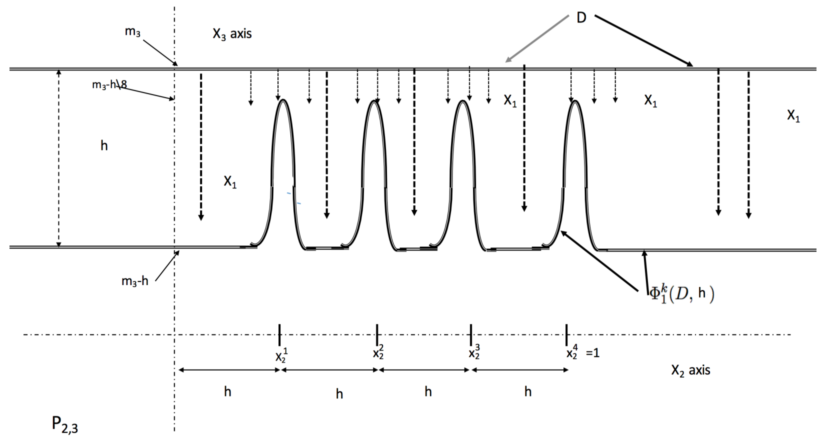

Figure 9: *The deformation of a line parallel to the x_2 axis by the flow generated by X_1^k, $k = 4$ at time h.*

$i, q = 1, \ldots, k$, and $c > 0$, we consider the curve $\mathfrak{L}_{i,q}^{k,\perp} - c\,\vec{\mathbf{e}}_3$ and its deformation $\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s)$ by the flow Φ_1^k, given by, for $s \in [0, h]$

$$\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s) \equiv \Phi_1^k(\mathfrak{L}_{i,q}^{k,\perp} - c\,\vec{\mathbf{e}}_3, s) \tag{4.24}$$

We will only be interested in the case $0 \leq c \leq 6+qh$. The shape of these curves is represented in figure 10. An analytical description is provided by the decomposition in the plane $P_{2,3}(ih)$

$$\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s) = \mathbb{S}_{6+qh}^{1,-}(A_{i,q}^{-}(c,s)) \cup \mathbb{S}_{6+qh}^{1,+}(A_{i,q}^{+}(c,s)) \cup {\rm G}_{i,q}^{\rm top}(c,s) \cup {\rm D}_{i,q}^{\rm bot}(c,s), \tag{4.25}$$

where $A_{i,q}^{-}(c,s) = (ih, -3+qh, -c-sh)$, $A_{i,q}^{-}(c,s) = (ih, 9+qh, -c-sh)$, where ${\rm D}_{i,q}^{\rm bot}$ denotes the segment at the bottom of $\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s)$ parallel to $\vec{\mathbf{e}}_2$, that is

$$\begin{cases} {\rm D}_{i,q}^{\rm bot}(c,s) = [B_{i,q}^{-}(c,s), B_{i,q}^{+}(c,s)] \text{ where} \\ B_{i,q}^{-}(c,s) = (ih, -3+qh, -6-qh-c-s)) \text{ and } B_{i,q}^{+}(c,s) = (ih, 9+qh, -6-qh-c-s)), \end{cases}$$

and where the set ${\rm G}_{i,q}^{\rm top}(c,s)$ has the form of a graph in the plane $P_{2,3}(h)$, namely

$${\rm G}_{i,q}^{\rm top}(c,s) = \left\{ \left(ih, x_2, 6+qh-c-\left[s - s\,\mathfrak{f}_1^k(x_2)\right]\right), x_2 \in [-3+qh, 9+qh] \right\}. \tag{4.26}$$

Remark 4.2. The set $\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s)$ may be considered as a perturbation of the set $\mathfrak{L}_{i,q}^{k,\perp} - (c+s)\vec{\mathbf{e}}_3$ in view of the relation

$$\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s) + (c+s)\vec{\mathbf{e}}_3 \setminus \mathfrak{L}_{i,q}^{k,\perp} \subset \{\left(ih, x_2, 6+qh+s\,\mathfrak{f}_1^k(x_2)\right), x_2 \in [0, 1+\frac{h}{4}]\}.$$

We have hence

$$\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s) \subset \left[\mathfrak{L}_{i,q}^{k,\perp} - (c+s)\vec{\mathbf{e}}_3\right] \cup V(c,s), \tag{4.27}$$

where $V(c,s)$ denotes the parallelipedic region

$$V(c,s) = -(c+s)\vec{\mathbf{e}}_3 + [\frac{h}{2}, 1+\frac{h}{2}]2 \times [0, \frac{3h}{4}]$$

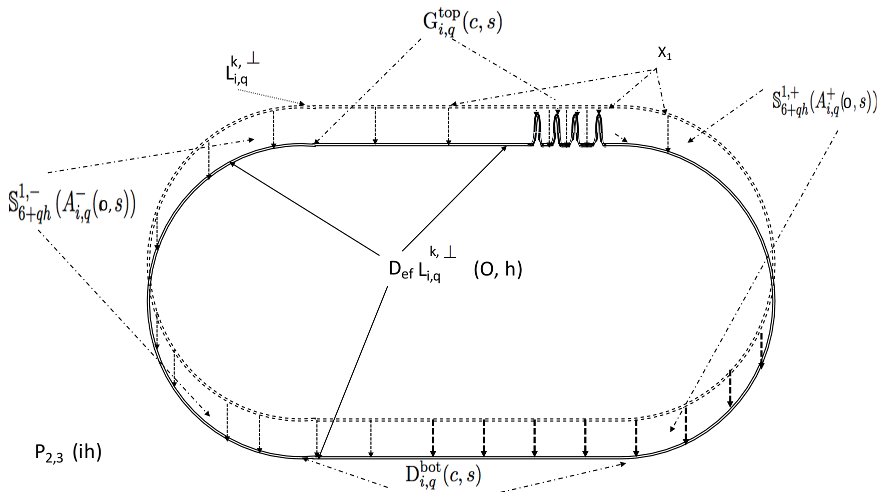

Figure 10: *The deformation of the curve* $\mathfrak{L}^{k,\perp}_{i,q}$ *by the flow* Φ^k_1, $k = 4$ *at time* h.

4.1.3 A variant of the Pontryagin construction

Whereas deformations of the domain act both on curves and functions, we have seen in (4.5) that it does not "commute" in general with the Pontryagin construction, that is the Pontryagin map of a deformed curve is not in general the deformation of the initial Pontryagin map. Concerning the curves which are of interest for us, namely the curves $\mathcal{D}_{\rm ef}\mathfrak{L}^{k,\perp}_{i,q}(c,s)$, we taylor a specific variant of the Pontryagin construction for our later use. Given $\varrho > 0$, $i, q = 1, \ldots, k$, $0 \leq c \leq 6 + qh$ and $s > 0$, our variant $\tilde{\mathbf{P}}^{\rm ontya}_{\varrho}[\mathcal{D}_{\rm ef}\mathfrak{L}^{k,\perp}_{i,q}(c,s)] : \mathbb{R}^3 \to \mathbb{S}^2$ will by different from the Pontryagin map only in a neighborhood of the top part $\mathrm{G}^{\rm top}_{i,q}(c,s)$. We introduce therefore the set

$$\mathrm{U}^{\rm top}_{i,q}(c,s,\varrho) = \underset{a \in \mathrm{G}^{\rm top}_{i,q}(c,s)}{\cup} \mathbb{D}^2_{1,3}(\varrho, a)$$

where, for $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, $\mathbb{D}^2_{1,3}(\varrho, a)$ denotes the disk in the plane $P_{1,3}(a_2)$ of radius ϱ centered at a, namely

$$\mathbb{D}^2_{1,3}(\varrho, a) = \{(x_1, a_2, x_3) \in \mathbb{R}^3, (x_1 - a_1)^2 + (x_2 - a_2)^2 \leq \varrho^2\} \subset P_{1,3}(a_2).$$

We then define the variant $\tilde{\mathbf{P}}^{\rm ontya}_{\varrho}[\mathcal{D}_{\rm ef}\mathfrak{L}^{k,\perp}_{i,q}(c,s),\, \mathfrak{e}^{\perp}_{\rm ref}]$ of the Pontryagin map in the following way: We set

$$\tilde{\mathbf{P}}^{\rm ontya}_{\varrho}[\mathcal{D}_{\rm ef}\mathfrak{L}^{k,\perp}_{i,q}(c,s)](x) = \mathbf{P}^{\rm ontya}_{\varrho}[\mathcal{D}_{\rm ef}\mathfrak{L}^{k,\perp}_{i,q}(c,s)](x) \text{ for } x \notin \mathrm{U}^{\rm top}_{i,q}(c,s,\varrho) \tag{4.28}$$

Otherwise, if $x \in \mathbb{D}^2_{1,3}(\varrho, a)$ for some $a = (a_1, a_2, a_3) \in \mathrm{G}^{\rm top}_{i,q}(c,s)$, we set

$$\tilde{\mathbf{P}}^{\rm ontya}_{\varrho}[\mathcal{D}_{\rm ef}\mathfrak{L}^{k,\perp}_{i,q}(c,s)](x_1, a_2, x_3) = \chi_{\varrho}\left(x_1 - a_1, x_3 - a_3\right). \tag{4.29}$$

where χ_ϱ is defined in (31) and (32). In other words, in the construction of $\tilde{\mathbf{P}}_\varrho^{\rm ontya}[\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s)]$, we replace the plane orthogonal to the curve by the plane parallel to $P_{1,3}$[7] on the part $\mathrm{G}_{i,q}^{\rm top}(c,s)$. We are going to rely on the following

Lemma 4.2. *We have, provided* $\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s) \subset \mathbb{R}^2 \times \mathbb{R}^+$

$$\tilde{\mathbf{P}}_\varrho^{\rm ontya}[\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s)] = P_{\zeta_1^k}(\mathbf{P}_\varrho^{\rm ontya}[\mathfrak{L}_{i,q}^{k,\perp} - c\vec{\mathbf{e}}_3, \mathfrak{e}_{\rm ref}^{\perp}]).$$

Sketch of the proof The proof follows from the observation that

$$\Phi_1^k(\mathbb{D}_{1,3}^2(\varrho,a),s) = \mathbb{D}_{1,3}^2(\varrho,\Phi_1^k(a,s))$$

a consequence of the fact that ζ_1^k depens only on the variable x_2 in the region considered and the fact that we consider only disks in planes orthogonal to $\vec{\mathbf{e}}_2$.

Concerning gradient estimates, we have:

Lemma 4.3. *We have, for some constant* $\mathrm{C}_{\rm def} > 0$ *and for any* $c \in \mathbb{R}^+$ *and* $s \in [0,h]$

$$|\nabla_3 \mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s)| + |\frac{\partial}{\partial s}\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(c,s)| \leq \mathrm{C}_{\rm def}k,$$

One may deduce these estimates from (4.20) or might be proven directly.

4.1.4 Cubic extensions

Whereas the previous construction works for quite general classes of maps and are hence not specific to the Sobolev framework, the extension method presented here induces singularities and hence is specially appropriate in the Sobolev setting.

The cube $\mathrm{Q}_r^4(\mathbf{a})$ *and its boundary.* We consider the ∞-norm on $\mathbb{R}^4$ given by

$$|\mathbf{x}|_\infty = \sup_{i=1,\dots,4} |x_1| \text{ for } \mathbf{x} = (x_1,x_2,x_3,x_4) \in \mathbb{R}^4$$

and the corresponding ∞-sphere Q_r^4 of radius $r > 0$ defined by

$$\mathrm{Q}_r^4 = \mathrm{Q}_r^4(0) \text{ where more generally } \mathrm{Q}_r^4(\mathbf{a}) \equiv \{\mathbf{x} \in \mathbb{R}^4, |x - \mathbf{a}|_\infty < r\} \text{ for } \mathbf{a} \in \mathbb{R}^4,$$

so that actually Q_r^4 corresponds the hypercube $\mathrm{Q}_r^4 = [-r,r]^4$. Given a 4-dimensional hypercube $\mathrm{Q}_r^4(\mathbf{a})$, its boundary $\partial\mathrm{Q}_r^4(\mathbf{a})$ is the union of 8 distinct three-dimensional cubes $\mathfrak{Q}_p^{3,\pm}(r,\mathbf{a})$ of size r defined, for $\mathbf{a} = (a_1,a_2,a_3,a_4)$ and $p = 1,\dots,4$ by

$$\mathfrak{Q}_p^{3,\pm}(r,\mathbf{a}) = \{\mathbf{x} = (x_1,x_2,x_3,x_4) \in \mathbb{R}^4, x_p = a_p \pm r, \sup_{i \neq p}|x_i - a_i| < r\}. \tag{4.30}$$

The sets $\mathfrak{Q}_p^{3,\pm}(r,\mathbf{a})$ are therefore included in a 3-dimensional hyperplane of $\mathbb{R}^4$ orthogonal to the vecteur $\vec{\mathbf{e}}_p$. We have

$$\partial\mathrm{Q}_r^4(\mathbf{a}) = \bigcup_{p=1}^4 \left(\mathfrak{Q}_p^{3,+}(r,\mathbf{a}) \cup \mathfrak{Q}_p^{3,-}(r,\mathbf{a})\right).$$

[7]these two planes coincide if $\mathfrak{g}'(x_2) = 0$.

Construction of the extension operator. Given a map $v : \partial \mathrm{Q}^4_r(a) \to \mathbb{R}^\ell$ defined on the boundary $\partial \mathrm{Q}^4_r(\mathbf{a})$ of a cube $\mathrm{Q}^4_r(\mathbf{a})$ we consider its cubic-radial extension $\mathfrak{Ext}_{r,\mathbf{a}}(v)$ defined on the full cube $\mathrm{Q}^4_r(\mathbf{a})$ for $v : \partial \mathrm{Q}^4_r(\mathbf{a}) \to \mathbb{R}^\ell$ by

$$\mathfrak{Ext}_{r,\mathbf{a}}(v)(\mathbf{x}) = v\left(\mathbf{a} + r\frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|_\infty}\right) \text{ for } \mathbf{x} \in \mathrm{Q}^4_r(\mathbf{a}), \tag{4.31}$$

so that $\mathfrak{Ext}_{r,\mathbf{a}}(v) = v$ on the boundary $\partial \mathrm{Q}^4_r(\mathbf{a})$. If v is Lipschitz, so is $\mathfrak{Ext}_{r,\mathbf{a}}(v)$, except near $\mathbf{a}$, where a singularity is created. However, if the map v has finite energy E_3 on the three-dimensional set $\partial \mathrm{Q}^4_r(\mathbf{a})$, then the same assertion holds for its extension $\mathfrak{Ext}_{r,\mathbf{a}}(v)$ on the cube $\mathrm{Q}^4_r(\mathbf{a})$ with the estimate

$$\mathrm{E}_3\left(\mathfrak{Ext}_{r,\mathbf{a}}(v),\, \mathrm{Q}^4_r(\mathbf{a})\right) \le \mathrm{K}_{\mathrm{ext}}\, r \mathrm{E}_3(v, \partial \mathrm{Q}^4_r(\mathbf{a})), \tag{4.32}$$

where $\mathrm{K}_{\mathrm{ext}}$ denotes some universal constant.

4.1.5 Creating Hopf singularities through the crossing of lines

We analyze next a situation which accounts for the creation of singularities in the construction of the Gordian cut $\mathbf{G}^k_{\mathbf{ord}}$. We restrict ourselves to cubes of radius $r = h/2$. The singularities are created applying the extension operator to maps $\Upsilon^h_{\mathbf{a}}$ defined on the boundary of cubes

$$\Upsilon^h_{\mathbf{a}} : \partial \mathrm{Q}^4_{h/2}(\mathbf{a}) \to \mathbb{S}^2, \text{ where } \mathbf{a} = (a_1, a_2, a_3, a_4) \equiv (a, a_4) \in \mathbb{R}^4, h \ge 0. \tag{4.33}$$

and which we are going to define next, using the Pontryagin construction or the variant that we have seen before. These construction are build on relevant curves on each of the faces $\mathfrak{Q}^{3,\pm}_p(h/2, \mathbf{a})$. We focus first on the top and bottom faces

$$\mathfrak{Q}^{3,\pm}_4(h/2, \mathbf{a}) = \mathrm{Q}^3_{h/2}(a) \times \{a_4 \pm \frac{h}{2}\}$$

and start the description working in the reference cube $\mathrm{Q}^3_{h/2}(a) = \{(x_1, x_2, x_3) \in \mathbb{R}^3, |x_i - a_i| \le h/2,\ i = 1, 2, 3,$ where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ is defined [8] in (4.33).

Some relevant curves in $\mathrm{Q}^3_{h/2}(a)$, $a = (a_1, a_2, a_3)$. We consider the two segments of $\mathrm{Q}^3_{h/2}(a)$ given by

$$\begin{cases} \mathrm{D}_{0,h}(a) = \{[a_1 - h/2, a_1 + h/2] \times \{(a_2, a_3)\} \text{ and} \\ \mathrm{D}^+_{\perp,h}(a) = \{a_1\} \times [a_2 - h/2, a_2 + h/2] \times \{a_3 - 3h/8\} \end{cases}$$

so that the two segments are parallel to $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ respectively, have hence orthogonal directions, each of them joining opposite faces of the cube: $\mathrm{D}_{0,h}(a) \subset P_{1,3}(a_3)$ and $\mathrm{D}^-_{\perp,h} \subset P_{2,3}(a_1)$. We consider also the smooth curve $\mathcal{C}^-_{\perp,h}(a)$ given as the following graph in the plane $P_{2,3}(a_1)$

$$\mathcal{C}^-_{\perp,h}(a) = \left\{ \left(a_1, x_2, a_3 + \left[h - hg_2(\frac{x_2 - a_2}{h})\right)\right], x_2 \in [a_2 - h/2, a_2 + h/2] \right\}, \tag{4.34}$$

where the function g_2 is defined in (4.18). This definition is consistent with (4.22) and (4.26): Indeed $\mathrm{D}_{0,h}(a)$ on one hand and $\mathcal{C}^-_{\perp,h}(a)$ and $\mathrm{D}^+_{\perp,h}(a)$ on the other are aimed to model suitable

[8] As a general rule roman bold characters as $\mathbf{a}$ correspond to points in $\mathbb{R}^4$ whereas symbols as a refer to points in $\mathbb{R}^3$

subsets of fibers $\mathfrak{L}^k_{i,j}$ and $\mathcal{D}_{\rm ef}\mathfrak{L}^{k,\perp}_{i,q}$ respectively, as we will discuss later in Remark 4.3 below. Notice that $\mathrm{D}^+_{\perp,h}(a)$ and $\mathcal{C}^-_{\perp,h}(a)$ both belong to the plane $P_{2,3}(a_1)$ and intersect along two segments parallel to $\vec{\mathbf{e}}_2$, namely we have

$$\mathrm{D}^+_{\perp,h}(a)\cap\mathcal{C}^-_{\perp,h}(a)=\{a_1\}\times([a_2-h/2,a_2+h/4]\cup[a_2+h/4,a_2+h/2])\times\{a_3-3h/8\} \quad (4.35)$$

In particular, their respective intersection with a suitably small neighborhood of the boundary coincide, see Figure 12.

Remark 4.3. *Relating $\mathcal{C}^-_{\perp,h}(a)$, $\mathrm{D}^+_{\perp,h}(a)$ and $\mathrm{D}_{0,h}(a)$ to $\mathfrak{L}^k$ and $\mathcal{D}_{\rm ef}\mathfrak{L}^{k,\perp}_{i,q}$.* As mentioned, the sets $\mathrm{D}^+_{\perp,h}(a)$, $\mathcal{C}^-_{\perp,h}(a)$ and $\mathrm{D}_{0,h}(a)$ are designed to represent suitable subsets of fibers $\mathfrak{L}^k_{i,j}$ and $\mathcal{D}_{\rm ef}\mathfrak{L}^{k,\perp}_{i,q}$. In the proof of Proposition 4, we will be led to consider points of the form

$$a=a_{i,j,q}\equiv h(i,j,q),\ \text{for some integers } i,j,q=1,\dots,k, \quad (4.36)$$

which belong to the cube $[0,1]^3$. We verify that

$$\mathrm{Q}^3_{h/2}(a_{i,j,q})\cap\mathfrak{L}^k=\mathrm{Q}^3_{h/2}(a_{i,j,q})\cap\mathfrak{L}^k_{j,q}=\mathrm{D}_{0,h}(a_{i,j,q}) \quad (4.37)$$

and that

$$\begin{cases}\mathcal{D}_{\rm ef}\mathfrak{L}^{k,\perp}_{i,p}(c,h)\cap\mathrm{Q}^3_{h/2}(a_{i,j,q})=\mathcal{C}^-_{\perp,h}(a_{i,j,q})\\ \left(\mathfrak{L}^{k,\perp}_{i,p}-(c+h)\,\vec{\mathbf{e}}_3\right)\cap\mathrm{Q}^3_{h/2}(a_{i,j,q})=\mathrm{D}_{\perp,h}(a),\end{cases} \quad (4.38)$$

provided we have the condition involving only the numbers c,p and q *but not* on the numbers i and j

$$c=5+\frac{3h}{8}+(p+k-q-1)h. \quad (4.39)$$

$\mathbb{S}^2$*-valued maps on* $\mathrm{Q}^3_{h/2}(a)$. Let $\varrho=10^{-3}h$ be given. We relate to the previously constructed curves $\mathbb{S}^2$-valued maps through the Pontryagin construction or its variant. In order to have orientations consistent with the constructions in subsection 4.2 in particular the framings on the sheaves, we choose on $\mathrm{D}_{0,h}$ the framing $\mathfrak{e}^\perp_0=(\vec{\mathbf{e}}_3,-\vec{\mathbf{e}}_2)$, whereas on $\mathrm{D}^+_{\perp,h}(a)$ we set $\mathfrak{e}^\perp_0=(\vec{\mathbf{e}}_1,-\vec{\mathbf{e}}_3)$. We first consider the map $\gamma^-_{\varrho,h}$ defined on $\mathrm{Q}^3_r(a)$ by

$$\gamma^{h,-}_a=\mathbf{P}^{\rm ontya}_\varrho[(\mathrm{D}_{0,h}(a),\mathfrak{e}^\perp_0)]\vee_3\tilde{\mathbf{P}}^{\rm ontya}_\varrho[\mathcal{C}^-_{\perp,h}(a)]. \quad (4.40)$$

The notation $\tilde{\mathbf{P}}^{\rm ontya}_\varrho[\mathcal{C}^+_{\perp,h}]$ which appears in (4.40) refers to the *variant of the Pontragyin construction* defined in Paragraph 4.1.3, for which the plane orthogonal to the curve is replaced by a plane parallel [9] to $P_{1,3}$. More explicitely, it is defined on $\mathrm{Q}^3_{h/2}(a)$ by

$$\tilde{\mathbf{P}}^{\rm ontya}_\varrho[\mathcal{C}^-_{\perp,h}](x_1,x_2,x_3)=\chi_\varrho\left(x_1,\left[x_3-[h-hg_2(\frac{x_2-a_2}{h})\right]\right),$$

where χ_ϱ is defined in (31) and (32). We define on $\mathrm{Q}^3_{h/2}(a)$ another map $\gamma^{h,+}_a$ as

$$\gamma^{h,+}_a=\mathbf{P}^{\rm ontya}_\varrho[(\mathrm{D}_{0,h}(a),\mathfrak{e}^\perp_0)]\vee_3\mathbf{P}^{\rm ontya}_\varrho[\mathrm{D}^+_{\perp,h}(a),\mathfrak{e}^\perp_0)], \quad (4.41)$$

[9] the corresponding framing would correspond then to the framing on $\mathrm{D}^-_{\perp,r}$ that is $(\vec{\mathbf{e}}_1,-\vec{\mathbf{e}}_3)$

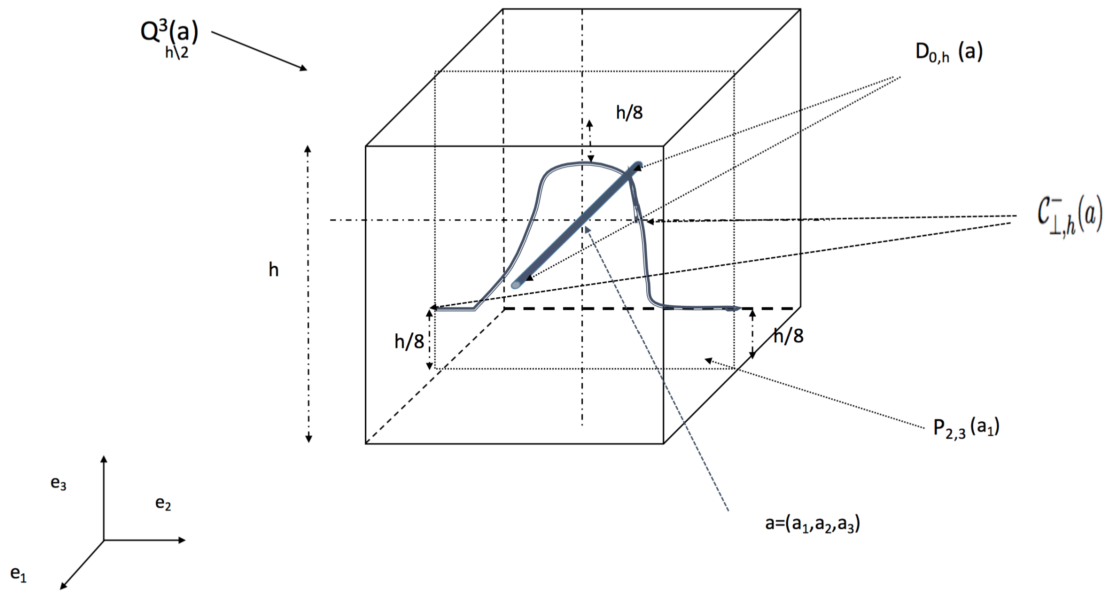

Figure 11: *The curve* $\mathcal{C}^-_{\perp,h}(a)$ *and the segment* $\mathrm{D}_{0,h}(a)$*. These curves yield by Pontryagin's construction and its variant the map* γ^h_a *on* $\mathfrak{Q}^{3,-}_4(h/2,\mathbf{a})$*.*

An important consequence of the definitions (4.41) and (4.40) as well as of (4.35) is that

$$\gamma^{h,-}_a(x) = \gamma^{h,+}_a(x) \text{ for } x \in \partial \mathrm{Q}^3_{h/2}(a), \tag{4.42}$$

Setting in analogy with (4.30), for $p = 1, 2, 3$

$$\mathfrak{Q}^{2,\pm}_p(r, a) = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3, x_p = a_p \pm r, \sup_{i \neq p} |x_i - a_i| < r\},$$

so that $\partial \mathrm{Q}^3_{h/2}(a) = \cup \mathfrak{Q}^{2,\pm}_p(r, a)$, we notice that

$$\begin{cases} \gamma^{h,+}_a(x) = \gamma^{h,-}_a(x) = \mathbb{P}_{\text{south}} \text{ for } x \in \mathfrak{Q}^{2,+}_3(h/2, a) \\ \gamma^{h,+}_a(x) = \gamma^{h,-}_a(x) = \chi_\varrho(x_1, -x_3 + \frac{3h}{8}) \text{ for } x \in \mathfrak{Q}^{2,\pm}_2(h/2, a) \\ \gamma^{h,+}_a(x) = \gamma^{h,-}_a(x) = \chi_\varrho\left((x_3 - a_3), -(x_2 - a_2)\right) \text{ for } x \in \mathfrak{Q}^{2,\pm}_1(h/2, a) \end{cases} \tag{4.43}$$

We notice also the symmetry properties on the boundary, for $p = 1, 2, 3$

$$\gamma^{h,+}_a(x) = \gamma^{h,-}_a(x) = \gamma^{h,+}_a(x \mp h\vec{\mathbf{e}}_p) = \gamma^{h,+}_a(x \mp h\vec{\mathbf{e}}_p) \text{ for } x \in \mathfrak{Q}^{2,\pm}_p(h/2, a). \tag{4.44}$$

$\mathbb{S}^2$*-valued maps on* $\partial \mathrm{Q}^4_{h/2}(\mathbf{a})$. Let $\mathbf{a} = (a, a_4) = (a_1, a_2, a_3, a_4)$. We take advantage of (4.42) and (4.43) to define on $\partial \mathrm{Q}^4_{h/2}(\mathbf{a})$ an $\mathbb{S}^2$ valued map $\Upsilon^h_{\mathbf{a}}$ whose restriction to the top face of the boundary is $\gamma^{h,+}_a$ and whose restriction to the botton face is $\gamma^{h,-}_a$. We define it as follows:

$$\begin{cases} \Upsilon^h_{\mathbf{a}}(x, a_4 + \frac{h}{2}) = \gamma^{h,+}_a(x), \text{ for } x \in \mathrm{Q}^3_{h/2}(a) \text{ i.e. } \mathbf{x} = (x, a_4 + \frac{h}{2}) \in \mathfrak{Q}^{3,+}_4(h/2, \mathbf{a}) \\ \Upsilon^h_{\mathbf{a}}(x, a_4 - \frac{h}{2}) = \gamma^{h,-}_a(x), x \in \mathrm{Q}^3_{h/2}(a) \text{ i.e. } \mathbf{x} = (x, a_4 - \frac{h}{2}) \in \mathfrak{Q}^{3,-}_4(h/2, \mathbf{a}) \text{ and} \\ \Upsilon^h_{\mathbf{a}}(x, x_4) = \gamma^{h,+}_a(x) = \gamma^{h,-}_a(x) \text{ for } x \in \partial(\mathrm{Q}^3_{h/2}(a)) \text{ and } x_4 \in [a_4 - \frac{h}{2}, a_4 + \frac{h}{2}]. \end{cases} \tag{4.45}$$

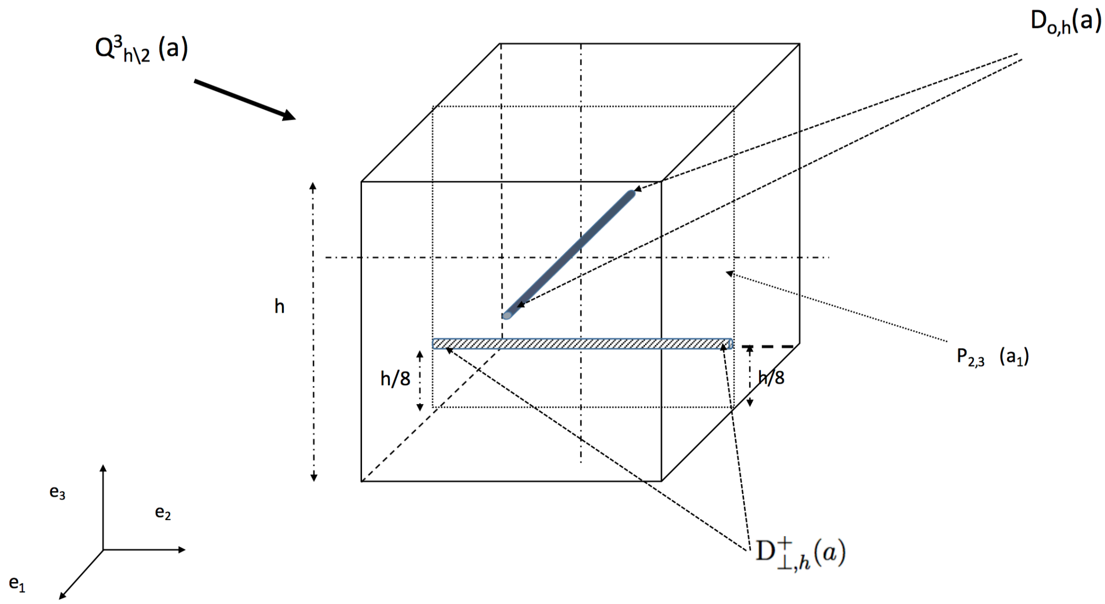

Figure 12: *The segment* $\mathrm{D}^{+}_{\perp,h}(a)$ *and the segment* $\mathrm{D}_{0,h}(a)$. *These curves yield by Pontryagin's construction and its variant the map* $\gamma_a^{h,+}$ *on* $\mathfrak{Q}_4^{3,+}(h/2,\mathbf{a})$.

It follows from (4.42) that $\Upsilon^h_{\mathbf{a}}$ is a Lipschitz $\mathbb{S}^2$-valued map on $\partial \mathrm{Q}^4_{h/2}(\mathbf{a})$ which has the topology of $\mathbb{S}^3$. As a matter of fact, the map $\Upsilon^h_{\mathbf{a}}$ can be constructed through the Pontragyin construction and its variant[10] related to a curve $\mathcal{L}^h_a$ we describe next.

The map $\Upsilon^h_{\mathbf{a}}$ *and the Pontryagin construction.* The map $\Upsilon^h_{\mathbf{a}}$ can be defined usinf Pontryagin constructions for a curve we define next. Consider the square

$$\mathrm{B}(\mathbf{a}) = \mathbf{a} + [-h/2, h/2] \times \{(0,0)\} \times [-h/2, h/2] \subset \mathrm{Q}^4_{h/2}(\mathbf{a}) \subset \mathbb{R}^4.$$

It is included in the two-dimensional subspace $\mathcal{P}_{1,4}(a_2, a_3)$ of $\mathbb{R}^4$ given by the equations $x_2 = a_2$ and $x_3 = a_3$. Set $\mathcal{L}_0(\mathbf{a}) = \partial \mathrm{B}(\mathbf{a})$, so that $\mathcal{L}_0(\mathbf{a})$ is composed of four segments of length h, two of then parallel to $\vec{\mathbf{e}}_1$, the two others to $\vec{\mathbf{e}}_4$. The vertices $M^{\pm}_{\mathrm{L}}(\mathbf{a}), M^{\pm}_{\mathrm{R}}(\mathbf{a})$ are given by

$$M^{\pm}_{\mathrm{L}}(\mathbf{a}) = \mathbf{a} + (-\frac{h}{2}, 0, 0, \pm\frac{h}{2}) \text{ and } M^{\pm}_{\mathrm{R}}(\mathbf{a}) = \mathbf{a} + (+\frac{h}{2}, 0, 0, \pm\frac{h}{2}),$$

so that

$$\mathcal{L}_0(\mathbf{a}) = \left[M^{+}_{\mathrm{L}}(\mathbf{a}), M^{+}_{\mathrm{R}}(\mathbf{a})\right] \cup \left[M^{-}_{\mathrm{L}}(\mathbf{a}), M^{-}_{\mathrm{R}}(\mathbf{a})\right] \cup \left[M^{-}_{\mathrm{L}}(\mathbf{a}), M^{+}_{\mathrm{L}}(\mathbf{a})\right] \cup \left[M^{-}_{\mathrm{R}}(\mathbf{a}), M^{+}_{\mathrm{R}}(\mathbf{a})\right].$$

Notice that $\left[M^{\pm}_{\mathrm{L}}(\mathbf{a}), M^{\pm}_{\mathrm{R}}(\mathbf{a})\right] = \mathrm{D}_{0,h}(a) \times \{a_4 \pm \frac{h}{2}\}$ and that the two other segments are

[10]in the sense that $\mathbf{P}^{\text{ontya}}$ is replaced by $\tilde{\mathbf{P}}^{\text{ontya}}$ in some parts

parallel to $\vec{\mathbf{e}}_4$. Ones verifies that $\mathcal{L}_0(\mathbf{a}) \subset \partial(\mathrm{Q}^4_{h/2}(\mathbf{a}))$ since

$$\begin{cases} F_{0,\mathrm{top}} \equiv \mathrm{D}_{0,h}(a) \times \{a_4 + h/2\} \subset \mathfrak{Q}^{3,+}_4(h/2, \mathbf{a}), \\ F_{0,\mathrm{bot}} \equiv \mathrm{D}_{0,h}(a) \times \{a_4 - h/2\} \subset \mathfrak{Q}^{3,-}_4(h/2, \mathbf{a}) \\ F_{0,\mathrm{L}} \equiv \left[M^-_{\mathrm{L}}(\mathbf{a}), M^+_{\mathrm{L}}(\mathbf{a})\right] \subset \mathfrak{Q}^{3,-}_1(h/2, \mathbf{a}) \text{ and} \\ F_{0,\mathrm{R}} \equiv \left[M^-_{\mathrm{R}}(\mathbf{a}), M^+_{\mathrm{R}}(\mathbf{a})\right] \subset \mathfrak{Q}^{3,+}_1(h/2, \mathbf{a}). \end{cases} \tag{4.46}$$

We consider another curve, the curve $\mathcal{L}_\perp(\mathbf{a})$, which is included in the hyperspace $x_1 = a_1$, defined by

$$\mathcal{L}_\perp(\mathbf{a}) = \mathcal{C}^-_{\perp,h}(a) \times \{a_4 - \frac{h}{2}\} \cup \mathrm{D}^+_{\perp,h}(a) \times \{a_4 + \frac{h}{2}\} \cup F_{\perp,1} \cup F_{\perp,2},$$

where $F_{\perp,1}$ and $F_{\perp,2}$ denote the segments parallel to $\vec{\mathbf{e}}_4$ given by

$$F_{\perp,1} \equiv \{a_1, a_2 - \frac{h}{2}, a_3 - \frac{3h}{8}\} \times [a_4 - \frac{h}{2}, a_4 + \frac{h}{2}] \text{ and } F_{\perp,2} \equiv \{a_1, a_2 + \frac{h}{2}, a_3 - \frac{3h}{8}\} \times [a_4 - \frac{h}{2}, a_4 + \frac{h}{2}].$$

We verify that $\mathcal{L}_\perp(\mathbf{a})$ is a connected closed curve in $\partial(\mathrm{Q}^4_{h/2}(\mathbf{a}))$ since $\mathrm{D}^+_{\perp,h}(a) \times \{a_4 + h/2\} \subset \mathfrak{Q}^{3,+}_4(h/2, \mathbf{a})$, $\mathcal{C}^-_{\perp,h}(a) \times \{a_4 - h/2\} \subset \mathfrak{Q}^{3,-}_4(h/2, \mathbf{a})$ $F_{\perp,1} \subset \mathfrak{Q}^{3,-}_2(h/2, \mathbf{a})$ and $F_{\perp,2} \subset \mathfrak{Q}^{3,+}_2(h/2, \mathbf{a})$ and that it does not intersect $\mathcal{L}_\perp(\mathbf{a})$. We set $\mathcal{L}(\mathbf{a}) = \mathcal{L}_0(\mathbf{a}) \cup \mathcal{L}_\perp(\mathbf{a})$. One may then verify that

$$\Upsilon^h_{\mathbf{a}} = \tilde{\mathbf{P}}^{\mathrm{ontya}}_{\varrho}[\mathcal{L}(\mathbf{a}), \mathfrak{e}^\perp_0]$$

where the frame $\mathfrak{e}^\perp_0$ corresponds to the frame defined in (4.40).

Remark 4.4. *First Properties of the map* $\Upsilon^h_{\mathbf{a}}$. In connection with Remark 4.3 we notice that, for a point $\mathbf{a}$ of the form $\mathbf{a} = (a_{i,j,q}, a_4)$ where $a_{i,j,q}$ is of the form given by (4.36), we have

$$\Upsilon^h_{\mathbf{a}}(\mathbf{x}) = \mathbf{P}^{\mathrm{ontya}}\left[\mathfrak{L}^k \cup (\mathfrak{L}^{k,\perp}_{j,p} - c\vec{\mathbf{e}}_3), \mathfrak{e}^\perp_{\mathrm{ref}}\right](\mathbf{x}) \text{ for } \mathbf{x} \in \partial\mathrm{Q}^3(a_{i,j,q}) \times [a_4 - \frac{h}{2}, a_4 + \frac{h}{2}], \tag{4.47}$$

provided the numbers c, p and q satisfy relation (4.39). Since

$$\partial\mathrm{Q}^3(a_{i,j,q}) \times [a_4 - \frac{h}{2}, a_4 + \frac{h}{2}] = \partial\mathrm{Q}^4_{h/2}((a_{i,j,q}), a_4) \setminus \left(\mathfrak{Q}^{3,+}_4(h/2, \mathbf{a}) \cup \mathfrak{Q}^{3,-}_4(h/2, \mathbf{a})\right)$$

It follows from (4.47) and (4.45) that

$$\Upsilon^h_{\mathbf{a}}(\mathbf{x}) = \mathbf{P}^{\mathrm{ontya}}\left[\mathfrak{L}^k \cup (\mathfrak{L}^{k,\perp} - c\vec{\mathbf{e}}_3), \mathfrak{e}^\perp_{\mathrm{ref}}\right](\mathbf{x}) \text{ for } \mathbf{x} \in \partial\mathrm{Q}^4_{h/2}(\mathbf{a}) \setminus \mathfrak{Q}^{3,-}_4(h/2, \mathbf{a}) \tag{4.48}$$

provided there exists some number $p \in \{1, \dots, k\}$ such that (4.39) holds, i.e. $c = 5 + 3h/8 + (p + k - q - 1)h$. Notice that the r.h.s of (4.47) does no longer depend on i, j, q. As a consequence we have the periodicity property

$$\Upsilon^h_{\mathbf{a}}(x, s) = \Upsilon^h_{\mathbf{a} \mp h\vec{\mathbf{e}}_\ell}(x, s) \text{ if } x \in \mathfrak{Q}^{2,\pm}_\ell(h/2, \mathbf{a}), \ell = 1, 2 \text{ and } s \in [a_4 - \frac{h}{2}, a_4 + \frac{h}{2}]. \tag{4.49}$$

Topological Properties of the map $\Upsilon^h_{\mathbf{a}}$. We first have:

Lemma 4.4. *The two curves $\mathcal{L}_0(\mathbf{a})$ and $\mathcal{L}_\perp(\mathbf{a})$ are linked in $\partial \mathrm{Q}^4_{h/2}(\mathbf{a})$ and*

$$\mathfrak{m}\left(\mathcal{L}_0(\mathbf{a}), \mathcal{L}_\perp(\mathbf{a})\right) = 1. \tag{4.50}$$

Proof. To establish (4.50), we deform the two curves in a continuous way so to obtain a simpler geometry. We may assume without loss of generally that $\mathbf{a} = 0$ and that $h = 1$ and set $\mathcal{L}_0 = \mathcal{L}_0(0)$ and $\mathcal{L}_\perp = \mathcal{L}_\perp(0)$. We introduce the three-dimensional sphere for the ∞-norm

$$\begin{cases} \mathbb{S}^{3,\mathrm{cub}}_{2,\perp} \equiv \left\{ \mathbf{x} = (x_1, 0, x_3, x_4) \text{ s.t. } |\mathbf{x}|_\infty = \frac{1}{2} \right\} = \{\mathbf{x} \in \partial \mathrm{Q}^4_{1/2}(0)\, x_2 = 0\} \text{ as well as} \\ \mathbb{S}^{3,\mathrm{cub},+}_{2,\perp} \equiv \mathbb{S}^{3,\mathrm{cub}}_{2,\perp} \cap \{x_3 \geq 0\} \text{ and } \mathbb{S}^{3,\mathrm{cub},-}_{2,\perp} \equiv \mathbb{S}^{3,\mathrm{cub}}_{2,\perp} \cap \{x_3 < 0\}, \end{cases}$$

so that we have $\mathcal{L}_0(0) \subset \mathbb{S}^{3,\mathrm{cub},+}_{2,\perp}$ and

$$\mathcal{L}_\perp \cap \mathbb{S}^{3,\mathrm{cub},+}_{2,\perp} = \{(0, 0, \frac{3}{8}, -\frac{1}{2})\} \text{ and } \mathcal{L}_\perp \cap \mathbb{S}^{3,\mathrm{cub},-}_{2,\perp} = \{(0, 0, -\frac{1}{8}, -\frac{1}{2}), (0, 0, -\frac{1}{8}, +\frac{1}{2})\}. \tag{4.51}$$

We first deform the curve $\mathcal{L}_0$ staying inside $\mathbb{S}^{3,\mathrm{cub},+}_{2,\perp} \subset \mathbb{S}^{3,\mathrm{cub}}_{2,\perp} \subset \partial \mathrm{Q}^4_{1/2}(0)$ and in such a way that, throughout the deformation, the deformed line has the shape of a rectangle, which, at the end of the deformation, lies inside the face $\mathfrak{Q}^{3,-}_4(1/2, 0) \subset \mathbb{R}^3 \times \{-1/2\}$. For $0 \leq t \leq 1$, denoting $M^\pm_{\mathrm{L}}(t)$ and $M^\pm_{\mathrm{R}}(t)$ the vertices of above mentioned the rectangle, we set

$$M^-_{\mathrm{L}}(t) = M^-_{\mathrm{L}}(0) = (-\frac{1}{2}, 0, 0, -\frac{1}{2}) \text{ and } M^-_{\mathrm{R}}(t) = M^-_{\mathrm{R}}(0) = (\frac{1}{2}, 0, 0, -\frac{1}{2}),$$

so that these vertices are not moved. For the two other vertices, we set, for $0 \leq t \leq 1/2$

$$M^+_{\mathrm{L}}(t) = M^+_{\mathrm{L}}(0) + t\vec{\mathbf{e}}_3 = (-\frac{1}{2}, 0, t, \frac{1}{2}) \text{ and } M^+_{\mathrm{R}}(t) = M^+_{\mathrm{R}}(0) + t\vec{\mathbf{e}}_3 = (\frac{1}{2}, 0, t, \frac{1}{2}),$$

whereas for $1/2 \leq t \leq 1$, we set

$$\begin{cases} M^+_{\mathrm{L}}(t) = M^+_{\mathrm{L}}(0) + \frac{1}{2}\vec{\mathbf{e}}_3 - (2t-1)\vec{\mathbf{e}}_4 = (-\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2} - 2t) \text{ and} \\ M^+_{\mathrm{R}}(t) = M^+_{\mathrm{L}}(0) + \frac{1}{2}\vec{\mathbf{e}}_3 - (2t-1)\vec{\mathbf{e}}_4 = (\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2} - 2t). \end{cases}$$

The functions $t \mapsto M^\pm_{\mathrm{L}}(t)$ and $t \mapsto M^\pm_{\mathrm{L}}(t)$ are hence continuous on $[0, 1]$, with values in $\partial \mathrm{Q}^4_{1/2}(0)$. We define, for $0 \leq t \leq 1$ the curve

$$\mathcal{L}_0(t) = F_{0,\mathrm{top}}(t) \cup F_{0,\mathrm{bot}}(t) \cup F_{0,\mathrm{L}}(t) \cup F_{0,\mathrm{R}}(t),$$

where we set $F_{\mathrm{bot}}(t) \equiv [M^-_{\mathrm{L}}(t), M^-_{\mathrm{L}}(t)] = F_{0,\mathrm{bot}}(0) \in \mathfrak{Q}^{3,-}_4(1/2, 0)$, $F_{\mathrm{top}}(t) = [M^+_{\mathrm{L}}(t), M^+_{\mathrm{L}}(t)]$, $F_{\mathrm{L}}(t) = [M^-_{\mathrm{L}}(t), M^+_{\mathrm{L}}(t)]$ and $F_{\mathrm{R}}(t) = [M^-_{\mathrm{R}}(t), M^+_{\mathrm{R}}(t)]$. We verify that $F_{\mathrm{top}}(t) \subset \mathfrak{Q}^{3,+}_4(1/2, 0)$ for $t \in [0, 1/2]$, $F_{\mathrm{top}}(t) \subset \mathfrak{Q}^{3,+}_3(1/2, 0)$, for $t \in [1/2, 1]$, $F_{\mathrm{L}}(t) \in \mathfrak{Q}^{1,-}_4(1/2, 0)$ and $F_{0,\mathrm{R}}(t) \in \mathfrak{Q}^{1,+}_4(1/2, 0)$ for $t \in [0, 1]$. It follows that $t \mapsto \mathcal{L}_0(t)$ is a continuous deformation of $\mathcal{L}_0$ and that for $t \in [0, 1]$, we have

$$\mathcal{L}_0(t) \subset \mathbb{S}^{3,\mathrm{cub},+}_{2,\perp} \subset \partial \mathrm{Q}^4_{1/2}(0) \cap \{x_3 \geq 0\} \text{ and } \mathcal{L}_0(t) \cap \mathcal{L}_\perp = \emptyset. \tag{4.52}$$

Hence, we have

$$\mathfrak{m}\left(\mathcal{L}_0(t), \mathcal{L}_\perp\right) = \mathfrak{m}\left(\mathcal{L}_0, \mathcal{L}_\perp\right) = \mathfrak{m}\left(\mathcal{L}_0(1), \mathcal{L}_\perp\right), \forall t \in [0,1]. \tag{4.53}$$

At time $t=1$, we observe that the points $M_{\mathrm{L}}^{+}(1) = (-\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2})$ and $M_{\mathrm{R}}^{+}(1) = (\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2})$ belong to $\mathfrak{Q}_4^{3,-}(1/2,0) \cap \{x_2 = 0\}$, so that all the points $M_{\mathrm{L}}^{\pm}(1)$ and $M_{\mathrm{R}}^{\pm}(1)$ belong to $\mathfrak{Q}_4^{3,-}(1/2,0) \cap \{x_2 = 0\}$. Hence the rectangle $\mathcal{L}_0(1)$ satisfies

$$\mathcal{L}_0(1) \subset \mathfrak{Q}_4^{3,-}(\frac{1}{2}, 0) \cap \{x_2 = 0\} \subset \mathbb{R}^3 \times \{-\frac{1}{2}\} \cap \{x_2 = 0\} = P_{1,3}(0) \times \{-\frac{1}{2}\}. \tag{4.54}$$

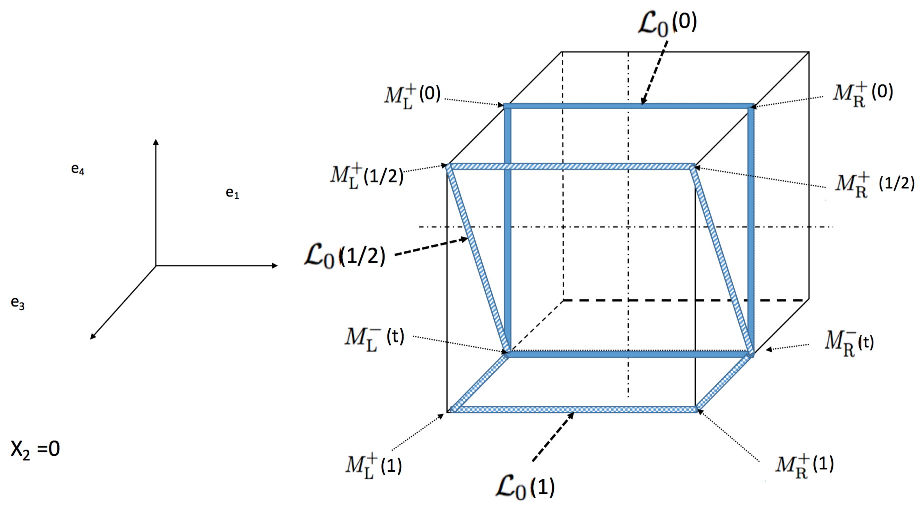

Figure 13: *The deformation of the curve $\mathcal{L}_0$ at times $t = 0, 1/2$ and $t = 1$.*

We perform a similar deformation on the curve $\mathcal{L}_\perp$ in particular the three segments which do not lie in $\mathfrak{Q}^{3,-}(\frac{1}{2}, 0)$. We set

$$N_{\mathrm{L}}^{-}(t) = N_{\mathrm{L}}^{-}(0) = (0, -\frac{1}{2}, -\frac{3}{8}, -\frac{1}{2}) \text{ and } N_{\mathrm{R}}^{-}(t) = N_{\mathrm{R}}^{-}(0) = (0, \frac{1}{2}, -\frac{3}{8}, -\frac{1}{2}),$$

The two other points composing the segments are moved as follows: For $0 \leq t \leq 1/2$, we set

$$N_{\mathrm{L}}^{+}(t) = (0, -\frac{1}{2}, -\frac{3}{8} - \frac{t}{4}, \frac{1}{2}) \text{ and } N_{\mathrm{R}}^{+}(t) = (0, \frac{1}{2}, 0, t, \frac{1}{2}),$$

so that $N_{\mathrm{L}}^{+}(\frac{1}{2}) = (0, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ and $N_{\mathrm{R}}^{+}(\frac{1}{2}) = (0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. For $1/2 \leq t \leq 1$, we set

$$\begin{cases} N_{\mathrm{L}}^{+}(t) = N_{\mathrm{L}}^{+}(\frac{1}{2}) - (2t-1)\vec{\mathbf{e}}_4 = (-\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2} - 2t) \text{ and} \\ N_{\mathrm{R}}^{+}(t) = N_{\mathrm{R}}^{+}(\frac{1}{2}) - (2t-1)\vec{\mathbf{e}}_4 = ((\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2} - 2t), \end{cases}$$

so that

$$N_{\mathrm{L}}^{+}(1) = (0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \text{ and } N_{\mathrm{R}}^{+}(1) = (0, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}). \tag{4.55}$$

All the points $N_{\mathrm{L}}^{\pm}(1), N_{\mathrm{R}}^{\pm}(1)$ hence belong the face $\mathfrak{Q}_4^{3,-}(1/2,0) \subset \mathbb{R}^3 \times \{-1/2\}$. We finally consider the curve $\mathcal{L}_\perp(t)$ defined, for $0 \leq t \leq 1$ by

$$\mathcal{L}_\perp(t) \equiv [N_{\mathrm{L}}^-(0), N_{\mathrm{L}}^+(t)] \cup [N_{\mathrm{R}}^-(0), N_{\mathrm{R}}^+(t)] \cup [N_{\mathrm{L}}^+(t), N_{\mathrm{R}}^+(t)] \cup \mathcal{C}_{\perp,h}^-(0) \times \{-\frac{1}{2}\}.$$

the deformation $t \mapsto \mathcal{L}_\perp(t)$ is a deformation of $\mathcal{L}_\perp(0) = \mathcal{L}_\perp$ such that, for $0 \leq t \leq 1$, we have

$$\mathcal{L}_\perp(t) \subset \mathbb{S}_{2,\perp}^{3,\mathrm{cub},-} \subset \partial \mathrm{Q}_{1/2}^4(0) \cap \{x_3 < 0\} \text{ and } \mathcal{L}_\perp(t) \cap \mathcal{L}_0(1) = \emptyset.$$

We notice that, for $t = 1$, we have

$$\mathcal{L}_\perp(1) \subset \mathfrak{Q}_4^{3,-}(\frac{1}{2}, 0) \cap \{x_1 = 0\} \subset \mathbb{R}^3 \times \{-\frac{1}{2}\} \cap \{x_1 = 0\} = P_{2,3}(0) \times \{-\frac{1}{2}\}. \tag{4.56}$$

By continuity of the linking number, we have $\mathfrak{m}(\mathcal{L}_0(1), \mathcal{L}_\perp(1)) = \mathfrak{m}(\mathcal{L}_0(1), \mathcal{L}_\perp(t)) = \mathfrak{m}(\mathcal{L}_0(1), \mathcal{L}_\perp(0)) = \mathfrak{m}(\mathcal{L}_0(1), \mathcal{L}_\perp)$, for all $t \in [0,1]$. Combining with (4.53) we deduce that

$$\mathfrak{m}(\mathcal{L}_0, \mathcal{L}_\perp) = \mathfrak{m}(\mathcal{L}_0(1), \mathcal{L}_\perp(1)). \tag{4.57}$$

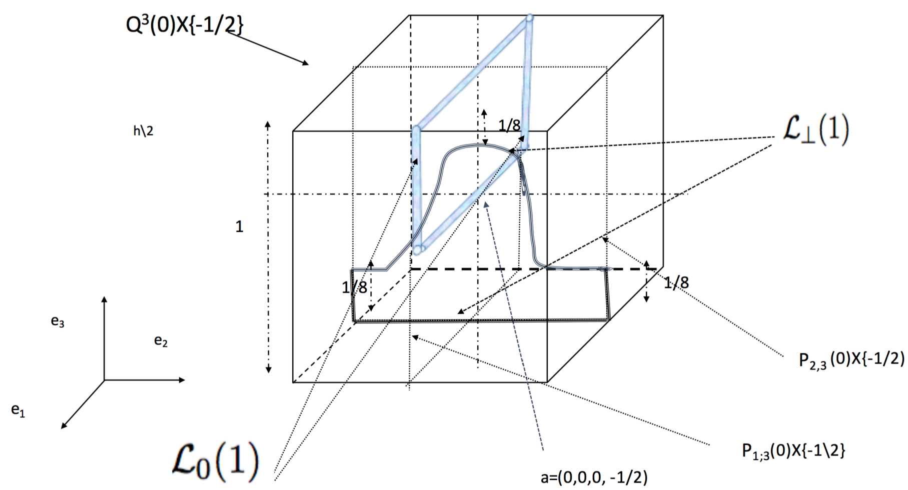

Figure 14: *The linking of the curves $\mathcal{L}_0(1)$ and $\mathcal{L}_\perp(1)$*

We may now take advantage, in view of (4.54) and (4.56), that the two curves are planar curves in the three dimensional affine space $\mathbb{R}^3 \times \{1/2\}$, included in planes which are not parallel, see Figure 14. Using the methof of crossing numbers, we may then show that

$$\mathfrak{m}(\mathcal{L}_0(1), \mathcal{L}_\perp(1)) = 1.$$

Combining this identity with identity (4.57), we deduce the desired result. □

Lemma 4.5. *We have $\mathbb{H}(\Upsilon_{\mathbf{a}}^h) = 2$.*

Proof. The proof follows immediately from Lemma 2.2 and identity (4.50). □

Energy estimates. Concerning the E_3 energy of $\Upsilon^h_{\mathbf{a}}$ we verify that

$$|\nabla_3 \Upsilon^h_{\mathbf{a}}| \leq Ch^{-1}$$

where $C > 0$ is some constant. By integration on the boundary $\partial \mathrm{Q}^4_{h/2}(a)$, whose measure is of order h^3, we deduce therefore that

$$\mathrm{E}_3\left(\Upsilon^h_{\mathbf{a}}, \partial \mathrm{Q}^4_{h/2}(a)\right) \leq C, \tag{4.58}$$

where $C > 0$ is some constant. Finally, we consider the extension $\boxdot_{h,\mathbf{a}}$ of $\Upsilon^h_{\mathbf{a}}$ to the cube $\mathrm{Q}^4_{h/2}(\mathbf{a})$ given by

$$\boxdot_{h,\mathbf{a}}(\mathbf{x}) = \mathfrak{E}\mathrm{xt}_{h/2,\mathbf{a}}\left(\Upsilon^h_{\mathbf{a}}(\mathbf{x})\right) \text{ for } \mathbf{x} \in \mathrm{Q}^4_{h/2}(\mathbf{a}). \tag{4.59}$$

Notice that $\boxdot_{h,\mathbf{a}}$ is Lipschitz on the cube $\mathrm{Q}^4_r(\mathbf{a})$ except at the origin, where, in view of Lemma 4.5 it possesses a point singularity of Hopf invariant equal to 2. Invoking scaling identities (in the spirit of (4.32))and (4.58), we obtain the energy identity

$$\mathrm{E}_3(\boxdot_{h,\mathbf{a}}) = \mathbf{K}_{\mathrm{box}} h, \text{ for any } h > 0, \tag{4.60}$$

where $\mathbf{K}_{\mathrm{box}}$ is a universal constant.

4.1.6 Deforming topologically trivial maps to constant maps

We assume here that we are given a map $w \in \mathrm{Lip} \cap W^{1,3}(\mathbb{R}^3, \mathbb{S}^2)$ such that we have

$$w(x) = \mathbb{P}_{\mathrm{south}} \text{ for } x \in \mathbb{R}^3 \setminus [-R, R]^3, \text{ for some } R > 0.$$

Hence, we may define a Hopf invariant of w. We have

Proposition 4.1. *Let w and $R > 0$ be as above and assume that $\mathbb{H}(w) = 0$. There exists a map $W \in C^0 \cap W^{1,3}([-R, R]^4, \mathbb{S}^2)$ such that the following holds:*

- $W(x, -R) = w(x)$ *for* $x \in [-R, R]^3$
- $W(x, R) = \mathbb{P}_{\mathrm{south}}$ *for* $x \in [-R, R]^3$
- $W(x, s) = \mathbb{P}_{\mathrm{south}}$ *for* $x \in \partial([-R, R]^3)$ *and* $s \in [-R, R]$
- $\mathrm{E}_3(W, [-R, R]^4) \leq 2\mathrm{C}_{\mathrm{ext}}\, R\, \mathrm{E}_3(v, [-R, R]^3)$.

Proof. We consider the continuous map $\tilde{w}$ from the boundary $\partial([-R, R]^4)$ to $\mathbb{S}^2$ defined by

$$\begin{cases} \tilde{w}(x, -R) = w(x) \text{ for } x \in [-R, R]^3 \text{ and} \\ \tilde{w}(\mathbf{x}) = \mathbb{P}_{\mathrm{south}} \text{ for } \mathbf{x} \in \partial([-R, R]^4) \setminus [-R, R]^3 \times \{-R\}, \end{cases}$$

so that $\tilde{w}$ is Lipschitz ans the homotopy class of $\tilde{w}$ is trivial. There exists therefore a Lipschitz map $\varphi : [-R, R]^4 \to \mathbb{S}^2$ such that

$$\varphi(\mathbf{x}) = \tilde{w}(\mathbf{x}) \text{ for } \mathbf{x} \in \partial([-R, R]^4).$$

Since φ is Lipschitz, we have

$$I_1 \equiv \int_{[-R,R]^4} |\nabla_4 \varphi|^3 < +\infty.$$

Let $0 < \rho \leq R$ be such that $\rho I_1 \leq \mathrm{C_{ext}}\, R^2 \mathrm{E}_3(v, [-R,R]^3)$. We define W as

$$\begin{cases} W(\mathbf{x}) = \tilde{w}\left(\dfrac{\mathbf{x}}{|x|_\infty}\right) \text{ if } |\mathbf{x}|_\infty \geq \rho \\ W(\mathbf{x}) = \varphi\left(\dfrac{R\mathbf{x}}{\rho}\right) \text{ if } |\mathbf{x}|_\infty \leq \rho, \end{cases}$$

so that W satisfies the three first condition in Proposition 4.1. For the energy extimate, we observe that, by (4.32), we have

$$\int_{|\mathbf{x}|_\infty \geq \rho} |\nabla_4 W|^3 \leq \mathrm{C_{ext}}\, R \mathrm{E}_3(v, [-R,R]^3).$$

On the other hand, by scaling we have

$$\mathrm{E}_3(W, [-\rho,\rho]^4) = \frac{\rho}{R} \mathrm{E}_3(\varphi, [-R,R]^4) = \frac{\rho I_1}{R}.$$

The conclusion follows combining the previous estimates. □

Remark 4.5. Related constructions can be found for instance in [10, 6].

4.2 Proof of Proposition 4

As mentioned in the introduction, the map $\mathbf{G}^k_{\mathbf{ord}}$, which is defined on the strip $\Lambda = \mathbb{R}^3 \times [0,50] \subset \mathbb{R}^4$ to $\mathbb{S}^2$, is a deformation of the spaghetton map to a constant map, the fourth space variable x_4 standing for a deformation or *time* parameter. The construction relies on corresponding deformations of the fibers of the sheaves $\mathfrak{L}^k$ and $\mathfrak{L}^{k,\perp}$, the value of $\mathbf{G}^k_{\mathbf{ord}}$ being then obtained thanks to the Pontryagin construction or its variant. The main part of the construction consist in deformed $\mathbf{S}^k_{\mathbf{pag}}$ to a map of trivial homotopy class. The guiding idea consists in "pushing down" along the x_3-axis the set $\mathfrak{L}^{k,\perp}$ while keeping the set $\mathfrak{L}^k$ fixed, singularities being created when two fibers meet and cross. When the sheave $\mathfrak{L}^{k,\perp}$ has been pushed down sufficiently, then the two sheaves are no longer linked, so that we obtain for the corresponding three dimensional "time" slices a map with trivial homotopy class. It remain to deform the later to a constant map, a task handled thanks to Proposition 4.1 present in subsection 4.1.6.

Concerning the main step, i.e. the deformation of the fibers, the main technical tools that we are going to use have been presented in Section 4.1, namely

- Deformations of curves and functions using the vector-fields $\vec{X}_0$ and $\vec{X}^k_1$
- The extension operator as presented in subsection 4.1.5. It is used on small cubes of size h near the crossing points of the fibers. It allows curves to cross while yielding singularities for $\mathbf{G}^k_{\mathbf{ord}}$.

Pushing down the sheave $\mathfrak{L}^{k,\perp}$ in the direction $\vec{\mathbf{e}}_3$, we see that it might meet $\mathfrak{L}^k$ in their respective straight parts[11], in the region $\mathcal{Q}_0^3 = [0,1]^3$. As a matter of fact, the points $\mathbb{R}^3$ where the fibers cross are given by

$$\mathrm{a}_{i,j,q}^k = (\frac{i}{k}, \frac{j}{k}, \frac{q}{k}) = (i\,h, j\,h, q\,h) = h(i,j,q) \text{ for } i,j,q = 1,\dots,k \text{ with } h = \frac{1}{k}, \tag{4.61}$$

so that the set of crossing points in $\mathbb{R}^3$ is given by

$$\{\mathrm{a}_{i,j,q}\}_{i,j,q=1,\dots,k} = (h\mathbb{I}_k)^3 = \boxplus_k^3(h) \subset [0,1]^3.$$

Given a time $s > 0$ will denote by $\mathfrak{L}_{j,q}^k(s)$ and $\mathfrak{L}_{i,q}^{k,\perp}(s)$ the deformations at time s of the curves $\mathfrak{L}_{j,q}^k$ and $\mathfrak{L}_{i,q}^{k,\perp}$ respectively, so that

$$\mathfrak{L}_{j,q}^k(0) = \mathfrak{L}_{j,q}^k \text{ and } \mathfrak{L}_{i,q}^{k,\perp}(0) = \mathfrak{L}_{i,q}^{k,\perp}.$$

As a matter of fact, the sheave $\mathfrak{L}^k$ will not be moves, so that we have throughout

$$\mathfrak{L}_{j,q}^k(s) = \mathfrak{L}_{j,q}^k \text{ for every } s \geq 0. \tag{4.62}$$

The construction will be divided in several distinct steps, where we use one of the above methods, i.e. either pushing down using fields $\vec{X}_0$ or $\vec{X}_1^k$ or using extensions creating singularities at the points $\mathrm{a}_{i,j,q}^k$. Each step n corresponds to a specific time interval $[\mathrm{T}_{n-1}^k, \mathrm{T}_n^k]$, with $\mathrm{T}_0^k = 0$,

$$\mathrm{T}_1^k = 5 + \frac{3h}{8} \text{ and } \mathrm{T}_{n+1}^k = \mathrm{T}_n^k + h = nh + \tau_h, \text{ for } n \in \{1,\dots,4k-2\}, \text{ where } \tau_h = 5 - \frac{5h}{8},$$

with $h = \dfrac{1}{k}$, so that all intervals, except the first and the last one, have size h. In each step k, we will construct the restriction of the map $\mathbf{G}_{\mathbf{ord}}^k$ to the corresponding strips in the $\mathbb{R}^4$ space, namely the strip Λ_n^k given by

$$\Lambda_n^k = \mathbb{R}^3 \times [\mathrm{T}_{n-1}^k, \mathrm{T}_n^k],$$

taking care that the construction yield the sam value on the intersections, that is on the time slices $\mathbb{R}^3 \times \{\mathrm{T}_n^k\}$. In some strips, the map $\mathbf{G}_{\mathbf{ord}}^k$ will have a finite number of point singularities: These space-time singularities will have the form

$$\begin{aligned}\mathbf{a}_{i,j,q,r}^k &\equiv (\mathrm{a}_{i,j,q}^k, \mathrm{T}_{2r+\frac{1}{2}}^k) = (\mathrm{a}_{i,j,q}^k, \mathrm{T}_{2r}^k + \frac{h}{2}) = (\mathrm{a}_{i,j,q}^k, \tau_h + \frac{h}{2} + 2h\,r) \\ &= (i\,h, j\,h, q\,h, \mathrm{T}_{2r}^k + \frac{h}{2}) = (ih, j\,h, q\,h, 5 - \frac{h}{8} + 2h\,r).\end{aligned} \tag{4.63}$$

Remark 4.6. Throughout the proof, the main focus is on the region $[0,1]^3$ and its close neighborhood. The restriction of the sheaves $\mathfrak{L}^k(s) = \mathfrak{L}^k$ and $\mathfrak{L}^{k,\perp}(s)$ as well as its translates are segments parallel to $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ respectively, as shown in Figures 3 to 8. The*curved parts* of the sheaves, which lie outside of this region are less relevant, except concerning the topological properties.

[11]consisting of segments parallel to $\vec{\mathbf{e}}_2$ and $\vec{\mathbf{e}}_3$ respectively

Step 1: Pushing $\mathfrak{L}^{k,\perp}$ downwards towards $\mathfrak{L}^k$.

We define in this step $\mathbf{G}^k_{\mathbf{ord}}$ on $\Lambda^k_1 = \mathbb{R}^3 \times [0, \mathrm{T}^k_1]$: We move $\mathfrak{L}^{k,\perp}$ along the constant vector field $\vec{X}_0 = -\vec{\mathbf{e}}_3$ while keeping $\mathfrak{L}^k$ fixed throughout. For $0 \leq x_4 \leq \mathrm{T}^k_1$ the fibers $\mathfrak{L}^{k,\perp}_{i,q}$ are hence translated with constant speed. We have therefore, for $i, q = 1, \ldots, k$,

$$\mathfrak{L}^{k,\perp}_{i,q}(x_4) \equiv P_{\zeta_0}(\mathfrak{L}^k_{i,q}, x_4) = \mathcal{L}^{k,\perp} - x_4\vec{\mathbf{e}}_3 \text{ for } x_4 \in [0, \mathrm{T}^k_1] = [0, 5 + \frac{5h}{8}] \tag{4.64}$$

We define the map $\mathbf{G}^k_{\mathbf{ord}}$ on the strip $\Lambda^k_1 \equiv \mathbb{R}^3 \times [0, \mathrm{T}^k_1]$ as

$$\begin{aligned} \mathbf{G}^k_{\mathbf{ord}}(x, s) &= \left[P_{\zeta_0}(\mathbf{P}^{\mathrm{ontya}}_{\varrho}[\mathcal{L}^{k,\perp}(s)]) \vee_3 \mathbf{P}^{\mathrm{ontya}}_{\varrho}[\mathcal{L}^k]\right](x) \\ &= \mathbf{P}^{\mathrm{ontya}}_{\varrho}\left[(\mathcal{L}^{k,\perp} - s\vec{\mathbf{e}}_3) \cup \mathcal{L}^k\right](x), \forall x \in \mathbb{R}^3. \end{aligned} \tag{4.65}$$

Here and in the sequel, the frame will be the reference frame, so that we omit to mention it in the operatot $\mathbf{P}^{\mathrm{ontya}}$. Notice that $\mathbf{G}^k_{\mathbf{ord}}$ defined as above is smooth on the strip Λ^k_1, that

$$\mathbf{G}^k_{\mathbf{ord}}(\cdot, 0) = \mathbf{S}^k_{\mathbf{pag}}(\cdot) \text{ on } \mathbb{R}^3, \tag{4.66}$$

that $\mathrm{dist}\left((\mathcal{L}^{k,\perp} - s\vec{\mathbf{e}}_3, \mathcal{L}^k\right) \geq \dfrac{5h}{8}$ for $s \in [0, \mathrm{T}^k_1]$ and that

$$\mathrm{dist}\left((\mathcal{L}^{k,\perp} - \mathrm{T}^k_1\vec{\mathbf{e}}_3, \mathcal{L}^k\right) = \frac{5h}{8}. \tag{4.67}$$

Turning to energy estimates, it follows from Remark 4.1 that

$$|\nabla_4 P_{\zeta_0}(\mathbf{P}^{\mathrm{ontya}}_{\varrho}(\mathcal{L}^{k,\perp}(s)(x)| \leq \mathrm{C}^0_{\mathrm{def}} k,$$

so that by integration on the set where $\mathbf{G}^k_{\mathbf{ord}}$ is not constant, we are led to

$$\int_{\Lambda^k_1} |\nabla_4 \mathbf{G}^k_{\mathbf{ord}}|^3 \leq \mathrm{K}^0_{\mathrm{def}} k^3 \text{ with } \mathrm{K}^0_{\mathrm{def}} = 240\left(\mathrm{C}^0_{\mathrm{def}}\right)^3. \tag{4.68}$$

In view of (4.67) we observe that the lowest fibers [12] of $\mathfrak{L}^{k,\perp}(\mathrm{T}^k_0)$, i.e. the curves $\mathfrak{L}^{k,\perp}_{i,1}(\mathrm{T}^k_0)$ for $i = 1, \ldots, k$ are at distance $5h/8$ of the upper fibers of $\mathfrak{L}^{k,\perp}$, i.e. the curves $\mathfrak{L}^{k,\perp}_{j,k}$ for $j = 1, \ldots, k$, the nearest points being the points in the set

$$\mathbf{A}^k_1 = \{\mathrm{a}_{i,j,k}, i, j = 1, \ldots, k\} = \boxplus^2_k(h) \times \{1\} \tag{4.69}$$

with

$$\mathrm{dist}(\mathrm{a}_{i,j,k}, \mathfrak{L}^{k,\perp}_{i,1}) = \frac{5h}{8} = \frac{h}{2} + \frac{h}{8} \text{ and } \mathrm{a}_{i,j,k} \in \mathfrak{L}^{k,\perp}_{j,k}.$$

Our next aim will be to continue to lower with the same speed "most" of the fibers of $\mathcal{L}^{k,\perp}$ while keeping $\mathfrak{L}^k$ fixed, without meeting the curves $\mathfrak{L}^k_{j,k}$, $j = 1, \ldots, k$ hence avoiding the singularities. In view of the above discussion, we are led to introduce the subset $\mathbf{L}^k_1$ of $\mathfrak{L}^{k,\perp}(\mathrm{T}^k_1)$ corresponding to the union of fibers of $\mathfrak{L}^{k,\perp}_k$ which are the closest to $\mathfrak{L}^k$, that is, we consider the sets

$$\mathbf{L}^{k,\perp}_1 = \bigcup_{i=1}^{k} \mathfrak{L}^{k,\perp}_{i,1} \text{ and } \mathbf{L}^{k,\perp}_1(s) = \bigcup_{i=1}^{k} \mathfrak{L}^{k,\perp}_{i,1}(s) \tag{4.70}$$

[12] according to the x_3 coordinate

as well as there complements

$$\mathbf{N}_1^{k,\perp} = \bigcup_{\substack{q\in\{2,\dots,k\}\\ i\in\{1,\dots,k\}}} \mathfrak{L}_{i,q}^{k,\perp} \text{ and } \mathbf{N}_1^{k,\perp}(s) = \bigcup_{\substack{q\in\{2,\dots,k\}\\ i\in\{1,\dots,k\}}} \mathfrak{L}_{i,q}^{k,\perp}(s) \tag{4.71}$$

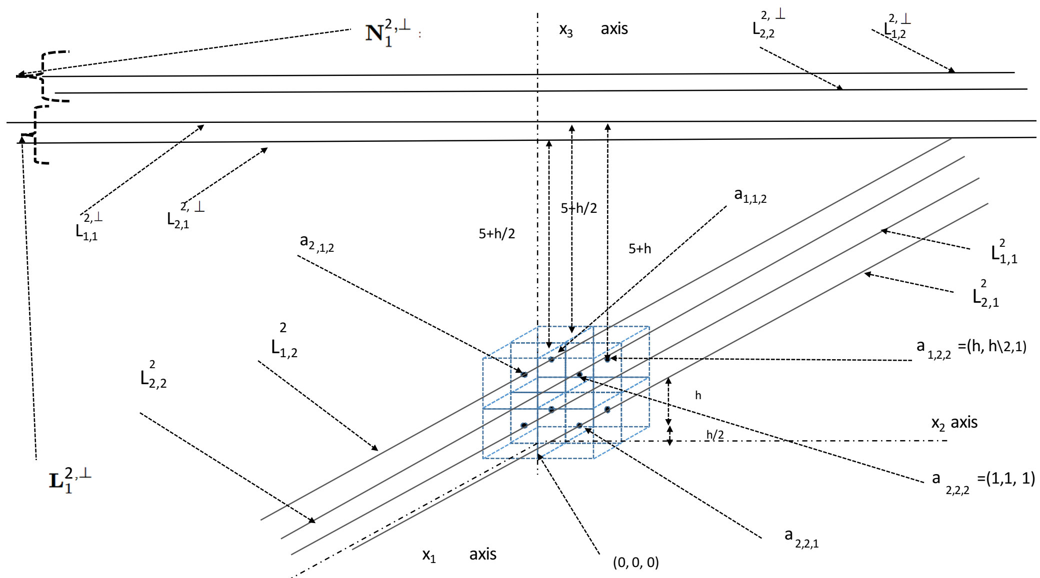

Figure 15: *A zoom on the crossing area at time* 0*,* $k = 2$*.*

Step 2: Avoiding the first crossings.

Defining $\mathbf{G}^k_{\mathbf{ord}}$ *on the strip* Λ_2^k. We define in this part $\mathbf{G}^k_{\mathbf{ord}}$ on the strip $\Lambda_2^k \equiv \mathbb{R}^3 \times [\mathrm{T}_1^k, \mathrm{T}_2^k] = \mathbb{R}^3 \times [\mathrm{T}_1^k, \mathrm{T}_1^k + h]$. We are going to invoke three different types of motions for the fibers:

- The fibers in $\mathbf{L}_1^k(\mathrm{T}_1^k)$ are moved according to the vector-field $\vec{X}_1^k = -\zeta_1 e_3$. In particular, they are not going to reach the points $\mathrm{a}_{i,j,k}$ in $\mathbf{A}_1^k$.
- The fibers in $\mathbf{N}_1^k(\mathrm{T}_1^k) = \mathfrak{L}^{k,\perp}(\mathrm{T}_1^k) \setminus \mathbf{L}_1^k(\mathrm{T}_1^k)$ are moved with constant speed according to the vector field $\vec{X}_0 = -\vec{\mathbf{e}}_3$. They hence will be translated down the $\vec{\mathbf{e}}_3$ direction by a lentgh equal to h at time T_2^k.
- The fibers in $\mathfrak{L}^k$ are *not* moved, as already mentioned.

We define hence[13] the map $\mathbf{G}^k_{\mathbf{ord}}$ on the strip $\Lambda_2^k \equiv \mathbb{R}^3 \times [\mathrm{T}_1^k, \mathrm{T}_2^k]$ as

$$\begin{aligned}\mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}_1^k + s) = P_{\zeta_0}\left(\mathbf{P}_\varrho^{\text{ontya}}[\mathbf{N}_1^{k,\perp}(\mathrm{T}_1^k)]\right)(s) \vee_3 P_{\zeta_1}(\mathbf{P}_\varrho^{\text{ontya}}[\mathbf{L}_1^{k,\perp}(\mathrm{T}_1^k)](s)\\ \vee_3 \mathbf{P}_\varrho^{\text{ontya}}[\mathfrak{L}^k].\end{aligned} \tag{4.72}$$

[13] As mentioned, we have omitted the frame, which is throughout reference frame, in the notation for $\mathbf{P}^{\text{ontya}}$

The reader may check that the three maps appearing in the definition (4.72) have, for any $s \in [0, h]$ disjoint supports, so that the gluing procedure in (4.72) is well-defined. Recall that we have

$$P_{\zeta_0}(\mathbf{P}_\varrho^{\rm ontya}[\mathbf{N}_1^{k,\perp}(\mathrm{T}_1^k)](s) = \mathbf{P}_\varrho^{\rm ontya}[P_{\zeta_0}(\mathbf{N}_1^{k,\perp}(\mathrm{T}_1^k)(s)] = \mathbf{P}_\varrho^{\rm ontya}[\mathbf{N}_1^{k,\perp} - (\mathrm{T}_1^k + s)\vec{\mathbf{e}}_3] \text{ and}$$
$$P_{\zeta_1}(\mathbf{P}_\varrho^{\rm ontya}[\mathbf{L}_1^k(\mathrm{T}_1^k)](s) = \tilde{\mathbf{P}}_\varrho^{\rm ontya}[\mathcal{D}_{\rm ef}\mathbf{L}_1^{k,\perp}(\mathrm{T}_1^k, s)] \text{ where } \mathcal{D}_{\rm ef}\mathbf{L}_1^{k,\perp}(\mathrm{T}_1^k, s) = \bigcup_{i=1}^{k} \mathcal{D}_{\rm ef}\mathfrak{L}_{i,1}^{k,\perp}(\mathrm{T}_1^k, s).$$

Going back to (4.72) we obtain hence

$$\mathbf{G}_{\mathbf{ord}}^k(\cdot, \mathrm{T}_1^k + s) = \mathbf{P}_\varrho^{\rm ontya}[\mathbf{N}_1^{k,\perp} - (\mathrm{T}_1^k + s)\vec{\mathbf{e}}_3] \vee_3 \tilde{\mathbf{P}}_\varrho^{\rm ontya}[\mathcal{D}_{\rm ef}\mathbf{L}_1^{k,\perp}(\mathrm{T}_1^k, s)] \vee_3 \mathbf{P}_\varrho^{\rm ontya}[\mathfrak{L}^k]. \tag{4.73}$$

The shape of the fibers for $s \in [\mathrm{T}_1^k, \mathrm{T}_2^k]$. We have already studied the effect of the flow Φ_1^k generated by the vector field X_1^k on the fibers in $\mathfrak{L}^{k,\perp}$ or there translates in Subsection 4.1.2. In view of the results there, we may write, for $s \in [0, h]$, concerning the transported fibers

$$\begin{cases} \mathfrak{L}_{i,1}^{k,\perp}(\mathrm{T}_1^k + s) = \Phi_1^k(\mathfrak{L}_{i,1}^{k,\perp} - \mathrm{T}_1^k\vec{\mathbf{e}}_3, s) = \mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(\mathrm{T}_1^k, s) \\ \mathfrak{L}_{i,q}^{k,\perp}(\mathrm{T}_1^k + s) = \Phi_0^k(\mathfrak{L}_{i,q}^{k,\perp} - \mathrm{T}_1^k\vec{\mathbf{e}}_3, s) = \mathfrak{L}_{i,k}^{k,\perp} - (\mathrm{T}_1^k + s)\vec{\mathbf{e}}_3 \text{ for } q \neq 1. \\ \mathfrak{L}_{j,q}^k(\mathrm{T}_1^k + s) = \mathfrak{L}_{j,q}^k. \end{cases} \tag{4.74}$$

The shape of the curve $\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(\mathrm{T}_1^k, s)$ is described in (4.25), see also Remark 4.2 and figures 9 and 10. Most of $\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}(\mathrm{T}_1^k, s)$ corresponds to the the translate $\mathfrak{L}_{i,k}^{k,\perp} - (\mathrm{T}_1^k + s)\vec{\mathbf{e}}_3$ except the part stuck above the points $\mathrm{a}_{i,j,k}$, at a vertical distance of $3h/8$ when $s = h$.

The energy on the strip Λ_2^k. It follows from the definition (4.72) of $\mathbf{G}_{\mathbf{ord}}^k$ on the strip $\Lambda_2^k \equiv \mathbb{R}^3 \times [\mathrm{T}_1^k, \mathrm{T}_2^k]$ and the gradient estimate for $\mathcal{D}_{\rm ef}\mathfrak{L}_{i,q}^{k,\perp}$ provided in Lemma 4.3 that

$$|\nabla_4 \mathbf{G}_{\mathbf{ord}}^k| \leq \mathrm{C}_{\rm def} k \text{ on } \Lambda_2^k,$$

so that by integration on the support of $\mathbf{G}_{\mathbf{ord}}^k$ which is included in $[-30, 30]^3$ restricted to Λ_1^k, we are led to the estimate

$$\int_{\Lambda_2^k} |\nabla_4 \mathbf{G}_{\mathbf{ord}}^k|^3 \leq 60^3\, h\mathrm{C}_{\rm def}^3 k^3 \leq \mathrm{K}_{\rm def} k^2. \tag{4.75}$$

where the constant $\mathrm{K}_{\rm def} = 60^3\mathrm{C}_{\rm def}^3$ does not depend on h.

On the shape of the fibers at time T_2^k. At time T_2^k, all fibers of $\mathfrak{L}^{k,\perp}$ have been translated by$-\mathrm{T}_2^k\vec{\mathbf{e}}_3 = -(\tau_h + h)\vec{\mathbf{e}}_3$, except the files in $\mathbf{L}_1^k(\mathrm{T}_2^k)$ which are rounded near the points $a_{i,j,k}$ in order to avoid collision with the fibers $\mathfrak{L}_{j,k}^k$, which they would otherwise have crossed. This situation is described in Figure 16. We have in view of the inclusion (4.27) of Remark 4.2

$$\mathfrak{L}^{k,\perp}(\mathrm{T}_2^k) \subset \Big(\mathfrak{L}^{k,\perp} - (\tau_h + h)\vec{\mathbf{e}}_3\Big) \cup [\frac{h}{2}, 1 + \frac{h}{2}]^2 \times [1 - \frac{3h}{8}, 1 + \frac{3h}{8}]. \tag{4.76}$$

We pay n special attention to the *first crossing region* defined by $\mathcal{O}_{\rm cross,1}^h$ defined by

$$\mathcal{O}_{\rm cross,1}^h = [\frac{h}{2}, 1 + \frac{h}{2}]^2 \times [1 - \frac{h}{2}, 1 + \frac{h}{2}] \subset \mathbb{R}^3,$$

since we already know that

$$\mathfrak{L}^{k,\perp}(\mathrm{T}_2^k) \setminus \mathcal{O}^h_{\mathrm{cross},1} = \Big(\mathfrak{L}^{k,\perp} - (\tau_h + h)\vec{\mathbf{e}}_3\Big) \setminus \mathcal{O}^h_{\mathrm{cross},1}. \tag{4.77}$$

We consider the points in $\mathbf{A}_1^k$ where fibers would have collided if transported by the constant vector fields [14] and the k^2 distinct cube $\mathrm{Q}^3_{\frac{h}{2}}(\mathrm{a}^k_{i,j,k})$ so that

$$\mathcal{O}^h_{\mathrm{cross},1} = \bigcup_{i,j=1}^{k} \mathrm{Q}^3_{\frac{h}{2}}(\mathrm{a}^k_{i,j,k}) \text{ where } \mathrm{Q}^3_{\frac{h}{2}}(\mathrm{a}^k_{i,j,k}) = \{x \in \mathbb{R}^3,\ \text{s.t } |x - \mathrm{a}^k_{i,j,k}|_\infty \leq \frac{h}{2}\}. \tag{4.78}$$

The intersection of two given cubes in the above collection is either void or included in the intersections of the boundaries. We have, in view of (4.73), for $s \in [0,h]$

$$\mathfrak{L}^{k,\perp}_{i,1}(\mathrm{T}_1^k + s) = \Phi_1^k(\mathfrak{L}^{k,\perp}_{i,1} - \mathrm{T}_1^k\vec{\mathbf{e}}_3, s) = \mathcal{D}_{\mathrm{ef}}\mathfrak{L}^{k,\perp}_{i,1}(\mathrm{T}_1^k, s) \text{ for } x \in \mathrm{Q}^3_{\frac{h}{2}}(\mathrm{a}_{i,j,k}).$$

Going back to Remark 4.3, we notice that (4.39) is fullfilled with $c = \mathrm{T}_1^k$, $p = 1$ and $q = 1$, so that (4.38) yields

$$\mathcal{D}_{\mathrm{ef}}\mathfrak{L}^{k,\perp}_{i,1}(\mathrm{T}_1^k, h) \cap \mathrm{Q}^3_{h/2}(\mathrm{a}_{i,j,k}) = \mathcal{C}^-_{\perp,h}(\mathrm{a}_{i,j,k}), \text{ for } i \in \{1,\dots,k\}$$

and hence by the definition of $\mathfrak{L}^{k,\perp}_{i,1}(\mathrm{T}_2^k)$, we have for any $i,j \in \{1,\dots,k\}$

$$\mathfrak{L}^{k,\perp}_{i,1}(\mathrm{T}_2^k) \cap \mathrm{Q}^3_{\frac{h}{2}}(\mathrm{a}_{i,j,k}) = \mathcal{C}^-_{\perp,h}(\mathrm{a}_{i,j,k}). \tag{4.79}$$

The valued of $\mathbf{G}^k_{\mathbf{ord}}$ *on the crossing regions* $\mathcal{O}^h_{\mathrm{cross},1}$ *at time* T_2^k. The value of $\mathbf{G}^k_{\mathbf{ord}}$ provided by (4.73) matches the definition of the map $\gamma^{h,-}_{\mathrm{a}_{i,j,k}}$ given in (4.40), that is we have, for $i,j = 1,\dots,k$

$$\mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}_2^k) = \gamma^{h,-}_{\mathrm{a}_{i,j,k}}(x) \text{ for } x = (x_1,x_2,x_3) \text{ for } x \in \mathrm{Q}^3_{\frac{h}{2}}(\mathrm{a}_{i,j,k}), \tag{4.80}$$

yields the value of $\mathbf{G}^k_{\mathbf{ord}}$ on $\mathcal{O}^h_{\mathrm{cross},1}$ thanks to (4.78). This follows combing (4.37), (4.79) together with (4.73) and the corresponding definition of the map we deduce the $\gamma^{h,-}_{\mathrm{a}_{i,j,k}}$.

Step 3: Allowing crossings of fibers thanks to singularities.

Aim and strategy. We define here the value of $\mathbf{G}^k_{\mathbf{ord}}$ on the strip $\Lambda_3^k = \mathbb{R}^3 \times [\mathrm{T}_2^k, \mathrm{T}_2^k + h = \mathrm{T}_3^k]$. Our aim, is to have at time T_3^k

$$\mathfrak{L}^{k,\perp}(\mathrm{T}_3^k) = \mathfrak{L}^{k,\perp} - (\tau_h + h)\vec{\mathbf{e}}_3, \tag{4.81}$$

and to define $\mathbf{G}^k_{\mathbf{ord}}$ accordingly using the operator $\mathbf{P}^{\mathrm{ontya}}_{\varrho}$. Notice that this is already achieved at time T_2^k off the set $\mathcal{O}^h_{\mathrm{cross},1}$ thanks to (4.77), so that we are not going to change the values on this set, that is, we will set, for $s \in [0,h]$ and for $x \in \mathbb{R}^3 \setminus \mathcal{O}^h_{\mathrm{cross},1}$

$$\mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}_2^k + s) = \mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}_2^k) = \mathbf{P}^{\mathrm{ontya}}_{\varrho}\Big[(\mathfrak{L}^{k,\perp} - \mathrm{T}_2^k\vec{\mathbf{e}}_3) \cup \mathfrak{L}^k, \mathfrak{e}^\perp_{\mathrm{ref}}\Big](x). \tag{4.82}$$

[14] as a matter of fact, we have $\{a_{i,j,k}\} = \mathfrak{L}^{k,\perp}_{j,k} \cap (\mathfrak{L}^{k,\perp}_{i,1} - (\tau_h + \frac{5h}{8})\vec{\mathbf{e}}_3)$, so that the collision would have occured at time $\tau_h + \frac{5h}{8}$ is the vector-field would have been $\vec{X}_0$ instead

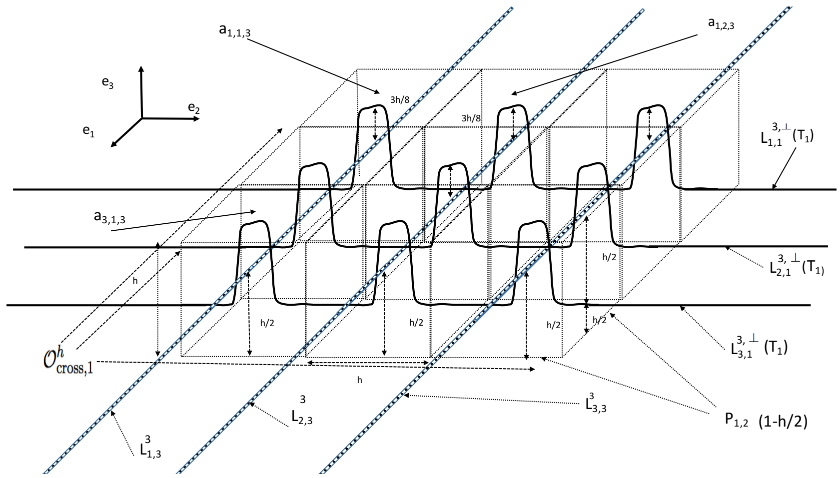

Figure 16: *The curves* $\mathfrak{L}^{k,\perp}_{i,1}(\mathrm{T}^k_2)$ *and* $\mathfrak{L}^k_{j,k}$ *at time* T^k_2*,* $k=3$ *as well as the set* $\mathcal{O}^h_{\mathrm{cross},1}$ *formed of* $k^2=9$ *cubes of edge* $h/2$*.*

It remains hence to defined the value of $\mathbf{G}^k_{\mathbf{ord}}$ on the *space-time crossing region*

$$\Theta^h_{\mathrm{cross},1} = \mathcal{O}^h_{\mathrm{cross},1} \times [\mathrm{T}^k_2, \mathrm{T}^k_2 + h = \mathrm{T}^k_3].$$

This region can be divided in four-dimensional cubes, so that we have

$$\Theta^h_{\mathrm{cross},1} = \bigcup_{i,j=1}^{k} \mathrm{Q}^4_{\frac{h}{2}}(\mathbf{a}^k_{i,j,k,1}), \tag{4.83}$$

where the points $\mathbf{a}^k_{i,j,q,1}$ are defined in (4.63). The intersection of two given cubes in the above collection is either void or included in the intersections of the boundaries. We first fixe the value of $\mathbf{G}^k_{\mathbf{ord}}$ on the boundary of each of the cubes as

$$\mathbf{G}^k_{\mathbf{ord}}(\mathbf{x}) = \Upsilon^h_{\mathbf{a}^k_{i,j,q,1}}(\mathbf{x}) \text{ on } \partial(\mathrm{Q}^4_{\frac{h}{2}}(\mathbf{a}^k_{i,j,k,1})),$$

where the maps $\Upsilon^h_{\mathbf{a}}$ are defined in (4.45). We then extend it inside by *cubic extension*

$$\mathbf{G}^k_{\mathbf{ord}}(\mathbf{x}) = \boxdot_{h,\mathbf{a}^k_{i,j,q,1}}(\mathbf{x}) = \mathfrak{E}\mathrm{xt}_{h/2,\mathbf{a}^k_{i,j,q,1}}\left(\Upsilon_{\mathbf{a}^k_{i,j,q,1}}(\mathbf{x})\right) \text{ for } \mathbf{x} \in \mathrm{Q}^4_{h/2}(\mathbf{a}^k_{i,j,q,1}). \tag{4.84}$$

It follows from this definition that the map $\mathbf{G}^k_{\mathbf{ord}}(\mathbf{x})$ is continuous on $\mathrm{Q}^4_{h/2}(\mathbf{a}^k_{i,j,q,1}) \setminus \{\mathbf{a}^k_{i,j,q,1}\}$, but singular at the point $\mathbf{a}^k_{i,j,q,1}$. At this stage, the map $\mathbf{G}^k_{\mathbf{ord}}$ is defined on the whole strip Λ^k_3 thanks to definitions (4.82) and (4.84). We show next that the two definitions are consistent and that the map is continuous near the boundaries of the cubes.

Continuity properties of $\mathbf{G}^k_{\mathbf{ord}}$ *on* $\Theta^h_{\mathrm{cross},1}$. We consider the set $\mathbb{A}^k_1$ of k^2 space-time singularities induced by definition (4.84) of $\mathbf{G}^k_{\mathbf{ord}}$ on Λ^k_2, namely

$$\mathbb{A}^k_1 = \bigcup_{i,j=1}^{k} \{\mathbf{a}^k_{i,j,k,1}\} = \boxplus^2_k(h) \times \{1\} \times \{\mathrm{T}^k_{3/2}\} \subset \Lambda^k_2, \text{with } \mathrm{T}^k_{3/2} = \frac{\mathrm{T}^k_2 + \mathrm{T}^k_3}{2} = \tau_h + \frac{3h}{2}. \tag{4.85}$$

We claim that:

$$\text{The restriction of } \mathbf{G}^k_{\mathbf{ord}} \text{ to } \Theta^h_{\rm cross,1} \text{ belongs to } C^0(\Theta^h_{\rm cross,1}\setminus \mathbb{A}^k_1, \mathbb{S}^2). \tag{4.86}$$

The only point to check is that the definition (4.84) yield the same value on the parts of the boundary of cubes which meet, that is for i, j, i, j' in $\{1, \dots, k\}$

$$\Upsilon^h_{\mathbf{a}^k_{i,j,q,1}}(x) = \Upsilon^h_{\mathbf{a}^k_{i',j',q,1}}(x) \text{ for } x \in \partial \mathrm{Q}^4_{h/2}(\mathbf{a}^k_{i,j,q,1}) \cap \partial \mathrm{Q}^4_{h/2}(\mathbf{a}^k_{i',j',q,1}), \tag{4.87}$$

the intersection being not empty if and only if $|i-i'| \leq 1$ and $|j-j'| \leq 1$ This is direct consequence of Remark 4.4 and specially identity (4.49) there, which yields for $\mathbf{a} \in \mathbb{A}^k_1$

$$\Upsilon^h_{\mathbf{a}}(x) = \mathbf{P}^{\rm ontya}_{\varrho}\left[(\mathfrak{L}^{k,\perp} - \mathrm{T}^k_2\vec{\mathbf{e}}_3) \cup \mathfrak{L}^k, \mathfrak{e}^\perp_{\rm ref}\right](x) \text{ on } \partial \mathrm{Q}^4_{h/2}(\mathbf{a}). \tag{4.88}$$

This yields (4.87), since the r.h.s of (4.88) does not depend on the choice of point $a \in \mathbb{A}^k_1$, and establishes the claim (4.86).

Continuity properties of $\mathbf{G}^k_{\mathbf{ord}}$ *on* $\Lambda^k_2 \cup \Lambda^k_3$. It follows from (4.88) again that the value given by (4.84) coincides with $\mathbf{P}^{\rm ontya}_{\varrho}\left[(\mathfrak{L}^{k,\perp} - \mathrm{T}^k_2\vec{\mathbf{e}}_3) \cup \mathfrak{L}^k, \mathfrak{e}^\perp_{\rm ref}\right]$ on $\partial\Theta^h_{\rm cross,1} \setminus \mathbb{R}^3 \times \{\mathrm{T}^k_2\}$ and coincides with the value of (4.82) and the definitions are consistent. Hence $\mathbf{G}^k_{\mathbf{ord}}$ is continuous near $\partial\Theta^h_{\rm cross,1} \setminus \mathbb{R}^3 \times \{\mathrm{T}^k_2\}$. To complete the continuity properties, it remains to verify that definition (4.84) and (4.114) are consistent and yield the same result: This is an immediat consequence of the definition of the map $\Upsilon^h_{\mathbf{a}}$.

We have hence established the $\mathbf{G}^k_{\mathbf{ord}}$ belongs to $C^0(\Lambda^k_2 \cup \Lambda^k_3 \setminus \mathbb{A}^k_1)$.

Energy of $\mathbf{G}^k_{\mathbf{ord}}$ *on the strip* Λ^k_3. In view of the decomposition (4.83) of $\Theta_{\rm cross,1}$, we have

$$\begin{aligned} \mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}, \Theta_{\rm cross,1}) &= \sum_{i,j=1}^{k} \mathrm{E}_3\left(\boxdot_{h,\mathbf{a}^k_{i,j,q,1}}, \mathrm{Q}^4_{h/2}(\mathbf{a}^k_{i,j,q,1})\right) \\ &= \sum_{i,j=1}^{k} \mathrm{K}_{\rm box}\, h = k^2 \mathrm{K}_{\rm box}\, h = \mathrm{K}_{\rm box} k. \end{aligned} \tag{4.89}$$

Next, we turn to the complement, i. e. the set $\Lambda^k_3 \setminus \Theta_{\rm cross,1}$. We first notice, that according to formula (4.82) we have $\dfrac{\partial \mathbf{G}^k_{\mathbf{ord}}}{\partial x_4} = 0$ on $\Lambda^k_3 \setminus \Theta_{\rm cross,1}$, so that

$$\begin{aligned} \mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}, \Lambda^k_3 \setminus \Theta_{\rm cross,1}) &= \int_{\mathrm{T}^k_2}^{\mathrm{T}^k_3} \mathrm{E}_3(\mathbf{P}^{\rm ontya}_{\varrho}\left[(\mathfrak{L}^{k,\perp} - \mathrm{T}^k_2\vec{\mathbf{e}}_3) \cup \mathfrak{L}^k, \mathfrak{e}^\perp_{\rm ref}\right], \mathbb{R}^3 \setminus \mathcal{O}_{\rm cross,1}) \\ &\leq h \mathrm{E}_3(\mathbf{S}^k_{\mathbf{pag}}, \mathbb{R}^3) \leq h \mathrm{C}^3_{\rm spg} k^3 = \mathrm{K}_{\rm spg} k^2. \end{aligned} \tag{4.90}$$

and hence we are led to the estimate

$$\int_{\Lambda^k_3} |\nabla \mathbf{G}^k_{\mathbf{ord}}|^3 \leq \mathbf{K}_{\rm spg} k^2 + \mathrm{K}_{\rm box} k. \tag{4.91}$$

The value of $\mathbf{G}^k_{\mathbf{ord}}$ *at time* T^k_3. Combining (4.82) with (4.88), we deduce that for any $x \in \mathbb{R}^3$, we have the identity

$$\begin{aligned} \mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}^k_3) &= \mathbf{P}^{\mathrm{ontya}}_{\varrho}\left[(\mathfrak{L}^{k,\perp} - \mathrm{T}^k_2\vec{\mathbf{e}}_3) \cup \mathfrak{L}^k, \mathfrak{e}^{\perp}_{\mathrm{ref}}\right](x) \\ &= \mathbf{P}^{\mathrm{ontya}}_{\varrho}\left[\mathfrak{L}^{k,\perp} - \mathrm{T}^k_2\vec{\mathbf{e}}_3, \mathfrak{e}^{\perp}_{\mathrm{ref}}\right] \vee_3 \mathbf{P}^{\mathrm{ontya}}_{\varrho}\left[\mathfrak{L}^k, \mathfrak{e}^{\perp}_{\mathrm{ref}}\right](x). \end{aligned} \tag{4.92}$$

We have also

$$\mathfrak{L}^{k,\perp}(\mathrm{T}^k_3) = \mathfrak{L}^{k,\perp} - \mathrm{T}^k_2\vec{\mathbf{e}}_3 \text{ and } \mathfrak{L}^k(\mathrm{T}^k_3) = \mathfrak{L}^k. \tag{4.93}$$

It corresponds hence to a downwards translation of the sheaf $\mathfrak{L}^{k,\perp}$, the absolute value of the total linking number being decreased by k^2.

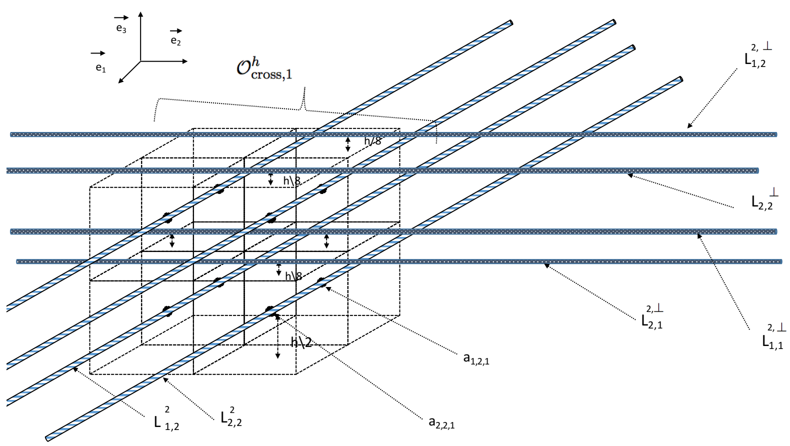

Figure 17: *The shape of the fibers* $\mathfrak{L}^{k,\perp}(\mathrm{T}^k_3)$ *at time* T^k_3 *for* $k = 2$. *The set of fibers* $\mathbf{L}^{k,\perp}_1(\mathrm{T}^k_3)$ *have crossed the upper layer of fibers on* $\mathfrak{L}^k$.

As in Steps 2 and 3, we will next use alternatively and in an iterative way the two previous construction : First pushing along the vector field $\vec{X}^k_1$ to circumvent singularities, and then crossing of the singularities using cubic extensions. However, the number of fibers which cross increases in a first stage of the process. We first show how this works on step 4 and 5, emphasing the few necessary adaptations, and then give the general scheme.

Step 4 : Avoiding the second crossings.

We define here $\mathbf{G}^k_{\mathbf{ord}}$ on the strip $\Lambda^k_4 = \mathbb{R}^3 \times [\mathrm{T}^k_3, \mathrm{T}^k_4]$. Step 4 is similar to step 2, that is we lower the fibers of $\mathcal{L}^{k,\perp}(\mathrm{T}^k_3) \equiv \mathcal{L}^{k,\perp} - \mathrm{T}^k_2\vec{\mathbf{e}}_3$ by a length equal to h far from singularities and circumventing the singularities $a^k_{i,j,q}$ which are on the way. These singularities are now twice as much as in Step 2, that is the $2k^2$ elements of set of points

$$\mathbf{A}^k_2 \equiv \bigcup_{i,j=1}^{k} \{\mathrm{a}_{i,j,k}, \mathrm{a}_{i,j,k-1}\} = \boxplus^2_k(h) \times \{1, 1-h\} \supset \mathbf{A}^k_1. \tag{4.94}$$

The main difference with Step 2 is that here we have *two layers* of fibers which are concerned by bypassing the singularities. We replace therefore the sets $\mathbf{L}_1^{k,\perp}$ and $\mathbf{N}_1^{k,\perp}$ defined in (4.70) and (4.71) respectively by the sets

$$\begin{cases} \mathbf{L}_2^{k,\perp}(\mathrm{T}_3^k) = \bigcup_{i=1}^{k} \Big(\mathfrak{L}_{i,1}^{k,\perp}(\mathrm{T}_3^k) \cup \mathfrak{L}_{i,2}^{k,\perp}(\mathrm{T}_3^k)\Big) \\ \mathbf{N}_2^{k,\perp}(\mathrm{T}_3^k) = \mathfrak{L}^{k,\perp}(\mathrm{T}_3^k) \setminus \mathbf{L}_2^{k,\perp}(\mathrm{T}_3^k) = (\mathfrak{L}^{k,\perp} - \mathrm{T}_2^k\vec{\mathbf{e}}_3) \setminus \mathbf{L}_2^{k,\perp}(\mathrm{T}_3^k) \end{cases}$$

and define $\mathbf{G}^k_{\mathbf{ord}}$ on the strip $\Lambda_4^k = \mathbb{R}^3 \times [\mathrm{T}_3^k, \mathrm{T}_4^k]$ in a way similar to (4.72), that is, for $s \in [0, h]$, we set

$$\mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}_3^k + s) = \mathbf{P}_\varrho^{\mathrm{ontya}}\Big[\mathbf{N}_2^{k,\perp}(\mathrm{T}_3^k + s)\Big] \vee_3 \tilde{\mathbf{P}}_\varrho^{\mathrm{ontya}}\Big[\mathbf{L}_2^{k,\perp}(\mathrm{T}_3^k + s)\Big] \vee_3 \mathbf{P}_\varrho^{\mathrm{ontya}}\Big[\mathfrak{L}^k\Big], \quad (4.95)$$

where we have set

$$\begin{cases} \mathbf{N}_2^{k,\perp}(\mathrm{T}_3^k + s) \equiv \mathbf{N}_2^{k,\perp}(\mathrm{T}_3^k) - s\vec{\mathbf{e}}_3 \\ \mathbf{L}_2^{k,\perp}(\mathrm{T}_3^k + s) \equiv \mathcal{D}_{\mathrm{ef}}\mathbf{L}_2^{k,\perp}(\mathrm{T}_3^k, s) = \bigcup_{i=1}^{k} \Big(\mathcal{D}_{\mathrm{ef}}\mathfrak{L}_{i,1}^{k,\perp}(\mathrm{T}_3^k, s) \cup \mathcal{D}_{\mathrm{ef}}\mathfrak{L}_{i,2}^{k,\perp}(\mathrm{T}_3^k, s)\Big), \end{cases}$$

so that the maps involved in the definition (4.95) have disjoint supports. It follows in view of definition (4.95) and invoking, as in Step 1, the gradient estimate for $\mathcal{D}_{\mathrm{ef}}\mathfrak{L}_{i,q}^{k,\perp}$ provided in Lemma 4.3 that is $|\nabla_4 \mathbf{G}^k_{\mathbf{ord}}| \leq \mathrm{C}_{\mathrm{def}} k$ on Λ_4^k, so that we are led, by integration, to the estimate

$$\int_{\Lambda_4^k} |\nabla \mathbf{G}^k_{\mathbf{ord}}|^3 \leq \mathrm{K}_{\mathrm{def}} k^2. \quad (4.96)$$

The shape of the fibers and the value of $\mathbf{G}^k_{\mathbf{ord}}$ *at time* T_4^k. At time T_4^k, all fibers of $\mathfrak{L}^{k,\perp}$ have been translated by$-(\tau_h + 2h)\vec{\mathbf{e}}_3$, except the files in $\mathbf{L}_2^k(\mathrm{T}_2^k)$ which have been rounded near the points in $\mathbf{A}_2$, in order to avoid collision with the fibers $\mathfrak{L}_{j,k}^k$ or $\mathfrak{L}_{j,k-1}^k$, which they would otherwise have crossed. We introduce the second *spatial crossing region* $\mathcal{O}^h_{\mathrm{cross},2}$ defined by

$$\mathcal{O}^h_{\mathrm{cross},2} = [\frac{h}{2}, 1 + \frac{h}{2}]^2 \times [1 - \frac{3h}{2}, 1 + \frac{h}{2}] \supset \mathcal{O}^h_{\mathrm{cross},1}$$

and deduce from the inclusion (4.27) in Remark 4.2, arguing as in Step 3, that

$$\mathfrak{L}^{k,\perp}(\mathrm{T}_4^k) \setminus \mathcal{O}^h_{\mathrm{cross},2} = \Big(\mathfrak{L}^{k,\perp} - (\tau_h + 2h)\vec{\mathbf{e}}_3\Big) \setminus \mathcal{O}^h_{\mathrm{cross},2}. \quad (4.97)$$

We decompose $\mathcal{O}^h_{\mathrm{cross},2}$ into cubes of edge of size h centered at the collisions points in $\mathbf{A}_2^k$ as

$$\mathcal{O}^h_{\mathrm{cross},2} = \bigcup_{a \in \mathbf{A}_2^k} \mathrm{Q}^3_{h/2}(a)$$

where the cubes may possibly touch only on their boundaries. We have, in view of (4.95), for any $a \in \mathbf{A}_2^k$, any $s \in [0, h]$, $q = 1, 2$ and any $i \in \{1, \ldots, k\}$

$$\mathfrak{L}_{i,q}^{k,\perp}(\mathrm{T}_3^k + s) = \Phi_1^k(\mathfrak{L}_{i,q}^{k,\perp} - \mathrm{T}_2^k\vec{\mathbf{e}}_3, s) = \mathcal{D}_{\mathrm{ef}}\mathfrak{L}_{i,q}^{k,\perp}(\mathrm{T}_1^k, s) \text{ on } \mathrm{Q}^3_{\frac{h}{2}}(a).$$

Going back to Remark 4.3, we notice that (4.39) is fullfilled with $c = \tau_h + 2h = \mathrm{T}_2^k$, $p = 1$ or $p = 2$ and $q = 1$ or $q = 2$ respectively, so that (4.38) yields

$$\begin{cases} \mathcal{D}_{\rm ef}\mathfrak{L}_{i,1}^{k,\perp}(\mathrm{T}_3^k, h) \cap \mathrm{Q}_{h/2}^3(\mathrm{a}_{i,j,k-1}) = \mathcal{C}_{\perp,h}^-(\mathrm{a}_{i,j,k-1}) \text{ and} \\ \mathcal{D}_{\rm ef}\mathfrak{L}_{i,2}^{k,\perp}(\mathrm{T}_3^k, h) \cap \mathrm{Q}_{h/2}^3(\mathrm{a}_{i,j,k}) = \mathcal{C}_{\perp,h}^-(\mathrm{a}_{i,j,k}), \end{cases}$$

and hence

$$\begin{cases} \mathfrak{L}_{i,1}^{k,\perp}(\mathrm{T}_4^k) \cap \mathrm{Q}_{\frac{h}{2}}^3(\mathrm{a}_{i,j,k-1}) = \mathcal{C}_{\perp,h}^-(a_{i,j,k-1}) \\ \mathfrak{L}_{i,2}^{k,\perp}(\mathrm{T}_4^k) \cap \mathrm{Q}_{\frac{h}{2}}^3(\mathrm{a}_{i,j,k}) = \mathcal{C}_{\perp,h}^-(a_{i,j,k}). \end{cases} \tag{4.98}$$

The value of $\mathbf{G}_{\mathbf{ord}}^k$ provided by (4.95) matches the definition of the map $\gamma_{\mathrm{a}_{i,j,k}}^{h,-}$ given in (4.40), that is we have, for $i, j = 1, \dots, k$ and $a \in \mathbf{A}_2^k$

$$\mathbf{G}_{\mathbf{ord}}^k(x, \mathrm{T}_4^k) = \gamma_a^{h,-}(x) \text{ for } x = (x_1, x_2, x_3) \in \mathrm{Q}_{\frac{h}{2}}^3(a), \tag{4.99}$$

Step 5: crossing once more through singularities.

This step is parallel to Step 3, our aim being to define the value of $\mathbf{G}_{\mathbf{ord}}^k$ on the strip $\Lambda_5^k = \mathbb{R}^3 \times [\mathrm{T}_4^k, \mathrm{T}_4^k + h = \mathrm{T}_5^k]$, so that at time T_5^k, we have $\mathfrak{L}^{k,\perp}(\mathrm{T}_5^k) = \mathfrak{L}^{k,\perp} - (\tau_h + 2h)\vec{\mathbf{e}}_3$, defining $\mathbf{G}_{\mathbf{ord}}^k$ accordingly using the operator $\mathbf{P}_\varrho^{\rm ontya}$. This is already achieved at time T_4^k off the set $\mathcal{O}_{\rm cross,2}^h$ thanks to (4.77), so that we are not going to change the values on this set. We defined next the value of $\mathbf{G}_{\mathbf{ord}}^k$ on the space-time crossing region $\Theta_{\rm cross,2}^h = \mathcal{O}_{\rm cross,2} \times [\mathrm{T}_4^k, \mathrm{T}_5^k]$ which can be divided in four-dimensional cubes of size h

$$\Theta_{\rm cross,2}^h = \overset{k}{\underset{i,j=1}{\cup}} \left(\mathrm{Q}_{\frac{h}{2}}^4(\mathbf{a}_{i,j,k,2}^k) \cup \mathrm{Q}_{\frac{h}{2}}^4(\mathbf{a}_{i,j,k-1,2}^k) \right) = \underset{\mathbf{a} \in \mathbb{A}_2^k}{\cup} \mathrm{Q}_{\frac{h}{2}}^4(\mathbf{a}), \tag{4.100}$$

where

$$\mathbb{A}_2^k = \overset{k}{\underset{i,j=1}{\cup}} \{\mathbf{a}_{i,j,k,2}^k, \mathbf{a}_{i,j,k-1,2}^k\} = \mathbf{A}_2^k \times \{\mathrm{T}_{9/2}^k\}, \text{ with } \mathrm{T}_{9/2}^k = \frac{\mathrm{T}_4^k + \mathrm{T}_5^k}{2},$$

the points $\mathbf{a}_{i,j,q,r}^k$ being defined in (4.63), the intersection of two given cubes in the above collection being either void or included in the intersections of the boundaries. We fix the value of $\mathbf{G}_{\mathbf{ord}}^k$ on the boundary of each of the cubes as

$$\mathbf{G}_{\mathbf{ord}}^k(\mathbf{x}) = \Upsilon_{\mathbf{a}}^h(\mathbf{x}) \text{ on } \partial(\mathrm{Q}_{\frac{h}{2}}^4(\mathbf{a})) \text{ for } \mathbf{x} \in \partial \mathrm{Q}_{h/2}^4(\mathbf{a}), \mathbf{a} \in \mathbb{A}_2^k,$$

where the maps $\Upsilon_{\mathbf{a}}^h$ are defined in (4.45). We then extend it inside by *cubic extension*

$$\mathbf{G}_{\mathbf{ord}}^k(\mathbf{x}) = \boxdot_{h,\mathbf{a}}(\mathbf{x}) = \mathfrak{Ext}_{h/2,\mathbf{a}}(\Upsilon_{\mathbf{a}}(\mathbf{x})) \text{ for } \mathbf{x} \in \mathrm{Q}_{h/2}^4(\mathbf{a}), \mathbf{a} \in \mathbb{A}_2^k. \tag{4.101}$$

It follows that $\mathbf{G}_{\mathbf{ord}}^k$ is continuous on $\mathrm{Q}_{h/2}^4(\mathbf{a}) \setminus \{\mathbf{a}\}$, $\mathbf{a} \in \mathbb{A}_2^k$, but singular at the point $\mathbf{a}$. The map $\mathbf{G}_{\mathbf{ord}}^k$ is now defined on the whole strip Λ_5^k thanks to definitions (4.82) and (4.101). As in Step 3, one may show that the definitions are consistent and that the restriction of $\mathbf{G}_{\mathbf{ord}}^k$ to $\Lambda_4^k \cup \Lambda_5^k$ is a map in $C^0(\Lambda_4^k \cup \Lambda_5^k \setminus \mathbb{A}_2^k, \mathbb{S}^2)$, each of the singularities in $\mathbb{A}_2^k$ having Hopf invariant equal to $+2$.

The value of $\mathbf{G}^k_{\mathbf{ord}}$ *at time* T^k_5. We verify, as in Step 3, that

$$\mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}^k_5) = \mathbf{P}^{\mathrm{ontya}}_{\varrho}\left(\left(\mathcal{L}^{k,\perp} - (\tau - h + 2h\vec{\mathbf{e}}_3)\right) \cup \mathcal{L}^k\right)(x) \text{ for } x \in \mathbb{R}3. \tag{4.102}$$

and that

$$\mathfrak{L}^{k,\perp}(\mathrm{T}^k_5) = \mathfrak{L}^{k,\perp} - (\tau_h + 2h)\vec{\mathbf{e}}_3 \text{ and } \mathfrak{L}^k(\mathrm{T}^k_5) = \mathfrak{L}^k.$$

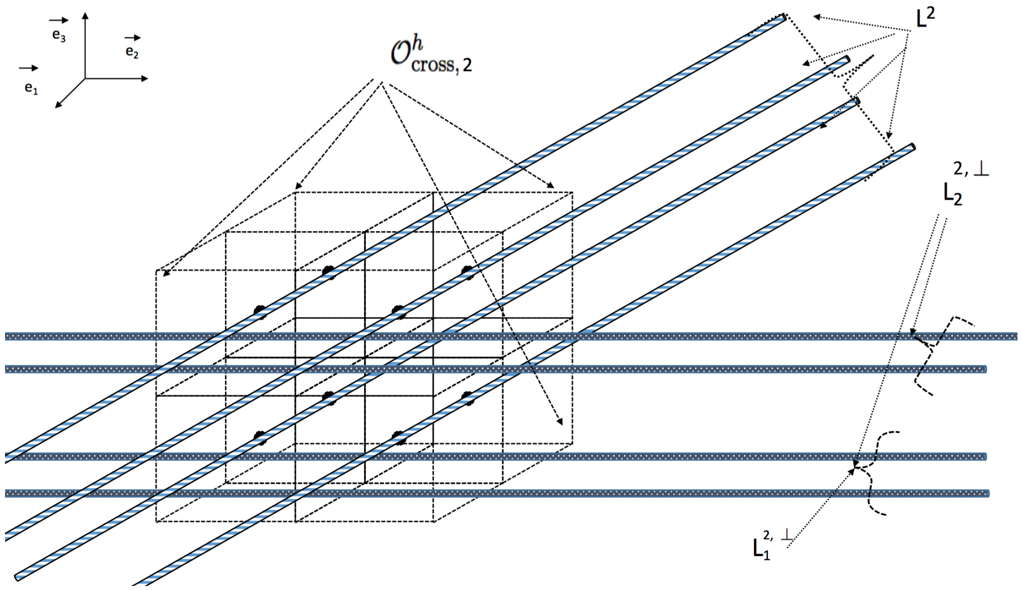

Figure 18: *The shape of the fibers* $\mathfrak{L}^{k,\perp}(\mathrm{T}^k_5)$ *at time* T^k_5 *for* $k = 2$. *The set of fibers* $\mathbf{L}^{k,\perp}_2(\mathrm{T}^k_5)$ *have crossed the upper layer of fibers on* $\mathfrak{L}^k$.

The energy of $\mathbf{G}^k_{\mathbf{ord}}$ *on the strip* Λ^k_5. In order to estimate the energy on the strip Λ^k_5 we argue as in Step 3. In view of (4.115), we have

$$\mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}, \Theta^h_{\mathrm{cross},2}) = \sum_{\mathbf{a}\in\mathbb{A}^k_2} \mathrm{E}_3\left(\boxdot_{h,\mathbf{a}}, \mathrm{Q}^4_{h/2}(\mathbf{a})\right) = \sharp(\mathbb{A}^k_2)\mathrm{K}_{\mathrm{box}}\, h = 2k^2\mathrm{K}_{\mathrm{box}}\, h = 2\mathrm{K}_{\mathrm{box}}k. \tag{4.103}$$

For the complement $\Lambda^k_5 \setminus \Theta^h_{\mathrm{cross},2}$ we have

$$\begin{aligned}\mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}, \Lambda^k_5 \setminus \Theta^h_{\mathrm{cross},2}) = &\int_{\mathrm{T}^k_2}^{\mathrm{T}^k_3} \mathrm{E}_3(\mathbf{P}^{\mathrm{ontya}}_{\varrho}\left[(\mathfrak{L}^{k,\perp} - \mathrm{T}^k_2\vec{\mathbf{e}}_3) \cup \mathfrak{L}^k, \mathfrak{e}^{\perp}_{\mathrm{ref}}\right], \mathbb{R}^3 \setminus \mathcal{O}^h_{\mathrm{cross},2}) \\ &\le h\mathrm{E}_3(\mathbf{S}^k_{\mathbf{pag}}, \mathbb{R}^3) \le h\mathrm{C}_{\mathrm{spg}}k^3 = \mathrm{K}_{\mathrm{spg}}k^2.\end{aligned} \tag{4.104}$$

Hence, we are led to the estimate

$$\int_{\Lambda^k_5} |\nabla_4 \mathbf{G}^k_{\mathbf{ord}}|^3 \le \mathbf{K}_{\mathrm{spg}}k^2 + 2\mathbf{K}_{\mathrm{box}}k. \tag{4.105}$$

We proceed now *iteratively* using the same constructions (namely pushing along the flow $\vec{X}_1^k$ and then using cubic extensions) up to step $4k+1$, after which the fibers will be no longer linked. As seen on the previous steps, we distinguish *even* and *odd* steps, in each of the two classes the construction follows the same pattern. Each *pair of even and odd steps* will be labelled by an integer ℓ running from 1 to $2k$. Similar to (4.94), we are led to consider for $\mathfrak{p} \in \{1, \dots, \ell\}$ the sets of "collisions points" of the fibers:

$$\begin{aligned} \mathbf{A}_{\mathfrak{p}}^k &\equiv \bigcup_{i,j=1}^{k} \bigcup_{q=k-\mathfrak{p}+1}^{k} \{\mathrm{a}_{i,j,q}^k\} = \boxplus_k^2(h) \times \{1, 1-h, \dots, 1-(\mathfrak{p}-1)\,h\} \\ &= \boxplus_k^2(h) \times \mathbb{J}_{\mathfrak{p}}^k \text{ where } \mathbb{J}_{\mathfrak{p}}^k \equiv \{1, 1-h, \dots, 1-(\mathfrak{p}-1)\,h\} \end{aligned} \tag{4.106}$$

so that $\mathbf{A}_{\mathfrak{p}}$ contains $\mathfrak{p}k^2$ elements and $\mathbf{A}_{\mathfrak{p}}^k \subset \mathbf{A}_{\mathfrak{p}+1}^k$ if $\mathfrak{p} \leq k-1$. We also generalize the definitions of $\mathbf{L}_1^{k,\perp}(s)$ and $\mathbf{N}_1^{k,\perp}(s)$ as follows:

$$\begin{cases} \mathbf{L}_{\mathfrak{p}}(s)^{k,\perp} = \bigcup_{i=1}^{k} \bigcup_{q=1}^{\mathfrak{p}} \mathfrak{L}_{i,q}^{k,\perp}(s) \\ \mathbf{N}_{\mathfrak{p}}(s)^{k,\perp} = \bigcup_{i=1}^{k} \bigcup_{q=\mathfrak{p}+1}^{k} \mathfrak{L}_{i,q}^{k,\perp}(s), \end{cases} \tag{4.107}$$

so that $\mathbf{L}_{\mathfrak{p}}(s)^{k,\perp} \cup \mathbf{N}_{\mathfrak{p}}(s)^{k,\perp} = \mathfrak{L}^{k,\perp}(s)$ and $\mathbf{L}_{\mathfrak{p}}(s)^{k,\perp} \cap \mathbf{N}_{\mathfrak{p}}(s)^{k,\perp} = \emptyset$.

We describe next more precisely the pattern of these steps, dividing the presentation into two periods.

Step 6 to Step $2k+1$.

Assume that at step $2\ell+1$, for $\ell \in \{1, \dots, k-1\}$, the map $\mathbf{G}_{\mathbf{ord}}^k$ has been constructed on $\mathbb{R}^3 \times [0, \mathrm{T}_{2\ell+1}^k]$ and satisfies for $x_4 = \mathrm{T}_{2\ell+1}^k$ we have

$$\mathbf{G}_{\mathbf{ord}}^k(x_1, x_2, x_3, \mathrm{T}_{2\ell+1}^k) = \mathbf{P}_{\varrho}^{\mathrm{ontya}}\left[\left(\mathfrak{L}^{k,\perp} - (\tau_h + \ell\,h)\vec{\mathbf{e}}_3\right) \cup \mathcal{L}^k\right](x_1, x_2, x_3). \tag{4.108}$$

This is indeed the case for $\ell = 1$ and $\ell = 2$, as seen in Step 3 and Step 5. We have hence

$$\mathfrak{L}^{k,\perp}(\mathrm{T}_{2\ell+1}^k) \equiv \mathcal{L}^{k,\perp} - (\tau_h + \ell\,h)\vec{\mathbf{e}}_3 \text{ and } \mathfrak{L}^k(\mathrm{T}_{2\ell+1}^k) = \mathfrak{L}^k.$$

In particular, the lowest fiber (according to the x_3 variable) in $\mathfrak{L}^{k,\perp}(\mathrm{T}_{2\ell+1}^k)$, that it the set $\mathbf{L}_1^{k,\perp}(\mathrm{T}_{2\ell+1}^k)$ has crossed the ℓ upper fibers of $\mathfrak{L}^k$, that is the fibers $\mathfrak{L}_{i,q}^k$ with $q \in \{k-\ell+1, k\}$. We have moreover

$$\mathrm{dist}(\mathbf{L}_1^{k,\perp}(\mathrm{T}_{2\ell}^k), \mathfrak{L}_{j,k-\ell}^k) = \frac{5h}{8},$$

$\mathbf{L}_1^{k,\perp}(\mathrm{T}_{2\ell+1}^k)$ being above $\mathfrak{L}_{j,k-\ell}^k$, according to the x_3 coordinate. Moreover, all fibers in $\mathbf{L}_\ell(\mathrm{T}_{2\ell+1}^k)$ have crossed at least one fiber of $\mathfrak{L}^k$, whereas none in $\mathbf{N}_\ell(\mathrm{T}_{2\ell+1}^k)$ has done it. At step $2\ell+3$, we wish to have

$$\mathfrak{L}^{k,\perp}(\mathrm{T}_{2\ell+3}^k) \equiv \mathcal{L}^{k,\perp} - (\tau_h + (\ell+1)\,h)\vec{\mathbf{e}}_3 \text{ and } \mathfrak{L}^k(\mathrm{T}_{2\ell+1}^k) = \mathfrak{L}^k, \tag{4.109}$$

For this purpose, we proceed exactly as seen in the previous steps.

Defining $\mathbf{G}_{\mathbf{ord}}^k$ on the strip $\Lambda_{2\ell+2}^k$. At step $2\ell+2$, we construct the map $\mathbf{G}_{\mathbf{ord}}^k$ on the strip $\Lambda_{2\ell+2}^k = \mathbb{R}^3 \times [\mathrm{T}_{2\ell+1}^k, \mathrm{T}_{2\ell+2}^k]$ lowering the fibers of $\mathcal{L}^{k,\perp}(\mathrm{T}_{2\ell+1}^k)$ by a length equal to

h circumventing the singularities $\mathrm{a} \in \mathbf{A}^k_{\ell+1}$ which are on the way. Similar to (4.73) and (4.95) we are led to define $\mathbf{G}^k_{\mathbf{ord}}$ on the strip $\Lambda^k_{2\ell+2}$ for $s \in [0,h]$ by the formula

$$\mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}^k_{2\ell+1}+s) = \mathbf{P}^{\mathrm{ontya}}_{\varrho}\left[\mathbf{N}^{k,\perp}_{\ell+1}(\mathrm{T}^k_{2\ell+1}+s)\right] \vee_3 \tilde{\mathbf{P}}^{\mathrm{ontya}}_{\varrho}\left[\mathbf{L}^{k,\perp}_{\ell+1}(\mathrm{T}^k_{2\ell+1}+s)\right] \vee_3 \mathbf{P}^{\mathrm{ontya}}_{\varrho}[\mathfrak{L}^k], \tag{4.110}$$

where we have set

$$\begin{cases} \mathbf{N}^{k,\perp}_{\ell+1}(\mathrm{T}^k_{2\ell+1}+s) \equiv \mathbf{N}^{k,\perp}_{\ell} - (\tau_h + \ell\, h + s)\vec{\mathbf{e}}_3 \\ \mathbf{L}^{k,\perp}_{\ell+1}(\mathrm{T}^k_{2\ell+1}+s) = \mathcal{D}_{\mathrm{ef}}\mathbf{L}^{k,\perp}_{\ell+1}(\mathrm{T}^k_{2\ell+1}, s) \equiv \overset{k}{\underset{i=1}{\cup}} \overset{\ell+1}{\underset{q=1}{\cup}} \mathcal{D}_{\mathrm{ef}}\mathfrak{L}^{k,\perp}_{i,q}(\mathrm{T}^k_{2\ell+1}, s), \end{cases}$$

It follows from the definition (4.110) of $\mathbf{G}^k_{\mathbf{ord}}$ on the strip $\Lambda^k_{2\ell+2} \equiv \mathbb{R}^3 \times [\mathrm{T}^k_{2\ell+1}, \mathrm{T}^k_{2\ell+2}]$ that $\mathbf{G}^k_{\mathbf{ord}}$ is Lipschitz on $\Lambda^k_{2\ell+2}$ and that $|\nabla_4 \mathbf{G}^k_{\mathbf{ord}}| \leq \mathrm{C}_{\mathrm{def}} k$ on $\Lambda^k_{2\ell+2}$, so that by integration on the support of $\mathbf{G}^k_{\mathbf{ord}}$ we have

$$\int_{\Lambda^k_{2\ell+2}} |\nabla_4 \mathbf{G}^k_{\mathbf{ord}}|^3 \leq \mathrm{K}_{\mathrm{def}} k^2. \tag{4.111}$$

The value of $\mathbf{G}^k_{\mathbf{ord}}$ *at time* $\mathrm{T}^k_{2\ell+2}$. At time $\mathrm{T}^k_{2\ell+2}$, all fibers of $\mathfrak{L}^{k,\perp}$ have been translated by$-(\tau_h + (\ell+1)h)\vec{\mathbf{e}}_3$, except the files in $\mathbf{L}^k_{\ell+1}(\mathrm{T}^k_2)$ which are rounded near the points in $\mathbf{A}_{\ell+1}$, in order to avoid collision with the fibers $\mathfrak{L}^k_{j,q}$ for $q = k, \dots, k-\ell$. We introduce the spatial crossing region of order $\ell+1$ defined by

$$\mathcal{O}^h_{\mathrm{cross},\ell+1} = [\frac{h}{2}, 1+\frac{h}{2}]^2 \times [1-(\ell+\frac{1}{2})h, 1+\frac{h}{2}] \supset \mathcal{O}^h_{\mathrm{cross},\ell}$$

and deduce from the inclusion (4.27) of Remark 4.2 and arguing as in Step 3, that

$$\mathfrak{L}^{k,\perp}(\mathrm{T}^k_{2\ell+2}) \setminus \mathcal{O}^h_{\mathrm{cross},\ell+1} = \Big(\mathfrak{L}^{k,\perp} - (\tau_h + (\ell+1)h)\vec{\mathbf{e}}_3\Big) \setminus \mathcal{O}^h_{\mathrm{cross},\ell+1}. \tag{4.112}$$

We decompose $\mathcal{O}^h_{\mathrm{cross},\ell+1}$ into cubes of edge of size h centered at the points in $\mathbf{A}^k_{\ell+1}$ as

$$\mathcal{O}^h_{\mathrm{cross},\ell+1} = \underset{a \in \mathbf{A}^k_{\ell+1}}{\cup} \mathrm{Q}^3_{h/2}(a)$$

where the cubes may possibly touch only on their boundaries. We have, in view of (4.95), for any $a \in \mathbf{A}^k_{\ell+2}$, any $s \in [0,h]$, $q = 1, \dots, \ell+1$ and any $i \in \{1, \dots, k\}$

$$\mathfrak{L}^{k,\perp}_{i,q}(\mathrm{T}^k_{2\ell+1}+s) = \Phi^k_1(\mathfrak{L}^{k,\perp}_{i,q} - (\tau_h + \ell + s)\vec{\mathbf{e}}_3, s) = \mathcal{D}_{\mathrm{ef}}\mathfrak{L}^{k,\perp}_{i,q}(\mathrm{T}^k_{2\ell+1}, s) \text{ on } \mathrm{Q}^3_{\frac{h}{2}}(a).$$

Given $q \in \{1, \ell+1\}$ and going back to Remark 4.3, we notice that (4.39) is fullfilled for $c = \tau_h + (\ell+1)h$, and $p = q$, so that (4.38) yields

$$\mathcal{D}_{\mathrm{ef}}\mathfrak{L}^{k,\perp}_{i,q}(\mathrm{T}^k_{2\ell+1}, h) \cap \mathrm{Q}^3_{h/2}(a_{i,j,k-\ell-1+q}) = \mathcal{C}^-_{\perp,h}(\mathrm{a}_{i,j,k-\ell-1+q}).$$

Hence, we have

$$\mathfrak{L}^{k,\perp}_{i,q}(\mathrm{T}^k_{2\ell+2}) \cap \mathrm{Q}^3_{\frac{h}{2}}(\mathrm{a}_{i,j,k-\ell-1+q}) = \mathcal{C}^-_{\perp,h}(a_{i,j,k-\ell-1+q}) \tag{4.113}$$

It follows that the value of $\mathbf{G}^k_{\mathbf{ord}}$ provided by (4.110) satisfies for $i, j = 1, \dots, k$ and $a \in \mathbf{A}^k_{\ell+1}$

$$\mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}^k_{2\ell+2}) = \gamma^{h,-}_a(x) \text{ for } x \in \mathrm{Q}^3_{\frac{h}{2}}(a), \tag{4.114}$$

Defining $\mathbf{G}^k_{\mathbf{ord}}$ *on* $\Lambda^k_{2\ell+3}$. This step, Step $2\ell+3$, is parallel to Step 3 and 5. We consider the space-time crossing region $\Theta^h_{\mathrm{cross},\ell+1} = \mathcal{O}_{\mathrm{cross},\ell+1} \times [\mathrm{T}^k_{2\ell+2}, \mathrm{T}^k_{2\ell+3}]$ which can be decomposed as

$$\Theta^h_{\mathrm{cross},\ell+1} = \bigcup_{i,j=1}^{k} \bigcup_{q=k-\ell}^{k} \mathrm{Q}^4_{\frac{h}{2}}(\mathbf{a}^k_{i,j,q,\ell+1}) = \bigcup_{\mathbf{a}\in\mathbb{A}^k_{\ell+1}} \mathrm{Q}^4_{\frac{h}{2}}(\mathbf{a}), \tag{4.115}$$

where

$$\begin{aligned} \mathbb{A}^k_{\ell+1} &= \bigcup_{i,j=1}^{k} \bigcup_{q=k-\ell}^{k} \{\mathbf{a}_{i,j,q,\ell+1}\} = \mathbf{A}^k_{\ell+1} \times \{\mathrm{T}^k_{2\ell+\frac{5}{2}}\}, \\ &= \boxplus^2_k(h) \times \mathbb{J}^k_{\ell+1} \times \{\mathrm{T}^k_{2\ell+\frac{5}{2}}\}. \end{aligned} \tag{4.116}$$

We fix the value of $\mathbf{G}^k_{\mathbf{ord}}$ on the boundary of each of the cubes as

$$\mathbf{G}^k_{\mathbf{ord}}(\mathbf{x}) = \Upsilon^h_{\mathbf{a}}(\mathbf{x}) \text{ on } \partial(\mathrm{Q}^4_{\frac{h}{2}}(\mathbf{a})) \text{ for } \mathbf{x} \in \partial\mathrm{Q}^4_{h/2}(\mathbf{a}), \mathbf{a} \in \mathbb{A}^k_{\ell+1},$$

and extend it inside by *cubic extension*

$$\mathbf{G}^k_{\mathbf{ord}}(\mathbf{x}) = \boxdot_{h,\mathbf{a}}(\mathbf{x}) = \mathfrak{E}\mathrm{xt}_{h/2,\mathbf{a}}\left(\Upsilon_{\mathbf{a}}(\mathbf{x})\right) \text{ for } \mathbf{x} \in \mathrm{Q}^4_{h/2}(\mathbf{a}), \mathbf{a} \in \mathbb{A}^k_{\ell+1}. \tag{4.117}$$

It follows $\mathbf{G}^k_{\mathbf{ord}}$ is continuous on $\mathrm{Q}^4_{h/2}(\mathbf{a}) \setminus \{\mathbf{a}\}$, $\mathbf{a} \in \mathbb{A}^k_{\ell+1}$, but singular at the point $\mathbf{a}$. The map $\mathbf{G}^k_{\mathbf{ord}}$ is now defined on the whole strip $\Lambda^k_{2\ell+2}$ thanks to definitions (4.110) and (4.117). As in Step 3, one may show that the definitions are consistent and that the restriction of $\mathbf{G}^k_{\mathbf{ord}}$ to $\Lambda^k_{2\ell+1} \cup \Lambda^k_{2\ell+2}$ is a map in $C^0(\Lambda^k_{2\ell+1} \cup \Lambda^k_{2\ell+2} \setminus \mathbb{A}^k_{\ell+1}, \mathbb{S}^2)$, each of the $(\ell+1)k^2$ singularities in $\mathbb{A}^k_{\ell+1}$ having Hopf invariant equal to $+2$.

The value of $\mathbf{G}^k_{\mathbf{ord}}$ *at time* $\mathrm{T}^k_{2\ell+3}$. We verify, as in Step 3, that (4.109) holds and that

$$\mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}^k_{2\ell+3}) = \mathbf{P}^{\mathrm{ontya}}_{\varrho}\left[\left(\mathcal{L}^{k,\perp} - (\tau_h + (\ell+1)h\vec{\mathbf{e}}_3)\right) \cup \mathcal{L}^k\right](x) \text{ for } x \in \mathbb{R}^3. \tag{4.118}$$

The energy of $\mathbf{G}^k_{\mathbf{ord}}$ *on the strip* $\Lambda^k_{2\ell+3}$. Arguing as in Step 3 and Step 5, we have

$$\begin{aligned} \mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}, \Theta_{\mathrm{cross},\ell+1}) &= \sum_{\mathbf{a}\in\mathbb{A}^k_{\ell+1}} \mathrm{E}_3\left(\boxdot_{h,\mathbf{a}}, \mathrm{Q}^4_{h/2}(\mathbf{a})\right) = \sharp(\mathbb{A}^k_{\ell+1})\mathrm{K}_{\mathrm{box}}\, h \\ &= \mathrm{K}_{\mathrm{box}} k^2(\ell+1)\, h = k^2 \mathrm{K}_{\mathrm{box}}\, h = \mathrm{K}_{\mathrm{box}}(\ell+1)k. \end{aligned}$$

The energy on the complement is computed as in Step 3 and Step 5, we that we finally obtain

$$\int_{\Lambda^k_{2\ell+3}} |\nabla_4 \mathbf{G}^k_{\mathbf{ord}}|^3 \leq \mathbf{K}_{\mathrm{spg}} k^2 + (\ell+1)\mathbf{K}_{\mathrm{box}} k. \tag{4.119}$$

The construction described above can go on as long as the fibers in $\mathbf{L}^k_1$ have to cross some layers of fibers in $\mathfrak{L}^k$: Consequently, the process stops at step $2k+1$, when $\mathbf{L}^k_1$ is now able to move down freely, since it it no longer linked to $\mathfrak{L}^k$. In order to complete the untying of the

Spaghetton map, and to " set free" the remaining fibers, we show next how to adapt slightly the description, in particular the definition (4.107).

Step $2k+2$ to Step $4k-1$.
The constructions follows the same patterns, the only difference being that at each additional pair of steps fibers are now leaving the crossing region, inducing modifications in the description of the labels and indices for the various sets which have been introduced before. In particular, the index ℓ runs from now from here from $\ell = k+1$ to $2k-1$.

As before, we assume that we are given some $\ell \in \{k, \dots, 2k-2\}$ and we assume that $\mathbf{G}^k_{\mathbf{ord}}$ has been constructed on $[0, \mathrm{T}^k_{2\ell+1}]$, and satisfies at time $\mathrm{T}^k_{2\ell+1}$ the identity (4.110): This is for instance the case for $\ell = k$, as seen before. The corresponding formulae for $\mathfrak{L}^{k,\perp}(2\ell+1)$ are hence also valid, so that the highest (according to the x_3 coordinate) fibers in $\mathfrak{L}^{k,\perp}(2\ell+1)$, that is the fibers $\mathfrak{L}^{k,\perp}_{i,k}(2\ell+1)$, $i = 1, \dots, k$ are now squeezed between the fibers $\mathfrak{L}^k_{j,2k-\ell}$ and $\mathfrak{L}^k_{j,2k-\ell+1}$, $j = 1, \dots, k$. Our aim, in Steps $2\ell+2$ and $2\ell+3$ will be to construct $\mathbf{G}^k_{\mathbf{ord}}$ on the strips $\Lambda^k_{2\ell+2}$ and $\Lambda^k_{2\ell+3}$ in such a way that identity (4.110) holds with ℓ replaced by $\ell+1$. This yields hence an iterative construction of $\mathbf{G}^k_{\mathbf{ord}}$.

The constructions in Steps $2\ell+2$ and $2\ell+3$ are essentially the same as in the construction for Steps 2 to $2\ell+1$, except that we need to modify a number of definitions. Firstly, we extend the definition of the sets $\mathbf{A}^k_{\mathfrak{p}}$ given in (4.106) for values of $\mathfrak{p} > k$ setting

$$\begin{aligned} \mathbf{A}^k_{\mathfrak{p}} &\equiv \bigcup_{i,j=1}^{k} \bigcup_{q=1}^{\ell-k} \{\mathrm{a}^k_{i,j,q}\} = \boxplus^2_k(h) \times \{h, 2h, \dots, (\mathfrak{p}-k)h\} \\ &= \boxplus^2_k(h) \times \mathbb{J}^k_{\mathfrak{p}} \text{ where } \mathbb{J}^k_{\mathfrak{p}} \equiv h\{1, \dots, \mathfrak{p}-k\} \text{ for } \mathfrak{p} > k. \end{aligned} \tag{4.120}$$

Likewise, we extend the definition of the sets $\mathbf{L}_{\mathfrak{p}}(s)^{k,\perp}$ and $\mathbf{N}_{\mathfrak{p}}(s)^{k,\perp}$ accordingly, for values of $\mathfrak{p} > k$ and $s \geq \mathrm{T}^k_{2k+2}$

$$\mathbf{L}_{\mathfrak{p}}(s)^{k,\perp} = \bigcup_{i=1}^{k} \bigcup_{\ell-k}^{k} \mathfrak{L}^{k,\perp}_{i,q}(s) \text{ and } \mathbf{N}_{\mathfrak{p}}(s)^{k,\perp} = \bigcup_{i=1}^{k} \bigcup_{q=\mathfrak{p}+1}^{k} \mathfrak{L}^{k,\perp}_{i,q}(s), \tag{4.121}$$

so that $\mathbf{L}_{\mathfrak{p}}(s)^{k,\perp} \cup \mathbf{N}_{\mathfrak{p}}(s)^{k,\perp} = \mathfrak{L}^{k,\perp}(s)$ and $\mathbf{L}_{\mathfrak{p}}(s)^{k,\perp} \cap \mathbf{N}_{\mathfrak{p}}(s)^{k,\perp} = \emptyset$. Finally, we define the spatial crossing region $\mathcal{O}_{\text{cross},\ell}$ and the space-time crossing region $\Theta_{\text{cross},\ell}$ as

$$\begin{cases} \mathcal{O}^h_{\text{cross},\ell} = [\dfrac{h}{2}, 1 + \dfrac{h}{2}]^2 \times [\dfrac{h}{2}, (2k-\ell+1)(\dfrac{h}{2})] \subset \mathcal{O}^h_{\text{cross},\ell+1} \\ \Theta^h_{\text{cross},\ell} = \mathcal{O}^h_{\text{cross},\ell} \times [\mathrm{T}^k_{2\ell+2}, \mathrm{T}^k_{2\ell+3}]. \end{cases}$$

Step $2\ell+2$: defining $\mathbf{G}^k_{\mathbf{ord}}$ on $\Lambda_{2\ell+2} = \mathbb{R}^3 \times [\mathrm{T}^k_{2\ell+1}, \mathrm{T}^k_{2\ell+2}]$. We define $\mathbf{G}^k_{\mathbf{ord}}$ again by formula (4.110), where the definitions of the various sets have been changed according to (4.121). It follows from the definition (4.110) of $\mathbf{G}^k_{\mathbf{ord}}$ is Lipschitz on $\Lambda^k_{2\ell+2}$ and that, as before, $|\nabla_4 \mathbf{G}^k_{\mathbf{ord}}| \leq \mathrm{C}_{\text{def}} k$ on $\Lambda^k_{2\ell+2}$, so that by integration on the support of $\mathbf{G}^k_{\mathbf{ord}}$ we have the estimate (4.111) remains valid. We observe also that at time $\mathrm{T}^k_{2\ell+2}$ identities (4.112) and (4.114) remain valid.

Step $2\ell+3$: defining $\mathbf{G}^k_{\mathbf{ord}}$ on $\Lambda^k_{2\ell+3} = \mathbb{R}^3 \times [\mathrm{T}^k_{2\ell+2}, \mathrm{T}^k_{2\ell+3}]$. We define again $\mathbf{G}^k_{\mathbf{ord}}$ by cubic extension by formula (4.117). One verifies that this definition of $\mathbf{G}^k_{\mathbf{ord}}$ yields a map whose restriction to $\Lambda^k_{2\ell+2}\cup\Lambda^k_{2\ell+3}$ is in $C^0(\Lambda^k_{2\ell+2}\cup\Lambda^k_{2\ell+3}\setminus\mathbb{A}^k_{\ell+1},\mathbb{S}^2)$, where $\mathbb{A}^k_{\ell+1}$ is defined in (4.116). Each singularity having Hopf invariant +2. Arguing as before, we obtain the energy estimate

$$\int_{\Lambda^k_{2\ell+2}} |\nabla_4 \mathbf{G}^k_{\mathbf{ord}}|^3 \leq \mathbf{K}_{\rm spg}k^2 + (2k-\ell)\mathbf{K}_{\rm box}k. \tag{4.122}$$

At time $\mathrm{T}^k_{2\ell+3}$, we notice that identity (4.110) holds with ℓ replaced by $\ell+1$, so that the iteration is complete.

The map $\mathbf{G}^k_{\mathbf{ord}}$ *at time* T^k_{4k-1}. The iteration is stopped when $\ell = 2k-1$, hence at step $4k-1$, that is at time $\mathrm{T}^k_{4k-1} = \tau_h + (4k-1)h$. Then, all fibers of $\mathfrak{L}^{k,\perp}(\mathrm{T}^k_{4k+1})$ have left the crossing region and the two sheafs $\mathfrak{L}^{k,\perp}(\mathrm{T}^k_{4k+1})$ and $\mathfrak{L}^k$ are not longer linked since

$$\mathfrak{L}^{k,\perp}(\mathrm{T}^k_{4k+1}) = \mathfrak{L}^{k,\perp} - (\tau_h + (2k-1)h)\vec{\mathbf{e}}_3 = \mathfrak{L}^{k,\perp} - (\tau_h + 2 - h)\vec{\mathbf{e}}_3 \subset \mathbb{R}^2 \times [-1, \frac{5h}{8}]$$

whereas $\mathfrak{L}^k((\mathrm{T}^k_{4k+1}) = \mathfrak{L}^k \subset \mathbb{R}^2 \times [h, 1]$. It follows, in view of (4.118) that

$$\mathbb{H}\left(\mathbf{G}^k_{\mathbf{ord}}(\cdot, \mathrm{T}^k_{4k+1})\right) = 0 \text{ and } \int_{\mathbb{R}^3} |\nabla_3 \mathbf{G}^k_{\mathbf{ord}}(\cdot, \mathrm{T}^k_{4k-1})|^3 \leq \mathrm{K}_{\rm spg}k^3. \tag{4.123}$$

and that the map $\mathbf{G}^k_{\mathbf{ord}}(\cdot, \mathrm{T}^k_{4k-1})$ is constant, equal to $\mathbb{P}_{\rm south}$ on $\mathbb{R}^3 \setminus [-20, 20]^3$.

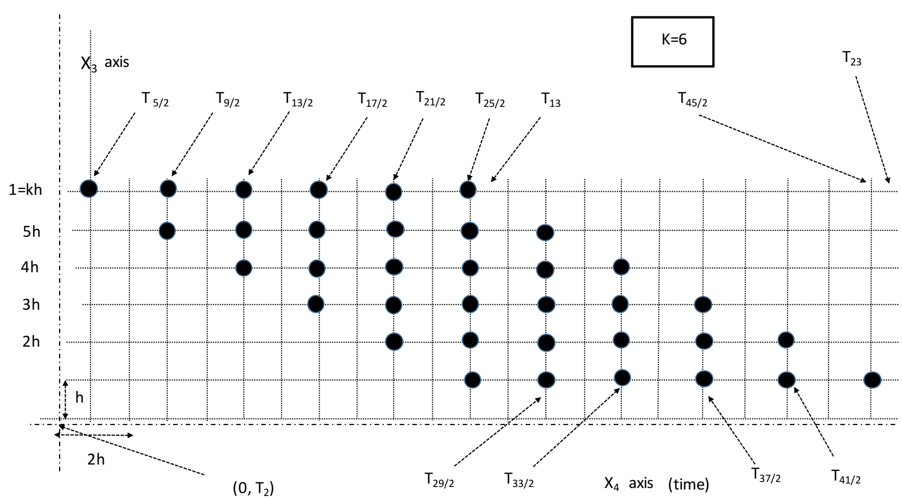

Figure 19: *The singularitie* $\mathbb{A}^k_{\rm sing}$ *for* $k = 6$ *projected on the* $(\vec{\mathbf{e}}_3, \vec{\mathbf{e}}_4)$ *plane, that is* $\mathbb{Y}^6_{\rm sing}$

Step $4k$: deforming to a constant map.

In this last step, we define the map $\mathbf{G}^k_{\mathbf{ord}}$ and the strip $\Lambda^{4k}_{\rm fin} = [\mathrm{T}^k_{4k-1}, \mathrm{T}^k_{4k}] \equiv \mathrm{T}^k_{4k-1}+40]$. We deform that that purpose the map $\mathbf{G}^k_{\mathbf{ord}}(\cdot, \mathrm{T}^k_{4k-1})$ which is Lipschitz and has trivial homotopy class to a constant map in Lipschitz way invoking Proposition 4.1 with $R = 20$, take as map $w = w^k$ the restriction of $\mathbf{G}^k_{\mathbf{ord}}(\cdot, \mathrm{T}^k_{4k-1})$ to the cube $[-20, 20]^3$: This yields a*Lipschitz map* $W = W^k$ defined on $[-20, 20]^4$ satisfying the four properties listed in Proposition 4.1. We set

$$\mathbf{G}^k_{\mathbf{ord}}(x, \mathrm{T}^k_{4k-1} + s) = W^k(x, s - 20) \text{ for } x \in \mathbb{R}^3 \text{ and } s \in [0, 40]. \tag{4.124}$$

It follows that $\mathbf{G}^k_{\mathbf{ord}}$ is *Lipschitz* on Λ^{4k}, continuous near $\mathbb{R}^3 \times \mathrm{T}^k_{4k-1}$ and such that

$$\mathbf{G}^k_{\mathbf{ord}}(x, s) = \mathbb{P}_{\mathrm{south}} \text{ on } (\mathbb{R}^3 \setminus [-20, 20]^3) \times [\mathrm{T}^k_{4k-1}, \mathrm{T}^k_{4k}] \cup \mathbb{R}^3 \times \{\mathrm{T}^k_{4k}\}. \tag{4.125}$$

The fourth property in Proposition 4.1 yields the energy estimate

$$\begin{aligned} \mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}, \Lambda^{4k}) &\leq 40\mathrm{C}_{\mathrm{ext}}\mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}(\cdot, \mathrm{T}^k_{4k-1}, \mathbb{R}^3) \\ &\leq 40\mathrm{C}_{\mathrm{ext}}\mathrm{K}_{\mathrm{spg}}k^3. \end{aligned} \tag{4.126}$$

Finally, we notice that $\mathrm{T}^k_{4k} = \mathrm{T}^k_{4k-1} + 40 = \mathrm{T}^k_1 + (4k-1)h + 40 = 5 + 3h/8 + 4 - h + 40 \leq 50$, so that we set

$$\mathbf{G}^k_{\mathbf{ord}}(\mathbf{x}) \equiv \mathbb{P}_{\mathrm{south}} \text{ for } x \in \Lambda^k_{4\mathrm{k}+1} = \mathbb{R}^3 \times [\mathrm{T}^k_{4k}, \mathrm{T}^k_{4k+1} = 50], \tag{4.127}$$

This definition yields a continuous map on a open neighborhood of $\Lambda^k_{4\mathrm{k}+1}$.

Proof of Proposition 4 completed.

First properties of $\mathbf{G}^k_{\mathbf{ord}}$*: Proof of Property* (44). So far, we have constructed the map $\mathbf{G}^k_{\mathbf{ord}}$ on each of the strips Λ^k_n, for $n = 1, \ldots, 4k+1$ by formulae (4.66), (4.74), (4.82), (4.84), (4.95), (4.101), (4.110), (4.117), (4.124) and (4.127). We notice that the definitions coincide on the intersections $\mathbb{R}^3 \times \{\mathrm{T}^k_n\}$ and are Lipschitz in an open neighborhood of these time slices. Moreover, for each n, the restriction of the map $\mathbf{G}^k_{\mathbf{ord}}$ to Λ^k_n belongs to $W^{1,3}(\Lambda^k_n, \mathbb{S}^2)$ with only a finite number of point singularities. Hence, we have defined the map $\mathbf{G}^k_{\mathbf{ord}}$ on the union $\Lambda = \overset{4k}{\underset{p=1}{\cup}} \Lambda^k_p = \mathbb{R}^3 \times [0, 50]$ in such a way that $\mathbf{G}^k_{\mathbf{ord}} : \Lambda \to \mathbb{S}^2$ is Lipschitz, having only a finite number of singularities. As a result of the definitions (4.66), (4.74), (4.82), (4.84), (4.95), (4.101), (4.110), (4.117), (4.124) and (4.127) we notice that $\mathbf{G}^k_{\mathbf{ord}}(x, 0) = \mathbf{S}^k_{\mathbf{pag}}(x)$ for $x \in \mathbb{R}^3$ and

$$\mathbf{G}^k_{\mathbf{ord}}(\mathbf{x}) = \mathbb{P}_{\mathrm{south}} \text{ for } \mathbf{x} = (x, x_4) \text{ with } |x| \geq 30 \text{ or } x_4 = 50.$$

This established property (44).

Energy estimate. Adding the energy estimates (4.68), (4.75), (4.91), (4.96), (4.105), (4.111), (4.119), (4.122) and (4.126), we are led to the estimate

$$\begin{aligned} \mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}, \Lambda) \leq \mathrm{K}^0_{\mathrm{def}}k^3 + 2k\mathrm{K}_{\mathrm{def}}k^2 + \left[\sum_{\ell=1}^{k} \ell\right] \mathrm{K}_{\mathrm{box}}k + \\ + 2k\mathbf{K}_{\mathrm{spg}}k^2 + \left[\sum_{\ell=k+1}^{2k-1} (2k - \ell)\right] \mathrm{K}_{\mathrm{box}}k + 40\mathrm{C}_{\mathrm{ext}}\mathrm{K}_{\mathrm{spg}}k^3. \end{aligned}$$

Since $\sum\limits_{\ell=1}^{k} \ell = \frac{k(k+1)}{2}$ and $\sum\limits_{\ell=k+1}^{2k-1} (2k - \ell) = \frac{k(k-1)}{2}$, we obtain

$$\mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}, \Lambda) \leq \left(\mathrm{K}^0_{\mathrm{def}} + 2\mathrm{K}_{\mathrm{def}} + 2\mathbf{K}_{\mathrm{spg}} + \mathrm{K}_{\mathrm{box}} + 40\mathrm{C}_{\mathrm{ext}}\mathrm{K}_{\mathrm{spg}}\right) k^3 \equiv \mathrm{K}_{\mathrm{gord}}k^3.$$

This establishes (45).

Properties of the singularities of $\mathbf{G}^k_{\mathbf{ord}}$. The singularities of $\mathbf{G}^k_{\mathbf{ord}}$ are those already described on each the strips $\Lambda^k_{2\ell+1}$, since the map is continuous near there intersections. Set

$$\begin{aligned}\mathbb{A}^k_{\rm sing} &= \bigcup_{\ell=1}^{2k-1} \mathbb{A}^k_\ell = \boxplus^2_k(h) \times \bigcup_{\ell=1}^{2k-1} \mathbb{J}^k_\ell \times \{\mathrm{T}^k_{2\ell+3/2}\} \subset \Lambda \\ &= \boxplus^2_k(h) \times \mathbb{Y}^k_{\rm sing} \text{ with } \mathbb{Y}^k_{\rm sing} = \bigcup_{\ell=1}^{2k-1} \mathbb{J}^k_\ell \times \{2\ell h + \delta_h\} \subset [0,1] \times [0,50],\end{aligned}$$

where we have set $\delta_h = \tau_h + h/2 = 5 - h/8$. It follows from our discussion that $\mathbf{G}^k_{\mathbf{ord}} \in C^0(\Lambda \setminus \mathbb{A}^k_{\rm sing}, \mathbb{S}^2)$, each singularity having Hopf invariant equal to +2. The set $\mathbb{Y}^k_{\rm sing}$ is represented in Figure 19 for $k = 6$. We show next, as the figure shows, that the points in $\mathbb{Y}^k_{\rm sing}$ are the vertices of a grid modelled on a parallelogram. We turn next to the proof of (47).

Proof of (47). We decompose $\mathbb{Y}^k_{\rm sing}$ as $\mathbb{Y}^k_{\rm sing} = \mathbb{Y}^{k,\rm up}_{\rm sing} \cup \mathbb{Y}^{k,\rm down}_{\rm sing}$, where

$$\begin{cases} \mathbb{Y}^{k,\rm up}_{\rm sing} = h \bigcup_{\ell=1}^{k} \bigcup_{j=k-\ell+1}^{k} \{j, 2\ell\} + (0, \delta_h) = h \bigcup_{\ell=1}^{k} \bigcup_{j=k-\ell+1}^{k} \{j, 2(\ell-k)\} + (0, 7-h/8) \text{ and} \\ \mathbb{Y}^{k,\rm down}_{\rm sing} = h \bigcup_{\ell=k+1}^{2k-1} \bigcup_{j=1}^{2k-\ell} \{j, 2(\ell-k)\} + (0, 7-h/8). \end{cases}$$

For the first set, we introduce the new indices ℓ' and j' such that $j = j'$ and $\ell - k = j' - \ell'$ so that

$$\mathbb{Y}^{k,\rm up}_{\rm sing} = \Phi_k\left(h \bigcup_{\ell'=1}^{k} \bigcup_{j'=\ell'}^{k} \{j', \ell'\}\right) \text{ and } \mathbb{Y}^{k,\rm up}_{\rm sing} = \Phi_k\left(h \bigcup_{\ell'=1}^{k} \bigcup_{j'=1}^{\ell'-1} \{j', \ell'\}\right).$$

which yields the desired result (47) and completes the proof of Proposition 4.

5 Proof of the main results

5.1 Proof of Proposition 2

5.1.1 Constructing the sequence $(\mathfrak{v}_k)_{k\in\mathbb{N}}$

The maps $\mathfrak{v}_k$ are directly deduced from the maps $\mathbf{G}^k_{\mathbf{ord}}$ performing some elementary transformations. Our main aim will be to transform the set of singularities given by (49), which are the nodes of a distorted grid into the nodes of a four dimensional orthonormal regular grid.

Transforming singularities into an orthonormal regular grid: The map $\tilde{\mathbf{G}}^k_{\mathbf{ord}}$. The map $\mathbf{G}^k_{\mathbf{ord}}$ is only defined on the strip Λ defined in (44). Given an integer $k \in \mathbb{N}^\star$, we first extend the map $\mathbf{G}^k_{\mathbf{ord}}$ to the whole space $\mathbb{R}^3 \times \mathbb{R}$ setting

$$\begin{cases} \mathbf{G}^k_{\mathbf{ord}}(x,s) = \mathbb{P}_{\rm south} \text{ for } x \in \mathbb{R}^3 \text{ and } s \geq 30, \\ \mathbf{G}^k_{\mathbf{ord}}(x,s) = \mathbf{S}^k_{\mathbf{pag}}(x) \text{ for } x \in \mathbb{R}^3 \text{ and } s \leq 0. \end{cases} \tag{5.1}$$

It follows from this definition that

$$\mathbf{G}^k_{\mathbf{ord}}(\mathbf{x}) = \mathbb{P}_{\rm south} \text{ for } \mathbf{x} \in \mathbb{R}^4 \setminus \mathcal{V} \text{ where } \mathcal{V} \equiv \{(x,s) \in \mathbb{R}^3 \times \mathbb{R} \text{ s.t. } |x| \leq 40 \text{ and } s \leq 50\}. \tag{5.2}$$

In view of the results in Proposition 4 and Proposition 3, we have the energy estimate, for any $a \geq 0$

$$\mathrm{E}_3(\mathbf{G}^k_{\mathbf{ord}}, \mathbb{R}^3 \times [-a, 0]) \leq (\mathbf{K}_{\mathrm{Gord}} + a\mathbf{K}_{\mathrm{spg}})\ k^3. \tag{5.3}$$

We introduce the map $\tilde{\mathbf{G}}^k_{\mathbf{ord}}$ defined on $\mathbb{R}^3 \times \mathbb{R}^+$ by

$$\tilde{\mathbf{G}}^k_{\mathbf{ord}}(\mathbf{x}) = \mathbf{G}^k_{\mathbf{ord}} \circ \Phi_k(\mathbf{x}) = \mathbf{G}^k_{\mathbf{ord}}\left(\Phi_k(\mathbf{x})\right) \text{ for } \mathbf{x} \in \mathbb{R}^3 \times [0, +\infty).$$

It follows from property (47) that $\tilde{\mathbf{G}}^k_{\mathbf{ord}} \in C^0(\mathbb{R}^4 \setminus \boxplus^4_k(h), \mathbb{S}^2)$ that is the set of singularities of $\tilde{\mathbf{G}}^k_{\mathbf{ord}}$ is $\boxplus^4_k(h)$, each of the k^4 singularities having Hopf invariant equal to $+2$. We claim that

$$\begin{cases} \tilde{\mathbf{G}}^k_{\mathbf{ord}}(\mathbf{x}) = \mathbb{P}_{\mathrm{south}} \text{ for } |\mathbf{x}| \geq 400 \text{ and} \\ \mathrm{E}_3(\tilde{\mathbf{G}}^k_{\mathbf{ord}}, \mathbb{R}^3 \times [0, +\infty)) \leq 5\sqrt{5}\,(\mathbf{K}_{\mathrm{Gord}} + 131\mathbf{K}_{\mathrm{spg}})\ k^3. \end{cases} \tag{5.4}$$

Indeed, consider the set

$$\Omega_k \equiv \Phi_k(\mathbb{R}^3 \times \mathbb{R}^+) = \{(x_1, x_2, x_3, -2x_3 + x_4 + 7 - h/8) \text{ with } x_4 \geq 0\}$$

and the intersection $\Omega_k \cap \mathcal{V}$. If $\mathbf{y} \in \Omega_k \cap \mathcal{V}$, then it is of the form

$$\begin{cases} \mathbf{y} = (x_1, x_2, x_3, -2x_3 + x_4 + 7 - \dfrac{h}{8}) \text{ with } x_4 \geq 0, |x_i| \leq 40, \text{ for } i = 1, 2, 3 \\ \qquad\qquad \text{and } -2x_3 + x_4 + 7 - \dfrac{h}{8} \leq 50. \end{cases}$$

Hence we deduce that $0 \leq x_4 \leq 51 + 2x_3 \leq 51 + 2 \times 40 = 131$, so that

$$\Omega_k \cap \mathcal{V} \subset \mathbb{B}^4(131).$$

The inverse Φ_k^{-1} of Φ_k, can be computed explicitely as

$$\Phi_k^{-1}(\mathbf{x}) = (x_1, x_2, x_3, 2x_3 + x_4) + (0, -7 + \frac{h}{8}), \text{ for } \mathbf{x} = (x_1, x_2, x_3, x_4)$$

so that $\Phi_k^{-1}(\mathbb{B}^4(131)) \subset \mathbb{B}^4(3 \times 131 + 7) = \mathbb{B}^4(400)$, and hence $\Phi_k^{-1}(\Omega_k \cap \mathcal{V}) \subset \mathbb{B}^4(400)$, which establishes the the first assertion of the claim (5.4). For the second assertion in (5.4), we have, by the chain rule

$$|\nabla_4 \tilde{\mathbf{G}}^k_{\mathbf{ord}}(\mathbf{x})|^2 \leq 5|\nabla_4 \tilde{\mathbf{G}}^k_{\mathbf{ord}}(\Phi_k(\mathbf{x}))|^2,$$

which yields the second assertions thanks to change of variables.

Extending $\tilde{\mathbf{G}}^k_{\mathbf{ord}}$ by symmetry. The extend the map $\tilde{\mathbf{G}}^k_{\mathbf{ord}}$ by symmetry to the whole on $\mathbb{R}^4$, setting

$$\tilde{\mathbf{G}}^k_{\mathbf{ord}}(x, s) = \tilde{\mathbf{G}}^k_{\mathbf{ord}}(x, -s) \text{ for } x \in \mathbb{R}^3 \text{ and } s \leq 0. \tag{5.5}$$

It follows from this construction and the trace theorem that this extension in in $W^{1,3}_{\mathrm{loc}}(\mathbb{R}^4, \mathbb{S}^2)$ and in $C^0(\mathbb{R}^4 \setminus \tilde{\mathbb{A}}^k_{\mathrm{sing}}, \mathbb{S}^2)$, where the set $\tilde{\mathbb{A}}^k_{\mathrm{sing}}$ is given by

$$\tilde{\mathbb{A}}^k_{\mathrm{sing}} = \boxplus^4_k(h) \cup \mathbb{S}_{\mathrm{sym}}\left(\boxplus^4_k(h)\right),$$

where $\mathbb{S}_{\mathrm{sym}}$ corresponds to the symmetry defined in (51). It follws from (5.4) that on

$$\begin{cases} \tilde{\mathbf{G}}^k_{\mathbf{ord}}(\mathbf{x}) = \mathbb{P}_{\mathrm{south}} \text{ for } |\mathbf{x}| \geq 400 \text{ and} \\ \mathrm{E}_3(\tilde{\mathbf{G}}^k_{\mathbf{ord}}, \mathbb{R}^4) \leq 10\sqrt{5}\,(\mathbf{K}_{\mathrm{Gord}} + 131\mathbf{K}_{\mathrm{spg}})\ k^3. \end{cases} \tag{5.6}$$

Rescaling $\tilde{\mathbf{G}}^k_{\mathbf{ord}}$. We now are in position to define the map $\mathfrak{v}_k$ as

$$\mathfrak{v}_k(\mathbf{x}) = \tilde{\mathbf{G}}^k_{\mathbf{ord}}(400\,\mathbf{x}) \text{ for } \mathbf{x} \in \mathbb{R}^4.$$

It follows then from (5.6) and scaling laws that

$$\begin{cases} \mathfrak{v}_k(\mathbf{x}) = \mathbb{P}_{\text{south}} \text{ for } |\mathbf{x}| \geq 1 \text{ and} \\ \mathrm{E}_3(\mathfrak{v}_k, \mathbb{R}^4) \leq \dfrac{1}{8\sqrt{5}} \left(\mathbf{K}_{\text{Gord}} + 131\mathbf{K}_{\text{spg}}\right) k^3. \end{cases} \tag{5.7}$$

Moreover $\mathfrak{v}_k \in C^0(\mathbb{R}^4 \setminus \Sigma_{\text{sing}}, \mathbb{S}^2)$ where Σ_{sing} is described in (50). In order to prove (29), we will rely on some additional notion related to branched transportation which are exposed in Appendix A, in particular the branched connection to the boundary $\mathfrak{L}^{\alpha}_{\text{brbd}}$, with the exponent α equal to the critical exponent in dimension 4, namely $\alpha_4 = \frac{3}{4}$. As a direct is a direct consequence of Proposition A.1 of the Appendix, we have:

Proposition 5.1. *We have the lower bound, for some universal constant $C > 0$*

$$\mathfrak{L}^{\alpha_4}_{\text{brbd}}(\boxplus^4_k(\mathrm{h}_{\text{scal}}), \partial([0, \frac{1}{400}]^3) \geq Ck^3 \log k, \quad \text{for any } k \in \mathbb{N}^*.$$

5.1.2 Proof of Proposition 2 completed

The only part of proposition 2 which has to be established is (29). For that purpose, we invoke the relationship between the functionals $\mathfrak{L}^{\alpha}_{\text{branch}}$ and $\mathfrak{L}^{\alpha_4}_{\text{brbd}}$ presented in Lemma A.11, choosing $\mathfrak{P} = \{1\}$ and $\Omega_1 = [0, 1/400]^4$. Since all singularities in $\boxplus^4_k(\mathrm{h}_{\text{scal}})$ have the same charge equal to $+2$, the conclusion applies showing that

$$\mathfrak{L}^{\alpha_4}_{\text{branch}}(\mathfrak{v}_k) \geq \mathfrak{L}^{\alpha_4}_{\text{brbd}}(\boxplus^4_k(\mathrm{h}_{\text{scal}}), [0, \partial([0, \frac{1}{400}]^4]) \geq Ck^3 \log k, \tag{5.8}$$

where we have used the result of Proposition 5.1 for the last inequality. On the other hand, combining (16) and (15) with the respective definitions (26) and (A.61) of $\mathrm{L}_{\text{branch}}$ and $\mathfrak{L}^{\alpha_4}_{\text{branch}}$ respectively, we are led, for general singularities (P_i, Q_j) to the inequality

$$\mathrm{L}_{\text{branch}}(P_i, Q_j) \geq \mathrm{C}_\nu \mathfrak{L}^{\alpha_4}_{\text{branch}}(P_i, Q_j), \tag{5.9}$$

where $\mathrm{C}_\nu > 0$ is the constant introduced in (15). Inequality (29) then follows, combining (5.8) and (5.9). The proof of Proposition 2 is hence complete.

5.2 Proof of Theorem 4

5.2.1 Sequences of radii and multiplicities

The following elementary observation will be used in our proof:

Lemma 5.1. *There exists a sequence of radii $(\mathfrak{r}_\mathrm{i})_{\mathrm{i}\in\mathbb{N}}$ and a sequence of integers $(\mathrm{k}_\mathrm{i})_{\mathrm{i}\in\mathbb{N}}$ such that the following properties are satisfied*

$$\sum_{\mathrm{i}\in\mathbb{N}} \mathfrak{r}_\mathrm{i} = \frac{1}{8}, \quad \sum_{\mathrm{i}\in\mathbb{N}} \mathfrak{r}_\mathrm{i} \mathrm{k}_\mathrm{i}^3 < +\infty \text{ and } \sum_{\mathrm{i}\in\mathbb{N}} \mathfrak{r}_\mathrm{i}\ \mathrm{k}_\mathrm{i}^3\ \log(\mathrm{k}_\mathrm{i}) = +\infty \tag{5.10}$$

We may assume furthermore that

$$3\mathfrak{r}_{\mathrm{i}+1} \geq \mathfrak{r}_\mathrm{i}. \tag{5.11}$$

Proof. Consider first the sequences $(\tilde{\mathfrak{r}}_{\mathfrak{i}})_{i\in\mathbb{N}}$ and $(\tilde{\mathrm{k}}_{\mathfrak{i}})_{\mathfrak{i}\in\mathbb{N}}$ given for $i \in \mathbb{N}\setminus\{0,1\}$ by

$$\tilde{\mathfrak{r}}_i = \frac{1}{\mathfrak{i}^4(\log \mathfrak{i})^2} \text{ and } \mathrm{k}_i = \mathfrak{i}, \text{ so that}$$

$$\sum_{\mathfrak{i}=2}^{+\infty} \tilde{\mathfrak{r}}_i = \sum_{\mathfrak{i}=2}^{+\infty} \frac{1}{\mathfrak{i}^4(\log \mathfrak{i})^2} < +\infty, \quad \sum_{\mathfrak{i}=2}^{+\infty} \tilde{\mathfrak{r}}_i\, \mathrm{k}_{\mathfrak{i}}^3 = \sum_{\mathfrak{i}=2}^{+\infty} \frac{1}{\mathfrak{i}\,(\log \mathfrak{i})^2} < +\infty, \text{ whereas}$$

$$\sum_{\mathfrak{i}=2}^{+\infty} \tilde{\mathfrak{r}}_i\, \mathrm{k}_{\mathfrak{i}}^3 \log(\mathrm{k}_{\mathfrak{i}}) = \sum_{\mathfrak{i}=2}^{+\infty} \frac{1}{\mathfrak{i}\,(\log \mathfrak{i})} = +\infty.$$

We then choose arbitrary values for $i = 0$ and $i = 1$ and finally set $\mathfrak{r}_i = \mathrm{c}\mathfrak{r}_i$, where the positive constant c is defined so that the first condition holds, that is satisfies $\mathrm{c}^{-1} = 8\sum_{\mathfrak{i}=0}^{+\infty}\tilde{\mathfrak{r}}_i$. □

5.2.2 Defining $\mathcal{U}$ gluing copies of the $\mathfrak{v}_k$'s

We introduce the set of points $\{\mathrm{M}_{\mathfrak{i}}\}_{\mathfrak{i}\in\mathbb{N}}$ in $\mathbb{R}^4$ defined by

$$\mathrm{M}_{\mathfrak{i}} = 4\left(\sum_{j=0}^{\mathfrak{i}} \mathfrak{r}_{\mathfrak{j}}\right)\vec{\mathbf{e}}_1 \text{ where } \vec{\mathbf{e}}_1 = (1,0,0,0), \text{ for } \mathfrak{i}\in\mathbb{N}, \tag{5.12}$$

so that the points $\mathrm{M}_{\mathfrak{i}}$ are all on the segment joining the origin to the point

$$\mathrm{M}_\star = \frac{1}{2}\vec{\mathbf{e}}_1 = (\frac{1}{2}, 0, \dots, 0),$$

converging thanks to the first identity in (5.10), to the point $\mathrm{M}_\star$ as $\mathfrak{i}\to+\infty$. We consider the collection of disjoint balls $(B_{\mathfrak{i}})_{i\in\mathbb{N}}$ defined by

$$B_{\mathfrak{i}} \equiv \mathbb{B}^4(\mathrm{M}_i, \mathfrak{r}_i) \text{ for } \mathfrak{i}\in\mathbb{N}, \quad \text{so that } \mathrm{dist}(B_{\mathfrak{i}}, B_{\mathfrak{j}}) \geq \mathfrak{r}_{\mathfrak{i}} + \mathfrak{r}_j \text{ if } \mathfrak{i}\neq\mathfrak{j},$$

The last assertion being a consequence of (5.11). We then define the map $\mathcal{U}$ on $\mathbb{B}^4(1)$ as

$$\mathcal{U}(x) = \mathfrak{v}_{\mathrm{k}_{\mathfrak{i}}}\left(\frac{x-\mathrm{M}_{\mathfrak{i}}}{\mathfrak{r}_{\mathfrak{i}}}\right) \quad \text{if } x\in B_{\mathfrak{i}}, \quad \mathcal{U}(x) = \mathbb{P}_{\mathrm{south}} \text{ if } x\in\mathbb{B}^4(1)\setminus \underset{\mathfrak{i}\in\mathbb{N}}{\cup} B_{\mathfrak{i}}. \tag{5.13}$$

We have in particular $\mathcal{U} = \mathbb{P}_{\mathrm{south}}$ on the boundary $\partial\mathbb{B}(1)$. Likewise we define a sequence of maps $(\mathcal{U}_{\mathfrak{i}})_{\mathfrak{i}\in\mathbb{N}}$ setting for $\mathfrak{i}\in\mathbb{N}$

$$\mathcal{U}_{\mathfrak{i}}(x) = \mathcal{U}(x) \text{ if } x\in \underset{\mathfrak{j}=0}{\overset{\mathfrak{i}}{\cup}} B_{\mathfrak{j}},\ \mathcal{U}_{\mathfrak{i}} = \mathbb{P}_{\mathrm{south}} \text{ otherwise.} \tag{5.14}$$

Since the map $\mathfrak{v}_k$ belongs to $\mathcal{R}_{\mathrm{ct}}(\mathbb{B}^4,\mathbb{S}^2)$ for any $k\in\mathbb{N}$, it follows that for any $\mathfrak{i}\in\mathbb{N}$, the map $\mathcal{U}_i$ belongs to $\mathcal{R}_{\mathrm{ct}}(\mathbb{B}^4,\mathbb{S}^2)$. Notice that the map $\mathcal{U}$ has an infinite countable number of singularities, whose only accumulation point is $\mathrm{M}_\star$. We notice also that $\mathcal{U} = \mathcal{U}_{\mathfrak{i}}$ on any compact subset K of $\bar{\mathbb{B}}^4\setminus\{\mathrm{M}_\star\}$ provided $\mathfrak{i}$ is choosen sufficiently large.

Lemma 5.2. *The map $\mathcal{U}$ belongs to $W^{1,3}_{\rm ct}(\mathbb{B}^4,\mathbb{S}^2)$. Moreover*

$$\mathcal{U}_{\mathfrak{i}} \to \mathcal{U} \text{ in } W^{1,3}(\mathbb{B}^4,\mathbb{R}^3) \text{ strongly} \quad \text{as } \mathfrak{i}\to+\infty. \tag{5.15}$$

Proof. It follows from the definition (5.13), the scaling properties (4) of the energy E_3 and inequality (28) of Proposition 2 that

$$\mathrm{E}_3(\mathcal{U},\mathbb{B}^4)=\sum_{\mathfrak{i}\in\mathbb{N}}\mathrm{E}_3(\mathcal{U},B_{\mathfrak{i}})=\sum_{\mathfrak{i}\in\mathbb{N}}\mathfrak{r}_{\mathfrak{i}}\mathrm{E}_3(\mathfrak{v}_{\mathrm{k}_{\mathfrak{i}}})\le \mathrm{C}_1\sum_{\mathfrak{i}\in\mathbb{N}}\mathfrak{r}_{\mathfrak{i}}\mathrm{k}_{\mathfrak{i}}^3<+\infty$$

and likewise for $\mathfrak{i}\in\mathbb{N}$, we have the estimate

$$\mathrm{E}_3\left(\mathcal{U}-\mathcal{U}_{\mathfrak{i}},\mathbb{B}^4\right)=\sum_{\mathfrak{j}=\mathfrak{i}+1}^{+\infty}\mathfrak{r}_{\mathfrak{j}}\mathrm{E}_3\left(\mathfrak{v}_{\mathrm{k}_{\mathfrak{j}}}\right)\le\sum_{\mathfrak{j}=\mathfrak{i}+1}^{+\infty}\mathfrak{r}_{\mathfrak{j}}\mathrm{k}_{\mathfrak{j}}^3\to 0 \text{ as } j\to+\infty,$$

which establishes the assertions of the Lemma. □

We turn to the description of the singular set $\Sigma^{\mathfrak{i}}_{\rm sing}=\Sigma^{\mathfrak{i}}_{\rm ing,+}\cup\Sigma^{\mathfrak{i}}_{\rm ing,-}$ of $\mathcal{U}_{\mathfrak{i}}$, where $\Sigma^{\mathfrak{i}}_{\rm ing,+}$ (resp. $\Sigma^{\mathfrak{i}}_{\rm ing,-}$) denotes the set of singularities of positive (resp. negative) topological charges, actually all equal to $+2$ (resp -2). We may write

$$\Sigma^{\mathfrak{i}}_{\rm ing,+}=\bigcup_{\mathfrak{j}=0}^{\mathfrak{i}}\left(\boxplus^m_{\mathrm{k}_{\mathfrak{j}}}(\frac{\mathfrak{r}_{\mathfrak{j}}\mathrm{h}_{\mathfrak{j}}}{400})+\mathrm{M}_{\mathfrak{j}}\right) \text{ and } \Sigma^{\mathfrak{i}}_{\rm ing,-}=\bigcup_{\mathfrak{j}=0}^{\mathfrak{i}}\left(\mathfrak{S}_{\rm sym}\left(\boxplus^m_{\mathrm{k}_{\mathfrak{j}}}\frac{\mathfrak{r}_{\mathfrak{j}}\mathrm{h}_{\mathfrak{j}}}{400}\right)+\mathrm{M}_{\mathfrak{j}}\right). \tag{5.16}$$

We have hence

$$\Sigma^{\mathfrak{i}}_{\rm ing,-}\cap\left(\left[0,\frac{\mathfrak{r}_{\mathfrak{j}}\mathrm{h}_{\mathfrak{j}}}{400}\right]^4+\mathrm{M}_{\mathfrak{j}}\right)=\emptyset,$$

so that arguing as in Proposition 5.1 and using the scaling properties of the branched transportation functional, we derive that, for any $\mathfrak{j}\in\mathbb{N}^*$, we have

$$\mathfrak{L}^{\alpha_4}_{\rm brbd}(\Sigma^{\mathfrak{i}}_{\rm ing,+},\partial\left(\left[0,\frac{\mathfrak{r}_{\mathfrak{j}}\mathrm{h}_{\mathfrak{j}}}{400}\right]^4+\mathrm{M}_{\mathfrak{j}}\right))\ge C\,\mathfrak{r}_{\mathfrak{j}}\,\mathrm{k}_{\mathfrak{j}}^3\,\log\mathrm{k}_{\mathfrak{j}}. \tag{5.17}$$

5.2.3 Proof of theorem 4 completed

In order to prove Theorem 4 we will invoke a variant of (24), which applies to maps which are not necessarily constant on the boundary.

A variant of (24). If Ω is a smooth domain in $\mathbb{R}^4$, w is a map in $\mathcal{R}(\Omega,\mathbb{S}^2)$ and $(\varphi_n)_{n\in\mathbb{N}^*}$ is a sequence of maps in $C^\infty(\bar\Omega,\mathbb{S}^2)$ such that $\varphi_n\rightharpoonup w$ in $W^{1,3}(\Omega)$ as $n\to+\infty$, then, we have the lower bound

$$\liminf_{n\to+\infty}\mathrm{E}_3(\varphi_n)\ge\mathrm{E}_3(w)+\mathrm{L}_{\rm brbd}(w,\partial\Omega). \tag{5.18}$$

The functional $\mathrm{L}_{\rm brbd}$ appearing on the r.h.s of (5.18) is defined, for an arbitrary $w\in\mathcal{R}(\Omega,\mathbb{S}^2)$ with ±1 singularities as

$$\mathrm{L}_{\rm brbd}(w,\partial\Omega)=\inf\{\mathbf{W}_2(G),G\in\mathcal{G}(\{P_i\}_{i\in J^+},\{Q\}_{j\in J^-},\partial\Omega)\},$$

where $\{P_i\}_{i\in J^+}$ denotes set of $+1$ singularities of w, $\{Q_i\}_{i\in J^-}$ the set of negative singularities and $\mathcal{G}(\{P_i\}_{i\in J^+}, \{Q_j\}_{j\in J^-}, \partial\Omega)$ represents the set of graphs satisfying conditions (A.1) and (**??**) in the Appendix. Notice that, in view of (15) and (16), we have, similar to (5.9)

$$\mathrm{C}_\nu \mathfrak{L}^{\alpha_4}_{\mathrm{brbd}}(P_i, Q_j) \leq \mathrm{L}_{\mathrm{brbd}}(P_i, Q_j). \tag{5.19}$$

Arguing by contradiction. We argue by contradiction and assume that there exist a sequence $(v_n)_{n\in\mathbb{N}}$ of maps in $C^\infty(\mathbb{B}^4, \mathbb{S}^2)$ such that

$$v_n \rightharpoonup \mathcal{U} \text{ weakly in } W^{1,p}(\mathbb{B}^4, \mathbb{S}^2) \text{ as } n \to +\infty, \tag{5.20}$$

so that in particular, by the Banach-Steinhaus Theorem

$$\gamma \equiv \limsup_{n\to+\infty} \mathrm{E}_3\left(v_n, \mathbb{B}^4\right) < +\infty. \tag{5.21}$$

Weak convergence to $\mathcal{U}_i$ on the sets $\Omega_{\mathfrak{i}}$. For given $\mathfrak{i} \in \mathbb{N}^*$, we consider the domain

$$\Omega_{\mathfrak{i}} = \mathbb{B}^4 \setminus \mathbb{B}^4(\mathrm{M}^\star, \varrho_{\mathfrak{i}}) \text{ where } \varrho_{\mathfrak{i}} = 4\left(\sum_{\mathfrak{j}=\mathfrak{i}+1}^{+\infty} \mathfrak{r}_{\mathfrak{j}}\right) + \mathfrak{r}_{\mathfrak{i}+1} \to 0 \text{ as } \mathfrak{i} \to +\infty,$$

so that, in particular $\mathbb{B}(\mathrm{M}_{\mathfrak{j}}, \mathfrak{r}_{\mathfrak{j}}) \subset \Omega_{\mathfrak{i}}$ if $\mathfrak{j} \leq \mathfrak{i}$ and $\Omega_{\mathfrak{i}} \cap \mathbb{B}(\mathrm{M}_{\mathfrak{j}}, \mathfrak{r}_{\mathfrak{j}}) = \emptyset$ if $\mathfrak{j} > \mathfrak{i}$. Let $v_n^{\mathfrak{i}}$ be the restriction of v_n to the set $\Omega_{\mathfrak{i}}$. It follows from (5.20) that

$$v_n^{\mathfrak{i}} \rightharpoonup \mathcal{U}_{\mathfrak{i}} \text{ weakly in } W^{1,3}(\Omega_{\mathfrak{i}}, \mathbb{S}^2) \text{ as } n \to +\infty,$$

so that by (5.18) and (5.21) we have for any $\mathfrak{i} \in \mathbb{N}^*$

$$\begin{aligned}\gamma &\geq \liminf_{n\to+\infty} \mathrm{E}_3(v_n^{\mathfrak{i}}) \geq \mathrm{E}_3(\mathcal{U}_1) + \mathrm{L}_{\mathrm{brbd}}(w, \partial\Omega)\\ &\geq \mathrm{C}_\nu \mathfrak{L}^{\alpha_4}_{\mathrm{brbd}}(\Sigma_{\mathrm{ing},+}, \Sigma_{\mathrm{ing},-}, \partial\Omega_{\mathfrak{i}}) = C_1 \mathfrak{L}^{\alpha_4}_{\mathrm{brbd}}\left(\bigcup_{\mathfrak{j}=1}^{\mathfrak{i}} \Sigma^{\mathfrak{j}}_{\mathrm{ing},+}, \bigcup_{\mathfrak{j}=1}^{\mathfrak{i}} \Sigma^{\mathfrak{j}}_{\mathrm{ing},-}, \partial\Omega_{\mathfrak{j}}\right).\end{aligned} \tag{5.22}$$

The contradiction. In view of (5.17), we may apply Lemma A.11 of the Appendix to the sets $\mathrm{B}^{\mathfrak{j}}_{\mathrm{ox}}$ and assert that

$$\begin{aligned}\mathfrak{L}^{\alpha_4}_{\mathrm{brbd}}\left(\bigcup_{\mathfrak{j}=1}^{\mathfrak{i}} \Sigma^{\mathfrak{j}}_{\mathrm{ing},+} \bigcup_{\mathfrak{j}=1}^{\mathfrak{i}} \Sigma^{\mathfrak{j}}_{\mathrm{ing},-}, \partial\Omega_{\mathfrak{j}}\right) &\geq \sum_{\mathfrak{j}=1}^{\mathfrak{i}} \mathfrak{L}^{\alpha_4}_{\mathrm{brbd}}(\Sigma^{\mathfrak{j}}_{\mathrm{ing},+}, \left(\left[0, \frac{\mathfrak{r}_{\mathfrak{j}} \mathrm{h}_{\mathfrak{j}}}{400}\right]^4 + \mathrm{M}_{\mathfrak{j}}\right)))\\ &\geq C \sum_{\mathfrak{j}=1}^{\mathfrak{i}} \mathfrak{r}_{\mathfrak{j}}\, \mathrm{k}_{\mathfrak{j}}^3 \, \log \mathrm{k}_{\mathfrak{j}},\end{aligned} \tag{5.23}$$

where we have invoked (5.17) for the last inequality. Combining (5.22) and (5.23) we obtain $C\sum_{\mathfrak{j}=1}^{+\infty} \mathfrak{r}_{\mathfrak{j}}\, \mathrm{k}_{\mathfrak{j}}^3 \, \log \mathrm{k}_{\mathfrak{j}} \leq \gamma$, which contradicts property (5.10), and hence completes the proof of Theorem 4.

5.3 Proof of Theorem 3

The main additional arguments leading for the proof of Theorem 3 are not specific to the sphere $\mathbb{S}^2$, so that we may consider a general a compact manifolf $\mathcal{N}$. We will invoke the folllowing:

Proposition 5.2. *Let $m_0 \in \mathbb{N}^*$, and assume that there exists a map u in $W^{1,p}_{\rm ct}(\mathbb{B}^{m_0}, \mathcal{N})$ which is not the weak limit of smooth maps between $\mathbb{B}^{m_0}$ and $\mathcal{N}$. That given any integer $m \geq m_0$ there exists a map v in $W^{1,p}_{\rm ct}(\mathbb{B}^{m}, \mathcal{N})$ which is not the weak limit of smooth maps between $\mathbb{B}^{m_0}$ and $\mathcal{N}$.*

The proof relies on two constructions we present next.

5.3.1 Adding dimensions

Let $m \in \mathbb{N}^*$ and consider a map $u : \mathbb{R}^m \to \mathbb{R}^\ell$ such that u is constant equal to some value c_0 outside the unit ball $\mathbb{B}^m$. We construct a map $\mathbb{I}^{m+1}_{\rm cyl}(u)$ from $\mathbb{R}^{m+1} \to \mathbb{R}^\ell$ constant equal to c_0 outside in the unit ball $\mathbb{B}^{m+1}$ as follows. First, we consider the translated map $u_{_A}$ defined on $\mathbb{R}^m$ by

$$u_{_A}(x) = u(x - A) \text{ where } A = (2, 0, \dots, 0),$$

so that $u_{_A}$ is equal to c_0 outside the ball $\mathbb{B}^m_1(A) \subset \mathbb{B}^m_3(0)$, in particular in the region $\{x_1 \leq 1\}$. We then introduce the map $T^{m+1}(u)$ defined for $(x_1, x_2, \dots, x_m, x_{m+1}) \in \mathbb{R}^{m+1}$ by

$$T^{m+1}(u)(x_1, x_2, \dots, x_m, x_{m+1}) = u_{_A}(\mathfrak{r}(x_1, x_2), x_3, x_4, \dots, x_{m+1}) \text{ with } \mathfrak{r}(x_1, x_2) = \sqrt{x_1^2 + x_2^2}.$$

The map $T^{m+1}(u)$ possesses hence cylindrical symmetry around the $m-1$ hypersurface $x_1 = x_2 = 0$. Moreover, It follows from the properties of u that $T^{m+1}(u)$ is equal to c_0 outside the ball $\mathbb{B}^{m+1}(3)$ and actually also on in region $\{\mathfrak{r}(x_1, x_2) \leq 1\}$, that is on the set $\mathbb{B}^2 \times \mathbb{R}^{m-1}$. Since we wish to obtain maps which are constant outside the unit ball $\mathbb{B}^{m+1}(1)$ we normalize $T^{m+1}(u)$ and consider the map ${\mathbb{I}_{\rm cyl}}^{m+1}(u)$ given, for $x \in \mathbb{R}^{m+1}$, by

$${\mathbb{I}_{\rm cyl}}^{m+1}(x) = T^{m+1}(u)(3x), \tag{5.24}$$

so that ${\mathbb{I}_{\rm cyl}}^{m+1}(u)$ equals c_0 outside $\mathbb{B}(1)$ and also

$${\mathbb{I}_{\rm cyl}}^{m+1}(u)(x) = \mathrm{c}_0 \text{ for } x \in \mathfrak{A}^{m+1} \equiv \mathbb{B}^2_{1/3} \times \mathbb{R}^{m-1}.$$

5.3.2 Restrictions to lower dimensional hyperplanes

For $\theta \in \mathbb{R}$, we consider the m-dimensional hyperplane $\mathcal{P}^m_\theta$ of $\mathbb{R}^{m+1}$ defined by

$$\mathcal{P}^m_\theta \equiv \mathrm{Vect}\,\{\cos\theta\,\vec{\mathbf{e}}_1 + \sin\theta\,\vec{\mathbf{e}}_2,\, \vec{\mathbf{e}}_3, \dots,\, \vec{\mathbf{e}}_{m+1}\}$$

and the half-hyperplane $\mathcal{P}^{m,+}_\theta$ defined by

$$\mathcal{P}^{m,+}_\theta = \{v \in \mathcal{P}^m_\theta, v.(\cos\theta\,\vec{\mathbf{e}}_1 + \sin\theta\,\vec{\mathbf{e}}_2) \geq 0\}. \tag{5.25}$$

Let $1 < p < +\infty$ and consider a map $v \in W^{1,p}(\mathbb{B}^{m+1}, \mathbb{R}^\ell)$. Its restriction to the intersection of the half-hyperplane $\mathcal{P}^{m,+}_\theta$ with the ball $\mathbb{B}^{m+1}(1)$ is in view of the trace theorem a map in $W^{1-\frac{1}{p},p}(\mathcal{P}_\theta^{m,+} \cap \mathbb{B}^{m+1})$. It yields a map $\mathbb{T}^m_{\mathtt{r},\theta}(v)$ defined on the m-dimensional half-ball $\mathbb{B}^{m,+} = \mathbb{B}^m \cap \{x_1 \geq 0\}$ setting for $(x_1, \ldots, x_m) \in \mathbb{B}^{m,+}$

$$\mathbb{T}^m_{\mathtt{r},\theta}(v)(x_1, \ldots, x_m) = v(x_1 \cos\theta, x_1 \sin\theta,\, x_2, \ldots, x_m). \tag{5.26}$$

Proposition 5.3. *Let* $c_0 \in \mathbb{R}^\ell$ *be given and let* U *be given in* $W^{1,p}_{\rm ct}(\mathbb{B}^{m+1}, \mathbb{R}^\ell)$ *such that* $U = c_0$ *on* $\mathfrak{A}^{m+1}$ *and let* $(W_n)_{n\in\mathbb{N}}$ *be a sequence converging weakly to* U *in* $W^{1,p}(\mathbb{B}^{m+1}, \mathbb{R}^\ell)$. *Then, there exists a subsequence* $\left(w_{\sigma(n)}\right)_{n\in\mathbb{N}}$ *and a sequence of angles* $(\theta_n)_{n\in\mathbb{N}}$ *converging to some limit* $\theta_\star$ *such that*

$$\mathbb{T}^m_{\mathtt{r},\theta_n}(W_{\sigma(n)})(\cdot) \rightharpoonup \mathbb{T}^m_{\mathtt{r},\theta_\star}(U) \text{ weakly in } W^{1,p}(\mathbb{B}^{m,+}, \mathbb{R}^\ell) \text{ as } n \to +\infty,$$

Proof. Since U is constant on the $\mathfrak{A}^{m+1}$, and since the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\mathbb{B}^{m+1}, \mathbb{R}^\ell)$ we have by Banach-Steinhaus theorem, for some constant $C > 0$ independent of n

$$\begin{aligned} C \geq \int_{\mathbb{B}^{m+1}} |\nabla W_n|^p \mathrm{d}x &= \int_0^{2\pi} \left(\int_{\mathcal{P}_\theta^{m,+}\cap\mathbb{B}^{m+1}} |\nabla W_n|^p |x_\theta| \right) \mathrm{d}\theta \text{ with } x_\theta = x \cdot (\cos\theta \vec{\mathbf{e}}_1 + \sin\theta \vec{\mathbf{e}}_2) \\ &\geq \frac{1}{2} \left(\int_{\mathcal{P}_\theta^{m,+}\cap\mathbb{B}^{m+1}} |\nabla W_n|^p \right) \mathrm{d}\theta. \end{aligned}$$

We may hence invoke Fubini's theorem to assert that given any $n \in \mathbb{N}$, there exists some angle $\tilde{\theta}_n \in \mathbb{R}$ such that

$$\int_{\mathcal{P}_{\tilde{\theta}_n}^{m,+}\cap\mathbb{B}^{m+1}} |\nabla W_n|^p \leq 2C,$$

It follows that the sequence $\mathbb{T}^m_{\mathtt{r},\theta_n}(W_n)$ is bounded in $W^{1,p}_{\rm ct}(\mathbb{B}^m, \mathbb{R}^\ell)$. By sequential weak compactness, we may extract a subsequence $(\sigma(n))_{n\in\mathbb{N}}$ such that $\theta_n = \tilde{\theta}_\sigma(n)$ converges to some limit $\theta_\star$ and such that $\mathbb{T}^m_{\mathtt{r},\theta_)}(W_{\sigma(n)})$ converges some map v in $W^{1,p}(\mathbb{B}^m)$. Since by the trace theorem we already now that the sequence converges to the map $\mathbb{T}^m_{\mathtt{r},\theta_\star} U$, the conclusion follows. □

Notice that the two operators $\mathbb{T}^m_{\mathtt{r},\theta}$ and $\mathbb{I}_{\rm cyl}{}^{m+1}$ we have introduce above are related through the identity

$$\mathbb{T}^m_{\mathtt{r},\theta} \circ \mathbb{I}_{\rm cyl}{}^{m+1}(v) = w, \text{ for any } v \text{ with compact support in } \mathbb{B}^m \text{ and any } \theta \in \mathbb{R}, \tag{5.27}$$

where the map w is defined by $w(x) = v(3x - A)$, for any $x \in \mathbb{B}^{m,+} \equiv \mathbb{B}^m \cap \{x_1 \geq 0\}$.

5.4 Proof of Proposition 5.2

For $m \in \mathbb{N}^*$, we define property $\mathcal{P}(m)$ as

$$\mathcal{P}(m) : \text{there exists } u_m \text{ in } W^{1,p}_{\rm ct}(\mathbb{B}^m, \mathcal{N}) \text{ which is not the weak limit of maps in } C^\infty(\mathbb{B}^m, \mathcal{N}).$$

We argue by induction and assume that $\mathcal{P}(m)$ holds. We claim that if $\mathcal{P}(m)$ holds, then

$$\mathbb{I}_{\rm cyl}{}^{m+1}(u) \text{ is not the weak limit in } W^{1,p}(\mathbb{B}^{m+1}, \mathcal{N}) \text{ of maps in } C^\infty(\mathbb{B}^{m+1}, \mathcal{N}). \tag{5.28}$$

In order to proof the claim (5.28), we argue by contradiction on assume that there exists a sequence of maps $(W_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{B}^{m+1},\mathcal{N})$ converging weakly to $U\equiv \mathbb{I}_{\rm cyl}(u)$.We apply Proposition 5.3 to the map U and the sequence $(W_n)_{n\in\mathbb{N}}$ so that, for some subsequence

$$\mathbb{T}^m_{\mathbb{r},\theta_n}(W_{\sigma(n)})(\cdot)\rightharpoonup \mathbb{T}^m_{\mathbb{r},\theta_\star}(U)=\mathbb{T}^m_{\mathbb{r},\theta_\star}\circ {\mathbb{I}_{\rm cyl}}^{m+1}(v)=u(3\cdot -A)$$

weakly in $W^{1,p}(\mathbb{B}^{m,+},\mathbb{R}^\ell)$ as $n\to+\infty$, where we have invoked (5.27). It follow that the map $v=u(3\cdot -A)$ is the weak limit of smooth maps between $\mathbb{B}^{m,+}$ and $\mathcal{N}$. Since $u(x)=v(\frac{x+A}{3})$ on $\mathbb{B}^m$ the same holds for u, but this contradicts our assumption and proves the claim (5.28) by contradiction.

It follows from (5.28) that, if $\mathcal{P}(m)$ holds then $\mathcal{P}(m+1)$ holds also, so that the proposition is proved by induction.

5.4.1 Proof of Theorem 3 completed

In Theorem 3, we have constructed a map $\mathcal{U}$ in $W^{1,3}_{\rm ct}(\mathbb{B}^4,\mathbb{S}^2)$ which is not the weak limit of smooth maps. Applying Proposition 5.2 with $m_0=4$and $\mathcal{N}=\mathbb{S}^2$), we deduce that for any given integer $m\geq 4$ there exists a map $\mathcal{V}_m$ in $W^{1,3}_{\rm Cte}(\mathbb{B}^m,\mathbb{S}^2)$ which is not the weak limit of maps in $C^\infty(\mathbb{B}^m,\mathbb{S}^2)$. This provides the proof of Theorem 3 in the special case $\mathcal{M}=\mathbb{B}^m$.

We extend next the result to an arbitrary smooth manifold $\mathcal{M}$ of dimension m. For that purpose, we choose an arbitrary point A on $\mathcal{M}$ and glue a suitably adapted copy of $\mathcal{V}_m$ at the point A. More precisely, we consider for $\rho>0$ the geodesic ball $\mathcal{O}_\rho(A)$ centered at A. If ρ is choosen sufficiently small, then there exist a diffeomorphism $\Phi:\mathcal{O}_\rho(A)\to\mathbb{B}^m$ and we may define a map $\mathcal{W}:\mathcal{M}\to\mathbb{S}^2$ setting

$$\mathcal{W}(x)=\mathcal{V}\left(\Phi(x)\right)\ \text{if}\ x\in\mathcal{O}_\rho(A),\ \mathcal{W}(x)=\mathbb{P}_{\rm south}\ \text{otherwise.}$$

One may then verify that $\mathcal{W}$ belongs $W^{1,3}(\mathcal{M},\mathbb{S}^2)$ and cannot be approximated weakly by maps in $C^\infty(\mathcal{M},\mathbb{S}^2)$, which completes the proof.

6 The lifting problem

6.1 Lifting the k-spaghetton map

Let $k\in\mathbb{N}^*$ be given and consider on $\mathbb{R}^3$ an arbitrary lifting U_k of the spaghetton map $\mathbf{S}^k_{\mathbf{pag}}$, that is a map $\mathrm{U}^k:\mathbb{R}^3\to SU(2)\simeq\mathbb{S}^3$ such that $\Pi\circ\mathrm{U}^k=\mathbf{S}^k_{\mathbf{pag}}$. Although the relationship between U^k and $\mathbf{S}^k_{\mathbf{pag}}$ has a genuine nonlocal nature, as suggestion by the relation (2.10), the peculiar geometry of the spaghetton map allows to recover some locality. This is expressed in the next lower bound.

Proposition 6.1. *Let U^k be any lipschitz lifting of the spaghetton map $\mathbf{S}^k_{\mathbf{pag}}$, that is such that $\mathbf{S}^k_{\mathbf{pag}}=\Pi\circ\mathrm{U}^k$. Then, we have, for every $1\leq p<+\infty$ and for some constant $C_p>0$ depending only on p*

$$\int_{\mathfrak{L}^k}|\nabla\mathrm{U}^k|^p\geq C_pk^{2p}. \tag{6.1}$$

The result is mainly a consequence of the following:

Lemma 6.1. *Let $a \in [0,1]$ and let $\Omega \subset P_{1,2}(a) \cap \{x_2 \geq 0\}$ be a smooth regular convex set such that $\Omega \supset [0,1] \times [8,11] \times \{a\}$. We have*

$$\left| \int_\Omega [\mathbf{S}^k_{\mathbf{pag}}]^*(\omega_{_{\mathbb{S}^2}}) \right| = 4\pi k^2. \tag{6.2}$$

If U^k *is as in Proposition 6.1 and* $\mathcal{C} = \partial\Omega$*, then we have*

$$\int_{\mathcal{C}} |\nabla_{_3} \mathrm{U}^k| \geq 2\pi k^2. \tag{6.3}$$

Proof of Lemma 6.1. Each of the k^2 fibers $\mathfrak{L}^{k,\perp}_{i,q}$ intersect the half-plane $\mathbb{R} \times [0,+\infty[\times\{a\}$ at a unique point $B_{i,q}(a)$ (see Figures 6 and 7). we notice that the points $B_{i,q}(0)$ belong to the square $[0,1] \times [9,11] \times \{0\}$, a little trigonometry shows that more generally $B_{i,q}(a)$ belongs to the rectangle $[0,1] \times [17/2, 11] \times \{a\}$. Our assumption on Ω hence implies that a neighborhood of the points $B_{i,q}(a)$ belongs to Ω. In view of the Pontryagin construction, near each point $M_{i,j}(a)$, the restriction of the spaghetton map $\mathbf{S}^k_{\mathbf{pag}}$ to the plane $P_{1,2}(a)$ maps a small neighborhood of $M_{i,j}(a)$ onto the sphere $\mathbb{S}^2$ yielding a contribution equal to the area $\mathbb{S}^2$, that is 4π to the integral in (6.1). Adding the contributions of the k^2 points, (6.2).

For the second assertion, we consider, as in subsection 2.2.1, the $su(2)$ valued 1-form $A^k \equiv (\mathrm{U}^k)^{-1} \cdot d\mathrm{U}^k$ and its first component the real-valued 1-form $A^k_1 = A^k.\sigma_1$, so that the curvature equation (2.5) leads to the relation

$$dA^k_1 = 2[\mathbf{S}^k_{\mathbf{pag}}]^*(\omega_{_{\mathbb{S}^2}}). \tag{6.4}$$

Integrating on Ω we deduce from (6.2) and (6.4) that $|\displaystyle\int_{\mathcal{C}} A^k_1| = 2\pi k^2$. Since $|\nabla_{_3} \mathrm{U}^k| \geq |A^k_1|$, we conclusion (6.3) follows. □

Proof of Proposition 6.1. Let $a \in [0,1]$. We choose as sets Ω the disks

$$\mathcal{D}(r,a) \equiv \mathbb{D}^2(r) \times \{0\} + N_0(a) \text{ where } N_0(a) = \{(1/2, 19/2\} \times \{a\}$$

so that for $r \geq 2$ we have $\mathcal{D}(r,a) \supset [0,1] \times [8,11] \times \{a\}$ and $\mathcal{D}(r,a) \subset \mathbb{R} \times [0,+\infty[\times\{a\}$ for $r \leq 8$. We may hence apply (6.3) to the circles $\mathcal{C}(r,a) \equiv \partial\mathcal{D}(r,a)$ for $2 \leq r \leq 8$. Integrating the obtain estimate with respect to the variable r, we are led to

$$\int_{\mathcal{D}(a,8)\setminus\mathcal{D}(a,2)} |\nabla_{_3} \mathrm{U}^k| \mathrm{d}x_1\, \mathrm{d}x_1 \geq 12\pi k^2 \tag{6.5}$$

We set $\mathcal{W} = \underset{a\in[0,1]}{\cup} (\mathcal{D}(a,8) \setminus \mathcal{D}(a,2))$. Integrating (6.5) with respect to a, we are led to

$$\int_{\mathcal{W}} |\nabla_{_3} \mathrm{U}^k| \mathrm{d}x \geq 12\pi k^2,$$

which leads directly to (6.1) in the case $p = 1$. The general case is deduced using Hölder's inequality. □

Notice that, in view of Proposition 3, we have $|\nabla \mathbf{S}^k_{\mathbf{pag}}| \leq \mathbf{C}_{\rm spg} k$ so that, for any $1 \leq p < +\infty$, we have

$$\int_{\mathfrak{L}^k} |\nabla \mathbf{S}^k_{\mathbf{pag}}|^p \leq C_p k^p, \tag{6.6}$$

which has to be compared with (6.1). The result in Lemma 6.1 carries over to some extend to Sobolev maps.

Proposition 6.2. *Let* $2 \leq p < +\infty$ *and* $\mathrm{U}^k \in W^{1,p}_{\rm loc}(\mathbb{R}^3, \mathbb{S}^3)$ *be such that* $\mathbf{S}^k_{\mathbf{pag}} = \Pi \circ \mathrm{U}^k$. *Then* (6.1) *holds.*

Proof. In the case $p \geq 3$ smooth maps are dense in $W^{1,p}_{\rm loc}(\mathbb{R}^3, \mathbb{S}^3)$ and a standard approximation result yields the result. In the case $2 \leq p < 3$ smooth maps are no longer dense, but one may prove that, since the spaghetton map is smooth, any $W^{1,p}$ lifting of the spaghetton map can be approximated by smooth maps, yielding hence a similar proof. □

Remark 6.1. In contrast, the result of Lemma 6.2 is no longer true for $1 \leq p < 2$. This observation related to the fact that there are lifting $W^{1,p}_{\rm loc}(\mathbb{R}^3, \mathbb{S}^3)$ which are singular, for instance on the fibers $\mathfrak{L}^k_{i,j}$ and $\mathfrak{L}^{k,\perp}_{i,j}$ (see e.g. the corresponding results in [8]). Moreover, it that case, it is difficult to give a meaning to (6.4).

6.2 Extension to higher dimensions

We add dimensions following the same scheme as in subsection 5.3.1. Since the spaghetton map $\mathbf{S}^k_{\mathbf{pag}}$ is constant outside the ball $\mathbb{B}^4(20)$ we renormalize it first so to obtain a constant map outside the unit ball, introducing the map $\tilde{\mathbf{S}}^k_{\mathbf{pag}}(\cdot) = \mathbf{S}^k_{\mathbf{pag}}(20\,\cdot)$, and then consider the map

$$\mathbf{S}^{k,5}_{\mathbf{pag}} = \mathbb{I}^5_{\rm cyl}(\tilde{\mathbf{S}}^k_{\mathbf{pag}}))$$

which is a lipschitz map on $\mathbb{R}^5$ which is constant outside the unit ball $\mathbb{B}^5$. More generally, given $m \geq 5$, we define iteratively the map $\mathbf{S}^{k,m}_{\mathbf{pag}}$ on the ball $\mathbb{B}^m$ as

$$\mathbf{S}^{k,m}_{\mathbf{pag}}(x) = \mathbb{I}^m_{\rm cyl}(\mathbf{S}^{k,m-1}_{\mathbf{pag}}(20x)) \text{ for } X \in \mathbb{B}^m$$

with the convention $\mathbf{S}^{k,3}_{\mathbf{pag}} = \tilde{\mathbf{S}}^k_{\mathbf{pag}}$. In view of (6.6), we obtain the bound

$$\int_{\mathbb{B}^m} |\nabla \mathbf{S}^{k,m}_{\mathbf{pag}}|^p \leq C k^p. \tag{6.7}$$

Lemma 6.2. *Let* $2 \leq p < +\infty$ *and* $\mathrm{U}^m_k \in W^{1,p}_{\rm loc}(\mathbb{B}^m, \mathbb{S}^3)$ *be such that* $\mathbf{S}^{k,m}_{\mathbf{pag}} = \Pi \circ \mathrm{U}^m_k$. *Then we have*

$$\int_{\mathbb{B}^m} |\nabla \mathrm{U}^m_k|^p \geq \mathrm{C}^m_p k^{2p}. \tag{6.8}$$

Proof. We establish inequality (6.8) arguing by induction on the dimension m. We first observe that the lower bound (6.8) has already been established for $m = 3$ in Lemma 6.2 with the choice of constant $\mathrm{C}^m_p = C_p$, where C_p refers to the constant in inequality (6.1). We next assume by induction that inequality (6.8) has been established some integer $m \geq 3$ and we are going to show that it then holds also in dimension $m + 1$. For that purpose, let $\mathrm{U}^{m+1}_k \in W^{1,p}_{\rm loc}(\mathbb{B}^{m+1}, \mathbb{S}^3)$ be an arbitrary lifting of the map $\mathbf{S}^{k,m+1}_{\mathbf{pag}}$. For $\theta \in [0, 2\pi)$, we consider the half-hyperplane $\mathcal{P}_\theta^{m,+}$ defined in (5.25) and the map $\mathbb{T}^m_{\mathbb{r},\theta}(\mathrm{U}_k^{m+1})$ defined on the m-dimensional ball $\mathbb{B}^m$ thanks to (5.26). It follows from these definitions that

$$\left(\Pi \circ \mathbb{T}^m_{\mathbb{r},\theta}(\mathrm{U}_k^{m+1})\right)(x) = \mathbf{S}^{k,m}_{\mathbf{pag}}(3x - A) \text{ for } x \in \mathbb{B}^{m,+}.$$

Hence, since by induction we assume that (6.8) holds in dimension m, we are led to the lower bound

$$\int_{\mathcal{P}_\theta^{m,+}\cap\mathbb{B}^m(\frac{1}{3}A,\frac{1}{3})} |\nabla \mathrm{U}_k^{m+1}|^p \geq \mathrm{C}_p^m \left(\frac{1}{3}\right)^{m-p} k^{2p}.$$

Integrating with respect to θ on the interval $(0, 2\pi)$ we obtain

$$\int_{\mathbb{B}^{m+1}} |\nabla \mathrm{U}_k^{m+1}|^p \geq \mathrm{C}_p^m \frac{2\pi}{3}\left(\frac{1}{3}\right)^{m-p} k^{2p},$$

so that the property (6.8) is established for the dimension $m+1$ choosing the constant C_p^{m+1} as $\mathrm{C}_p^{m+1} = 2\pi\left(\frac{1}{3}\right)^{m+1-p}\mathrm{C}_p^m$. □

6.3 Proof of Theorem 5

We first construct a map $\mathcal{V} = \mathcal{V}_0$ in the special case $\mathcal{M} = \mathbb{B}^m$, imposing moreover the additional condition $\mathcal{V}_0 = \mathbb{P}_{\text{south}}$ on $\partial\mathbb{B}^m$.

6.3.1 Construction of $\mathcal{V}_0$ on $\partial\mathbb{B}^m$.

Gluing copies of the $\mathbf{S}^{k,m}_{\mathbf{pag}}$*'s.* We construct as in subsection 5.2.1 a sequence of radii $(\mathfrak{r}_{\mathrm{i},p})_{\mathrm{i}\in\mathbb{N}^*}$ and a sequence of integers $(\mathrm{k}_{\mathrm{i},p})_{\mathrm{i}\in\mathbb{N}}$ such that the following properties are satisfied:

$$\sum_{\mathrm{i}\in\mathbb{N}} \mathfrak{r}_{\mathrm{i},p} = \frac{1}{8}, \quad \sum_{\mathrm{i}\in\mathbb{N}} \mathfrak{r}_{\mathrm{i},p}^{m-p}\mathrm{k}_{\mathrm{i},p}^p < +\infty \text{ and } \sum_{\mathrm{i}\in\mathbb{N}} \mathfrak{r}_{\mathrm{i},p}^{m-p}\ \mathrm{k}_{\mathrm{i},p}^{2p} = +\infty. \tag{6.9}$$

In the case $m-p > 1$, a possible choice for these sequences is given, for $\mathfrak{i} \geq 2$, by $\mathfrak{r}_\mathfrak{i} = \dfrac{\mathrm{c}}{\mathfrak{i}(\log \mathfrak{i})^2}$ and $\mathrm{k}_\mathrm{i} = \left[\mathfrak{i}^{\frac{m-p-1}{p}}\right]$, where $\mathrm{c} = \dfrac{1}{8}\sum_{\mathrm{i}\in\mathbb{N}}\dfrac{1}{\mathfrak{i}(\log \mathfrak{i})^2}$. In the case $0 < m-p \leq 1$, we may choose instead for $\mathfrak{i} \geq 2$, $\mathfrak{r}_i = \mathfrak{i}^{-\frac{3}{m-p}}$ and $\mathrm{k}_\mathrm{i} = \mathfrak{i}^{\frac{1}{p}}$.

We define as above the set of points $\{\mathrm{M}_\mathrm{i}\}_{\mathrm{i}\in\mathbb{N}}$ in $\mathbb{B}^m$ by $\mathrm{M}_\mathrm{i} = 4\left(\sum_{j=0}^{\mathrm{i}} \mathfrak{r}_\mathrm{i}\right)\vec{\mathbf{e}}_1$ where $\vec{\mathbf{e}}_1 = (1, 0, \ldots, 0)$, for $\mathfrak{i} \in \mathbb{N}$, so that these points converge to $\mathrm{M}_\star = \frac{1}{2}\vec{\mathbf{e}}_1$ as $\mathfrak{i} \to +\infty$, and consider the collection of disjoint balls $(B_\mathrm{i})_{i\in\mathbb{N}}$ defined by $B_\mathrm{i} \equiv \mathbb{B}^m(\mathrm{M}_i, \mathfrak{r}_i)$ for $\mathfrak{i} \in \mathbb{N}$. We then define the map $\mathcal{V}_0$ on $\mathbb{B}^m$ as

$$\mathcal{V}_0(x) = \mathbf{S}^{k,m}_{\mathbf{pag}}\left(\frac{x - \mathrm{M}_\mathrm{i}}{\mathfrak{r}_\mathrm{i}}\right) \text{ if } x \in B_\mathrm{i}, \ \ \mathcal{U}(x) = \mathbb{P}_{\text{south}} \text{ if } x \in \mathbb{B}^4(1) \setminus \underset{\mathrm{i}\in\mathbb{N}}{\cup} B_\mathrm{i}. \tag{6.10}$$

so that $\mathcal{V}_0 = \mathbb{P}_{\text{south}}$ on the boundary $\partial\mathbb{B}^m$. Invoking the scaling properties (4) of the p-energy, we are led to

$$\mathrm{E}_p(\mathcal{V}_0, \mathbb{B}^m) = \sum_{\mathrm{i}\in\mathbb{N}} \mathrm{E}_p(\mathcal{V}_0, B_\mathrm{i}) = \sum_{\mathrm{i}\in\mathbb{N}} \mathfrak{r}_\mathfrak{i}^{m-p}\mathrm{E}_p(\mathfrak{S}_{\mathrm{k}_\mathrm{i}}) \leq \mathrm{C}\sum_{\mathrm{i}\in\mathbb{N}} \mathfrak{r}_\mathfrak{i}^{m-p}\mathrm{k}_\mathfrak{i}^p < +\infty,$$

so that $\mathcal{V}_0$ belongs to $W^{1,p}(\mathbb{B}^m, \mathbb{S}^2)$. Next assume that there exists a lifting U_0 of $\mathcal{V}_0$ in $W^{1,p}(\mathbb{B}^m, \mathbb{S}^3)$ and consider its restriction $\mathrm{U}_\mathfrak{i}$ to the ball $B_\mathfrak{i}$. It follows from Lemma 6.2 and the scaling properties of the energy that

$$\mathrm{E}_p(\mathrm{U}_0, B_\mathfrak{i}) \geq C_p^m \mathfrak{r}_\mathfrak{i}^{m-p} \mathrm{k}_\mathfrak{i}^{2p} \text{ and hence } \mathrm{E}_p(\mathrm{U}_0, \mathbb{B}^m) \geq C_p^m \sum_{\mathfrak{i} \in \mathbb{N}} \mathfrak{r}_\mathfrak{i}^{m-p} \mathrm{k}_\mathfrak{i}^{2p} = +\infty,$$

leading to a contradiction, which established the proof of the theorem in the special case considered in this subsection.

6.3.2 Proof of Theorem 5 completed for a general manifold $\mathcal{M}$

The argument is somewhat parallel to the argument in subsection 5.4.1. With the same notation, we set

$$\mathcal{V} = \mathcal{V}_0 \left(\Phi(x)\right) \text{ if } x \in \mathcal{O}_a, \mathcal{V} = \mathbb{P}_{\mathrm{south}} \text{ otherwise,}$$

and we verify that the map $\mathcal{V}$ has the desired property.

Appendix: related notions on branched transportation

In this Appendix we recall and recast some aspects of branched transportation, an optimization problem which is involved in a wide area of applications, including practical ones, for instance leafs growth, or network design. We focus on questions directly related to our main problem, trying to keep however this part *completely* self-contained.

Branched transportation appears when one seeks to optimize transportation costs when the average cost decreases with density. Consider a finite set A of points belonging to the closure of a bounded open domain Ω of $\mathbb{R}^m$: We wish to connect (or transport) them to the boundary $\partial\Omega$. The total cost be to be minimized is the sum of the length of paths joining the given points to the boundary multiplied by a *density function* φ, depending on the density representing the number of points using the same portion of paths. For minimizers, such paths are unions of segments, but possibly with varying densities. The intuitive idea is that it is cheaper to share the same path than to travel alone, so that high densities are selected by the minimization process. This induces branching points, i.e. points where segments join to induce higher multiplicity. The density function appearing in our context, as well as in a large of part of the literature, is given by the power law $\varphi(d) = d^\alpha$, with given parameter $0 < \alpha < 1$. Notice that φ is sublinear, $(d_1 + d_2)^\alpha << d_1^\alpha + d_2^\alpha$ for large numbers. Our aim is to describe the behavior of minimal branched transportation when the number of *points increases and ultimately goes to* $+\infty$. A special emphasis is put on the critical case $\alpha = \alpha_m = 1 - 1/m$. Our presentation closely follows [30, 31] and also the general presentation in [4]: We perform however the necessary adaptation for connections to the boundary, which have been less considered so far. As far as we are aware of, the main result of this Appendix, presented in Theorem A.1, is new.

A.1 Directed graphs connecting a finite set to the boundary

A.1.1 Directed graphs and charges

The *theory of oriented graphs* offers an appropriate framework to describe the object we have in mind[15]. Such oriented graphs involve:

- **Points**. These points are of two kinds: The points in A we wish to connect to the boundary, but also additional points, the branching points and points on the boundary.
- **Oriented segments**. They join the points above. Orientation is important, as well as multiplicity which is a positive integer.

A general directed graphs G is defined by a *finite* set $E(G)$ of *oriented segments with endpoints belonging to* $\overline{\Omega}$: If e is a segment $E(G)$, then we denote by e^- and e^+ the endpoints of e, e^- (resp e^+) denoting the entrance point (resp the exit point), so that $e = [e^-, e^+]$ and $\partial e = \{e^-, e^+\}$. We assume that for any segment e in the additional condition that

$$\text{if } [e^-, e^+] \in E(G) \text{ then } [e^+, e^-] \notin E(G) \tag{A.1}$$

holds, i.e. if an oriented segment belongs to the graph, the segment with opposite direction does not. Segments may be repeated with multiplicity. If $e \in E(G)$, we denote by $d(e, G) \in \mathbb{N}^\star$ its multiplicity[16] and simply write $d(e)$ is this is not a source of confusion. We denote by $\mathcal{G}(\Omega)$ the set of graphs having the previous properties, namely

$$\mathcal{G}(\Omega) = \{\text{graphs } G \text{ such that (A.1) holds}\}.$$

We denote by $V(G)$ be set of vertex of the graph, i.e.

$$V(G) = \underset{e \in E(G)}{\cup} \partial e = \underset{e \in E(G)}{\cup} \{e^-, e^+\} \subset \overline{\Omega}.$$

Given a vertex $\sigma \in V(G)$, we set

$$E^\pm(\sigma, G) = \{e \in E(G), e^\mp = a\} \text{ and } E(\sigma, G) = E^+(\sigma, G) \cup E^-(\sigma, G),$$

so that $E^+(\sigma, G)$ (resp $E^-(\sigma, G)$) represents the sets of segments of the graph G having σ as entrance point (resp. as exit point) and $E(\sigma, G)$ the subset of segments having σ as endpoint. We set

$$\sharp\left(E^\pm(\sigma, G)\right) = \sum_{e \in E^\pm(\sigma, G)} d(e) \in \mathbb{N}^\star$$

and introduce the notion of *charge* of a point $\sigma \in V(G)$ as

$$\mathrm{Ch_g}(\sigma, G) = \sharp\left(E^+(\sigma, G)\right) - \sharp\left(E^-(\sigma, G)\right) \in \mathbb{Z}. \tag{A.2}$$

We consider the subsets $V_0(G)$, $V_{\mathrm{chg}}(G)$ and $V_{\mathrm{bd}}(G)$ of $V(G)$ defined by

$$\begin{cases} V_0(G) = \{\sigma \in V(G), \mathrm{Ch_g}(\sigma, G) = 0\} \\ V_{\mathrm{chg}}(G) = \{\sigma \in V(G), \mathrm{Ch_g}(\sigma, G) \neq 0, \sigma \in V(G) \setminus (V_0(G) \cup \partial\Omega)\} \\ V_{\mathrm{bd}}(G) = \{\sigma \in V(G), \sigma \in \partial\Omega\}. \end{cases} \tag{A.3}$$

[15] we might also invoke the theory of 1-dimensional integer currents, which is however more abstract

[16] This is of course an essential feature for branched transportation

A point $\sigma \in V_0(G)$ will be termed *a pure branching point*, a point in $V_{\rm chg}(G)$ a charged point or simply a *charge*[17]. The set of graphs with only *positive charges* plays a distinguished role in the later analysis. We set

$$\begin{cases} \mathcal{G}^+(\Omega) = \{G \in \mathcal{G}(\Omega),\ \text{s.t. } \mathrm{Ch_g}(\sigma, G) \geq 0\ \forall \sigma \in G\} \\ \mathcal{G}_0(\Omega) = \{G \in \mathcal{G}(\Omega),\ \text{s.t. } \mathrm{Ch_g}(\sigma, G) = 0\ \forall \sigma \in G\}. \end{cases} \tag{A.4}$$

In several places, we will invoke the fact that, if $G \in \mathcal{G}^+(\Omega)$, then

$$E^+(\sigma, G) \neq \emptyset \text{ for any } \sigma \in V(G). \tag{A.5}$$

Indeed, by definition $E(\sigma, G)$ contains at least one element, and since the charge is positive there are at least as many elements in $E^+(\sigma, G)$ as in $E^-(\sigma, G)$.

A.1.2 Elementary operations on directed graphs

Gluing graphs. Let G_1 and G_2 be two graphs in $\mathcal{G}(\Omega)$. We assume furthermore that

$$\text{if } e_1 \in E(G_1),\ e_2 \in E(G_2) \text{ then } e_1 = e_2 \text{ or } e_1 \cap e_2 \text{ contains at most one point.} \tag{A.6}$$

If condition (A.6) is not met one may add new points and divide some segments in two so that the transformed graph satisfy the condition. Given a segment e, we denote $-e$ the segment with opposite orientation, i.e. if $e = [e^-, e^+]$, then $-e \equiv [e^+, e^-]$. We consider the following subsets of $E(G_1) \cup E(G_2)$

$$\begin{cases} E_0(G_1, G_2) \equiv \{e \in G_1 \text{ s.t. }, -e \in G_2 \text{ with } d(e, G_1) = d(-e, G_2)\} \\ E^+(G_1, G_2) \equiv \{e \in G_1 \text{ s.t. } -e \notin G_2\} \cup \{e \in G_2 \text{ s.t. } -e \notin G_1\} \\ E^\pm(G_1, G_2) \equiv \{e \in G_1 \text{ s.t. } -e \in G_2 \text{ with } d(e, G_1) > d(-e, G_2)\} \\ E^\mp(G_1, G_2) \equiv \{e \in G_2 \text{ s.t. } -e \in G_1 \text{ with } d(e, G_2) > d(-e, G_1)\} \end{cases}$$

We define the *glued* graph

$$G = G_1 \curlyvee G_2 \in \mathcal{G}(\Omega), \tag{A.7}$$

given by the set of its directed segments

$$\begin{aligned} E(G) &\equiv E(G_1) \cup E(G_2) \setminus E_0(G_1, G_2) \\ &= E^+(G_1, G_2) \cup E^\pm(G_1, G_2) \cup E^\mp(G_1, G_2). \end{aligned} \tag{A.8}$$

with multiplicities given by

$$\begin{cases} d(e, G) = d(e, G_1) + d(e, G_2) \text{ if } e \in E^+(G_1, G_2) \\ d(e, G) = d(e, G_1) - d(e, G_2) \text{ if } e \in E^\pm(G_1, G_2) \\ d(e, G) = d(e, G_2) - d(e, G_1) \text{ if } e \in E^\mp(G_1, G_2), \end{cases} \tag{A.9}$$

where we have used the convention, for $i = 1, 2$, that $d(e, G_i) = 0$ if $e \notin G_i$. His vertex set is then provided by the endpoints of the segments, so that $V(G) \subset V(G_1) \cup V(G_2)$. The inclusion might be strict in the general case. We have:

[17] Notice that a charged point may however also be a branching point

Proposition A.1. *Let* $\sigma \in V(G)$*. We have*

$$\mathrm{Ch_g}(\sigma, G_1 \curlyvee G_2) = \mathrm{Ch_g}(\sigma, G_1) + \mathrm{Ch_g}(\sigma, G_2), \tag{A.10}$$

with the convention, for $i = 1, 2$*, that* $\mathrm{Ch_g}(\sigma, G_i) = 0$ *if* $\sigma \notin G_i$*. If* $G_i \in \mathcal{G}^+(\Omega)$ *for* $i = 1, 2$*, then we have*

$$V_{\mathrm{chg}}(G) = V_{\mathrm{chg}}(G_1) \cup V_{\mathrm{chg}}(G_2). \tag{A.11}$$

The result is a direct consequence of (A.9). We reader may check also that the gluing operation $\curlyvee$ enjoies classical properties as commutativity and associativity. Finally we write

$$G = G_1 \overset{\star}{\curlyvee} G_2 \tag{A.12}$$

in the case when, if a segment e belongs to $E(G_1)$, then the opposite segment does not belong to G_2, so that no cancellations for segments occur in the gluing process. The set $E(G)$ is in that case the union $E(G_1) \cup E(G_2)$, the multiplicities being sumed.

Subgraphs. Let G_1 and G be two graphs in $\mathcal{G}(\Omega)$. We say that G_1 is a subgraph of G if $E(G_1) \subset E(G)$ and if the multiplicities satisfy the conditions

$$d(e, G_1) \leq d(e, G) \text{ for } e \in E(G_1). \tag{A.13}$$

If the two conditions above are satisfied, then we write $G_1 \Subset G$. We introduce next the complement G_2 of G_1 with respect to G. We define the set of oriented segments of G_2 as

$$E(G_2) = [E(G) \setminus E(G_1)] \cup E_{\mathrm{comp}}(G_1, G)$$

where $E_{\mathrm{comp}}(G_1, G)$ is defined as

$$E_{\mathrm{comp}}(G_1, G) \equiv \{e \in E(G_1), d(e, G_1) < d(e, G)\},$$

and with multiplicities given by

$$\begin{cases} d(e, G_2) = d(e, G) \text{ if } e \in E(G) \setminus E(G_1) \\ d(e, G_2) = d(e, G) - d(e, G_1) \text{ if } e \in E_{\mathrm{comp}}(G_1, G). \end{cases} \tag{A.14}$$

Notice that there are no segments in G_1 and G_2 with opposite orientations. It follows from these definitions that

$$G = G_1 \overset{\star}{\curlyvee} G_2, \text{ so that we may write } G_2 = G \smallsetminus G_1.$$

We observe that, in view of Proposition A.1 , if G and G_1 belong to $\mathcal{G}^+(\Omega)$ and if furthermore

$$\mathrm{Ch_g}(\sigma, G_1) \leq \mathrm{Ch_g}(\sigma, G), \text{ for any } \sigma \in V_{\mathrm{chg}}(G) \tag{A.15}$$

then $G_2 \in \mathcal{G}^+(\Omega)$. If G_1 and G are two graphs in $\mathcal{G}(\Omega)$ such that $G_1 \Subset G$ and such that condition (A.15) is satifies, then we write $G_1 \overset{\star}{\Subset} G$.

Restrictions of graphs to subdomains. Let $\Omega_1 \subset \Omega$ be a subdomain of Ω and assume for the sake of simplicity (and also for further applications) that both Ω_1 and Ω are polytopes. Let

G a graph in $\mathcal{G}(\Omega)$. We define the restriction G_1 of G to Ω_1 as the graph such that its set of segments is given by

$$E(G_1) = \{e \cap \bar{\Omega}_1, e \in E(G)\}.$$

Its set of vertices is then given by

$$V(G_1) = (V(G) \cap \Omega_1) \cup \left(\underset{e \in E(G)}{\cup} \partial \left(\bar{e} \cap \bar{\Omega}_1 \right) \right).$$

We use the notation $G_1 = G \llcorner \Omega_1$. One may check that $G_1 \in \mathcal{G}(\Omega_1)$ and also $G_1 \in \mathcal{G}(\Omega)$; If we assume moreover that $G \in \mathcal{G}^+(\Omega)$, then we have $G \in \mathcal{G}^+(\Omega_1)$, but it does not belong, in general to $\mathcal{G}^+(\Omega)$, since negative charges may be created on $\partial\Omega_1$.

A.1.3 The single path property

The next property, termed *the single path property*, has been considered in [30, 31, 4].

Definition A.1. *Let $G \in \mathcal{G}(\Omega)$. We say that G possesses the single path property, if for any vertex $\sigma \in V(G) \cap \Omega$ there is at most one segment e in $E(G)$, possibly repeated with multiplicity, such that σ is the entrance point of e, that is $E^+(\sigma, G)$ is a singleton or empty.*

In other words, if G possesses the single path property, then there might be several segments ending at the same vertex, but at most one starting from it. This property possibly models some intuitive features, as for instance in river networks. We denote by $\mathcal{G}_{\rm sp}(\Omega)$ (resp. $\mathcal{G}^+_{\rm sp}(\Omega)$) the set of all graphs in $\mathcal{G}(\Omega)$ (resp. $\mathcal{G}^+(\Omega)$) which possess the single path property. Notice that if $G \in \mathcal{G}^+(\Omega)$, then $E^+(\sigma, G)$ can not be empty for $\sigma \in \Omega$, so that it is necessarily a singleton.

A.1.4 Threads, loops and bridges

A heuristic image of the notion of thread we describe next, is provided by a a curve for with one end in given by a point in A, reaching to the boundary $\partial\Omega$, and constructed using only segments in $E(G)$. This suggest the following definition.

Definition A.2. *A directed graph G is said to be a polygonal curve in Ω, in short a P_Ω-curve, if and only if there exists an ordered collection $B = (b_1, \dots, b_{\rm q})$ of q not necessary distinct points in $\bar{\Omega}$ such that G satisfies $V(G) = B$, relation* (A.1) *holds, and*

$$E(G) = \{[b_i, b_{i+1}], \text{ with multiplicity } 1, i = 1, \dots, \mathrm{q}\} \text{ and } b_{\rm q} \in \partial\Omega \text{ or } b_{\rm q} = b_1. \tag{A.16}$$

Since the P_Ω-curve G is completely determined by the orderet set B, we may set

$$G = \mathrm{G}_{\rm rp}(B).$$

Notice that, even if in (A.16) each segment $[b_i, b_{i+1}]$ appears with multiplicity one, the same segment may appear possibly in a further part of the sequence, so that its final multiplicity might be larger that one.

Definition A.3. *Let $G = \mathrm{G}_{\rm rp}(B)$ be a P_Ω-curve. We say that $\mathrm{G}_{\rm rp}(B)$ is*

- *a loop if either $b_1 = b_{\rm q}$.*

- *a bridge if* $b_1 \in \partial\Omega$ *and* $b_{\mathrm{q}} \in \partial\Omega$.
- *A thread emanating from a point* $p \in \Omega$ *if* $p = b_1$ *and* $b_{\mathrm{q}} \in \partial\Omega$.

We denote by $\mathcal{T}_{\mathrm{hread}}(p, \Omega)$ the set of all threads emanating from p. We notice that

$$\begin{cases} V_{\mathrm{chg}}(\mathrm{G_{rp}}(B) = \emptyset \text{ when } \mathrm{G_{rp}}(B) \text{ is a loop or a bridge} \\ V_{\mathrm{chg}}(\mathrm{G_{rp}}(B)) = \{a\} \text{ with } \mathrm{Ch_g}(a) = 1 \text{ if } \mathrm{G_{rp}}(B) \text{ is in } \mathcal{T}_{\mathrm{hread}}(p, \Omega). \end{cases} \tag{A.17}$$

Notice that loops and bridges are elements in $\mathcal{G}_0(\Omega)$. We denote by $\mathcal{L}_{\mathrm{oop}}(\Omega)$ the set of loops. We say that a graph G has a loop if there exists a loop L such that $L \Subset G$. In particular a thread $G = \mathrm{G_{rp}}(B)$ has *a loop* if there exists a subset formed of consecutive points in B yielding a loop. Given a point $p \in \Omega$ we denote by

$$\mathbb{T}_{\mathrm{hread}}(p, \Omega) \subset \mathcal{T}_{\mathrm{hread}}(p, \Omega)$$

the set of all threads *without loops emanating* from p. It follows a a quite straightforward way from the definitions above that the segments of a thread in $\mathbb{T}_{\mathrm{hread}}(p, \Omega)$ have exactly *multiplicity* one and that, if a thread has the single path property, then it has not loops. One may moreover verify:

Lemma A.1. *Let* $\sigma \in \Omega$ *and let* $T \in \mathcal{T}_{\mathrm{hread}}(\sigma, \Omega)$. *There exists a finite family* $(L_j)_{j\in J}$ *of loops such that*

$$T = T_p \overset{\star}{\curlyvee} \left(\overset{\star}{\underset{j\in J}{\curlyvee}} L_j \right) \text{ with } T_p \in \mathbb{T}_{\mathrm{hread}}(p, \Omega). \tag{A.18}$$

Proof. We may write $T = \mathrm{G_{rp}}(B)$ where B denotes an ordered set $B = \{b_1 = \sigma, b_2, \ldots, b_{\mathrm{q}}\}$, with $b_{\mathrm{q}} \in \partial\Omega$. If all points in B are distinct, then $T \in \mathbb{T}_{\mathrm{hread}}(\sigma, \Omega)$ and there is nothing to prove. Otherwise there are two points, say b_{i_1} and b_{i_2} with $1 \leq i_1 < i_2 < b_{\mathrm{q}}$ which are identical. Then we set $L_1 = \mathrm{G_{rp}}\{b_{i_1}, \ldots, b_{i_2} = b_{i_1}\}$ and $\tilde{T}_1 = \mathrm{G_{rp}}\{b_1, \ldots, b_{i_1}, b_{i_2+1}, \ldots, b_{\mathrm{q}}\}$. We verify that

$$T = \tilde{T}_1 \overset{\star}{\curlyvee} L_1 \text{ with } \tilde{T}_1 \in \mathcal{T}_{\mathrm{hread}}(p, \Omega) \text{ and } L_1 \text{ is a loop.}$$

If $\tilde{T}_1$ has no loop, then we are done. Otherwise, we start the process again with $\tilde{T}_1$. It stops in a finite number of iterations, since the number of points is finite. □

A.1.5 Subthreads and subloops

Consider a graph G in $\mathcal{G}^+(\Omega)$ and an ordered set $B = (b_1, \ldots, b_{\mathrm{q}})$ of elements of $V(G)$.

Definition A.4. *The* P_Ω*-curve* $\mathrm{G_{rp}}(B)$ *is said to be a maximal subcurve of* G *if* $\mathrm{G_{rp}}(B) \Subset G$ *and if* $b_i \in \Omega$ *for* $i = 1, \ldots, b_{\mathrm{q}-1}$ *and*

- *either* $b_{\mathrm{q}} \in \partial\Omega$ *or*
- *either there does not exist any point* $b_{\mathrm{q}+1} \in V(G)$ *such that* $\mathrm{G_{rp}}(b_1, \ldots, b_{\mathrm{q}}, b_{\mathrm{q}+1}) \Subset G$.

Our next result readily follows from the definition:

Lemma A.2. *Let* $G \in \mathcal{G}^+(\Omega)$ *and* $\sigma \in V(G)$. *There exists an ordered set* $B = (b_1, \ldots, b_{\mathrm{q}})$ *such that* $b_1 = \sigma$ *and such that* $\mathrm{G_{rp}}(B)$ *is a maximal subcurve of* G.

Proof. We construct the maximal subcurve inductively. Since $G \in \mathcal{G}^+(\Omega)$, it follows from (A.5) that $E^+(\sigma, G)$ is not empty, hence there exists some point $b_2 \in V(G)$ such that $[\sigma, b_2] \in E^+(\sigma, G)$ and therefore $\mathrm{G_{rp}}\{\sigma, b_2\} \Subset G$. If $b_2 \in \partial\Omega$ then $B \equiv \{\sigma, b_2\}$ is maximal and be are done. Otherwise, we notice as above that $E^+(b_2, G)$ is not empty so that there exists some point $b_3 \in V(G)$ such that $[b_2, b_3] \in E^+(\sigma, G)$ and therefore $\mathrm{G_{rp}}\{\sigma, b_2, b_3\} \Subset G$. If $b_3 \in \partial\Omega$ then $B \equiv \{\sigma, b_2, b_3\}$ is maximal and we are done. Otherwise we go on, until we reach the boundary or have no more segments available to go on. □

Lemma A.3. *Let $G \in \mathcal{G}^+(\Omega)$. A maximal subcurve $\mathrm{G_{rp}}(B)$ of of G is either a thread emanating from b_1 or a maximal loop.*

Proof. If $b_\mathrm{q} \in \partial\Omega$, then $\mathrm{G_{rp}}(B)$ is a thread emanating from σ and the statement is proved. We consider next the case when $b_\mathrm{q} \in \Omega$ and show that in this case $\mathrm{G_{rp}}(B)$ is a loop. To that aim, we claim first that there exist some index $i_0 \in \{1, \ldots, \mathrm{q}-1\}$ such that

$$b_\mathrm{q} = b_{i_0}. \tag{A.19}$$

Since $G \in \mathcal{G}^+(\Omega)$, it follows from (A.5) that the set $E^+(b_\mathrm{q}, G)$ contains at least one segment, say $[b_1, b_{\mathrm{q}+1}]$, where $b_{\mathrm{q}+1} \in V(G)$. if (A.19) were not true, then we would have $\mathrm{G_{rp}}(b_1, \ldots, b_\mathrm{q}, b_{\mathrm{q}+1}) \Subset G$ leading to a contradiction. It remains to show that

$$i_0 = 1. \tag{A.20}$$

Assume by contradiction that (A.20) is not true and let k be the number of times the point b_{i_0} appears in the ordered set B. Since it is both an exit and an entrance point for the segments in $E(\mathrm{G_{rp}}(B))$ except for the segment $[b_{\mathrm{q}-1}, b_\mathrm{q}]$ for which it it only an exit point, we deduce that

$$E^+(b_{i_0}, \mathrm{G_{rp}}(B)) = E^-(b_{i_0}, \mathrm{G_{rp}}(B)) - 1.$$

On the other hand, we deduce from (A.13) that

$$E^-(b_{i_0}, \mathrm{G_{rp}}(B)) \leq E^-((b_{i_0}, G) \leq E^+((b_{i_0}, G).$$

so that $E^+(b_{i_0}, \mathrm{G_{rp}}(B)) \leq E^+((b_{i_0}, G) - 1$. Hence we may choose some point $b_{\mathrm{q}+1}$ such that $\mathrm{G_{rp}}(b_1, \ldots, b_\mathrm{q}, b_{\mathrm{q}+1}) \Subset G$, which leads is a contradiction with the definition of maximal subcurves, so that (A.20) is established. This finally show that $\mathrm{G_{rp}}(B)$ is a loop in the case considered. □

Lemma A.4. *Let $B = (b_1, \ldots, b_\mathrm{q})$ such that $\mathrm{G_{rp}}(B)$ is a maximal subcurve of G. If $\mathrm{Ch_g}(b_1) > 0$ then $\mathrm{G_{rp}}(B)$ is a thread emanating from b_1.*

Proof. We have to show that $\mathrm{G_{rp}}(B)$ is not a loop, that is $b_\mathrm{q} \neq b_1$. Assume by contradiction that $b_\mathrm{q} = b_1$. Arguing as in the proof of Lemma A.3, we obtain

$$\begin{cases} E^+(b_{i_0}, \mathrm{G_{rp}}(B)) = E^-(b_{i_0}, \mathrm{G_{rp}}(B)) \text{ and} \\ E^-(b_{i_0}, \mathrm{G_{rp}}(B)) \leq E^-((b_{i_0}, G) = E^+((b_{i_0}, G) - \mathrm{Ch_g}(\sigma, G) \end{cases}$$

so that $E^+(b_{i_0}, \mathrm{G_{rp}}(B)) \leq E^+((b_{i_0}, G) - 1$. Hence we may choose some point $b_{\mathrm{q}+1}$ such that $\mathrm{G_{rp}}(b_1, \ldots, b_\mathrm{q}, b_{\mathrm{q}+1}) \Subset G$, which leads to a contradiction. □

Combining Lemmas A.1, A.2, A.3 and A.4, we deduce:

Corollary A.1. *Let $\sigma \in V_{\mathrm{chg}}G$. There exists a thread $T_\sigma \in \mathbb{T}(\sigma, \Omega)$ such that $T_\sigma \overset{\star}{\Subset} G$.*

A.1.6 Decomposing graphs into threads and loops and bridges

Consider a graph $G \in \mathcal{G}^+(\Omega)$. Since $V(G)$ is a finite set, we may write

$$V_{\rm chg}(G) = \{p_1, \ldots, p_{\ell_{\rm c}}\},$$

each point p_i in the collection having multiplicity $\mathrm{M}_i \equiv \mathrm{Ch_g}(\mathrm{p}_1, G) \in \mathbb{N}^*$. The following result emphasizes the importance of threads in this context:

Proposition A.2. *Let $G \in \mathcal{G}^+(\Omega)$. We may decompose the graph G as*

$$G = \overset{\star}{\underset{i\in\{1,\ldots,\ell_{\rm c}\}}{\curlyvee}} \left(\overset{\star}{\underset{j\in\{1,\ldots,\mathrm{M}_i\}}{\curlyvee}} T_{i,j} \right) \overset{\star}{\curlyvee} T_0 \text{ with } T_{i,j} \in \mathbb{T}_{\rm hread}(p_i, \Omega) \text{ and } T_0 \in \mathcal{G}_0(\Omega). \tag{A.21}$$

If moreover $G \in \mathcal{G}^+_{\rm sg}(\Omega)$, that is if G posssesses the single path property, then decomposition (A.21) *is unique and, for any $i \in \{1, \ldots, \ell_{\rm c}\}$, we have*

$$T_{i,j} = T_{i,j'} \text{ for } j \text{ and } j' \text{ in } \{1, \ldots, \mathrm{M}_i\}. \tag{A.22}$$

Proof. We present first the construction of the subgraphs $T_{1,1}$ and then proceed in an recursive way.

Step 1: construction of $T_{1,1}$. Since the point p_1 has positive charge M_i with respect to G, we may apply Corollary A.1 and choose $T_{1,1} = T_{p_1}$, so that define the graph

$$G_{1,1} = G \setminus T_{1,1}, \text{and hence } G = G_{1,1} \overset{\star}{\curlyvee} T_{1,1} \text{ with } G_1 \in \mathcal{G}^+(\Omega).$$

the total charge of G_1 has now decreases by 1. More precisely, it follows from the rules (A.14) and (A.17) for charges that for $i = 2, \ldots, \ell_{\rm c}$, we have

$$\mathrm{Ch_g}(p_i, G_{1,1}) = \mathrm{Ch_g}(p_i, G) \text{ for } i = 2, \ldots, \ell_{\rm c} \text{ and } \mathrm{Ch_g}(p_1, G_{1,1}) = \mathrm{Ch_g}(p_1, G) - 1,$$

in case $p_1 \in V(G_{1,1})$, which occurs in particular in p_1 has multiplicity. In the case $\ell_{\rm c} = 1$ and $\mathrm{M}_1 = 1$, we deduce that $G_{1,1} \in \mathcal{G}_0(\Omega)$, so that setting $T_0 = G_{1,1}$, we obtain (A.21). Otherwise, we proceed recursively.

Step 2: iterating the construction. We proceed as in step 1, but with G replaced by G_1. If $\mathrm{M}_1 > 1$, when invoke Corollary A.1 again to assert that there exists a thread $\tilde{T}_{1,2} \in \mathbb{T}_{\rm hread}(p_1, \Omega)$ which is a subgraph of $G_{1,1}$. We set $G_{1,2} = G_{1,1} \setminus T_{1,2}$ so that we have $G_{1,1} = G_{1,2} \overset{\star}{\curlyvee} T_{1,1}$ and

$$\mathrm{Ch_g}(p_i, G_{1,2}) = \mathrm{Ch_g}(p_i, G) \text{ for } i = 2, \ldots, \ell_{\rm c} \text{ and } \mathrm{Ch_g}(p_1, G_1) = \mathrm{Ch_g}(p_1, G) - 2,$$

If $\mathrm{M}_1 = 2$ and $\ell_{\rm c}$, then we are done, then we obtain (A.21) with $T_0 = G_{1,2}$. Otherwise, we proceed with $G_{1,2}$ and construct iteratively the threads $\tilde{T}_{1,3}, \ldots, \tilde{T}_{1,\mathrm{M}_1}$, and then $\tilde{T}_{2,1}, \ldots, T_{2,\mathrm{M}_2}$, ... $\tilde{T}_{\ell_{\rm c},1}, \ldots, \tilde{T}_{\ell_{\rm c},\mathrm{M}_{\ell_{\rm c}}}$. Setting $T_0 = G_{\ell_{\rm c},\mathrm{M}_{\ell_{\rm c}}}$ we obtain formula (A.21). □

Remark A.1. In (A.21), we may impose additionnally that $G_{\rm thread}$ has no loop.

A.1.7 Prescribing charges and the Kirchhoff law

We are now in position to model connections of a given set to the boundary with possible branching points . Consider a finite set $A \subset \overline{\Omega}$, with points possibly repeated with multiplicity $\mathrm{M}(a) \in \mathbb{N}^{\star}$, so that $\sharp(A) = \sum_{a \in A} \mathrm{M}(a)$. We restrict our attention to graphs $G \in \mathcal{G}^{+}(\Omega)$ satisfying the additional conditions

$$V_{\mathrm{chg}}(G) = A \text{ and } \mathrm{Ch_g}(a, G) = \mathrm{M}(a),\ \forall a \in A \cap \Omega. \tag{A.23}$$

This is equivalent to *Kirchhoff's law*

$$\begin{cases} \sharp\left(E^{+}(\sigma, G)\right) = \sharp\left(E^{-}(\sigma, G)\right) + \mathrm{M}(a) & \text{for } a \in A \cap \Omega \subset V(G) \\ \sharp\left(E^{+}(\sigma, G)\right) = \sharp\left(E^{-}(\sigma, G)\right) & \text{for any } \sigma \in V(G) \cap \Omega \setminus A, \end{cases} \tag{A.24}$$

We introduce the class of graphs aimed to model connections of points in A to the boundary, namely the set

$$\mathcal{G}(A, \partial\Omega) = \{G \in \mathcal{G}^{+}(\Omega) \text{ such that } A \subset V(G) \text{ and (A.24) holds}\}. \tag{A.25}$$

It follows that if G belongs to $\mathcal{G}(A, \partial\Omega)$, then the points of A are the only "source" points of the graph inside Ω, with charge $\mathrm{M}(a)$, whereas all the other points have charge 0. The simplest example G_0 of an element in $\mathcal{G}(A, \partial\Omega)$ when Ω is convex is provided by the graph for which each element a in A is connected by a segment to an element of the boundary b so that in this case $V(G_0) = \underset{a \in A}{\cup} \{a, b\}$ and $E(G) = \underset{a \in A}{\cup} \{[a, b]\}$. Notice that, going back to (A.21), if $G \in \mathcal{G}(A, \partial\Omega)$ then we have likewise

$$G_{\mathrm{thread}} \equiv \overset{\star}{\underset{i \in \{1, \ldots, \ell_c\}}{\curlyvee}} \left(\overset{\star}{\underset{j \in \{1, \ldots, \mathrm{M}_i\}}{\curlyvee}} T_{i,j} \right) \in G(A, \partial\Omega).$$

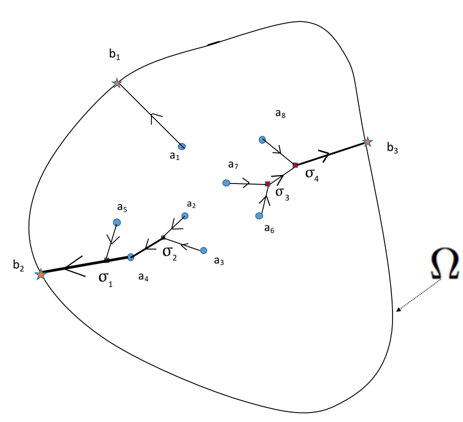

Figure 20: *Branched transport of the points a_i.*

Remark A.2. In the definition above, we allowed points in A to be on the boundary $\partial\Omega$: This, perhaps unnatural aspect of the definition, is motivated by the fact that we will face such a situation in Subsection A.4, and this convention simplifies somewhat the presentation. However, one may verify that

$$\mathcal{G}(A, \partial\Omega) = \mathcal{G}(A \setminus \partial\Omega, \partial\Omega). \tag{A.26}$$

A.2 The functional and minimal branched connections to the boundary

Given $0 \leq \alpha \leq 1$, we consider the functional W_α defined on the set $\mathcal{G}(\Omega)$

$$\mathrm{W}_\alpha(G) = \sum_{e \in E(G)} (d(e))^\alpha \mathcal{H}^1(e) \quad \text{for } G \in \mathcal{G}(\Omega). \tag{A.27}$$

and the non-negative quantity

$$\mathfrak{L}^\alpha_{\mathrm{brbd}}(A, \partial\Omega) = \inf \left\{ \mathrm{W}_\alpha(G), G \in \mathcal{G}(A, \partial\Omega) \right\}, \tag{A.28}$$

which we will term the branched connection of order α of the set A to the boundary $\partial\Omega$. Notice that the case $\alpha = 1$ has already been introduced in [11] as minimal connection to the boundary. Using, among other arguments, the fact that

$$\mathrm{W}_\alpha(G) \leq \mathrm{W}_\alpha(G_{\mathrm{thread}})$$

with equality if and only if T_0 in (A.21) is empty, it can be proved, as in [30]:

Lemma A.5. *The infimum in* (A.28) *is achieved by some graph* $G^\alpha_{\mathrm{opt}} \in \mathcal{G}(A, \partial\Omega)$*. Moreover* G^α_{opt} *has no loops and we may therefore write*

$$G^\alpha_{\mathrm{opt}} = G = \overset{\star}{\underset{a \in A}{\curlyvee}} \left(\overset{\star}{\underset{j \in \{1, \dots, \mathrm{M}(a)\}}{\curlyvee}} T_{a,j} \right) \text{ with } T_{a,j} \in \mathbb{T}_{\mathrm{hread}}(a, \Omega). \tag{A.29}$$

Moreover, we have $d(e) \leq \sharp(A)$ *for any* $e \in E(G^\alpha_{\mathrm{opt}})$.

We notice that, as a straightforward consequence of follow, we have

$$\mathfrak{L}^\alpha_{\mathrm{brbd}}(A, \partial\Omega) = \mathfrak{L}^\alpha_{\mathrm{brbd}}(A \setminus \partial\Omega, \partial\Omega). \tag{A.30}$$

We next show, similar to results in [30, 31, 4]:

Lemma A.6. *The graph* G^α_{opt} *possesses the single path property.*

Proof. We argue by contradiction and assume that there exists some vertex $\sigma_0 \in V(G)$ and two distinct vertices σ_1 and σ_2 in $V(G)$ such that $[\sigma_0, \sigma_i] \in E(G)$ for $i = 1, 2$. In view of the decomposition (A.29) we may find then two charges a_1 and a_2 in A such that, for $i = 1, 2$, the segment $[\sigma_0, \sigma_i]$ belongs to $E(T_i)$ where T_i is a thread of the form T_{a_i, j_i} appearing in (A.29). In the case the two threads have no vertex in common past the vertex σ_0, we may write them under the form

$$T_i = \mathrm{G}_{\mathrm{rp}}(B_i) = \mathrm{G}_{\mathrm{rp}}(B_{0,i}) \overset{\star}{\curlyvee} \mathrm{G}_{\mathrm{rp}}(B_{1,i}) \tag{A.31}$$

where $B_{0,i} = \{a_i, \dots, \sigma_0\}$ and $B_{1,i} = \{\sigma_0, \sigma_i, b_{i,2}, \dots, b_{i,\ell_i}\}$, with $b_{i,\ell_i} \in \partial\Omega$, the vertex σ_0 being the only common point of the sets $B_{1,1}$ and $B_{1,2}$. In the case the two threads have a common vertex b_{com} past σ_0 we write

$$T_i = \mathrm{G_{rp}}(B_i) = \mathrm{G_{rp}}(B_{0,i}) \overset{\star}{\curlyvee} \mathrm{G_{rp}}(B_{1,i}) \overset{\star}{\curlyvee} \mathrm{G_{rp}}(B_{2,i}) \tag{A.32}$$

where $B_{0,i}$ is as above and $B_{1,i} = \{\sigma_0, \sigma_i, b_{i,2}, \dots, b_{i,\ell_i} = b_{\mathrm{com}}\}$. In order to obtain a contradiction, we compare the energy of the graph G with the energy of two comparison graphs $\tilde{G}_1$ and $\tilde{G}_2$, which we construct next, and which corresponds, roughly speaking, to an interchange of the threads T_1 and T_2. We first consider the modified threads

$$\tilde{T}_1 = \mathrm{G_{rp}}(B_{0,1}) \overset{\star}{\curlyvee} \mathrm{G_{rp}}(B_{1,2}) \overset{\star}{\curlyvee} \mathrm{G_{rp}}(B_{2,1}) \text{ and } \tilde{T}_2 = \mathrm{G_{rp}}(B_{0,2}) \overset{\star}{\curlyvee} \mathrm{G_{rp}}(B_{1,1}) \overset{\star}{\curlyvee} \mathrm{G_{rp}}(B_{2,2}).$$

We then define

$$\tilde{G}_1 = (G \setminus T_1) \overset{\star}{\curlyvee} \tilde{T}_1 \text{ and } \tilde{G}_2 = (G \setminus T_2) \overset{\star}{\curlyvee} \tilde{T}_2.$$

One verifies that, for $i = 1, 2$, $\tilde{G}_i \in \mathcal{G}(A, \partial\Omega)$. For $i = 1, 2$ and $j = 0, \dots, \ell_i - 1$ we set $e_{i,j} \equiv [b_{i,j}, b_{i,j+1}]$ where $b_{i,0} = \sigma_0$ and $b_{i,1} = \sigma_i$, for $i = 1, 2$. We observe that

$$\begin{cases} d(e_{1,j}, \tilde{G}_1) = d_{1,j} - 1 \text{ for } j = 1, \dots, \ell_1 \text{ and } d(e_{2,j}, \tilde{G}_2) = d_{2,j} + 1 \text{ for } j = 1, \dots, \ell_2 \\ d(e_{1,j}, \tilde{G}_1) = d_{i,j} + 1 \text{ for } j = 1, \dots, \ell_1 \text{ and } d(e_{1,j}, \tilde{G}_2) = d_{2,j} - 1 \text{ for } j = 1, \dots, \ell_2, \end{cases}$$

where we have set $d_{i,j} = d(e_{i,j}, G)$. All other segments have the same density as for G. It follows :

$$\begin{cases} W_\alpha(\tilde{G}_1) - W_\alpha(G) = \displaystyle\sum_{j=0}^{\ell_1 - 1} \left[(d_{1,j} + 1)^\alpha - d_{1,j}^\alpha\right] |e_{1,j}| + \sum_{j=0}^{\ell_2 - 1} \left[(d_{2,j} - 1)^\alpha - d_{2,j}^\alpha\right] |e_{2,j}| \geq 0 \\ W_\alpha(\tilde{G}_2) - W_\alpha(G) = \displaystyle\sum_{j=0}^{\ell_1 - 1} \left[(d_{1,j} - 1)^\alpha - d_{1,j}^\alpha\right] |e_{1,j}| + \sum_{j=0}^{\ell_2 - 1} \left[(d_{2,j} + 1)^\alpha - d_{2,j}^\alpha\right] |e_{2,j}| \geq 0. \end{cases}$$

Adding these inequalities we obtain

$$\sum_{i=1}^{2} \sum_{j=0}^{\ell_i - 1} \left[(d_{i,j} + 1)^\alpha + (d_{i,j} - 1)^\alpha - 2d_{i,j}^\alpha\right] \geq 0.$$

By concavity of the density function $\phi(d) = d^\alpha$, we have however for $d \leq 1$

$$(d + 1)^\alpha + (d - 1)^\alpha - 2d^\alpha < 0$$

so that we have reached a contradiction which establishes the announced result. □

Remark A.3. Using simple comparison arguments, one may easily prove that if A and B are disjoint finite subsets of Ω then

$$\mathfrak{L}^\alpha_{\mathrm{brbd}}(A \cup B, \partial\Omega) \leq \mathfrak{L}^\alpha_{\mathrm{brbd}}(A, \partial\Omega) + \mathfrak{L}^\alpha_{\mathrm{brbd}}(B, \partial\Omega) \tag{A.33}$$

and if $0 < \alpha' \leq \alpha$ on $\Omega \subset \Omega'$ then we have

$$\mathfrak{L}^\alpha_{\mathrm{brbd}}(A, \partial\Omega) \leq \mathcal{L}^{\alpha'}_{\mathrm{branch}}(A, \partial\Omega) \text{ and } \mathfrak{L}^\alpha_{\mathrm{brbd}}(A, \partial\Omega) \leq \mathfrak{L}^\alpha_{\mathrm{brbd}}(A, \partial\Omega'). \tag{A.34}$$

Remark A.4. In our further analysis, we will be led to consider the case Ω is a polytope, $\bar{\Omega} = \Omega_1 \cup \Omega_2$, where $\Omega_1 \cap \Omega_2 = \emptyset$, Ω_1 and Ω_2 being polytopes. Given a graph $G \in \mathcal{G}^+(\Omega)$, one verifies that

$$W_\alpha(G) = W_\alpha(G_1) + W_\alpha(G_2), \text{ where } G_\mathfrak{p} = G \llcorner \Omega_\mathfrak{p} \text{ for } \mathfrak{p} = 1, 2. \tag{A.35}$$

Assume next that $G \in \mathcal{G}(A, \Omega)$, where $A \subset \Omega$ is a finite set. We have, for $\mathfrak{p} = 1, 2$

$$G_\mathfrak{p} \in \mathcal{G}(A_\mathfrak{p}, \Omega_\mathfrak{p}) \text{ so that } W_\alpha(G_\mathfrak{p}) \geq \mathfrak{L}^\alpha_{\text{brbd}}(A_\mathfrak{p}, \partial\Omega_\mathfrak{p}) \text{ where } G_\mathfrak{p} = G \llcorner \Omega_\mathfrak{p}. \tag{A.36}$$

In the next subsection, we will be mainly concerned with the asymptotic behavior of $\mathfrak{L}^\alpha_{\text{brbd}}(A, \partial\Omega)$ as the number of elements in A tends to $+\infty$, specially in the case they are equi-distributed. Our methods rely on various decomposition, as presented next.

A.3 Decomposing the domain and the graphs

We discuss here issues related to partitions of the domain Ω, assuming it is a polytope. We consider the case where the set Ω is decomposed as a finite union

$$\bar{\Omega} = \underset{\mathfrak{p} \in \mathfrak{P}}{\cup} \bar{\Omega}_\mathfrak{p}, \text{ where the sets } \Omega_\mathfrak{p} \text{ are disjoint polytopes i.e. } \Omega_\mathfrak{p} \cap \Omega'_\mathfrak{p} = \emptyset \text{ for } \mathfrak{p} \neq \mathfrak{p}'. \tag{A.37}$$

Given a finite subset A of Ω which does not intersect the boundaries $\partial\Omega_\mathfrak{p}$, we have the lower bound

$$\mathfrak{L}^\alpha_{\text{brbd}}(A, \partial\Omega) \geq \sum_{\mathfrak{p} \in \mathfrak{P}} \mathfrak{L}^\alpha_{\text{brbd}}(A_\mathfrak{p}, \partial\Omega_\mathfrak{p}) \text{ where } A_\mathfrak{p} = \bar{\Omega}_\mathfrak{p} \cap A. \tag{A.38}$$

Indeed, if G is a graph in $\mathcal{G}(A, \partial\Omega)$, then the restriction $G_\mathfrak{p}$ to the subset $\Omega_\mathfrak{p}$ belongs to $\mathcal{G}(A_\mathfrak{p}, \partial\Omega_\mathfrak{p})$. On the other hand, we have

$$\mathrm{W}_\alpha(G) = \sum_{\mathfrak{p} \in \mathfrak{P}} \mathrm{W}_\alpha(G_\mathfrak{p}),$$

from which the conclusion (A.38) is deduced. We assume next that $\mathfrak{P} = \{1, 2\}$, that is $\bar{\Omega} = \Omega_1 \cup \Omega_2$, where $\Omega_1 \cap \Omega_2 = \emptyset$, Ω_1 and Ω_2 being polytopes. Our next result, is an improvement of (A.38) for this case.

Proposition A.3. *Assume that $\mathfrak{P} = \{1, 2\}$. Then, we have the lower bound*

$$\mathfrak{L}^\alpha_{\text{brbd}}(A, \partial\Omega) \geq \mathfrak{L}^\alpha_{\text{brbd}}(A_1, \partial\Omega_1) + \mathfrak{L}^\alpha_{\text{brbd}}(A_2, \partial\Omega_2) + \kappa_\alpha \frac{\sharp(A_1)}{\sharp(A)} (\mathrm{N_{el}})^\alpha \text{dist}\,(\Omega_1, \partial\Omega), \tag{A.39}$$

where $\kappa_\alpha > 0$ is some constant depending only on α and where $\mathrm{N_{el}} = \sharp(A)$ denotes the number of elements in A.

The previous result is obviously only of interest in the case $\text{dist}\,(\Omega_1, \partial\Omega) \neq 0$, that is when $\bar{\Omega}_1 \subset \Omega$. The proof involves concavity properties, in particular the next elementary result.

Lemma A.7. *Let $0 < \alpha \leq 1$, $a \geq 1$ and $b \geq 1$ be two given numbers. There exists some universal constant $\kappa_\alpha > 0$ depending only on α such that*

$$(a + b)^\alpha \geq a^\alpha + \kappa_\alpha \inf\{b^\alpha, b\, a^{\alpha-1}\}.$$

Proof of Lemma A.7. We distinguish three cases.

Case 1: $b \leq \frac{a}{2}$. We rely on the Taylor expansion of the expression $(1+s)^\alpha$ for s in $(0,1]$ which leads to the concavity estimate

$$(1+s)^\alpha \geq 1 + \alpha s + \frac{1}{2}\alpha(\alpha-1)s^2 \geq 1 + \frac{1}{2}\alpha s(1+(\alpha-1)s) \geq 1 + \frac{1}{2}\alpha s(1-s). \tag{A.40}$$

We apply (A.40) with $s = \frac{b}{a} \leq \frac{1}{2}$, so that $1 - s \geq \frac{1}{2}$, leading to the inequality

$$(a+b)^\alpha \geq a^\alpha(1 + \frac{1}{4}\alpha s) \geq a^\alpha + \frac{1}{4}\,\alpha b\, a^{\alpha-1}. \tag{A.41}$$

Case 2: $8a \geq b \geq \frac{1}{2}a$. In this case, we obtain invoking (A.40) once more

$$(a+b)^\alpha \geq (\frac{3}{2}a)^\alpha \geq (1+\frac{1}{8}\alpha)a^\alpha \geq a^\alpha + \frac{1}{8}\alpha\left(\frac{b}{8}\right)^\alpha \geq a^\alpha + \alpha\left(\frac{1}{8}\right)^{\alpha+1} b^\alpha. \tag{A.42}$$

Case 3: $8a \leq b$. In this case we write

$$(a+b)^\alpha \geq b^\alpha \geq \frac{1}{8^\alpha}b^\alpha + (1-\frac{1}{8^\alpha})b^\alpha \geq a^\alpha + (1-\frac{1}{8^\alpha})b^\alpha. \tag{A.43}$$

We set $\kappa_\alpha = \inf\{\alpha/4, \alpha\,(1/8)^{\alpha+1}, (1-1/8^\alpha)\}$. Combining (A.41), (A.42) and (A.43) in the three cases, we complete the proof of the lemma. □

We use Lemma A.7 in the case we have the additional assumption

$$a + b \leq \mathrm{N}_{\mathrm{ber}}, \tag{A.44}$$

where $\mathrm{N}_{\mathrm{ber}} \gg 1$ is some large number. It follows from (A.44) that $b^\alpha \geq b(\mathrm{N}_{\mathrm{ber}})^{\alpha-1}$ and $a^{\alpha-1} \geq (\mathrm{N}_{\mathrm{ber}})^{\alpha-1}$ so that in this case, (A.41) leads to the inequality

$$(a+b)^\alpha \geq a^\alpha + \kappa_\alpha\, b(\mathrm{N}_{\mathrm{ber}})^{\alpha-1}, \tag{A.45}$$

and hence the right hand side of (A.45) *behaves linearily* with respect to b.

Proof of Proposition A.3. As mentioned, we may assume that $\bar{\Omega}_1 \subset \Omega$, since otherwise the result (A.39) is a immediate consequence of (A.38). In this situation we have $\Omega_2 = \Omega\backslash\bar{\Omega}_1$. Let G^α_{opt} be an optimal graph for $\mathfrak{L}^\alpha_{\mathrm{brbd}}(A, \partial\Omega)$. We amuse for simplicity that all multiplicities in A are equal to one. We proceed first with a spatial decomposition of this graph, introducing the subgraphs $G_\mathfrak{p} = G \llcorner \Omega_\mathfrak{p}$. Going back to Remark A.4, we have

$$W_\alpha(G) = W_\alpha(G_1) + W_\alpha(G_2) \geq \mathfrak{L}^\alpha_{\mathrm{brbd}}(A_1, \partial\Omega_1) + W_\alpha(G_2). \tag{A.46}$$

To estimate $W_\alpha(G_2)$, we rely on the next Lemma:

Lemma A.8. *We have the improved lower bound for* $W_\alpha(G_2)$

$$W_\alpha(G_2) \geq \mathfrak{L}^\alpha_{\text{brbd}}(A_2, \partial\Omega_2) + \kappa_\alpha \frac{\sharp(A_1)}{\sharp(A)} (\mathrm{N_{el}})^\alpha \mathrm{dist}\,(\Omega_1, \partial\Omega). \tag{A.47}$$

where $\alpha > 0$ *is the constant provided by Lemma A.7.*

Combining Lemma (A.8) with inequality (A.46), we obtain inequality (A.39) which completes the proof of Proposition A.3. □

Proof of Lemma A.8. In view of Lemma A.5 we may decompose G^α_{opt} as in (A.29), so that we may decompose the graph G_2 as $G_2 = G_{2,1} \overset{\star}{\curlyvee} G_{2,2}$ with for $\mathfrak{q} = 1, 2$

$$G_{2,\mathfrak{q}} = \left(\curlyvee_{a \in A_\mathfrak{q}} T_a \right) \llcorner \Omega_2, \text{ where the threads } T_a \in \mathbb{T}_{\text{hread}}(a, \Omega) \text{ satisfy (A.6).}$$

We notice that $G_{2,2} \in \mathcal{G}(A_2, \Omega_2)$, so that

$$W_\alpha(G_{2,2}) \geq \mathfrak{L}^\alpha_{\text{brbd}}(A_2, \partial\Omega_2), \tag{A.48}$$

whereas $G_{2,1} \in \mathcal{G}_0(\Omega_2)$. Given a segment e of the graph G_2, we denote by $d_{2,2}(e)$ (resp. $d_{1,2}(e)$) its multiplicity according to the graph $G_{2,1}$ (resp $G_{2,2}$), with the convention that $d_{2,1}(e) = 0$ (resp. $d_{2,2}(e) = 0$) if the segment does not belong to $E(G_{2,1})$ (resp. $E(G_{2,2})$). It follows from the last statement in Lemma A.5 that

$$d(E, G) = d(E, G_2) = d_{2,2}(e) + d_{2,1}(e) \leq \mathrm{N_{el}}, \tag{A.49}$$

and the definition of W_α leads to the identity

$$W_\alpha(G_2) = W_\alpha(G_{2,2} \curlyvee G_{1,2}) = \sum_{e \in E(G_2)} (d_{2,2}(e) + d_{2,1}(e))^\alpha \mathcal{H}^1(e).$$

We split the remaining of the proof into three steps.

Step 1. *We have the lower bound*

$$W_\alpha(G_2) = W_\alpha(G_{2,2} \curlyvee G_{2,1}) \geq W_\alpha(G_{2,2}) + \kappa_\alpha \, (\mathrm{N_{el}})^{\alpha-1} \sum_{e \in \tilde{E}(G_2)} d_{2,1}(e) \mathcal{H}^1(e), \tag{A.50}$$

Proof of (A.50). We invoke next inequality (A.45) of Lemma A.7 with $\mathrm{N_{ber}} = \mathrm{N_{el}}$, $a = d_{2,2}(e)$ and $b = d_{2,1}(e)$. Since (A.49) yields (A.44) in the case considered, we obtain

$$\begin{aligned} W_\alpha(G_{2,2} \curlyvee \ G_{1,2}) &\geq \sum_{e \in E(G_2)} \left(d_{2,2}(e)^\alpha + \kappa_\alpha d_{2,1}(e) \, (\mathrm{N_{el}})^{\alpha-1} \right) \mathcal{H}^1(e) \\ &\geq \sum_{e \in E(\tilde{G}_2)} d_{2,2}(e)^\alpha \mathcal{H}^1(e) + \kappa_\alpha \, (\mathrm{N_{el}})^{\alpha-1} \sum_{e \in \tilde{E}(G_2)} d_{2,1}(e) \mathcal{H}^1(e). \end{aligned} \tag{A.51}$$

Since, by definition, we have $W_\alpha(G_{2,2}) = \sum_{e \in E(\tilde{G}_2)} d_{2,2}(e)^\alpha \mathcal{H}^1(e)$, we obtain (A.50).

Step 2. We have the lower bound

$$\sum_{e\in E(G_2)} d_{2,1}(e)\mathcal{H}^1(e) \geq \sharp(A_1)\mathrm{dist}(\Omega_1, \partial\Omega). \tag{A.52}$$

Proof of (A.52). We take advantage of the linearity of the l.h.s with respect to multiplicity. Indeed, we notice that

$$\sum_{e\in E(G_2)} d_{2,1}(e)\mathcal{H}^1(e) = \sum_{a\in A_1} \mathcal{H}^1(\mathcal{C}_a \cap \Omega_2),$$

where $\mathcal{C}_a$ denotes the polygonal curve related to the thread T_a. Since any thread T_a, joins a point in Ω_1 to the boundary $\partial\Omega$, we have

$$H^1(\mathcal{C}_a \cap \Omega_2) \geq \mathrm{dist}(\Omega_1, \partial\Omega),$$

so that the conclusion (A.52) follows combining the two previous relations.

Step 3. *Proof of Lemma A.8 completed.* Combining the lower-bound (A.50), (A.52) with (A.48), we derive the lower bound (A.47), which completes the proof of Lemma A.8.

□

A.4 Estimates for minimal branched connections

An important observation made[18] in Xia is:

Proposition A.4. *Assume that $\alpha \in (\alpha_m, 1]$, where $\alpha_m = 1 - \frac{1}{m}$. Then we have, for some constant $C(\Omega, \alpha)$ depending only on Ω and α,*

$$\mathfrak{L}^{\alpha}_{\mathrm{brbd}}(A, \partial\Omega) \leq C(\Omega, \alpha) \left(\sharp(A)\right)^{\alpha}, \tag{A.53}$$

The proof is obvious for $\alpha = 1$. Indeed in this case, one may obtain an upper bound for $\mathcal{L}^1_{\mathrm{branch}}(A, \partial\Omega)$ estimating $W_1(G_0)$ where G_0 is constructed as in subsection A.1 connecting each point in A to its nearest point on the boundary. We obtain

$$W_1(G_0) \leq \mathrm{diam}(\Omega) \left(\sharp(A)\right),$$

yielding the result in the case considered. In the case $\alpha_m \leq \alpha < 1$, estimate (A.53) yields an improvement on the growth in terms of $\sharp A$. This is achieved in [30] replacing the elementary comparison graph G_0 by graphs having branching points obtained through a dyadic decomposition.

Remark A.5. The result of Proposition (A.4) is optimal in the sense that one may find simple distributions of points for which the asymtotic behavior in of order $\left(\sharp(A)\right)^{\alpha}$,

[18] Here we refer to Proposition 3.1 in [30]. Although the statement there is slightly different from ours, the reader may easily adapt the proof.

A.5 The case of a uniform grid

We next focus on behavior of $\mathfrak{L}^{\alpha}_{\mathrm{brbd}}$ in the special case Ω is the m-dimensional unit cube that is $\Omega = (0,1)^m$ and the points of A are located on an uniform grid. We consider therefore for an integer k in $\mathbb{N}^*$ the distance $h = \frac{1}{\mathrm{k}}$ and the set of points

$$\mathbf{A}^{\mathrm{k}}_m \equiv \boxplus^{\mathrm{k}}_m(h) = \left\{ a^k_{\mathrm{I}} \equiv h\, I = h\,(i_1, i_2, \ldots, i_m), \text{ for } I \in \{1, \ldots, \mathrm{k}\}^m \right\},$$

so that $\sharp(\mathbf{A}^{\mathrm{k}}_m) = \mathrm{k}^m$. Notice that $\mathbf{A}^{\mathrm{k}}_m \cap \partial\left((0,1)^m\right) \neq \emptyset$ (see Remark A.2). We set

$$\Lambda^{\alpha}_m(\mathrm{k}) = \mathfrak{L}^{\alpha}_{\mathrm{brbd}}(\mathbf{A}^{\mathrm{k}}_m, \partial(0,1)^m) \text{ and } \Lambda^{m,\alpha}_{\mathrm{norm}}(\mathrm{k}) \equiv \mathrm{k}^{-m\alpha}\Lambda^{\alpha}_m(\mathrm{k})$$

and are interested in the asymptotic behavior of the quantities $\Lambda^{\alpha}_m(\mathrm{k})$ and $\Lambda^{m,\alpha}_{\mathrm{norm}}(\mathrm{k})$ as $\mathrm{k} \to +\infty$. We observe first that it follows from Proposition A.4 that, if $\alpha < \alpha_m$ then, we have the upper bound

$$\Lambda^{\alpha}_m(\mathrm{k}) \leq C_{\alpha}\mathrm{k}^{m\alpha} \text{ i.e. } \Lambda^{m,\alpha}_{\mathrm{norm}}(\mathrm{k}) \leq C_{\alpha}, \tag{A.54}$$

where the constant $C_{\alpha} > 0$ does not depend on k. In the critical case $\alpha = \alpha_m$, the upper bound (A.54) no longer holds as our next result shows.

Theorem A.1. *There exists some constant* $\mathrm{C}_m > 0$ *such that for all* $\mathrm{k} \in \mathbb{N}^*$*, we have the lower bound*

$$\Lambda^{\alpha_m}_m(\mathrm{k}) \geq \mathrm{C}_m\mathrm{k}^{m\alpha_m}\log\mathrm{k} = \mathrm{C}_m\mathrm{k}^{m-1}\log\mathrm{k},$$

that is

$$\Lambda^{m,\alpha_m}_{\mathrm{norm}}(\mathrm{k}) \geq \mathrm{C}_m\log\mathrm{k}.$$

Remark A.6. The fact that the quantity $\Lambda^{m,\alpha_m}_{\mathrm{norm}}(\mathrm{k}) = \mathrm{k}^{1-m}\Lambda^{\alpha_m}_m(\mathrm{k})$ does not remain bounded as $\mathrm{k} \to +\infty$ is related to and may also presumably be deduced from the fact that the Lebesgue measure *is not irrigible* for the critical value $\alpha = \alpha_m$, a result proved in [12] (see also [4]).

The proof of Theorem A.1 will rely on several preliminary results we present first, starting with elementary scaling laws. Let $\mathrm{q} \in \mathbb{N}^*$ be given, and consider for $\mathrm{k} \in \mathbb{N}$ the set

$$\frac{1}{\mathrm{q}}\mathbf{A}^k_m = \mathbf{A}^{\mathrm{q}k}_m \cap [0, \frac{1}{\mathrm{q}}]^m = \boxplus^{\mathrm{k}}_m(\frac{h}{\mathrm{q}}) = \left\{ a^k_{\mathrm{I}} \equiv \frac{1}{\mathrm{qk}}\, I, I \in \{1, \ldots, \mathrm{k}\}^m \right\},$$

so that $\frac{1}{\mathrm{q}}\mathbf{A}^k_m$ contains k^m elements. The scaling law writes as

$$\mathfrak{L}^{\alpha}_{\mathrm{brbd}}\left(\frac{1}{\mathrm{q}}\mathbf{A}^k_m, \partial\left([0, \frac{1}{\mathrm{q}}]^m\right)\right) = \mathrm{q}^{-1}\mathfrak{L}^{\alpha}_{\mathrm{brbd}}\left(\mathbf{A}^k_m, \partial\left([0,1]^m\right)\right) = \mathrm{q}^{-1}\Lambda^{\alpha}_m(\mathrm{k}). \tag{A.55}$$

The main ingredient in the proof of Theorem A.1 is a consequence of Proposition A.3:

Lemma A.9. *Let* $\mathrm{q} \in \mathbb{N}^*$ *be given. There exists some constant* $\mathrm{C}^{\alpha}_{\mathrm{q}} > 0$ *such that*

$$\Lambda^{m,\alpha}_{\mathrm{norm}}(\mathrm{q\,k}) \geq \mathrm{q}^{m(\alpha_m - \alpha)}\,\Lambda^{m,\alpha}_{\mathrm{norm}}(\mathrm{k}) + \mathrm{C}^{\alpha}_{\mathrm{q}}, \quad \text{for any } \mathrm{k} \in \mathbb{N}^*.$$

Proof. we consider the set $\mathbf{A}_m^{\mathrm{q}k}$ and decompose the domain $\overline{\Omega} = [0,1]^m$ as an union of cubes $\mathrm{Q}_{\mathbf{J}}$ with $\mathbf{J} \equiv (j_1, j_2, \dots, j_m) \in \mathfrak{J} \equiv \{0, \dots, \mathrm{q}-1\}^m$ and

$$\mathrm{Q}_{\mathbf{J}} = \frac{1}{\mathrm{q}}\mathbf{J} + (0, \frac{1}{\mathrm{q}})^m = \frac{1}{\mathrm{q}}(j_1, j_2, \dots, j_m) + (0, \frac{1}{\mathrm{q}})^m,$$

so that $\mathrm{Q}_{\mathbf{J}} \cap \mathrm{Q}_{\mathbf{J}'} \neq \emptyset$ if $\mathbf{J} \neq \mathbf{J}'$ and $[0,1]^m = \underset{\mathbf{J}\in\mathfrak{J}}{\cup} \bar{\mathrm{Q}}_{\mathbf{J}}$. We set

$$A_{\mathbf{J}} \equiv A_m^{\mathrm{qk}} \cap \mathrm{Q}_{\mathbf{J}}, \text{ so that } A_{\mathfrak{p}} = \frac{1}{\mathrm{q}}\mathbf{J} + \frac{1}{\mathrm{q}}\mathbf{A}_m^k.$$

It follows from the scaling law (A.55) and translation invariance that

$$\mathfrak{L}_{\mathrm{brbd}}^{\alpha}(A_{\mathbf{J}}, \partial \mathrm{Q}_{\mathbf{J}}) = \mathrm{q}^{-1}\Lambda_m^{\alpha}(\mathrm{k}) \text{ for } \mathbf{J} \in \mathfrak{J}. \tag{A.56}$$

We next single out a cube $\mathrm{Q}_{\mathbf{J}_0}$ which is far from the boundary. For that purpose, we consider the integer $\mathrm{q}_0 \equiv \left[\frac{\mathrm{q}}{2}\right]$, the multi-index $\mathbf{J}_0 = (\mathrm{q}_0, \mathrm{q}_0, \dots, \mathrm{q}_0)$ and the sets

$$\Omega_1 = \mathrm{Q}_{\mathbf{J}_0} \text{ and } \Omega_2 = \underset{\mathbf{J}\in\mathfrak{J}\setminus\{\mathbf{J}_0\}}{\cup} \mathrm{Q}_{\mathbf{J}} \text{ so that } \mathrm{dist}(\Omega_1, \partial\Omega) \geq \frac{1}{4} \text{ for } \mathrm{q} \geq 3.$$

Applying inequality (A.39) of Proposition A.53, we are led to

$$\begin{aligned}\Lambda_m^{\alpha}(\mathrm{qk}) = \mathfrak{L}_{\mathrm{brbd}}^{\alpha}(\mathbf{A}_m^{\mathrm{qk}}, \partial(0,1)^m) \geq &\mathfrak{L}_{\mathrm{brbd}}^{\alpha}(A_{\mathbf{J}_0}, \partial \mathrm{Q}_{J_0}) + \mathfrak{L}_{\mathrm{brbd}}^{\alpha}(\Omega_2 \cap \mathbf{A}_m^{\mathrm{qk}}, \partial(0,1)^m) \\ &+ \frac{1}{4}\kappa_{\alpha}\mathrm{k}^{m\alpha}\mathrm{q}^{m(\alpha-1)}.\end{aligned} \tag{A.57}$$

We deduce from inequality (A.38) and (A.55) that

$$\mathfrak{L}_{\mathrm{brbd}}^{\alpha}(\Omega_2 \cap A_m^{\mathrm{qk}}, \partial(0,1)^m) \geq \sum_{\mathbf{J}\in\mathfrak{J}\setminus\{\mathbf{J}_0\}} \mathfrak{L}_{\mathrm{brbd}}^{\alpha}(A_{\mathbf{J}}, \partial \mathrm{Q}_{\mathbf{J}}) = [\mathrm{q}^m - 1]\,\mathrm{q}^{-1}\Lambda_m^{\alpha}(\mathrm{k}). \tag{A.58}$$

Combining (A.57), (A.58) and (A.55) again for $\mathbf{J} = \mathbf{J}_0$, we are led to the lower bound

$$\Lambda_m^{\alpha}(\mathrm{qk}) \geq \mathrm{q}^{m-1}\Lambda_m^{\alpha}(\mathrm{k}) + \frac{1}{4}\kappa_{\alpha}\mathrm{k}^{m\alpha}\mathrm{q}^{m(\alpha-1)}.$$

Multiplying both sides by $(\mathrm{qk})^{-m\alpha}$, we obtain the desired result with $\mathrm{C}_{\mathrm{q}}^{\alpha} = \frac{1}{4}\kappa_{\alpha}\mathrm{q}^{-1}$. □

Lemma A.10. *We have for any integer* $1 \leq \mathrm{k}' \leq \mathrm{k}$

$$\Lambda_m^{\alpha}(\mathrm{k}') \leq \frac{\mathrm{k}}{\mathrm{k}'}\Lambda_m^{\alpha}(\mathrm{k}) \quad \text{and hence} \quad \Lambda_{\mathrm{norm}}^{m,\alpha}(\mathrm{k}') \leq \left(\frac{\mathrm{k}}{\mathrm{k}'}\right)^{m\alpha+1} \Lambda_{\mathrm{norm}}^{m,\alpha}(\mathrm{k}).$$

Proof. consider the cube $\mathcal{Q}'_{\mathrm{k}} = (0, \frac{\mathrm{k}'}{\mathrm{k}})^m \subset (0,1)^m$ and the set $A' = \mathbf{A}_m^{\mathrm{k}} \cap \mathcal{Q}'_{\mathrm{k}}$. It follows from inequality (A.38) that

$$\mathfrak{L}_{\mathrm{brbd}}^{\alpha}(A', \partial\mathcal{Q}'_{\mathrm{k}}) \leq \mathfrak{L}_{\mathrm{brbd}}^{\alpha}(A_m^{\mathrm{k}}, \partial(0,1)^m) = \Lambda_m^{\alpha}(\mathrm{k}),$$

whereas the scaling property yields

$$\mathfrak{L}_{\mathrm{brbd}}^{\alpha}(A', \partial\mathcal{Q}'_{\mathrm{k}}) = \frac{\mathrm{k}}{\mathrm{k}'}\Lambda_m^{\alpha}(\mathrm{k}').$$

The conclusion follows combining the previous inequalities. □

Proof of Theorem A.1 completed. In the special case $\alpha = \alpha_m$, the exponent of q in the r.h.s of the inequality of Lemma A.9 vanishes, so that we obtain

$$\Lambda^{m,\alpha_m}_{\rm norm}({\rm q\,k}) \geq \Lambda^{m,\alpha_m}_{\rm norm}({\rm k}) + {\rm C}^{\alpha_m}_{\rm q}, \text{ for any integer } {\rm q} \geq 3.$$

Iterating this lower bound, we obtain, for any any integer $\ell > 0$, to the lower bound

$$\Lambda^{m,\alpha_m}_{\rm norm}({\rm q}^\ell) \geq {\rm C}^{\alpha_m}_{\rm q}\ell. \tag{A.59}$$

On the other hand, it follows from Lemma A.10 that for any ${\rm q}^\ell \leq {\rm k} \leq \mathfrak{q}^{\ell+1}$ we have

$$\Lambda^{m,\alpha_m}_{\rm norm}({\rm k}) \geq {\rm q}^{-m}\ \Lambda^{m,\alpha_m}_{\rm norm}({\rm q}^\ell). \tag{A.60}$$

so that, combining with (A.59) we deduce that, for any ${\rm k} \in \mathbb{N}^*$, we obtain the inequality

$$\Lambda^{m,\alpha_m}_{\rm norm}({\rm k}) \geq {\rm q}^{-m}{\rm C}^{\alpha_m}_{\mathfrak{q}}\left[\frac{\log {\rm k}}{\log {\rm q}}\right],$$

which leads immediately to the conclusion, fixing the value of q for instance ${\rm q} = 5$. □

Remark A.7. For $\alpha < \alpha_m$ the same type of argument show that

$$\Lambda^{m,\alpha}_{\rm norm}({\rm k}) \to +\infty \text{ as } {\rm k} \to +\infty.$$

A.6 Relating $\mathfrak{L}^\alpha_{\rm brbd}$ and $\mathfrak{L}^\alpha_{\rm branch}$

We introduce here the possibility of having points with negative charges, and consider as in the introduction a collection of points $(P_i)_{i\in J^+}$ in Ω with positive charge $+1$, a collection of points $(N_j)_{j\in J^-}$ in Ω, with negative charge -1. We define the set $\mathcal{G}(P_i, N_i, \Omega)$ of graph satisfying

$$\left(\underset{i\in J^+}{\cup}\{N_i\}\right) \cup \left(\underset{j\in J^-}{\cup}\{N_j\}\right) \subset V(G) \subset \bar{\Omega}.$$

and with (A.23) replaced by a modified version including the possibility of having negative charges. For $0 \leq \alpha \leq 1$ we set

$$\mathfrak{L}^\alpha_{\rm brbd}(P_i, N_i, \partial\Omega) = \inf\left\{{\rm W}_\alpha(G), G \in \mathcal{G}\left(\{P_i\}_{i\in J^+}, \{N_j\}_{j\in J^-}, \Omega\right)\right\}, \tag{A.61}$$

where the functional ${\rm W}_\alpha(G)$ is the weighted length of the graph connection defined by

$${\rm W}_\alpha(G) = \sum_{e\in E(G)} (d(e))^\alpha\, \mathcal{H}^1(e) \text{ for } G \in \mathcal{G}\left(\{P_i\}_{i\in J}, \{N_i\}_{i\in J}, \Omega\right).$$

Next assume that we are given a family $(\Omega_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{P}}$ of disjoint domains in $\mathbb{R}^m$, that is satisfying $\Omega_\mathfrak{p} \cap \Omega_{\mathfrak{p}'} = \emptyset$ if $\mathfrak{p} \neq \mathfrak{p}'$. We assume moreover that

$$\underset{i\in J^-}{\cup}\{N_i\} \cap \Omega_\mathfrak{p} = \emptyset \text{ and set } \mathfrak{A}_\mathfrak{p} = \{P_i\}_{i\in J} \cap \Omega_\mathfrak{p}. \tag{A.62}$$

Lemma A.11. *If* (A.62) *is satisfied, then we have the inequality*

$$\mathfrak{L}^\alpha_{\rm brbd}(P_i, N_j, \Omega) \geq \sum_{\mathfrak{p}\in\mathfrak{P}} \mathfrak{L}^\alpha_{\rm brbd}(\mathfrak{A}_\mathfrak{p}, \partial\Omega_\mathfrak{p}).$$

Proof. Let G be a graph in $\mathcal{G}\left(\{P_i\}_{i\in J^+}, \{N_i\}_{i\in J^-}, \Omega\right)$ and set $G_\mathfrak{p} = G \cap \Omega_\mathfrak{p}$. Since there are no negative charges in $\Omega_\mathfrak{p}$, It turns out that $G_\mathfrak{p} \in \mathcal{G}(\mathfrak{A}_\mathfrak{p}, \partial\Omega_\mathfrak{p})$, so that

$$\mathrm{W}_\alpha(G_\mathfrak{p}) \geq \mathfrak{L}^\alpha_{\mathrm{brbd}}(\mathfrak{A}_\mathfrak{p}, \partial\Omega_\mathfrak{p}).$$

On the other hand, we have $\mathrm{W}_\alpha(G) \geq \sum_{\mathfrak{p}\in\mathfrak{P}} \mathrm{W}_\alpha(G_\mathfrak{p})$ so that the conclusion follows. □

Notice that, in the case $\Omega = \mathbb{R}^m$, then $\mathfrak{L}^\alpha_{\mathrm{brbd}}\left(P_i, Q_i, \mathbb{R}^m\right) = +\infty$, except in the case $\sharp(J^+) = \sharp(J^-)$, i.e. there are the same number of $+1$ charges as -1 charges. In that case, we may choose $J^+ = J^- \equiv J$ and set

$$\mathfrak{L}^\alpha_{\mathrm{branch}}\left(\, P_i, Q_i\right) = \mathfrak{L}^\alpha_{\mathrm{brbd}}\left(P_i, Q_j, \mathbb{R}^m\right)$$

which is in the case $m = 4$ and $\alpha = \frac{3}{4} = \alpha_4$, is related to functional $\mathrm{L}_{\mathrm{branch}}$ presented in the Introduction.

References

[1] F.Almgreen, F.Browder and E.Lieb *Co-area, liquid crystals, and minimal surfaces*, in Partial differential equations (Tianjin, 1986), Springer (1988), 1–22.

[2] G.Alberti, S.Baldo and G.Orlandi *Variational convergence for functionals of Ginzburg-Landau type* Indiana Univ. Math. J. **54** (2005), 1411–1472.

[3] D.Auckly and L. Kapitanski, *The Pontrjagin-Hopf invariants for Sobolev maps*, Commun. Contemp. Math **12** (2010), 121–181.

[4] M.Bernot, V.Caselles, and J.M.Morel, *Optimal transportation networks. Models and theory.* Springer (2009). 200 p.

[5] F. Bethuel, *A characterization of maps in $H^1(\mathbb{B}^3, \mathbb{S}^2)$ which can be approximated by smooth maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (1990), 269–286.

[6] F. Bethuel, *The approximation problem for Sobolev maps between two manifolds*, Acta Math. **167** (1991), 153–206.

[7] F. Bethuel, H. Brezis and J.M. Coron, *Relaxed energies for harmonic maps*, *in* Variational methods (Paris, 1988), Springer (1990), 37–52.

[8] F.Bethuel and D. Chiron, *Some questions related to the lifting problem in Sobolev spaces*, *in* Perspectives in nonlinear partial differential equations, Contemp. Math., **446** (2007), 125–152.

[9] F. Bethuel, J.M. Coron, F.Demengel and F. Hélein, *A cohomological criterion for density of smooth maps in Sobolev spaces between two manifolds*, *in* Nematics (Orsay, 1990), Kluwer Acad. (1991) 15–23.

[10] F. Bethuel and X. Zheng, *Density of smooth functions between two manifolds in Sobolev spaces*, J. Funct. Anal. **80** (1988), 60–75.

[11] H.Brezis, J.M. Coron, and E.H. Lieb, *Harmonic maps with defects*, Comm. Math. Phys. **107** (1986), 649–705.

[12] G.Devillanova and S.Solimini, *On the dimension of an irrigable measure*, Rend. Semin. Mat. Univ. Padova **117** (2007), 1–49.

[13] M.Giaquinta, G.Modica and J. Soucek, *The Dirichlet energy of mappings with values into the sphere*, Manuscripta Math. **65** (1989), 489–507.

[14] M.Giaquinta, G.Modica and J. Soucek,*Cartesian currents in the calculus of variations. I. Cartesian currents*, Springer (1998).

[15] P.Hajlasz, *Approximation of Sobolev mappings*, Nonlinear Anal. **22** (1994), 1579–1591.

[16] F.Hang and F.H. Lin, *Topology of Sobolev mappings*, Math. Res. Lett. **8** (2001), 321–330.

[17] F.Hang and F. H.Lin, *Topology of Sobolev mappings II*, Acta Math. **191** (2003), 55–107.

[18] F.Hang and F. H.Lin, *Topology of Sobolev mappings III*, Comm. Pure Appl. Math.**56** (2003), 13831415.

[19] R.Hardt and T. Rivière, *Connecting topological Hopf singularities*, Ann. Sc. Norm. Super. Pisa Cl. Sci.**2** (2003),287–344.

[20] R.Hardt and T. Rivière, *Ensembles singuliers topologiques dans les espaces fonctionnels entres variétes* Séminaire EDP, Ecole Polytechnique (2001-2002), 14p.

[21] R.Hardt and T. Rivière, *Connecting rational homotopy type singularities*, Acta Math. **200** (2008), 15–83.

[22] R.Hardt and T. Rivière, *Sequential weak approximation of maps of finite Hessian energy*, preprint (2014).

[23] T. Isobe, *Characterization of the strong closure of* $C^\infty(\mathbb{B}^4; S^2)$ *in* $W^{1,p}(\mathbb{B}^4, \mathbb{S}^2)$ *for* $\frac{16}{5}p <$ 4. J. Math. Anal. Appl.**190** (1995), 361–372.

[24] A. Kosinski, *Differential manifolds*, Pure and Applied Mathematics, Academic Press, (1993).

[25] M, Monastyrsky, *Topology of gauge fields and condensed matter*, Plenum Press, New York, (1993)

[26] M.Pakzad and Rivière, *Weak density of smooth maps for the Dirichlet energy between manifolds*, Geom. Funct. Anal. **13** (2003), 223–257.

[27] L. Pontrjagin, A classification of mappings of the three-dimensional complex into the two-dimensional sphere, Rec. Math. [Mat. Sbornik] N.S. 9(51) (1941) 331–363.

[28] T.Rivière, *Minimizing fibrations and p-harmonic maps in homotopy classes from* $\mathbb{S}^3$ *into* $\mathbb{S}^2$. Comm. Anal. Geom. **6** (1998), 427–483.

[29] R.Schoen and K.Uhlenbeck, *Boundary regularity and the Dirichlet problem for harmonic maps*, J. Differential Geom. **18** (1983), 253–268.

[30] Q.Xia, *Optimal paths related to transport problems*, Commun. Contemp. Math.**5** (2003), 251–279.

[31] Q.Xia and A.Vershynina, *On the transport dimension of measures*, SIAM J. Math. Anal. **41** (2009/10) 2407–2430.